Homology, Homotopy and Applications, vol. 22(1), 2020, pp.333–341

NONTRIVIAL EXAMPLE OF THE COMPOSITION OF THE BRANE PRODUCT AND COPRODUCT ON GORENSTEIN SPACES

SHUN WAKATSUKI

(*communicated by Charles A. Weibel*)

Abstract

We give an example of a space with nontrivial composition of the brane product and the brane coproduct, which we introduced in a previous article.

1. Introduction

In this article, we give an example where the composition $\mu \circ \delta$ of the brane product and the brane coproduct is nontrivial.

Theorem 1.1. *Let k be a positive even integer. Let M be the Eilenberg–MacLane space $K(\mathbb{Z}, 2n)$ with $n > k/2$. Then the composition $\mu_{S^k} \circ \delta_{S^k}$ of the S^k-brane product*

$$\mu_{S^k} : H_*(\mathrm{Map}(S^k, M)) \to H_{*+2n-1}(\mathrm{Map}(S^{k-1} \times S^1, M)),$$

*analogous to [**3**, Part I, Chapter 5], and the S^k-brane coproduct*

$$\delta_{S^k} : H_*(\mathrm{Map}(S^{k-1} \times S^1, M)) \to H_{*-2n+k-1}(\mathrm{Map}(S^k, M)),$$

*introduced in our previous paper [**10**], is nontrivial.*

In some cases, the S^k-brane coproduct is trivial. Now we recall the definition of a pure Sullivan algebra. A Sullivan algebra $(\wedge V, d)$ with $\dim V < \infty$ is called *pure* if $d(V^{\mathrm{even}}) = 0$ and $d(V^{\mathrm{odd}}) \subset \wedge V^{\mathrm{even}}$ (cf. [**5**, Section 32]). Here we denote $V^{\mathrm{even}} = \bigoplus_n V^{2n}$ and $V^{\mathrm{odd}} = \bigoplus_n V^{2n+1}$.

Theorem 1.2. *Let k be a positive even integer and M a k-connected (Gorenstein) space with $\bigoplus_n \pi_n(M) \otimes \mathbb{K}$ of finite dimension. Assume that the minimal Sullivan model of M is pure and has at least one generator of odd degree. Then the S^k-brane coproduct is trivial for M.*

For a connected Lie group G and its closed connected subgroup H, the homogeneous space $M = G/H$ satisfies the assumption if the canonical map $\pi_*(H) \otimes \mathbb{K} \to \pi_*(G) \otimes \mathbb{K}$ is *not* surjective.

Received June 16, 2019, revised July 8, 2019; published on January 8, 2020.
2010 Mathematics Subject Classification: 55P50, 55P62.
Key words and phrases: string topology, brane topology, rational homotopy theory.
Article available at `http://dx.doi.org/10.4310/HHA.2020.v22.n1.a19`

The following corollary follows from Theorems 1.1 and 1.2.

Corollary 1.3. *Let k be a positive even integer and M a k-connected (Gorenstein) space with $\bigoplus_n \pi_n(M) \otimes \mathbb{K}$ of finite dimension. Assume that the minimal Sullivan model of M is pure. Then the composition $\mu_{S^k} \circ \delta_{S^k}$ is nontrivial if and only if M is a finite product $\prod_i K(\mathbb{Z}, 2n_i)$ of Eilenberg–MacLane spaces of even degrees.*

Now we explain the background of the above theorems. Chas and Sullivan [**1**] introduced the loop product $\mu\colon H_*(LM \times LM) \to H_{*-m}(LM)$ on the homology of the free loop space $LM = \mathrm{Map}(S^1, M)$ of a connected closed oriented manifold M of dimension m. Constructing a 2-dimensional topological quantum field theory without counit, Cohen and Godin [**2**] generalized this product to other string operations, including the loop coproduct $\delta\colon H_*(LM) \to H_{*-m}(LM \times LM)$. But Tamanoi [**9**] showed that any string operation corresponding to a positive genus surface is trivial. In particular, the composition $\mu \circ \delta$ is trivial. In contrast, Theorem 1.1 shows that this is not the case for brane operations and hence they give richer structures on the homology of mapping spaces.

