

GIGANTIC RANDOM SIMPLICIAL COMPLEXES

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Abstract

We provide a random simplicial complex by applying standard constructions to a Poisson point process in Euclidean space. It is gigantic in the sense that—up to homotopy equivalence—it almost surely contains infinitely many copies of every compact topological manifold, both in isolation and in percolation.

1. Introduction

This article presents a naturally constructed random simplicial complex, which is rich enough to realize any compact topological manifold, at least up to homotopy equivalence. Recall that a result of John Milnor [28, Theorem 1] states that any topological manifold is homotopy equivalent to a countable locally finite simplicial complex. Examples of noncompact topological manifolds like a connected sum of infinitely many tori show that finiteness cannot be expected in general. However, a refinement due to Kirby and Siebenmann [21, 22] states that any compact topological manifold is homotopy equivalent to a finite simplicial complex. Hence from the viewpoint of homotopy theory, finite simplicial complexes constitute a rich and highly interesting class of topological spaces. Note furthermore that Manolescu, based on work of Galewski-Stern [15] (among others), disproved the Triangulation Conjecture: For every integer $d \geq 5$, there exists a compact topological manifold of dimension d which is not homeomorphic to a finite simplicial complex [24]. This perhaps unexpected result may be viewed as an argument towards considering topological manifolds up to homotopy equivalence. Up to dimension three, every topological manifold is homeomorphic to a simplicial complex [29]. In the notoriously exotic dimension four, the Casson invariant proves that Freedman’s E_8 manifold [14] is a compact topological manifold not homeomorphic to a simplicial complex, as explained for example in [5].

Our work provides a natural construction of a random simplicial complex associated with Euclidean space \mathbb{R}^d which contains all finite simplicial complexes (hence, in particular, all compact topological manifolds, at least up to homotopy equivalence) in the following two different ways:

Lonely Complex Almost surely, it contains infinitely many copies of any given d -embeddable finite simplicial complex as isolated components.

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Giant Beast Almost surely, it contains an unbounded connected component having infinitely many copies of any given d -embeddable finite simplicial complex as wedge summands.

Here a simplicial complex \mathcal{K} is called *d-embeddable* if there exists a piecewise linear embedding $\mathcal{K} \rightarrow \mathbb{R}^d$. Recall that a finite simplicial complex of dimension d is $(2d + 1)$ -embeddable. Recall also that the wedge sum of two pointed simplicial complexes (or general pointed topological spaces) is the one-point union at the respective basepoints. Up to homotopy it is the same as a “whiskered one-point union”: joining the two basepoints via an additional edge. Before giving both the construction and a precisely formulated theorem, a consequence on homology groups is readily obtained.

Theorem. *There exists a random simplicial complex with the following properties: the homology of the simplicial complex almost surely contains the homology of any finite simplicial complex of dimension at most n , for every $n \in \mathbb{N}$, infinitely many times as a direct summand.*

This theorem is weaker than the statements on the lonely complex and the giant beast in the sense that homology is determined by homotopy type, but not vice versa. Now to its construction, coming in two flavours: Any subset—which in our case is locally finite—of \mathbb{R}^d gives rise to at least two types of abstract simplicial complexes, a *Vietoris-Rips* complex and a *Čech* complex, depending on the ambient Euclidean metric and a real positive parameter ρ . The subset itself is a random one, given by a stationary Poisson point process η in \mathbb{R}^d with intensity measure $t \cdot \Lambda_d$, where $0 < t \in \mathbb{R}$ and Λ_d refers to the Lebesgue measure. Precise definitions are given in Section 2, namely Definitions 2.2, 2.3, and Section 3. The resulting theorems on random Vietoris-Rips and Čech complexes, respectively, also come in two flavours: isolation and percolation. For the sake of brevity, only the Čech version will be stated in this introduction; see Theorems 4.3 and 4.4 for the statements on the random Vietoris-Rips complex.

Theorem 1.1 (Lonely complexes). *Let \mathcal{L} be an arbitrary d -embeddable simplicial complex. For every stationary Poisson point process in \mathbb{R}^d , the accompanying Čech complex almost surely contains infinitely many isolated simplicial complexes that are homotopy equivalent to \mathcal{L} .*

Theorem 1.2 (The giant beast). *Let \mathcal{K} be an arbitrary d -embeddable simplicial complex. Assume that the random Čech complex of a stationary Poisson point process on \mathbb{R}^d contains an unbounded connected simplicial complex \mathcal{B}_∞ . Then the random Čech complex contains almost surely infinitely many subcomplexes that are homotopy equivalent to \mathcal{K} and only connected to \mathcal{B}_∞ by an edge.*

The proof of each of these theorems consists of two steps. In the first step we prove that any d -embeddable simplicial complex can be generated in \mathbb{R}^d in such a way that it occurs with positive probability. For Theorem 1.1 the second step is immediate: either one can use an ergodic theorem combined with a method called ‘trick of small discs’ introduced by Cowan [10] and recently in a similar context used by Schneider [33], or a direct application of the Borel-Cantelli lemma does the job what we prefer for the sake of simplicity. For Theorem 1.2 the second step is more complicated: we prove some kind of mixing property of the unbounded connected component (which also would imply ergodicity), and then use a modified Borel-Cantelli lemma. To the

best of our knowledge the mixing property of the unbounded connected component is new.

Observe that by the groundbreaking work of Meester and Roy [25, 26] the question whether there is an unbounded connected complex and, in particular, its uniqueness is well understood. For each $\rho > 0$ there exists $t_{\text{perc}}(\rho) > 0$, such that for every stationary Poisson point process η on \mathbb{R}^d with intensity $t > t_{\text{perc}}(\rho)$ the random Čech complex $C(\eta, \rho)$ contains one unbounded connected simplicial complex \mathcal{B}_∞ , see also Theorem 4.2.

It is worth noting that every finite simplicial complex \mathcal{K} admits a geometric realization $\text{real}(\mathcal{K})$ in Euclidean space and a real number $\rho > 0$ such that the Čech complex associated with a suitable finite subset of $\text{real}(\mathcal{K})$ and ρ is homotopy equivalent to \mathcal{K} . Hence, for a suitable d , the preceding results apply to any finite simplicial complex, up to homotopy equivalence.

One distinguishing feature of our model, in comparison with the extensive list of random simplicial complexes given for example in [23, 27] or [17] (see Kahle’s chapter in [34] for an overview and further references, also to other survey articles), is its infinite size. Another example of a random infinite-dimensional cell complex is the plaquette complex constructed by Aizenman et al. [4] where large (but still finite) boundary cycles are investigated. But most of the previous investigations and results concentrated on finite random simplicial complexes which may be obtained by restricting to some bounded subset in \mathbb{R}^d . Important results in this direction which we want to emphasize in our context are results on Betti numbers of random simplicial complexes due to Kahle [18], Kahle and Meckes (proving limit theorems) [19, 20], Decreusefond et al. [11], and recently Adler, Subag and Yogeshwaran [2], see also [3] for results on more general point processes. These results should be compared with the following immediate consequence of our main results.

Corollary. *Any finite list of natural numbers is realized infinitely many times as Betti numbers of both isolated and wedge summand subcomplexes of our random simplicial complex.*

Two notable constructions of “large” mathematical objects are the Rado graph (which contains every finite or countably infinite graph as an induced subgraph) and the Urysohn universal space (which contains an isometric copy of every separable metric space). Both spaces are closely connected to random constructions. The Rado graph is the limit of the Erdős-Renyi graph, and the Urysohn universal space is the limit of the random metric spaces introduced by Vershik [35]. Yet for the Rado graph it is immediate that—although it contains every finite or infinite graph—the induced clique complex (the analogue of the Vietoris-Rips complex in that setting) is trivial in the sense that almost surely there is a vertex which generates a cone over the subgraph. In contrast to this, in our construction the subcomplexes are wedge summands and thus our simplicial complex has interesting homology not only on the level of subcomplexes but also as a whole.

The research on geometric random simplicial complexes goes back to work of Gilbert [16] who introduced the random geometric graph in the background of the construction of the Čech, resp. Vietoris-Rips complex. A thorough treatment of this random geometric graph was given in the seminal book of Penrose [31] where subgraph counts are at the core of the investigations. Concentration inequalities for

subgraph counts have very recently been obtained by Bachmann [6] and Bachmann and Reitzner [7], and it would be interesting if these concentration results can be extended to random Betti numbers.

In recent years there have been prominent activities on topological data analysis and persistent homology, where reconstructions of topological structure from data sets [13, 12, 36] are discussed. Roughly stated, one viewpoint here is the following: The data, a point cloud in Euclidean space, stems from sampling a suitable topological manifold. A natural question is to detect topological features which may occur in a totally random point cloud in Euclidean space. The richness of the random point cloud presented here could indicate that it is rather the absence than the presence of certain topological properties which characterizes non-random data.

The article is structured as follows. Section 2 provides information on simplicial complexes relevant to the present discussion, and Section 3 achieves the same for Poisson point processes. The final section 4 contains proofs of slightly stronger versions of the theorems mentioned in this introduction. We tried to present this work such that it may be accessible to both stochastic geometers and algebraic topologists.

