

## ON $d$ -CATEGORIES AND $d$ -OPERADS

TOMER M. SCHLANK AND LIOR YANOVSKI

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### Abstract

We extend the theory of  $d$ -categories, which are a strict model for  $(d, 1)$ -categories introduced by Lurie, by providing an explicit description of the right mapping spaces of the  $d$ -homotopy category of an  $\infty$ -category. Using this description, we deduce an invariant  $\infty$ -categorical characterization of the  $d$ -homotopy category. We then proceed to develop an analogous theory of  $d$ -operads, which model  $\infty$ -operads with  $(d - 1)$ -truncated multi-mapping spaces, and prove analogous results for them.

## 1. Introduction

### Overview & organization

The notion of a  $d$ -category was introduced by Lurie in [Lur09, 2.3.4], as a strict model for what we call an *essentially  $d$ -category*: an  $\infty$ -category<sup>1</sup> all of whose mapping spaces are homotopically  $(d - 1)$ -truncated. To any  $\infty$ -category  $\mathcal{C}$ , Lurie associates a  $d$ -category  $h_d\mathcal{C}$ , which we call the  *$d$ -homotopy category* of  $\mathcal{C}$ . While this  $d$ -category is shown to be universal in the 1-categorical (simplicially enriched) sense among  $d$ -categories that  $\mathcal{C}$  is mapped to [Lur09, 2.3.4.12], the question of how  $h_d\mathcal{C}$  relates to  $\mathcal{C}$  as an  $\infty$ -category is left unaddressed. The goal of this work is to fill this gap and to give an analogous treatment for operads.

In Section 2, we begin by showing that the right mapping spaces of  $h_d\mathcal{C}$  are given, up to *isomorphism*, by applying  $h_{d-1}$  to the right mapping spaces of  $\mathcal{C}$  (Proposition 2.13). This is the main technical result of this note, the proof of which goes through the comparison with the “middle mapping spaces”. From this we deduce that  $h_d\mathcal{C}$  is obtained from  $\mathcal{C}$  by  $(d - 1)$ -truncation of the mapping spaces. More precisely, we show that  $h_d$  can be promoted to a functor of  $\infty$ -categories, which is left adjoint to the inclusion of the full subcategory spanned by essentially  $d$ -categories into  $\mathbf{Cat}_\infty$ . And furthermore, that the unit map of this adjunction is essentially surjective and is given on mapping spaces by the  $(d - 1)$ -truncation map (Theorem 2.15).

In Section 3, we develop a parallel theory for operads. We call an  $\infty$ -operad an *essentially  $d$ -operad* if all of its multi-mapping spaces are  $(d - 1)$ -truncated. We

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<sup>1</sup>In the sense of [Lur09], i.e. a quasi-category.

begin by defining a notion of a  $d$ -operad (Definition 3.3) that relates to essentially  $d$ -operads in the same way that  $d$ -categories relate to essentially  $d$ -categories. We then define the  $d$ -homotopy operad functor (Definition 3.6), again by analogy with (and by means of) the  $d$ -homotopy category functor. This is achieved by analyzing the behavior of the  $d$ -homotopy category functor on inner and coCartesian fibrations (Proposition 3.5). Finally, we bootstrap the results of Section 2, to obtain analogous results for (essentially)  $d$ -operads (Theorem 3.12) and some corollaries.

This work grew out of a project whose goal is to generalize the classical Eckmann-Hilton argument to the  $\infty$ -categorical setting [SY].

**Conventions**

We work in the setting of  $\infty$ -categories (a.k.a. quasi-categories) and  $\infty$ -operads, relying heavily on the results of [Lur09] and [Lur]. Since we have numerous references to these two foundational works, references to [Lur09] are abbreviated as T.? and those to [Lur] as A.?. As a rule, we follow the notation of [Lur09] and [Lur] whenever possible. However, we supplement this notation and deviate from it in several cases in which we believe this enhances readability:

1. We abuse notation by identifying an ordinary category  $\mathcal{C}$  with its nerve  $N(\mathcal{C})$ .
2. We abbreviate the data of an  $\infty$ -operad  $p: \mathcal{O}^\otimes \rightarrow \mathbf{Fin}_*$  by  $\mathcal{O}$  and reserve the notation  $\mathcal{O}^\otimes$  for the  $\infty$ -category that is the source of  $p$ . Similarly, given two  $\infty$ -operads  $\mathcal{O}$  and  $\mathcal{U}$ , we write  $f: \mathcal{O} \rightarrow \mathcal{U}$  for a map of  $\infty$ -operads from  $\mathcal{O}$  to  $\mathcal{U}$ . The underlying  $\infty$ -category of  $\mathcal{O}$ , which in [Lur] is denoted by  $\mathcal{O}_{(1)}^\otimes$ , is here denoted by  $\underline{\mathcal{O}}$ .
3. Given two  $\infty$ -operads  $\mathcal{O}$  and  $\mathcal{U}$ , we denote by  $\text{Alg}_{\mathcal{O}}(\mathcal{U})$  the  $\infty$ -operad  $\text{Alg}_{\mathcal{O}}(\mathcal{U})^\otimes \rightarrow \mathbf{Fin}_*$  from Example A.3.2.4.4. This is the internal mapping object induced from the closed symmetric monoidal structure on  $\mathbf{Op}_\infty$  (see A.2.2.5.13). The underlying  $\infty$ -category  $\underline{\text{Alg}}_{\mathcal{O}}(\mathcal{U})$  is the usual  $\infty$ -category of  $\mathcal{O}$ -algebras in  $\mathcal{U}$  (which in [Lur] is denoted by  $\text{Alg}_{\mathcal{O}}(\mathcal{U})$ ). Moreover, the maximal Kan subcomplex  $\underline{\text{Alg}}_{\mathcal{O}}(\mathcal{U})^\simeq$  is the space of morphisms  $\text{Map}_{\mathbf{Op}_\infty}(\mathcal{O}, \mathcal{U})$  from  $\mathcal{O}$  to  $\mathcal{U}$  as objects of the  $\infty$ -category  $\mathbf{Op}_\infty$ .

**2.  $d$ -Categories**

Recall the following definition from classical homotopy theory.

**Definition 2.1.** For  $d \geq 0$ , a space  $X \in \mathcal{S}$  is called  $d$ -truncated if  $\pi_i(X, x) = 0$  for all  $i > d$  and all  $x \in X$ . In addition, a space is called  $(-2)$ -truncated if and only if it is contractible and it is called  $(-1)$ -truncated if and only if it is either contractible or empty. We denote by  $\mathcal{S}_{\leq d}$  the full subcategory of  $\mathcal{S}$  spanned by the  $d$ -truncated spaces. The inclusion  $\mathcal{S}_{\leq d} \hookrightarrow \mathcal{S}$  admits a left adjoint and we call the unit of the adjunction the  $d$ -truncation map.

This leads to the following definition in  $\infty$ -category theory.

**Definition 2.2.** Let  $d \geq -1$  be an integer. An *essentially  $d$ -category* is an  $\infty$ -category  $\mathcal{C}$  such that for all  $X, Y \in \mathcal{C}$ , the mapping space  $\text{Map}_{\mathcal{C}}(X, Y)$  is  $(d - 1)$ -truncated. We denote by  $\mathbf{Cat}_d$  the full subcategory of  $\mathbf{Cat}_\infty$  spanned by essentially  $d$ -categories.

*Example 2.3.* An  $\infty$ -category  $\mathcal{C}$  is an essentially 1-category if and only if it lies in the essential image of the nerve functor  $N: \mathbf{Cat} \rightarrow \mathbf{Cat}_\infty$  and it is an essentially 0-category if and only if it is equivalent to the nerve of a poset.

One might hope that an  $\infty$ -category  $\mathcal{C}$  is an essentially  $d$ -category if and only if  $\mathcal{C}$  is a  $(d - 1)$ -truncated object of the presentable  $\infty$ -category  $\mathbf{Cat}_\infty$  in the sense of T.5.5.6.1. This turns out to be *false*. The latter condition is equivalent to both spaces  $\text{Map}(\Delta^0, \mathcal{C})$  and  $\text{Map}(\Delta^1, \mathcal{C})$  being  $(d - 1)$ -truncated, while the former to the  $(d - 1)$ -truncatedness of the projection map

$$\text{Map}(\Delta^1, \mathcal{C}) \rightarrow \text{Map}(\Delta^{\{0\}}, \mathcal{C}) \times \text{Map}(\Delta^{\{1\}}, \mathcal{C}).$$

It can be deduced that a  $(d - 1)$ -truncated object of  $\mathbf{Cat}_\infty$  is an essentially  $d$ -category and that an essentially  $d$ -category is a  $d$ -truncated object of  $\mathbf{Cat}_\infty$ . To see that both converses are false, consider on the one hand a  $d$ -truncated space as an  $\infty$ -groupoid, and on the other, an  $\infty$ -category with two objects and a  $d$ -truncated space of maps from the first to the second (and no other non-trivial maps).

In T.2.3.4, Lurie develops the theory of  $d$ -categories, which are a strict model for essentially  $d$ -categories. We begin by recalling some basic definitions and properties. First, we introduce the following definition/notation (which is a variation on notation T.2.3.4.11).