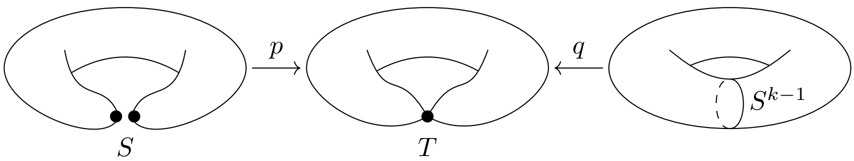
The brane product was introduced in Cohen, Hess, and Voronov [**3**, Part I, Chapter 5] as a generalization of the loop product to the sphere space $S^kM = \mathrm{Map}(S^k, M)$ for $k \geqslant 1$. In [**10**], the author generalized the brane product to the mapping spaces from manifolds, by means of connected sums.

The brane coproduct, a generalization of the loop coproduct to the mapping spaces from manifolds, is constructed by the author [**10**] in the case where the rational homotopy group $\bigoplus_n \pi_n(M) \otimes \mathbb{Q}$ is of finite dimension. This assumption can be considered as the "finiteness" of the dimension of the $(k-1)$-fold based loop space $\Omega^{k-1}M$ as a Gorenstein space. A Gorenstein space is a generalization of a manifold in the point of view of Poincaré duality, which was introduced to string topology by Félix and Thomas [**6**]. For example, connected closed oriented manifolds, classifying spaces of connected Lie groups, and their Borel constructions are Gorenstein spaces. Moreover, any 1-connected space M with $\bigoplus_n \pi_n(M) \otimes \mathbb{Q}$ of finite dimension is a Gorenstein space. In spite of this huge generalization, string operations still tend to be trivial. For example, the loop product μ is trivial over a field of characteristic zero for the classifying space of a connected Lie group [**6**, Theorem 14]. Moreover, it is an open problem to find a Gorenstein space with nontrivial composition $\mu \circ \delta$ of the loop product and coproduct.

Here we briefly review the construction of the brane product and coproduct. See Section 2 for details. Let $\mathbb{K}$ be a field of characteristic zero, S an oriented manifold of dimension k with two disjoint base points, and M a k-connected m-dimensional $\mathbb{K}$-Gorenstein space with $\bigoplus_n \pi_n(M) \otimes \mathbb{K}$ of finite dimension. Denote the "connected sum" and "wedge sum" of S with itself along the two base points by $S_\#$ and $S_\vee$, respectively. Note that, by the definition of the connected sum, we have the canonical inclusion $S^{k-1} \hookrightarrow S_\#$ and the quotient map $q\colon S_\# \to (S_\#)/S^{k-1} = S_\vee$. Similarly we have $S^0 = \mathrm{pt} \coprod \mathrm{pt} \hookrightarrow S$ and $p\colon S \to S/S^0 = S_\vee$. Hence we have the diagram

$$M^S \xleftarrow{\mathrm{incl}} M^{S_\vee} \xrightarrow{\mathrm{comp}} M^{S_\#} \tag{1.1}$$

as a dual of the following diagram:



$$S \xrightarrow{\quad p \quad} S_\vee \xleftarrow{\quad q \quad} S_\#$$

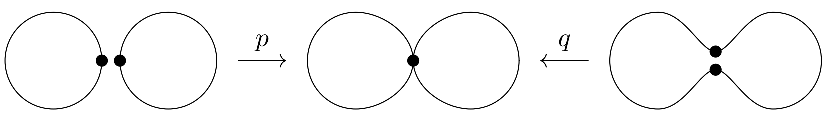
Using this diagram, we can construct two operations, the S-brane product μ_S and coproduct δ_S:

$$\mu_S \colon H_*(M^S) \to H_{*-m}(M^{S_\#}),$$
$$\delta_S \colon H_*(M^{S_\#}) \to H_{*-\bar{m}}(M^S).$$

Note that, if T and U are oriented k-manifolds and we take $S = T \coprod U$ with one base point on T and the other on U, then μ_S and δ_S have the form

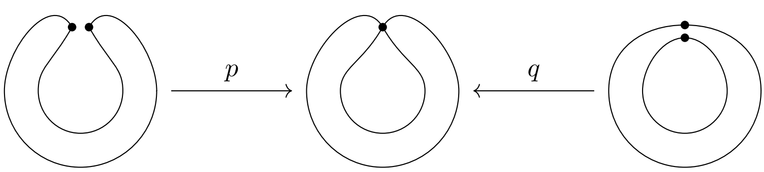
$$\mu_{T \coprod U} \colon H_*(M^T \times M^U) \to H_{*-m}(M^{T\#U}),$$
$$\delta_{T \coprod U} \colon H_*(M^{T\#U}) \to H_{*-\bar{m}}(M^T \times M^U).$$