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2. Simplicial complexes

Simplicial complexes have various incarnations. In the sequel, we will consider both abstract simplicial complexes and geometric simplicial complexes, as described for example in [30, §2, §3]. Any geometric simplicial complex can be viewed as an abstract simplicial complex by using its vertex scheme. Conversely, any abstract simplicial complex \mathcal{K} admits a geometric realization $\text{real}(\mathcal{K})$. We note that any two geometric realizations of \mathcal{K} are homeomorphic. In what follows, the term “simplicial complex” may be used without further adjectives; the reader should be able to specify that from the context. The types of constructions employed here produce abstract simplicial complexes with points of vertices being embedded in Euclidean space.

Definition 2.1. Two simplicial complexes \mathcal{K} and \mathcal{L} on vertex sets $V(\mathcal{K})$ and $V(\mathcal{L})$, respectively, are *combinatorially equivalent* if there exists a bijection $\varphi: V(\mathcal{K}) \rightarrow V(\mathcal{L})$ such that $F \in \mathcal{K}$ if and only if $\varphi(F) \in \mathcal{L}$.

Every finite simplicial complex is combinatorially equivalent to a subcomplex of a standard N -simplex Δ^N [30, Corollary 2.9]. Examples of simplicial complexes arise naturally from metric spaces. Two such instances are provided by Vietoris-Rips and Čech complexes.

Definition 2.2 (Vietoris-Rips complex). Let $X = (X, \mathbf{d})$ be a metric space (usually a locally finite subset of \mathbb{R}^d) and $0 < \rho \in \mathbb{R}$. The *Vietoris-Rips complex* of (X, \mathbf{d}) with respect to ρ

$$\text{VR}(X, \mathbf{d}, \rho) = \text{VR}(X, \rho)$$

is the abstract simplicial complex on vertex set X whose k -simplices are all subsets $\{x_0, \dots, x_k\} \subset X$ with $\mathbf{d}(x_i, x_j) \leq \rho$ for all $0 \leq i, j \leq k$.

The Vietoris-Rips complex of X (w.r.t. ρ) is determined by its 1-skeleton in the sense that a subset $\{x_0, \dots, x_k\} \subset X$ is a k -simplex if and only if all its subsets $\{x_i, x_j\}$ with two elements are 1-simplices. In other words, the Vietoris-Rips complex equals the clique complex of the graph on vertex set X whose edges are those pairs of vertices with distance smaller than or equal to ρ .

Definition 2.3 (Čech complex). Let X be a subset of a metric space (Y, \mathbf{d}) (usually a locally finite subset of \mathbb{R}^d) and $0 < \rho \in \mathbb{R}$. The Čech complex of $X \subset (Y, \mathbf{d})$ with respect to ρ

$$C(X \subset Y, \mathbf{d}, \rho) = C(X, \rho)$$

is the abstract simplicial complex whose k -simplices are all subsets $\{x_0, \dots, x_k\} \subset X$ admitting a point $y \in Y$ with $\mathbf{d}(x_i, y) \leq \frac{\rho}{2}$ for all $0 \leq i \leq k$.

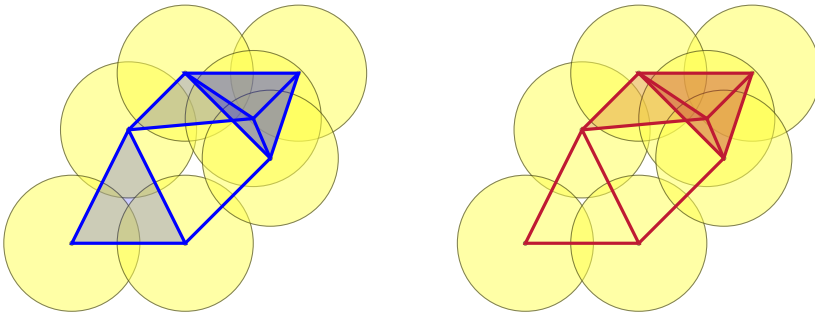


Figure 1: Vietoris-Rips and Čech complex of the same 7-element set.

It directly follows from the definitions that the Čech complex is a subcomplex of the Vietoris-Rips w.r.t. the same parameter ρ and that their 1-skeleta coincide. Moreover, it is well-known that the Vietoris-Rips complexes can be squeezed between two Čech complexes in the following way:

Lemma 2.4. *Let X be a subset of \mathbb{R}^d and $0 < \rho \in \mathbb{R}$. Then*

$$C(X, \rho) \subseteq VR(X, \rho) \subseteq C(X, 2\rho).$$

Moreover, $C(C, \rho)$ and $VR(X, \rho)$ have the same 1-skeleton.

Let $0 < \rho \in \mathbb{R}$. We say that a finite simplicial complex \mathcal{K} admits a (d -dimensional) Čech representation with respect to ρ if there exists a finite subset $X \subset \mathbb{R}^d$ such that \mathcal{K} is combinatorially equivalent to $C(X, \rho)$. In this case, we call X a (d -dimensional) Čech representation of \mathcal{K} . The definition of a Vietoris-Rips representation is analogous.

Remark 2.5. Given those definitions, one of the first questions one might ask is, which simplicial complexes \mathcal{K} do admit a Čech or a Vietoris-Rips representation. In the latter

case, \mathcal{K} necessarily has to be a *flag* complex, i.e., a simplicial complex whose inclusion-minimal missing faces are of cardinality 2. Moreover, the Vietoris-Rips complex is known to have a free fundamental group. As a consequence, it is shown in [9] that if in the definition of Vietoris-Rips complexes the parameter ρ is allowed to lie in a small interval, the resulting complex might not be a Vietoris-Rips complex in the original sense. But besides these conditions, no other results yielding an answer to the posed question exist.

It is therefore natural to relax the original question and to only ask for such a classification up to homotopy equivalence. In fact, for Čech representations a complete characterization is known: On the one hand, if $\phi: \mathcal{K} \rightarrow \mathbb{R}^d$ is a piecewise linear embedding of a simplicial complex, then it follows from the Nerve Lemma (see e.g., [8, Theorem 10.6]) that the Čech complex of a sufficiently dense point set in $\phi(\mathcal{K})$ with respect to a small enough distance parameter ρ is homotopy equivalent to $\text{real}(\mathcal{K}) = \phi(\mathcal{K})$. On the other hand, another application of the Nerve Lemma shows that any Čech complex of a finite point set in \mathbb{R}^d is homotopy equivalent to a d -embeddable finite simplicial complex. Hence, Čech complexes of finite point sets in \mathbb{R}^d model precisely the homotopy types of d -embeddable finite simplicial complexes. In particular, the homology groups and the Betti numbers of such vanish in degrees above $d - 1$.

For Vietoris-Rips complexes, the situation is more complicated. On the one hand, it is shown in [1, Theorem II] that every d -embeddable simplicial complex \mathcal{K} admits a d -dimensional Vietoris-Rips representation up to homotopy equivalence, i.e., there exists a point set $X \subseteq \mathbb{R}^d$ and a distance parameter ρ such that $\text{VR}(X, \rho)$ has the same homotopy type as \mathcal{K} . On the other hand, it is easy to see that the boundary complex of a cross-polytope of dimension n admits even a 2-dimensional Vietoris-Rips presentation. Indeed, the Vietoris-Rips complex of $2n$ points that are equidistantly distributed on the unit sphere with respect to parameter $\rho = 1 - \epsilon$ (for ϵ small enough) is combinatorially equivalent to the boundary complex of the n -dimensional cross-polytope. Hence, any wedge sum of spheres of arbitrary dimensions occurs as the homotopy type of a simplicial complex admitting a d -dimensional Vietoris-Rips representation. Considering homology and topological Betti numbers instead of homotopy equivalence, this also means that every list of positive integers can occur as the Betti numbers of a simplicial complex with a d -dimensional Vietoris-Rips representation, in contrast to the situation for Čech complexes.

Given one Čech representation $X \subset \mathbb{R}^d$ of \mathcal{K} with respect to ρ , it is natural to ask whether X may be perturbed slightly (while staying a Čech representation). This question will be answered for both Čech and Vietoris-Rips complexes in Theorem 2.9 after introducing two auxiliary notions.