Moreover, if we take $T = U = S^1$, then $\mu_{S^1 \coprod S^1}$ and $\delta_{S^1 \coprod S^1}$ coincide with the usual loop product and coproduct, respectively. Hence the S-brane product and coproduct are generalizations of the loop product and coproduct.



$$S = S^1 \textstyle\coprod S^1 \xrightarrow{\quad p \quad} S_\vee = S^1 \vee S^1 \xleftarrow{\quad q \quad} S_\# = S^1$$

Remark 1.4. Here it should be remarked that the composition $\mu_{S^k} \circ \delta_{S^k}$ corresponds to a cobordism without "genus". In fact, if we take $k = 1$, the composition $\mu_{S^1} \circ \delta_{S^1}$ is equal to the composition $\delta \circ \mu$, not $\mu \circ \delta$, of the loop product μ and coproduct δ.



$$S = S^1 \xrightarrow{\quad p \quad} S_\vee = S^1 \vee S^1 \xleftarrow{\quad q \quad} S_\# = S^1 \textstyle\coprod S^1$$

Section 2 contains brief background material on brane operations. In Section 3, we construct rational models of the S^k-brane product and coproduct, which gives a method of computation. Finally, in Section 4, we prove Theorems 1.1 and 1.2 using the above models.

Acknowledgments

I would like to express my gratitude to Katsuhiko Kuribayashi and Takahito Naito for productive discussions and valuable suggestions. Moreover, I would like to thank my supervisor Nariya Kawazumi for the enormous support and comments. Furthermore, I am grateful to the referee for helpful comments which made the article clearer. This work was supported by JSPS KAKENHI Grant Number 16J06349 and the Program for Leading Graduate School, MEXT, Japan.

2. Brane operations for the mapping space from manifolds

In this section we review the constructions of the S-brane product and coproduct from [**10**]. Since the cochain models work well for fibrations, we define the duals of the S-brane product and coproduct at first, and then we define the S-brane product and coproduct as the duals of them.

Let $\mathbb{K}$ be a field of characteristic zero. This assumption enables us to make full use of rational homotopy theory. For the basic definitions and theorems on homological algebra and rational homotopy theory, we refer the reader to [**5**].

Let k be a positive integer and M a k-connected space such that $\bigoplus_n \pi_n(M) \otimes \mathbb{K}$ is finite dimensional and the minimal Sullivan model of M is pure. Then M and $\Omega^{k-1}M$ are Gorenstein spaces by [**4**, Proposition 3.4]. Here we refer the reader to [**4**] for the definition and basic properties of Gorenstein spaces.

We use the following theorem to construct the brane operations.

Theorem 2.1 ([**6**, Theorem 12] for $k = 1$, [**10**, Corollary 3.2] for $k \geqslant 2$)**.** *Under the above assumptions, we have an isomorphism*

$$\mathrm{Ext}^*_{C^*(S^{k-1}M)}(C^*(M), C^*(S^{k-1}M)) \cong H^{*-\bar{m}}(M),$$

where $\bar{m}$ is the dimension of $\Omega^{k-1}M$ as a Gorenstein space.

Here, $\mathrm{Ext}_A(L, N)$ is defined using a semifree resolution of (L, d) over (A, d), for a dga (A, d) and (A, d)-modules (L, d) and (N, d). $\mathrm{Tor}_A(L, N)$ is defined similarly. See [**5**, Section 1] for details of semifree resolutions.

Now we can define the S-brane coproduct for an oriented manifold S with two distinct base points. Consider the diagram, extending (1.1),

$$\begin{tikzcd} M^{S_\#} \arrow[d, "\mathrm{res}"'] & M^{S_\vee} \arrow[l, "\mathrm{comp}"'] \arrow[r, "\mathrm{incl}"] \arrow[d] & M^S \\ S^{k-1}M & M. \arrow[l, "c"] \end{tikzcd} \tag{2.1}$$

Here, the square is a pullback diagram, the map res is the restriction map to S^{k-1}, and c is the embedding as the constant maps. By Theorem 2.1, we have an isomorphism $\mathrm{Ext}^{\bar{m}}_{C^*(S^{k-1}M)}(C^*(M), C^*(S^{k-1}M)) \cong H^0(M) \cong \mathbb{K}$, hence the generator

$$c_! \in \mathrm{Ext}^{\bar{m}}_{C^*(S^{k-1}M)}(C^*(M), C^*(S^{k-1}M))$$

is well-defined up to the multiplication by a non-zero scalar. Using the map $c_!$ and the diagram (2.1), we can define the shriek map $\mathrm{comp}_!$ as the composition