Definition 2.6. Let $X, Y \subset \mathbb{R}^d$ be finite nonempty subsets, with cardinalities $|X|$ and $|Y|$, respectively. If $|X| = |Y|$, then the *distance* between X and Y is the maximum distance of the pairs of points $(x, y) \in X \times Y$ with respect to the optimal bijective map $f: X \rightarrow Y$. Otherwise, X and Y are said to be of distance ∞ :

$$d(X, Y) = \begin{cases} \min_{\substack{f: X \rightarrow Y \\ f \text{ bijective}}} \max_{x \in X} \|f(x) - x\|_2, & \text{if } |X| = |Y|, \\ \infty, & \text{if } |X| \neq |Y|. \end{cases}$$

This notion of distance for finite subsets of \mathbb{R}^d is well suited for our purposes, and may be compared with the classical Hausdorff distance

$$d_{\text{Haus}}(X, Y) = \max\left\{\sup_{x \in X} \inf_{y \in Y} \|x - y\|_2, \sup_{y \in Y} \inf_{x \in X} \|x - y\|_2\right\}$$

of subsets $X, Y \subset \mathbb{R}^d$ as follows. For finite subsets $X, Y \subset \mathbb{R}^d$ one always has an inequality $d_{\text{Haus}}(X, Y) \leq d(X, Y)$, and equality holds for subsets with $1 \leq |X| = |Y| \leq 2$. The triangles $X = \{(0, 0), (1, 0), (1, 1)\}$ and $Y = \{(-1, 1), (0, 1), (-1, 0)\}$ in \mathbb{R}^2 satisfy $d_{\text{Haus}}(X, Y) = \sqrt{2} < 2 = d(X, Y)$.

Definition 2.7. Let $0 < \rho \in \mathbb{R}$ and let \mathcal{K} be a finite simplicial complex. A d -dimensional Čech representation $X \subset \mathbb{R}^d$ of \mathcal{K} with respect to ρ is called *generic* if there exists a parameter $\delta > 0$ such that for every $Y \subset \mathbb{R}^d$ with $d(X, Y) < \delta$, the simplicial complex $C(Y, \rho)$ is combinatorially equivalent to \mathcal{K} . A generic d -dimensional Vietoris-Rips representation is defined analogously.

Remark 2.8. By Definition 2.7, every vertex of a generic Čech representation $X \subset \mathbb{R}^d$ may be moved in a small neighborhood, in particular, a δ -ball (for δ sufficiently small), without affecting the combinatorial properties of the resulting simplicial complex. This allows us to reduce the occurrence of a specific simplicial complex in our randomized model to the event that, in a sufficiently large window, every point of our Poisson point process lies in a δ -ball around a vertex of a given simplicial complex and every such ball contains exactly one point of our Poisson point process.

Theorem 2.9. *Let $0 < \rho \in \mathbb{R}$. If a finite simplicial complex \mathcal{K} admits a d -dimensional Čech representation with respect to ρ , then it admits a generic d -dimensional Čech representation with respect to ρ . The analogous statement is true for Vietoris-Rips representations.*

Proof. We show the claim for Čech complexes. Lemma 2.4 implies that the claim for Vietoris-Rips complexes directly follows from the one for Čech complexes.

Let $X \subset \mathbb{R}^d$ be a d -dimensional Čech representation of \mathcal{K} with respect to ρ , i.e., \mathcal{K} is combinatorially equivalent to $C(X, \rho)$. We show that there exists a Čech representation $Y \subset \mathbb{R}^d$ and $\delta > 0$ such that any points set $Z \subset \mathbb{R}^d$ with $d(Y, Z) < \delta$ is a Čech representation of \mathcal{K} .

For a finite point set $A \subset \mathbb{R}^d$ we denote by S_A its minimal bounding sphere, i.e., the d -dimensional sphere with minimal radius that contains A . Let further r_A respectively m_A denote the radius respectively the center of S_A . It is straightforward to show that for $A \subseteq X$ we have $A \in C(X, \rho)$ if and only if $r_A \leq \rho$. We distinguish two cases.

Case 1: $r_A < \rho$ for all $A \in C(X, \rho)$.

Let

$$\delta := \min\{|r_A - \rho| : A \subseteq X\}.$$

Note that, by assumption, we have $\delta > 0$ and the minimum is attained for a face A of $C(X, \rho)$. Let $Z \subseteq \mathbb{R}^d$ be a point set with $d(X, Z) \leq \frac{\delta}{2}$ and let $f : X \rightarrow Z$ be a bijection such that $d(X, Z) = \max_{x \in X} \|f(x) - x\|_2$. We show that $C(X, \rho)$ is combinatorially equivalent to $C(Z, \rho)$ via the map f , i.e., $A \in C(X, \rho)$ if and only if

$f(A) \in C(Z, \rho)$. If $A \in C(X, \rho)$, then $r_A \leq \rho - \delta$. Since

$$\|f(x) - m_A\| \leq \|x - m_A\| + \frac{\delta}{2} \leq r_A + \frac{\delta}{2} \leq \rho - \frac{\delta}{2} < \rho$$

for all $x \in A$, we have $r_{f(A)} \leq \rho$ and hence $f(A) \in C(Z, \rho)$. If $A \notin C(X, \rho)$, then $r_A \geq \rho + \delta$. This implies

$$r_{f(A)} \geq r_A - \frac{\delta}{2} \geq \rho + \frac{\delta}{2} > \rho$$

and therefore $f(A) \notin C(Z, \rho)$. The claim follows. In particular, we have shown that the Čech representation X is already generic.

Case 2: There exists $A \in C(X, \rho)$ such that $r_A = \rho$.

Let

$$\delta := \min\{r_A - \rho : A \notin C(X, \rho)\}.$$

As $r_A > \rho$ for $A \notin C(X, \rho)$, we have $\delta > 0$. Let

$$Y := \left\{ \frac{\rho}{\rho + \frac{9}{10}\delta} \cdot x : x \in X \right\} \subseteq \mathbb{R}^d$$

be a “scaled version” of X . We claim that $C(X, \rho)$ is combinatorially equivalent to $C(Y, \rho)$ via the natural bijection $f : X \rightarrow Y : x \mapsto \frac{\rho}{\rho + \frac{9}{10}\delta} \cdot x$. Let $A \in C(X, \rho)$, i.e., $r_A \leq \rho$. As Y is obtained from X by scaling it with $\frac{\rho}{\rho + \frac{9}{10}\delta} < 1$, we conclude

$$r_{f(A)} = r_A \cdot \frac{\rho}{\rho + \frac{9}{10}\delta} < r_A \leq \rho$$

and hence $f(A) \in C(Y, \rho)$. If $A \notin C(X, \rho)$, then $r_A \geq \rho + \delta$ and thus

$$r_{f(A)} = \frac{\rho}{\rho + \frac{9}{10}\delta} \cdot r_A \geq \frac{\rho}{\rho + \frac{9}{10}\delta} \cdot (\rho + \delta) > \rho,$$

which implies $f(A) \notin C(Y, \rho)$. It now follows from Case 1 that Y is a generic Čech representation of \mathcal{K} . □

3. Poisson point processes

Let $(\mathbb{X}, \mathcal{X})$ be a measurable space and $(\Omega, \mathcal{F}, \mathbb{P})$ a fixed underlying probability space.

Definition 3.1. We denote by $N_\sigma := N_\sigma(\mathbb{X})$ the space of all σ -finite measures χ on \mathbb{X} , with $\chi(B) \in \mathbb{N}_0 \cup \{\infty\}$ for all $B \in \mathcal{X}$, and by $\mathcal{N}_\sigma := \mathcal{N}_\sigma(\mathbb{X})$ the smallest σ -algebra on the set N_σ , such that the mappings $\chi \rightarrow \chi(B)$ are measurable for all $B \in \mathcal{X}$.

Definition 3.2. A point process is a measurable mapping from our underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$ to the space of all counting measures $(N_\sigma, \mathcal{N}_\sigma)$, i.e. for all $B \in \mathcal{X}$ and all $k \in \mathbb{N}_0$ it holds that

$$\{\omega \in \Omega : \eta(\omega)(B) = k\} \in \mathcal{F}.$$

To shorten our notation we write $\eta(B)$ for the random variable $\eta(\omega)(B)$. Thus a point process η is a discrete measure having mass concentrated at random points in the underlying space \mathbb{X} . To simplify our Notation we will often handle η as a random set of points given by

$$x \in \eta \Leftrightarrow x \in \{y \in \mathbb{X} : \eta(\{y\}) > 0\}.$$

The intensity measure μ of a point process η on the space $(\mathbb{X}, \mathcal{X})$ is defined as the expected number of points of η contained in the set B . Hence

$$\mu(B) := \mathbb{E}[\eta(B)], \quad B \in \mathcal{X}.$$

Definition 3.3. Consider a σ -finite measure μ on \mathbb{X} . Then a Poisson point process η with intensity measure μ on \mathbb{X} satisfies the following properties:

- (i) For all $B \in \mathcal{X}$ and all $k \in \mathbb{N}_0$ it holds, that $\eta(B) \stackrel{d}{\sim} \text{Po}_{\mu(B)}$, i.e.,

$$\mathbb{P}(\eta(B) = k) = \frac{\mu(B)^k}{k!} e^{-\mu(B)},$$

and for $\mu(B) = \infty$, we set $\frac{\infty^k}{k!} e^{-\infty} = 0$ for all k .

- (ii) For all $m \in \mathbb{N}$ and all pairwise disjoint measurable sets $B_1, \dots, B_m \in \mathcal{X}$, the random variables $\eta(B_1), \dots, \eta(B_m)$ are independent.

In the case $\mathbb{X} = \mathbb{R}^d$ in which we are interested in this paper, a point process is called stationary if its intensity measure μ is invariant under translations. This implies that $\mu = t \cdot \Lambda_d$, where $t > 0$ and Λ_d denotes the d -dimensional Lebesgue measure.

Remark 3.4 (Cube construction). We consider the space $(\mathbb{R}^d, \mathcal{B}^d)$, where \mathcal{B}^d denotes the Borel- σ -algebra on \mathbb{R}^d and divide the space \mathbb{R}^d into countable many cubes along the lattice \mathbb{Z}^d , i.e.

$$\mathbb{R}^d = \bigcup_{v \in \mathbb{Z}^d} (v + [0, 1]^d),$$

where we used $v + [0, 1]^d$ as short notation for the set $\{v + x \in \mathbb{R}^d : x \in [0, 1]^d\}$. This allows us to construct a stationary Poisson point process η on \mathbb{R}^d with intensity measure $\mu := t\Lambda_d$ where $t > 0$ and Λ_d denotes the d -dimensional Lebesgue measure. We are taking a sequence $(P_v)_{v \in \mathbb{Z}^d}$ of independent Poisson distributed random variables with mean value $\mu([0, 1]^d) = t$ and a double indexed sequence $(X_{(v,j)})_{(v,j) \in \mathbb{Z}^d \times \mathbb{N}}$ of independent uniformly distributed random variables on the unit cube $[0, 1]^d$. Note that we assume independence of all random variables used, especially we assume that $(X_{(v,j)})$ and $(P_v)_{v \in \mathbb{Z}^d}$ are independent. Now for every $v \in \mathbb{Z}^d$ we use the Poisson random variable to determine the number of points we load into the Cube $v + [0, 1]^d$ and we use the uniform random variables $X_{(v,j)}$, $j \in \mathbb{N}$ to determine the position of the points to add in the Cube. The Poisson point process η on \mathbb{R}^d is given by

$$\eta := \sum_{v \in \mathbb{Z}^d} \sum_{j=1}^{P_v} \delta_{v+X_{(v,j)}},$$

where δ_w denotes the Dirac-measure with mass concentrated in the point $w \in \mathbb{R}^d$. The above defined measure η is a Poisson point process on \mathbb{R}^d with intensity measure $\mu = t\Lambda_d$.

4. Random simplicial complexes

In this section we apply the Vietoris-Rips and the Čech complex construction to the random set of points given by a Poisson point process η .

Definition 4.1. Let η be a stationary Poisson point process on \mathbb{R}^d . We define the Poissonized versions of the Vietoris-Rips complex and Čech complex by replacing the fixed point set in Definitions 2.2 and 2.3 by the random point set that is given by η , i.e.

$$\text{VR}(\eta, \rho) = \text{VR}(\{x \in \mathbb{R}^d : \eta(\{x\}) > 0\}, \rho),$$

and

$$\text{C}(\eta, \rho) = \text{C}(\{x \in \mathbb{R}^d : \eta(\{x\}) > 0\}, \rho).$$

In a first step one should be interested in the question whether this construction yields isolated bounded components or one or several connected unbounded components. This was answered by Meester and Roy [25, 26].

Theorem 4.2 (Percolation). *For each $\rho > 0$ there exists $t_{\text{perc}}(\rho) > 0$, such that for every stationary Poisson point process η on \mathbb{R}^d with intensity $t < t_{\text{perc}}(\rho)$ the random Čech complex $\text{C}(\eta, \rho)$ contains no unbounded connected simplicial complex with probability one. If the intensity satisfies $t > t_{\text{perc}}(\rho)$ then with probability one the random Čech complex $\text{C}(\eta, \rho)$ contains precisely one unbounded connected simplicial complex \mathcal{B}_∞ , a unique giant component in $\text{C}(\eta, \rho)$.*

To simplify notation, we use $\mathcal{S}_{\text{VR},d}$ to denote the class of all simplicial complexes that admit a d -dimensional Vietoris-Rips representation. Similarly, $\mathcal{S}_{\text{C},d}$ denotes the class of all simplicial complexes having a d -dimensional Čech representation. As it will usually be clear from the context, which dimension d we are considering, we will mostly omit d from the notation and just write \mathcal{S}_{VR} and \mathcal{S}_{C} in the following.

Theorem 4.3. *For every simplicial complex $K \in \mathcal{S}_{\text{VR},d}$ with vertex set V and parameter $\rho > 0$ and every stationary Poisson point process η on \mathbb{R}^d with intensity $t > 0$ the infinite Vietoris-Rips complex $\text{VR}(\eta, \rho)$ contains almost surely infinitely many isolated simplicial complexes K_n , $n \in \mathbb{N}$ that are combinatorially equivalent to K .*

Theorem 4.4. *For every simplicial complex $K \in \mathcal{S}_{\text{VR},d}$ with vertex set V and parameter $\rho > 0$ and every stationary Poisson point process η on \mathbb{R}^d with intensity $t > t_{\text{perc}}(\rho)$ the infinite Vietoris-Rips complex $\text{VR}(\eta, \rho)$ contains a giant unbounded connected simplicial complex \mathcal{B}_∞ which contains almost surely infinitely many sub complexes K_n , $n \in \mathbb{N}$ that are combinatorially equivalent to K and only connected to \mathcal{B}_∞ by an edge.*

By replacing the VR operator with the C operator, both theorems can be formulated for the class of Čech complexes too.

Theorem 4.5. *For every simplicial complex $K \in \mathcal{S}_{\text{C},d}$ with vertex set V and parameter $\rho > 0$ and every stationary Poisson point process η on \mathbb{R}^d with intensity $t > 0$ the infinite Čech complex $\text{C}(\eta, \rho)$ contains almost surely infinitely many isolated simplicial complexes K_n , $n \in \mathbb{N}$ that are combinatorially equivalent to K .*

Theorem 4.6. *For every simplicial complex $K \in \mathcal{S}_{\text{VR},d}$ with vertex set V and parameter $\rho > 0$ and every stationary Poisson point process η on \mathbb{R}^d with intensity $t > t_{\text{perc}}(\rho)$ the infinite Čech complex $\mathcal{C}(\eta, \rho)$ contains a giant unbounded connected simplicial complex \mathcal{B}_∞ which contains almost surely infinitely many sub complexes K_n , $n \in \mathbb{N}$ that are combinatorially equivalent to K and only connected to \mathcal{B}_∞ by an edge.*

Proof of Theorems 4.3 and 4.5. Let $K \in \mathcal{S}_{\text{VR},d}$ with vertex set $V = \{v_0, \dots, v_n\}$ and parameter $\rho > 0$, then $K = \text{VR}(V, \rho)$ by definition of $\mathcal{S}_{\text{VR},d}$. Further let η be a stationary Poisson point process on \mathbb{R}^d with intensity $t > 0$. We denote by $\alpha > 0$ the smallest distance of two vertices of K , i.e.,

$$\alpha = \min_{\substack{v, v' \in V \\ v \neq v'}} \|v - v'\|_2.$$

It follows from Theorem 2.9 that K is generic and therefore by Definition 2.6 there exists a $\delta_0 > 0$ such that for every point set X with $d(X, V) < \delta_0$ the complex $\text{VR}(X, \rho)$ is combinatorially equivalent to K .

We set $\delta := \min(\frac{\alpha}{2}, \delta_0)$ to ensure, that all balls with radius δ around the vertices of V are pairwise disjoint.

We define the surrounding box W of the set V and its coordinate width β by

$$W = \prod_{i=1}^d [a_i, b_i], \quad \beta = \max_{i=1}^d |a_i - b_i|,$$

where $a_i, b_i \in \mathbb{R}$ are the minimum resp. the maximum value of the i -th entries of the position vectors of all coordinates, i.e.,

$$a_i = \min_{v \in V} v_i, \quad b_i = \max_{v \in V} v_i.$$

For $\theta \in \gamma\mathbb{Z}^d$ with $\gamma := \beta + 2(\delta + \rho)$ we construct the two translated and extended boxes

$$\begin{aligned} \theta + W_I &= \{\theta\} + W + \delta B_1^d, & \text{inner box,} \\ \theta + W_O &= \{\theta\} + W + (\delta + \rho) B_1^d, & \text{outer box,} \end{aligned}$$

as Minkowski addition of W with the scaled d -dimensional open unit ball δB_1^d and the translation direction θ . Further we denote by $\theta + V$ the translated vertex set $\{\theta + v : v \in V\}$.

Now the randomly generated Vietoris-Rips complex $\text{VR}(\eta \cap (\theta + W_I), \rho)$ will be investigated, in order to show that it is combinatorially equivalent to K with a positive probability that is not depending on the translation $\theta \in \gamma\mathbb{Z}^d$.

Lemma 4.7. *Let A_θ be the event that the Vietoris-Rips complex $\text{VR}(\eta \cap (\theta + W_I), \rho)$ is combinatorially equivalent to K , i.e.,*

$$A_\theta := \{\text{VR}(\eta \cap (\theta + W_I)) \text{ is combinatoric equivalent to } K\}.$$

Then there exists a constant $c_A \in (0, 1]$ such that $\mathbb{P}(A_\theta) \geq c_A$ for all $\theta \in \gamma\mathbb{Z}^d$.