$$H^*(M^{S_\vee}) \xleftarrow[\cong]{\mathrm{EM}} \mathrm{Tor}^*_{C^*(S^{k-1}M)}(C^*(M), C^*(M^{S_\#}))$$
$$\xrightarrow{\mathrm{Tor}_{\mathrm{id}}(c_!,\mathrm{id})} \mathrm{Tor}^{*+\bar{m}}_{C^*(S^{k-1}M)}(C^*(S^{k-1}M), C^*(M^{S_\#})) \xrightarrow[\cong]{} H^{*+\bar{m}}(M^{S_\#}),$$

where the map EM is the Eilenberg–Moore map, which is an isomorphism since $S^{k-1}M$ is 1-connected (see [**5**, Theorem 7.5] for details). By this, we define the dual of the S-brane coproduct as the composition

$$\delta_S^\vee \colon H^*(M^S) \xrightarrow{\mathrm{incl}^*} H^{*+\bar{m}}(M^{S_\vee}) \xrightarrow{\mathrm{comp}_!} H^{*+\bar{m}}(M^{S_\#}).$$

Similarly we can define the S-brane product using the generator

$$\Delta_! \in \mathrm{Ext}^m_{C^*(M^2)}(C^*(M), C^*(M^2))$$

and the diagram

$$\begin{array}{ccccc} M^S & \xleftarrow{\mathrm{incl}} & M^{S_\vee} & \xrightarrow{\mathrm{comp}} & M^{S_\#} \\ \downarrow & & \downarrow & & \\ M \times M & \xleftarrow{\Delta} & M. & & \end{array}$$

3. Models of the brane operations

In this section, we consider the case $S = S^k$ and give rational models of the S^k-brane operations, for an integer $k \geqslant 1$. In Section 4, we will prove Theorem 1.1 and Theorem 1.2 using these models.

Naito [**8**] constructed a rational model of the duals of the loop product and coproduct in terms of Sullivan models using the torsion functor description of [**7**]. The author [**10**] constructed a rational model of the duals of the brane product and coproduct as a generalization of it. Here we give a rational model of the S^k-brane operations by a similar method.

3.1. Models of spaces

Take a Sullivan model $(\wedge V, d)$ of M with $V^{\leqslant k} = 0$ and $\dim V < \infty$. For simplicity, we sometimes denote $(\wedge V, d)$ by $\mathcal{M}$. Denote $(S^k)_\# = S^{k-1} \times S^1$ by $T^{(k)}$ and $(S^k)_\vee = (S^{k-1} \times S^1)/S^{k-1}$ by $U^{(k)}$. For an integer $l \in \mathbb{Z}$, let $s^l V$ be a graded module defined by $(s^l V)^n = V^{n+l}$ and $s^l v$ denotes the element in $s^l V$ corresponding to an element $v \in V$.

(3.1) Consider s as an derivation on the algebra $\wedge V^{\otimes 2} \otimes \wedge sV$ with $s \circ s = 0$. Define a derivation d on the algebra by

$$d(sv) = 1 \otimes v - v \otimes 1 - \sum_{i=1}^{\infty} \frac{(sd)^i}{i!}(v \otimes 1),$$

inductively. Denote the dga $(\wedge V^{\otimes 2} \otimes \wedge sV, d)$ by $\mathcal{M}(I)$. This is a Sullivan model of the path space M^I ($\simeq M$). Moreover, define a map $\bar{\varepsilon} \colon \mathcal{M}(I) \to \mathcal{M}$ by $\bar{\varepsilon}(v \otimes 1) = \bar{\varepsilon}(1 \otimes v) = v$ and $\bar{\varepsilon}(sv) = 0$ for $v \in V$. Then it is a relative Sullivan model (resolution) of the product map. See [**5**, Section 15 (c)] or [**11**, Appendix A] for details.

(3.2) Assume $k \geqslant 2$. Define derivations $s^{(k-1)}$ and d on the graded algebra $\wedge V \otimes \wedge s^{k-1}V$ by

$$\begin{aligned} s^{(k-1)}(v) &= s^{k-1}v, & s^{(k-1)}(s^{k-1}v) &= 0, \\ d(v) &= dv, \quad \text{and} & d(s^{k-1}v) &= (-1)^{k-1}s^{(k-1)}dv. \end{aligned}$$

Denote the dga $\wedge V \otimes s^{k-1}V$ by $\mathcal{M}(S^{k-1})$. This is a Sullivan model of the space $M^{S^{k-1}}$. See [**10**, Section 5] for details.

(3.3) Assume $k \geqslant 2$. Define derivations $s^{(k)}$ and d on the graded algebra $\wedge V \otimes \wedge s^{k-1}V \otimes \wedge s^k V$ by

$$\begin{aligned} s^{(k)}(v) &= s^k v, & s^{(k)}(s^{k-1}v) &= s^{(k)}(s^k v) = 0, \\ d(v) &= dv, & d(s^{k-1}v) &= d(s^{k-1}v), \quad \text{and} \quad d(s^k v) = s^{k-1}v + (-1)^k s^{(k)}dv. \end{aligned}$$

Denote the dga $\wedge V \otimes \wedge s^{k-1}V \otimes \wedge s^k V$ by $\mathcal{M}(D^k)$. This is a Sullivan model of the space M^{D^k} ($\simeq M$). Moreover, define a map $\tilde{\varepsilon}\colon \mathcal{M}(D^k) \to \mathcal{M}$ by $\tilde{\varepsilon}(v) = v$, $\tilde{\varepsilon}(s^{k-1}v) = \tilde{\varepsilon}(s^k v) = 0$ for $v \in V$. Then it is a relative Sullivan model (resolution) of the map $\varepsilon\colon \mathcal{M}(S^{k-1}) \to \mathcal{M}$, where $\varepsilon(v) = v$ and $\varepsilon(s^{k-1}v) = 0$. In particular, $\tilde{\varepsilon}$ is a quasi-isomorphism. See [**10**, Section 5] for details.

Next we construct models of mapping spaces which appear in the definition of brane operations, using the above models.

(3.4) Since $M^{T^{(k)}} = (M^{S^{k-1}})^{S^1}$, we have a Sullivan model $\mathcal{M}(T^{(k)}) = (\wedge V \otimes \wedge s^{k-1}V \otimes \wedge sV \otimes \wedge s s^{k-1}V, d)$ of $M^{T^{(k)}}$ iterating the construction in (3.2).

(3.5) Since $U^{(k)}$ is homotopy equivalent to $S^k \vee S^1$, the mapping space $M^{U^{(k)}}$ is homotopy equivalent to $M^{S^k} \times_M M^{S^1}$, and hence we have a Sullivan model $\mathcal{M}(U^{(k)}) = (\wedge V \otimes \wedge s^k V, d) \otimes (\wedge V \otimes \wedge sV, d)$.

3.2. Models of operations

Here we give a model of the S^k-brane product and coproduct in a similar way to [**8**] and [**10**].

First we give a model of the S^k-brane coproduct. Recall that the dual $\delta^\vee_{S^k}$ of the S^k-brane coproduct is the composition

$$\delta^\vee_{S^k}\colon H^*(M^{S^k}) \xrightarrow{\text{incl}^*} H^{*+\bar{m}}(M^{U^{(k)}}) \xrightarrow{\text{comp}_!} H^{*+\bar{m}}(M^{T^{(k)}}).$$

The first map $\text{incl}^*\colon H^*(M^{S^k}) \to H^{*+\bar{m}}(M^{U^{(k)}})$ is induced by the canonical inclusion $\mathcal{M}(S^k) \to \mathcal{M}(U^{(k)})$, which we also denote by incl^*. The second map $\text{comp}_!\colon H^{*+\bar{m}}(M^{U^{(k)}}) \to H^{*+\bar{m}}(M^{T^{(k)}})$ is computed as follows. Let

$$\gamma \in \hom_{\mathcal{M}(S^{k-1})}(\mathcal{M}(D^k), \mathcal{M}(S^{k-1}))$$

be a representative of the nontrivial element (see Theorem 2.1)

$$c_! \in \operatorname{Ext}^{\bar{m}}_{C^*(S^{k-1}M)}(C^*(M), C^*(S^{k-1}M)) \cong H^{\bar{m}}(\hom_{\mathcal{M}(S^{k-1})}(\mathcal{M}(D^k), \mathcal{M}(S^{k-1}))).$$