Proof. From Definition 2.6 it follows directly that if $d(\eta \cap (\theta + W_I), \theta + V) < \delta$ holds,

then the simplicial complexes $\text{VR}(\eta \cap (\theta + W_I), \rho)$ and K are combinatorially equivalent. Therefore

$$A_\theta \supseteq \{d(\eta \cap (\theta + W_I), \theta + V) < \delta\},$$

where the right hand side denotes the event that every open ball with radius δ around every translated vertex $v \in \theta + V$ contains exactly one point of η , and that there are no points in $\eta \cap (\theta + W_I)$ outside of these balls. Further by the choice of δ all balls are pairwise disjoint,

$$A_\theta \supseteq \left\{ \eta(B_\delta^d(v)) = 1, \forall v \in V \text{ and } \eta\left(W_I \setminus \bigcup_{v \in V} B_\delta^d(v)\right) = 0 \right\},$$

where we used the translation invariance of η to set $\theta = 0$. By the independence property of the Poisson point process η it follows that

$$\begin{aligned} \mathbb{P}(A_\theta) &\geq \left(\prod_{v \in V} \mathbb{P}(\eta(B_\delta^d(v)) = 1) \right) \cdot \mathbb{P}\left(\eta\left(W_I \setminus \bigcup_{v \in V} B_\delta^d(v)\right) = 0\right) \\ &= \prod_{v \in V} (t\kappa_d \delta^d \exp(-t\kappa_d \delta^d)) \cdot \exp(-(t\Lambda_d(W_I) - |V|t\kappa_d \delta^d)) \\ &= (t\kappa_d \delta^d)^{n+1} \exp(-(n+1)t\kappa_d \delta^d - t\Lambda_d(W_I) + (n+1)t\kappa_d \delta^d) \\ &= (t\kappa_d \delta^d)^{n+1} \exp(-t\Lambda_d(W_I)) =: c_A > 0, \end{aligned}$$

where κ_d denotes the volume of the d -dimensional unit ball. \square

In the next step we will define the event B_θ to ensure that the simplicial complex given by $\text{VR}(\eta \cap (\theta + W_I), \rho)$ is isolated. Again we will show that this event has a positive probability not depending on the translation $\theta \in \gamma\mathbb{Z}^d$.

Lemma 4.8. *Let B_θ be the event that the Poisson point process η has no points in the ρ -neighborhood around $\theta + W_I$, i.e.,*

$$B_\theta := \{\eta((\theta + W_O) \setminus (\theta + W_I)) = 0\}.$$

Then there exists a constant $c_B \in (0, 1]$ such that $\mathbb{P}(B_\theta) = c_B$ for all $\theta \in \gamma\mathbb{Z}^d$.

Proof. By definition we have $W_I \subset W_O$ and therefore it follows from the translation in-variance that

$$\mathbb{P}(B_\theta) = \exp(-t(\Lambda_d(W_O) - \Lambda_d(W_I))) =: c_B > 0,$$

for all $\theta \in \gamma\mathbb{Z}^d$. \square

To prove our main result, it is sufficient to show that there existing infinitely many translations $\theta \in \gamma\mathbb{Z}^d$ such that $E_\theta := A_\theta \cap B_\theta$ occurs. The event A_θ states that $\text{VR}(\eta \cap (\theta + W_I), \rho)$ is combinatorially equivalent to K and B_θ ensures that this simplicial complex can not be connected to any other point that is not already in the vertex set $\eta \cap (\theta + W_I)$. Note that for all $y \in \eta \setminus (\theta + W_I)$ and for all $x \in \eta \cap (\theta + W_I)$ we have $\|y - x\|_2 > \rho$.

Further it follows from the independence property of η that the events A_θ and B_θ are independent and therefore by Lemmas 4.7 and 4.8 we have

$$\mathbb{P}(E_\theta) = \mathbb{P}(A_\theta)\mathbb{P}(B_\theta) \geq c_A c_B =: c \in (0, 1]$$

for all $\theta \in \gamma\mathbb{Z}^d$. Now we are ready to use the well known Borel-Cantelli lemma:

Lemma 4.9 (Borel-Cantelli). *Let E_1, E_2, \dots be a sequence of pairwise independent events on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with $\sum_{i=1}^\infty \mathbb{P}(E_i) = \infty$.*

$$\text{Then } \mathbb{P}\left(\limsup_{i \rightarrow \infty} E_i\right) = 1.$$

For $\theta_1, \theta_2 \in \gamma\mathbb{Z}^d$ with $\theta_1 \neq \theta_2$ the equality $(\theta_1 + W_O) \cap (\theta_2 + W_O) = \emptyset$ follows from the definition of γ . Hence the events E_{θ_1} and E_{θ_2} are independent. Further there exists a constant $c := c_A c_B \in (0, 1]$ such that $\mathbb{P}(E_\theta) \geq c$ for all $\theta \in \gamma\mathbb{Z}^d$.

Since \mathbb{Z}^d is countable, choosing a bijection $\varphi: \mathbb{N} \rightarrow \gamma\mathbb{Z}^d$ implies that

$$\sum_{\theta \in \gamma\mathbb{Z}^d} \mathbb{P}(E_\theta) = \sum_{i=1}^\infty \mathbb{P}(E_{\varphi(i)}) \geq \sum_{i=1}^\infty c = \infty.$$

Using the Borel-Cantelli Lemma on the series $E_{\varphi(1)}, E_{\varphi(2)}, \dots$ yields

$$\mathbb{P}\left(\limsup_{i \rightarrow \infty} E_{\varphi(i)}\right) = 1,$$

and thus

$$\begin{aligned} \mathbb{P}(E_{\varphi(i)} \text{ for infinitely many } i \in \mathbb{N}) &= \mathbb{P}(E_\theta \text{ for infinitely many translations } \theta \in \gamma\mathbb{Z}^d) \\ &= 1, \end{aligned}$$

which completes the proof of Theorem 4.3. The proof of Theorem 4.5 is similar. \square

The proofs of Theorem 4.3 and Theorem 4.5 use the Borel-Cantelli Lemma in its original form and thus rely essentially on the strong independence property of the Poisson point process. In the following proof of Theorem 4.4 and Theorem 4.6 this is no longer possible. If two complexes are both joined via the giant component of the simplicial complex, they are not independent and a careful analysis of their dependency structure is necessary.

Proof of Theorems 4.4 and 4.6. Let η be a stationary Poisson point process on \mathbb{R}^d with intensity $t > t_{\text{perc}}(\rho)$. Note that in this case the lower bound for the intensity is depending on the distance parameter of the simplicial complex, to ensure that the 1-skeleton of $\text{VR}(\eta, \rho)$ percolates and thus there exists a giant unbounded connected component, denoted as \mathcal{B}_∞ , in $\text{VR}(\eta, \rho)$, see Theorem 4.2.

For $K \in \mathcal{S}_{\text{VR},d}$ with vertex set $V = \{v_0, \dots, v_n\}$ and parameter $\rho > 0$, we have $K = \text{VR}(V, \rho)$ by definition of $\mathcal{S}_{\text{VR},d}$. Denote by $\text{conv}(V)$ the convex hull of the set V and choose one vertex on the boundary of the convex hull as the connection vertex $v_A \in \partial \text{conv}(V) \cap V$ to \mathcal{B}_∞ . Let \mathbb{H} be a $(d-1)$ -dimensional supporting hyperplane such that $\mathbb{H} \cap \text{conv}(V) \subseteq \partial \text{conv}(V)$ and denote by \mathbb{H}^\pm the corresponding open half spaces such that $\mathbb{H}^+ \cap \text{conv}(V) = \emptyset$. Let $\vec{u} \in \mathbb{R}^d$ be the normal vector of \mathbb{H} pointing into the half space \mathbb{H}^+ with $\|\vec{u}\|_2 = 1$. To shorten our notation, we set $V^* := V \setminus \{v_A\}$.

For $\varepsilon \in (0, \frac{\rho}{4})$ we define the linkage vertex $v_L \in \mathbb{H}^+$ by $v_L := v_A + (\rho - 2\varepsilon)\vec{u}$ and for all $x \in B_\varepsilon^d(v_A)$ and $y \in B_\varepsilon^d(v_L)$ it follows directly by the Triangle inequality, that $\|x - y\|_2 < \rho$. Note that, by continuity, there exists an $\varepsilon_0 > 0$ small enough, such that for all $y \in B_{\varepsilon_0}^d(v_L)$ it holds that $B_\rho^d(y) \cap B_{\varepsilon_0}^d(v) = \emptyset$ for all $v \in V^*$.

At this point, we conclude three important consequences of our construction by choosing the parameter δ in a proper way:

Remark 4.10. Let $\delta := \min(\frac{\alpha}{2}, \delta_0, \varepsilon_0)$, where α and δ_0 are as in the proof of Theorem 4.3 before, then

- (i) all balls with radius δ around all vertices of V are pairwise disjoint,
- (ii) for all sets X with $d(X, V) < \delta$ it follows that $\text{VR}(X, \rho)$ is combinatorially equivalent to K , and
- (iii) every point $x \in B_\delta^d(v_L)$ is forced to connect to every point $y \in B_\delta^d(v_A)$, but not allowed to connect to any other point $z \in B_\delta^d(v)$ for all $v \in V^*$.

We continue with the geometric construction to prove our main result by defining the extended convex hulls given by

$$\begin{aligned} U_I &= \text{conv}(V) + \delta B_1^d, & \text{inner hull,} \\ U_O &= \text{conv}(V) + (\delta + \rho)B_1^d, & \text{outer hull,} \end{aligned}$$

and the two surrounding boxes $W_I \subset W_O$ of the outer hull U_O that have their vertices on the lattice $s\mathbb{Z}^d$ with $s := \frac{\rho}{4\sqrt{d}}$ given by

$$W_I = \prod_{i=1}^d [a_i, b_i], \quad W_O = \prod_{i=1}^d [-sk + a_i, b_i + sk],$$

where $k \in \mathbb{N}$, $k > 4\sqrt{d}$ and $a_i, b_i \in s\mathbb{Z}$ are given by

$$a_i = \max_{y \in s\mathbb{Z}} \left\{ y : y \leq \min_{x \in U_O} \{x_i\} \right\}, \quad b_i = \min_{y \in s\mathbb{Z}} \left\{ y : y \geq \max_{x \in U_O} \{x_i\} \right\}.$$

Further we define the “hitbox” W_H as extension of the box W_O by

$$W_H = W_O + \frac{\rho}{2\sqrt{d}}B_1^d.$$

We denote by γ_i the coordinate width of W_H in direction of the i -th standard basis vector \vec{e}_i . Clearly, by definition of W_H it holds that $\gamma_i > \rho + \frac{\rho}{\sqrt{d}}$ for all $i \in \{1, \dots, d\}$ and additionally γ_i is an integer multiple of the lattice width $s = \frac{\rho}{4\sqrt{d}}$.

Now we can conclude two important properties of our lattice $s\mathbb{Z}^d$ in the following lemmas:

Lemma 4.11. *For all $x, y \in s\mathbb{Z}^d$ with $x \neq y$ the events*

$$\left\{ \eta\left(B_{\frac{\delta}{2}}^d(x)\right) = 1 \right\} \quad \text{and} \quad \left\{ \eta\left(B_{\frac{\delta}{2}}^d(y)\right) = 1 \right\}$$

are independent.

Proof. It follows directly from $x \neq y$ that the distance between the lattice points is bounded from below by the lattice width. Thus $\|x - y\|_2 \geq s$ and $B_{\frac{\delta}{2}}^d(x) \cap B_{\frac{\delta}{2}}^d(y) = \emptyset$.

The claim then follows directly from the independence property of the Poisson point process η . \square

Lemma 4.12. *Let L^* be the set of all lattice points $z \in s\mathbb{Z}^d$ that are contained in the interior of the shell $W_O \setminus W_I$. If $X \subset \mathbb{R}^d$ is a set of points with $d(X, L^*) < \frac{s}{2} = \frac{\rho}{8\sqrt{d}}$, then the balls with radius ρ around the points of X are covering the inner shell $W_O \setminus W_I$ and the outer shell $W_H \setminus W_O$, i.e.*

$$W_H \setminus W_I = (W_O \setminus W_I) \cup (W_H \setminus W_O) \subset \bigcup_{x \in X} B_\rho^d(x).$$

Proof. First, let $y \in W_O \setminus W_I$ be a point in the inner shell, then by definition of L^* there exists a lattice point $z \in L^*$ such that $y = z + v$ for some v with $\|v\|_2 \leq \sqrt{d}s = \frac{\rho}{4}$. Further, for every lattice point $z \in L^*$ there exists a unique point $x \in X$ such that $\|z - x\|_2 < \frac{s}{2} = \frac{\rho}{8\sqrt{d}}$, and thus it follows by the Triangle inequality that $\|y - x\|_2 \leq \frac{\rho}{4} + \frac{\rho}{8\sqrt{d}} < \rho$. Second, let $y' \in W_H \setminus W_O$ be a point in the outer shell, then by definition of W_H there exists a point $y \in W_O \setminus W_I$ such that $\|y - y'\|_2 \leq \frac{\rho}{2\sqrt{d}}$. Thus $\|y' - x\|_2 \leq \|y' - y\|_2 + \|y - x\|_2 < \frac{\rho}{2\sqrt{d}} + \frac{\rho}{4} + \frac{\rho}{8\sqrt{d}} < \rho$. Finally, the desired result follows directly from these two steps. \square

For $\theta \in \gamma\mathbb{Z}^d := \gamma_1\mathbb{Z} \times \dots \times \gamma_d\mathbb{Z}$ we define the translated sets $(\theta + U_I)$, $(\theta + U_O)$, $(\theta + W_I)$, $(\theta + W_O^{(k)})$, $V(\theta)$ and $(\theta + V^*)$ as Minkowski addition of $\{\theta\}$ to the corresponding set. Also we define the translated connection point $\theta + v_A$ and linkage point $\theta + v_L$, but to shorten the Notation we will omit to specify θ in the following, when all sets and points are considered to be translated with the same vector $\theta \in \gamma\mathbb{Z}^d$. Further, by using the translation in-variance of our Poisson point process η the following results are not depending on the choice of $\theta \in \gamma\mathbb{Z}^d$, thus w.l.o.g. we can set $\theta = 0$ to simplify the following proofs.

To make use of Lemma 4.12 we will now define the event that the set of points in the Poisson point process contained in the shell $W_O \setminus W_I$ has a distance smaller than $\frac{s}{2}$ to the set L^* . The task is to show that this event has a positive probability not depending on the translation $\theta \in \gamma\mathbb{Z}^d$.

Lemma 4.13. *Let B_θ be the event that the set $\eta \cap (\theta + (W_O \setminus W_I))$ has distance smaller than $\frac{s}{2}$ to the set $\theta + L^*$, i.e.,*

$$B_\theta := \left\{ d(\eta \cap (\theta + (W_O \setminus W_I)), \theta + L^*) < \frac{s}{2} \right\}.$$

Then there exists a constant $c_B \in (0, 1]$ such that $\mathbb{P}(B_\theta) \geq c_B$ for all $\theta \in \gamma\mathbb{Z}^d$.

Proof. Note that L^* is a finite set of points and denote by $l := |L^*|$ the cardinality of L^* . Similar to the proof of Lemma 4.7 we can rewrite the event B_θ and by using the independence gained from Lemma 4.11 we have

$$\begin{aligned} \mathbb{P}(B_\theta) &= \mathbb{P}(B_0) = \left(\prod_{x \in L^*} \mathbb{P}\left(\eta(B_{\frac{s}{2}}^d(x)) = 1\right) \right) \cdot \mathbb{P}\left(\eta\left((W_O \setminus W_I) \setminus \bigcup_{x \in L^*} B_{\frac{s}{2}}^d(x)\right)\right) \\ &= (t\kappa_d(\frac{s}{2})^d)^l \exp(-lt\kappa_d(\frac{s}{2})^d - t\Lambda_d(W_O \setminus W_I) + lt\kappa_d(\frac{s}{2})^d) \\ &= (t\kappa_d(\frac{s}{2})^d)^l \exp(-t\Lambda_d(W_O \setminus W_I)) =: c_B > 0. \end{aligned} \quad \square$$

As in the proof of Theorem 4.3 we can show that the randomly generated simplicial complex $\text{VR}(\eta \cap (\theta + U_I), \rho)$ is combinatorially equivalent to K with a positive probability that is not depending on the translation $\theta \in \gamma\mathbb{Z}^d$.

Lemma 4.14. *Let A_θ be the event that the Vietoris-Rips complex $\text{VR}(\eta \cap (\theta + U_I), \rho)$ is combinatorially equivalent to K , i.e.,*

$$A_\theta := \{\text{VR}(\eta \cap (\theta + U_I), \rho) \text{ is combinatorially equivalent to } K\}.$$

Then there exists a constant $c_A \in (0, 1]$ such that $\mathbb{P}(A_\theta) \geq c_A$ for all $\theta \in \gamma\mathbb{Z}^d$.

Proof. Note that $\delta < \frac{\rho}{4}$ ensures that $B_\delta^d(v_L) \cap U_I = \emptyset$ and therefore, by using the same arguments as in the proof of Lemma 4.7, we obtain the desired result for

$$c_A := (t\kappa_d\delta^d)^{n+1} \exp(-t\Lambda_d(U_I)). \quad \square$$

In the next step we will ensure, that the randomly generated simplicial complex $\text{VR}(\eta \cap U_I, \rho)$ is forced to connect to $\eta \cap (W_O \setminus W_I)$ by the attachment vertex v_A and the linkage vertex v_L . Therefore we define the line $v_L(\zeta) = v_L + \zeta\vec{u}$, $\zeta \in [0, \infty)$ as continuation of the direct path from v_A to v_L . Note that we can choose a finite number of points $v_L(0) = v_L(\zeta_0), v_L(\zeta_1), \dots, v_L(\zeta_N)$ on the line such that

- (i) $v_L(\zeta_i) \in W_I$ for all $i = 0, \dots, N$,
- (ii) it holds that $\|v_L(\zeta_i) - v_L(\zeta_{i-1})\|_2 < \rho - 2\delta$ for all $i = 1, \dots, N$,
- (iii) $0 < \|v_L(\zeta_N) - z\|_2 < \frac{\rho}{2\sqrt{d}}$ for some $z \in \partial W_I$.