Then the map

$$\begin{aligned} \operatorname{Tor}_{\mathrm{id}}(c_!, \mathrm{id})\colon \operatorname{Tor}^*_{C^*(S^{k-1}M)}&(C^*(M), C^*(M^{S_\#})) \\ &\longrightarrow \operatorname{Tor}^{*+\bar{m}}_{C^*(S^{k-1}M)}(C^*(S^{k-1}M), C^*(M^{S_\#})) \end{aligned}$$

is induced by the cochain map

$$\gamma \otimes \mathrm{id}\colon \mathcal{M}(D^k) \otimes_{\mathcal{M}(S^{k-1})} \mathcal{M}(T^{(k)}) \to \mathcal{M}(S^{k-1}) \otimes_{\mathcal{M}(S^{k-1})} \mathcal{M}(T^{(k)}),$$

since $\mathcal{M}(D^k)$ is a resolution of $\mathcal{M}$ over $\mathcal{M}(S^{k-1})$. The map $\mathrm{comp}_!$ is computed by this combined with the quasi-isomorphism

$$\tilde{\varepsilon} \otimes \mathrm{id}\colon \mathcal{M}(D^k) \otimes_{\mathcal{M}(S^{k-1})} \mathcal{M}(T^{(k)}) \xrightarrow[\simeq]{} \mathcal{M} \otimes_{\mathcal{M}(S^{k-1})} \mathcal{M}(T^{(k)}). \tag{3.6}$$

Hence the dual of the S^k-brane coproduct is induced by the composition

$$\begin{aligned} \mathcal{M}(S^k) \xrightarrow{\mathrm{incl}^*} \mathcal{M}(U^{(k)}) &\xrightarrow{\cong} \mathcal{M} \otimes_{\mathcal{M}(S^{k-1})} \mathcal{M}(T^{(k)}) \\ &\xleftarrow[\simeq]{\tilde{\varepsilon}\otimes\mathrm{id}} \mathcal{M}(D^k) \otimes_{\mathcal{M}(S^{k-1})} \mathcal{M}(T^{(k)}) \\ &\xrightarrow{\gamma\otimes\mathrm{id}} \mathcal{M}(S^{k-1}) \otimes_{\mathcal{M}(S^{k-1})} \mathcal{M}(T^{(k)}) \xrightarrow{\cong} \mathcal{M}(T^{(k)}). \end{aligned} \tag{3.7}$$

Similarly, the dual of the S^k-brane product is induced by the composition

$$\begin{aligned} &\mathcal{M}(T^{(k)}) \xrightarrow{\mathrm{comp}^*} \mathcal{M}(U^{(k)}) \xrightarrow{\cong} \mathcal{M} \otimes_{\mathcal{M}^{\otimes 2}} (\mathcal{M}(I) \otimes_{\mathcal{M}} \mathcal{M}(S^k)) \\ &\xleftarrow[\simeq]{\bar{\varepsilon}\otimes\mathrm{id}} \mathcal{M}(I) \otimes_{\mathcal{M}^{\otimes 2}} (\mathcal{M}(I) \otimes_{\mathcal{M}} \mathcal{M}(S^k)) \xrightarrow{\eta\otimes\mathrm{id}} \mathcal{M}^{\otimes 2} \otimes_{\mathcal{M}^{\otimes 2}} (\mathcal{M}(I) \otimes_{\mathcal{M}} \mathcal{M}(S^k)) \\ &\quad \xrightarrow{\cong} \mathcal{M}(I) \otimes_{\mathcal{M}} \mathcal{M}(S^k) \xrightarrow{\bar{\varepsilon}\otimes\mathrm{id}} \mathcal{M} \otimes_{\mathcal{M}} \mathcal{M}(S^k) \xrightarrow{\cong} \mathcal{M}(S^k) \end{aligned} \tag{3.8}$$

Here $\eta \in \hom_{\mathcal{M}^{\otimes 2}}(\mathcal{M}(I), \mathcal{M}^{\otimes 2})$ is a representative of the nontrivial element $\Delta_! \in \mathrm{Ext}^m_{C^*(M^2)}(C^*(M), C^*(M^2))$ and $\mathrm{comp}^*\colon \mathcal{M}(T^{(k)}) \to \mathcal{M}(U^{(k)})$ is the canonical quotient map.

4. Proof of Theorem 1.1 and Theorem 1.2

In this section, we give a proof of Theorem 1.1 and Theorem 1.2 using the models constructed above. First we recall the description of $c_!$ in [**10**].

Proposition 4.1 ([**10**, Proposition 6.2])**.** *Assume that k is even. Define an element*

$$\gamma \in \hom_{\mathcal{M}(S^{k-1})}(\mathcal{M}(D^k), \mathcal{M}(S^{k-1}))$$

by $\gamma(s^k y_1 \cdots s^k y_q) = s^{k-1}x_1 \cdots s^{k-1}x_p$ and $\gamma(s^k y_{j_1} \cdots s^k y_{j_l}) = 0$ for $l < q$. Then γ defines a non-trivial element in $\mathrm{Ext}_{\mathcal{M}(S^{k-1})}(\mathcal{M}, \mathcal{M}(S^{k-1}))$.