We will now use the translated versions of these points to construct the linkage from the random simplicial complex to the inner shell by defining the event that every ball with a positive but small radius δ' around these linkage points contains exactly one point of the Poisson point process and that the remaining space in the inner box (that is not already covered by U_I or by these balls) is empty. Denote by $VL = \{v_L(0), v_L(\zeta_1), \dots, v_L(\zeta_N)\}$ the set of linkage points and choose

$$\delta' := \min \left\{ \delta, \min_{z \in \partial W_I} \|v_L(\zeta_N) - z\|_2, \frac{1}{2} \min_{\substack{x, y \in VL \\ x \neq y}} \|x - y\|_2 \right\},$$

to ensure that all balls with radius δ' around the points in VL are pairwise disjoint and contained completely in the inner shell $W_I \setminus U_I$. Then we can define the event A'_θ to ensure that if A'_θ occurs, the connection from $\theta + v_A$ to the boundary of $\theta + W_I$ is established and we show that this event has a positive probability that is not depending on the translation $\theta \in \gamma\mathbb{Z}^d$:

Lemma 4.15. *Let A'_θ denote the event that all points of η that are contained in the space $\theta + (W_I \setminus U_I)$ are δ' close to the set $\theta + VL$, i.e.*

$$A'_\theta := \{d(\eta \cap (\theta + (W_I \setminus U_I)), \theta + VL) < \delta'\}.$$

Then there exists a constant $c'_A \in (0, 1]$ such that $\mathbb{P}(A'_\theta) \geq c'_A$ for all $\theta \in \gamma\mathbb{Z}^d$.

Proof. The cardinality of VL is given by $N + 1$ and the choice of δ' ensures, that we can rewrite this event in a similar manner like in the proof of the lemma before. It

follows that

$$\begin{aligned} \mathbb{P}(A'_\theta) &= \mathbb{P}(A'_0) = \left(\prod_{x \in VL} \mathbb{P}(\eta(B_{\delta'}^d(x)) = 1) \right) \cdot \mathbb{P} \left(\eta \left((W_I \setminus U_I) \setminus \bigcup_{x \in VL} B_{\delta'}^d(x) \right) \right) \\ &= (t\kappa_d \delta'^d)^{N+1} \exp(-(N+1)t\kappa_d \delta'^d - t\Lambda_d(W_I \setminus U_I) + (N+1)t\kappa_d \delta'^d) \\ &= (t\kappa_d \delta'^d)^{N+1} \exp(-t\Lambda_d(W_I \setminus U_I)) =: c'_A. \quad \square \end{aligned}$$

We conclude the results of the previous construction in the following lemma:

Lemma 4.16. *For every $\theta \in \gamma\mathbb{Z}^d$, let T_θ denote the event that the simplicial complex $\text{VR}(\eta \cap (\theta + U_I), \rho)$ is combinatorially equivalent to K and connected to a set of points in the outer shell $\theta + (W_O \setminus W_I)$; moreover, these points are the center points of balls with radius ρ which cover the hitbox $\theta + W_H$. Then there exists a constant $c_T \in (0, 1]$ such that $\mathbb{P}(T_\theta) \geq c_T$ for all $\theta \in \gamma\mathbb{Z}^d$.*

Proof. It follows directly from our construction that the occurrence of A_θ , A'_θ and B_θ at the same translation θ implies that also T_θ occurs at this translation θ . Therefore it follows that

$$T_\theta \supseteq A_\theta \cap A'_\theta \cap B_\theta.$$

Further the events A_θ , A'_θ and B_θ are defined on the pairwise disjoint sets $\theta + U_I$, $\theta + (W_I \setminus U_I)$ and $\theta + (W_O \setminus W_I)$, thus it follows directly by the independence property of the Poisson point process η that

$$\mathbb{P}(T_\theta) \geq \mathbb{P}(A_\theta \cap A'_\theta \cap B_\theta) = \mathbb{P}(A_\theta) \cdot \mathbb{P}(A'_\theta) \cdot \mathbb{P}(B_\theta) \geq c_A c'_A c_B =: c_T. \quad \square$$

To continue with the proof, we will investigate the event that the giant unbounded component \mathcal{B}_∞ “meets” the box W_H in a way, such that it is forced to connect to the points inside the box $\theta + W_H$, given T_θ occurs. We can show, that the event H_θ has a positive probability that is not depending on the translation $\theta \in \gamma\mathbb{Z}^d$:

Lemma 4.17. *Denote by H_θ the event that there exists a vertex of \mathcal{B}_∞ contained in $\theta + (W_H \setminus W_O)$. Then there exists a constant $c_H \in (0, 1]$ such that $\mathbb{P}(H_\theta) \geq c_H$ for all $\theta \in \gamma\mathbb{Z}^d$.*

Proof. Assume, that $c_H = 0$, then it follows from the translation in-variance of η that $\mathbb{P}(H_\theta) = 0$ for all translations $\theta \in \mathbb{R}^d$. Further we can find a countable index set $I \subset \mathbb{R}^d$ of translations such that $\bigcup_{\theta \in I} (\{\theta\} + (W_H \setminus W_O)) = \mathbb{R}^d$, thus it follows

$$\begin{aligned} \mathbb{P}(\exists x \in \mathcal{B}_\infty) &= \mathbb{P} \left(\exists x \in \mathcal{B}_\infty \cap \bigcup_{\theta \in I} (\{\theta\} + (W_H \setminus W_I)) \right) \\ &\leq \mathbb{P} \left(\exists x \in \bigcup_{\theta \in I} (\mathcal{B}_\infty \cap (\{\theta\} + (W_H \setminus W_I))) \right) \\ &\leq \sum_{\theta \in I} \mathbb{P}(\exists x \in \mathcal{B}_\infty \cap (\{\theta\} + (W_H \setminus W_I))) = \sum_{\theta \in I} 0 = 0, \end{aligned}$$

which implies, that there does not exist a giant unbounded component in \mathbb{R}^d which is false, because we are in the case of percolation. \square

The construction of $\theta + W_H$ implies, that if H_θ and T_θ occur for the same translation $\theta \in \gamma\mathbb{Z}^d$, then the simplicial complex $\text{VR}(\eta \cap (\theta + U_I), \rho)$ is combinatorially equivalent to K and that it is connected to the giant unbounded component \mathcal{B}_∞ . Note that the event H_θ depends only on the points of η not contained in $\theta + W_O$, and the event T_θ depends only on the points of η contained in $\theta + W_O$. Thus the events H_θ and T_θ are independent. In the remaining part of this proof, we will show that the event $E_\theta = H_\theta \cap T_\theta$ occurs for infinitely many translations $\theta \in \gamma\mathbb{Z}^d$. To accomplish this, we will employ the following modified version of Borel-Cantelli stated in [32].

Lemma 4.18 (Modified Borel-Cantelli). *Let E_1, E_2, \dots be a sequence of events on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with $\sum_{i=1}^\infty \mathbb{P}(E_i) = \infty$ and*

$$\liminf_{n \rightarrow \infty} \frac{\sum_{i,j=1, i \neq j}^n [\mathbb{P}(E_i \cap E_j) - \mathbb{P}(E_i)\mathbb{P}(E_j)]}{(\sum_{i=1}^n \mathbb{P}(E_j))^2} = 0. \tag{4.1}$$

Then $\mathbb{P}\left(\limsup_{i \rightarrow \infty} E_i\right) = 1$.

Note that E_{θ_1} and E_{θ_2} are not independent for $\theta_1, \theta_2 \in \gamma\mathbb{Z}^d$, but there exists a series of translations $(\theta_i)_{i \in \mathbb{N}_0}$ such that (4.1) holds for the Events $(E_{\theta_i})_{i \in \mathbb{N}_0}$.

Lemma 4.19. *There exist a series of integers $k_i \in \mathbb{N}$ such that the following holds: For $\theta_0 := 0$ and $\theta_i := \theta_{i-1} + \gamma e_1 k_i$ we have*

$$|\mathbb{P}(E_{\theta_i} \cap E_{\theta_j}) - \mathbb{P}(E_{\theta_i})\mathbb{P}(E_{\theta_j})| \leq \frac{1}{i^2},$$

for all $i, j \in \mathbb{N}_0$ with $j > i$.

Proof. Set $\theta_0 = 0$ and assume that k_1, \dots, k_i and thus $\theta_1, \dots, \theta_i$ have already been chosen. For sufficiently large $R > 0$, consider the translation $\theta_R := \theta_i + R e_1$ and the two half-spaces

$$\mathbb{H}^- := \left\{ \langle x, e_1 \rangle \leq \frac{\|\theta_i + \theta_R\|}{2} \right\} \quad \text{and} \quad \mathbb{H}^+ := \left\{ \langle x, e_1 \rangle > \frac{\|\theta_i + \theta_R\|}{2} \right\},$$

where e_1 denotes the first element of the standard basis of \mathbb{R}^d . Note that \mathbb{H}^- and \mathbb{H}^+ are bounded by the hyperplane that is orthogonal to the connecting line from θ_i to θ_R and contains the center of this line.