Note that, although the proposition is proved only when $k = 2$ in [**10**], the same proof also applies when $k > 2$ as long as k is even.

Now we give a proof of Theorem 1.1 using the above description.

Proof of Theorem 1.1. We compute the S^k-brane coproduct using (3.7). Since $M = K(\mathbb{Z}, 2n)$, we take the Sullivan model $(\wedge V, d) = (\wedge x, 0)$ where x is the generator of degree $2n$. Note that, in this case, the differentials in $\mathcal{M}(S^k)$ and $\mathcal{M}(T^{(k)})$ are zero, and hence they are identified with the cohomology groups $H^*(M^{S^k})$ and $H^*(M^{T^{(k)}})$.

By Proposition 4.1, we have a representative γ of the shriek map $c_!$ defined by $\gamma(1) = s^{k-1}x$ and $\gamma((s^k x)^l) = 0$ for $l \geqslant 1$.

Since any Sullivan algebra satisfies the lifting property for a surjective quasi-isomorphism, there is a section φ of $\tilde{\varepsilon} \otimes \mathrm{id}$ in (3.6), which is also a quasi-isomorphism.

It is given explicitly by $\varphi(1\otimes x) = 1\otimes x$, $\varphi(1\otimes s^k x) = 1\otimes ss^{k-1}x$, and $\varphi(1\otimes sx) = 1\otimes sx$.

Using these maps, we compute the composition (3.7). Since all maps in the composition are $\wedge V$-linear, it is enough to compute the image for the elements $(s^k x)^n$ for $n \geqslant 0$. Applying incl^* and the section φ to the element, we have that it is mapped to $1\otimes (ss^{k-1}x)^n \in \mathcal{M}(D^k)\otimes_{\mathcal{M}(S^{k-1})}\mathcal{M}(T^{(k)})$. Then the map $\gamma\otimes\mathrm{id}$ send it to $s^{k-1}x\otimes(ss^{k-1}x)^n \in \mathcal{M}(S^{k-1})\otimes_{\mathcal{M}(S^{k-1})}\mathcal{M}(T^{(k)})$. Hence the S^k-brane coproduct $\delta^\vee_{S^k}$ is the map determined by $\delta^\vee_{S^k}(\alpha) = s^{k-1}x\iota(\alpha)$, where $\iota\colon \mathcal{M}(S^k)\to\mathcal{M}(T^{(k)})$ is the algebra map defined by $\iota(x) = x$ and $\iota(s^k x) = ss^{k-1}x$.

Similarly we can compute the S^k-brane product. Define a $\wedge V^{\otimes 2}$-linear map $\eta\colon \mathcal{M}(I)\to\wedge V^{\otimes 2}$ by $\eta(1) = 0$ and $\eta(sx) = 1$. By [**11**, Theorem 5.6 (2)] or a straightforward computation, η is a representative of the shriek map $\Delta_!$. We have a section ψ of $\bar\varepsilon\otimes\mathrm{id}$ in (3.8), which is defined by $\psi(x\otimes 1) = 1\otimes(x_1\otimes 1)$, $\psi(1\otimes sx) = 1\otimes(sx\otimes 1) - sx\otimes 1$, and $\psi(1\otimes s^k x) = 1\otimes(1\otimes s^k x)$. Here we denote the element $x\otimes 1\in\mathcal{M}(I)$ by x_1.

As a result, the S^k-brane product $\mu^\vee_{S^k}$ is the map determined by $\mu^\vee_{S^k}(\beta) = 0$, $\mu^\vee_{S^k}(sx\cdot\beta) = -\rho(\beta)$, and $\mu^\vee_{S^k}(s^{k-1}x\cdot\beta) = \mu^\vee_{S^k}(sx\cdot s^{k-1}x\cdot\beta) = 0$, for $\beta\in\wedge x\otimes\wedge ss^{k-1}x$. Here $\rho\colon \wedge x\otimes\wedge ss^{k-1}x\to\mathcal{M}(S^k)$ is the algebra map defined by $\rho(x) = x$ and $\rho(ss^{k-1}x) = s^k x$.

Composing these two, we have $\delta_{S^k}\circ\mu_{S^k}\neq 0$. In fact, $\delta_{S^k}\circ\mu_{S^k}(sx) = -s^{k-1}x \neq 0\in\mathcal{M}(T^{(k)})\cong H^*(M^{T^{(k)}})$. This proves the theorem. □

Next we prove Theorem 1.2.

Proof of Theorem 1.2. Let $(\wedge V, d)$ be the minimal Sullivan model of M, $\{x_1,\dots,x_p\}$ a basis of V^{even}, and $\{y_1,\dots,y_q\}$ a basis of V^{odd}. Consider the part

$$\mathcal{M}\otimes_{\mathcal{M}(S^{k-1})}\mathcal{M}(T^{(k)}) \xleftarrow[\simeq]{\tilde\varepsilon\otimes\mathrm{id}} \mathcal{M}(D^k)\otimes_{\mathcal{M}(S^{k-1})}\mathcal{M}(T^{(k)}) \xrightarrow{\gamma\otimes\mathrm{id}} \mathcal{M}(S^{k-1})\otimes_{\mathcal{M}(S^{k-1})}\mathcal{M}(T^{(k)})$$

in (3.7). Define a section φ of $\tilde\varepsilon\otimes\mathrm{id}$ by $\varphi(1\otimes v) = 1\otimes v$, $\varphi(1\otimes sv) = 1\otimes sv$, for $v\in V$, $\varphi(1\otimes ss^{k-1}x_i) = 1\otimes ss^{k-1}x_i$, and $\varphi(1\otimes ss^{k-1}y_j) = 1\otimes ss^{k-1}y_j + (-1)^k s\sigma(dy_j\otimes 1)$. Here, in the last term $s\sigma(dy_j\otimes 1)$, σ is the derivation which sends $v\otimes 1$ to $s^k v\otimes 1$, for $v\in V$, and the other generators to 0. The map s is also the derivation which sends v to sv, $s^{k-1}v$ to $ss^{k-1}v$, and others to 0. Then we have $\operatorname{Im}\varphi\subset\mathcal{N}\otimes_{\mathcal{M}(S^{k-1})}\mathcal{M}(T^{(k)})$, where $\mathcal{N} = \wedge V\otimes\wedge s^{k-1}V\otimes\wedge s^k\{x_1,\dots,x_p\}\subset\mathcal{M}(D^k)$. Let γ be the representative of $c_!$ given by Proposition 4.1. Since V has at least one generator of odd degree, γ is zero on $\mathcal{N}$. This implies that the composition $(\gamma\otimes 1)\circ\varphi$ is zero, and hence the brane coproduct δ_{S^k} is zero. □

References

[**1**] M. Chas and D. Sullivan, String topology, 1999, arXiv:math/9911159.

[**2**] R.L. Cohen and V. Godin, A polarized view of string topology, In *Topology, geometry and quantum field theory*, volume 308 of *London Math. Soc. Lecture Note Ser.*, pages 127–154. Cambridge Univ. Press, Cambridge, 2004.

[3] R.L. Cohen, K. Hess and A.A. Voronov, *String topology and cyclic homology*, *Adv. Courses Math. CRM Barcelona.* Birkhäuser Verlag, Basel, 2006. Lectures from the Summer School held in Almería, September 16–20, 2003.

[4] Y. Félix, S. Halperin and J.-C. Thomas, Gorenstein spaces, *Adv. Math.*, 71(1):92–112, 1988.

[5] Y. Félix, S. Halperin and J.-C. Thomas, *Rational homotopy theory*, volume 205 of *Grad. Texts in Math.* Springer-Verlag, New York, 2001.

[6] Y. Félix and J.-C. Thomas, String topology on Gorenstein spaces, *Math. Ann.*, 345(2):417–452, 2009.

[7] K. Kuribayashi, L. Menichi and T. Naito, Derived string topology and the Eilenberg-Moore spectral sequence, *Israel J. Math.*, 209(2):745–802, 2015.

[8] T. Naito, String operations on rational Gorenstein spaces, 2013, arXiv: 1301.1785.

[9] H. Tamanoi, Loop coproducts in string topology and triviality of higher genus TQFT operations, *J. Pure Appl. Algebra*, 214(5):605–615, 2010.

[10] S. Wakatsuki, Coproducts in brane topology, *Algebr. Geom. Topol.*, 19(6):2961–2988, 2019.

[11] S. Wakatsuki, Description and triviality of the loop products and coproducts for rational Gorenstein spaces, 2016, arXiv:1612.03563.

Shun Wakatsuki `swaka@ms.u-tokyo.ac.jp`

Graduate School of Mathematical Sciences, University of Tokyo, 3-8-1 Komaba Meguro-ku, Tokyo 153-8914, Japan