We denote by $\eta^\pm := \eta \cap \mathbb{H}^\pm$ the restrictions of η to the half-spaces. Observe that η^+ and η^- are two independent Poisson point processes and since there is percolation, we have an unbounded connected component C^\pm in each halfspace. Further, these unbounded components C^\pm connect to the single giant component \mathcal{B}_∞ in the space $\mathbb{R}^d = \mathbb{H}^+ \cup \mathbb{H}^-$.

Denote by D^- the event that $\theta_i + W_H$ connects to C^- within \mathbb{H}^- , and by D^+ the event that $\theta_R + W_H$ connects to C^+ within \mathbb{H}^+ . Then the independence property of the Poisson point process first shows that D^+ and D^- are independent, and further that

$$\mathbb{P}(E_{\theta_i} \cap E_{\theta_R} \cap (D^- \cap D^+)) = \mathbb{P}(E_{\theta_i} \cap D^-)\mathbb{P}(E_{\theta_R} \cap D^+). \tag{4.2}$$

Furthermore the event E_{θ_i} occurs if $\theta_i + W_H$ connects to $C^- \cup C^+$. If in addition D^- holds, then it connects even within C^- . In other words, $E_{\theta_i} \cap (D^-)^c$ only occurs if

$\theta_i + W_H$ connects to $C^- \cup C^+$ via C^+ . But this implies, that there is a component \tilde{C} connecting $\theta_i + W_H$ to \mathbb{H}^+ and avoiding C^- . For $R \rightarrow \infty$ this means that \tilde{C} is a second unbounded connected component in the half space \mathbb{H}^- which avoids C^- , which has probability zero. Hence

$$\lim_{R \rightarrow \infty} \mathbb{P}(E_{\theta_i} \cap (D^-)^C) = 0 \quad (4.3)$$

and by the translation and rotation invariance it follows, that

$$\lim_{R \rightarrow \infty} \mathbb{P}(E_{\theta_R} \cap (D^+)^C) = 0. \quad (4.4)$$

Combining these results yields

$$\begin{aligned} & |\mathbb{P}(E_{\theta_i} \cap E_{\theta_R}) - \mathbb{P}(E_{\theta_i})\mathbb{P}(E_{\theta_R})| \\ &= |\mathbb{P}(E_{\theta_i} \cap E_{\theta_R} \cap (D^- \cap D^+)) + \mathbb{P}(E_{\theta_i} \cap E_{\theta_R} \cap (D^- \cap D^+)^C) \\ &\quad - (\mathbb{P}(E_{\theta_i} \cap D^-) + \mathbb{P}(E_{\theta_i} \cap (D^-)^C))(\mathbb{P}(E_{\theta_R} \cap D^+) + \mathbb{P}(E_{\theta_R} \cap (D^+)^C))| \\ &\leq |\mathbb{P}(E_{\theta_i} \cap E_{\theta_R} \cap (D^- \cap D^+)) - \mathbb{P}(E_{\theta_i} \cap D^-)\mathbb{P}(E_{\theta_R} \cap D^+)| \\ &\quad + \mathbb{P}(E_{\theta_i} \cap E_{\theta_R} \cap (D^- \cap D^+)^C) + \mathbb{P}(E_{\theta_i} \cap D^-)\mathbb{P}(E_{\theta_R} \cap (D^+)^C) \\ &\quad + \mathbb{P}(E_{\theta_i} \cap (D^-)^C)\mathbb{P}(E_{\theta_R} \cap D^+) + \mathbb{P}(E_{\theta_i} \cap (D^-)^C)\mathbb{P}(E_{\theta_R} \cap (D^+)^C). \end{aligned}$$

Using (4.2) eliminates the first term. Further we can bound one factor in each of the last three terms by 1 and rewrite the complement in the second term which leads us to

$$\begin{aligned} & |\mathbb{P}(E_{\theta_i} \cap E_{\theta_R}) - \mathbb{P}(E_{\theta_i})\mathbb{P}(E_{\theta_R})| \\ &\leq \mathbb{P}(E_{\theta_i} \cap E_{\theta_R} \cap ((D^-)^C \cup (D^+)^C)) + \mathbb{P}(E_{\theta_R} \cap (D^+)^C) + 2\mathbb{P}(E_{\theta_i} \cap (D^-)^C) \\ &\leq \mathbb{P}(E_{\theta_i} \cap E_{\theta_R} \cap (D^-)^C) + \mathbb{P}(E_{\theta_i} \cap E_{\theta_R} \cap (D^+)^C) + \mathbb{P}(E_{\theta_R} \cap (D^+)^C) \\ &\quad + 2\mathbb{P}(E_{\theta_i} \cap (D^-)^C) \\ &\leq \mathbb{P}(E_{\theta_i} \cap (D^-)^C) + \mathbb{P}(E_{\theta_R} \cap (D^+)^C) + \mathbb{P}(E_{\theta_R} \cap (D^+)^C) + 2\mathbb{P}(E_{\theta_i} \cap (D^-)^C) \\ &\leq 2\mathbb{P}(E_{\theta_R} \cap (D^+)^C) + 3\mathbb{P}(E_{\theta_i} \cap (D^-)^C). \end{aligned}$$

Thus (4.3) and (4.4) imply

$$\lim_{R \rightarrow \infty} |\mathbb{P}(E_{\theta_i} \cap E_{\theta_R}) - \mathbb{P}(E_{\theta_i})\mathbb{P}(E_{\theta_R})| = 0.$$

Therefore, for every $i \in \mathbb{N}_0$ there exists a constant $R_i \in \mathbb{R}_{>0}$ such that

$$|\mathbb{P}(E_{\theta_i} \cap E_{\theta_R}) - \mathbb{P}(E_{\theta_i})\mathbb{P}(E_{\theta_R})| < \frac{1}{i^2}$$

for $R > R_i$. Set $k_{i+1} := \left\lceil \frac{R_i}{\gamma} \right\rceil$ and thus $\theta_{i+1} = \theta_i + \gamma e_1 k_{i+1}$. This proves the claim. \square

Finally, we can apply Lemma 4.18, the modified version of Borel-Cantelli, to the sequence of events $(E_{\theta_i})_{i \in \mathbb{N}_0}$ given by the translation sequence $(\theta_i)_{i \in \mathbb{N}_0}$ constructed in Lemma 4.19. To shorten the notation, set $E_i := E_{\theta_i}$.

Let $c_E := c_T c_H$. It follows directly from the independence of T_θ and H_θ together with Lemma 4.16 and Lemma 4.17 that

$$\sum_{i=1}^n \mathbb{P}(E_i) \geq n c_E,$$

and thus

$$\sum_{i=1}^{\infty} \mathbb{P}(E_i) = \infty.$$

Further, using Lemma 4.19 and the symmetry in i and j we have

$$\begin{aligned} \sum_{i,j=1, i \neq j}^n |\mathbb{P}(E_i \cap E_j) - \mathbb{P}(E_i)\mathbb{P}(E_j)| &= 2 \sum_{i=1}^n \sum_{j=i+1}^n |\mathbb{P}(E_i \cap E_j) - \mathbb{P}(E_i)\mathbb{P}(E_j)| \\ &\leq 2 \sum_{i=1}^n \sum_{j=i+1}^n \frac{1}{i^2} \leq 2n \sum_{i=1}^n \frac{1}{i^2} \leq \frac{\pi^2}{3}n. \end{aligned}$$

Thus for $c_E := c_T c_H$ it follows that

$$\frac{\sum_{i,j=1, i \neq j}^n |\mathbb{P}(E_i \cap E_j) - \mathbb{P}(E_i)\mathbb{P}(E_j)|}{(\sum_{i=1}^n \mathbb{P}(E_j))^2} \leq \frac{\pi^2 n}{3(nc_E)^2} = \frac{\pi^2}{3c_E^2} \cdot \frac{1}{n}$$

and therefore

$$\lim_{n \rightarrow \infty} \left| \frac{\sum_{i,j=1, i \neq j}^n [\mathbb{P}(E_i \cap E_j) - \mathbb{P}(E_i)\mathbb{P}(E_j)]}{(\sum_{i=1}^n \mathbb{P}(E_j))^2} \right| \leq \lim_{n \rightarrow \infty} \frac{\pi^2}{3c_E^2} \cdot \frac{1}{n} = 0,$$

which implies (4.1) and Lemma 4.18 yields

$$\mathbb{P}\left(\limsup_{i \rightarrow \infty} E_i\right) = 1,$$

and thus

$$\mathbb{P}(E_{\theta_i} \text{ for infinitely many } i \in \mathbb{N}) = 1$$

which completes the proof of Theorem 4.4. The proof of Theorem 4.6 is similar. \square

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