**Notation 2.4.** (1) Let  $A \subseteq B$  and  $D$  be simplicial sets. We define  $B \rtimes_A D$  by the following pushout diagram

$$\begin{array}{ccc} A \times D & \longrightarrow & B \times D \\ \downarrow & & \downarrow \\ A & \longrightarrow & B \rtimes_A D. \end{array}$$

- (2) Let  $A \subseteq B$  and  $X$  be simplicial sets. Given two maps  $f, g: B \rightarrow X$  such that  $f|_A = g|_A$  we obtain a map  $f \cup g: B \rtimes_A \partial\Delta^1 \rightarrow X$ . A homotopy relative to  $A$  (or “rel.  $A$ ” for short) is an extension of  $f \cup g$  to  $B \rtimes_A \Delta^1$ .
- (3) Given inclusions of simplicial sets  $A \subseteq B \subseteq C$  and a simplicial set  $X$ , let  $[B, C; X]$  be the set of maps  $B \rightarrow X$  for which there exists an extension to  $C$ . We denote by  $[A, B, C; X]$  the set obtained from  $[B, C; X]$  by identifying maps that are homotopic rel.  $A$ .

*Remark 2.5.* Let  $\mathcal{C}$  be an  $\infty$ -category, let  $A \subseteq B$  be an inclusion of simplicial sets, and consider  $f, g: B \rightarrow \mathcal{C}$  such that  $f|_A = g|_A$ . By the discussion at the beginning of T.2.3.4, a homotopy from  $f$  to  $g$  rel.  $A$  is the same as an equivalence from  $f$  to  $g$  as objects of the  $\infty$ -category  $\mathcal{D}$  that is given as a pullback

$$\begin{array}{ccc} \mathcal{D} & \longrightarrow & \mathcal{C}^B \\ \downarrow & & \downarrow \\ \Delta^0 & \longrightarrow & \mathcal{C}^A. \end{array}$$

Therefore, the existence of a homotopy rel.  $A$  is an *equivalence relation*. We note that the above diagram is also a *homotopy pullback* in the Joyal model structure as the right vertical map is a categorical fibration and all objects are fibrant.

**Definition 2.6** (T.2.3.4.1). Let  $\mathcal{C}$  be a simplicial set and let  $d \geq -1$  be an integer. We will say that  $\mathcal{C}$  is a  $d$ -category if it is an  $\infty$ -category and the following additional conditions are satisfied:

- (1) Given a pair of maps  $f, f': \Delta^d \rightarrow \mathcal{C}$ , if  $f$  and  $f'$  are homotopic relative to  $\partial\Delta^d$ , then  $f = f'$ .
- (2) Given  $m > d$  and a pair of maps  $f, f': \Delta^m \rightarrow \mathcal{C}$ , if  $f|_{\partial\Delta^m} = f'|_{\partial\Delta^m}$ , then  $f = f'$ .

*Example 2.7.* By T.2.3.4.5, an  $\infty$ -category  $\mathcal{C}$  is a 1-category if and only if it is *isomorphic* to the nerve of an ordinary category. By T.2.3.4.3, it is a 0-category if and only if it is *isomorphic* to the nerve of a poset (compare Example 2.3).

Next, we shall recall the definition of the  $d$ -homotopy category  $h_d\mathcal{C}$  of an  $\infty$ -category  $\mathcal{C}$ . Using the notation  $K^d = \text{sk}^d K$  for the  $d$ -th skeleton of a simplicial set  $K$ , we recall the following construction.

**Lemma 2.8** (T.2.3.4.12). *For  $d \geq 1$ , given an  $\infty$ -category  $\mathcal{C}$ , there exists an essentially unique simplicial set  $h_d\mathcal{C}$ , such that for every simplicial set  $K$ , we have a bijection*

$$\text{hom}(K, h_d\mathcal{C}) \simeq [K^{d-1}, K^d, K^{d+1}; \mathcal{C}]$$

that is natural in  $K$ . We denote the canonical map by  $\theta_d: \mathcal{C} \rightarrow h_d\mathcal{C}$ .

Using the above construction, we have the following definition:

**Definition 2.9.** Given an  $\infty$ -category  $\mathcal{C}$  and an integer  $d \geq -2$ , we define the  $d$ -homotopy category of  $\mathcal{C}$  to be  $h_d\mathcal{C}$  of Lemma 2.8 when  $d \geq 1$  and

- (1) For  $d = -2$  we set  $h_{-2}\mathcal{C} = \Delta^0$ .
- (2) For  $d = -1$  we set  $h_{-1}\mathcal{C} = \begin{cases} \emptyset & \mathcal{C} = \emptyset \\ \Delta^0 & \mathcal{C} \neq \emptyset \end{cases}$  with the unique map  $\theta_{-1}: \mathcal{C} \rightarrow h_{-1}\mathcal{C}$ .
- (3) For  $d = 0$ , we first define a pre-ordered set  $\tilde{h}_0\mathcal{C}$  with the same objects as  $\mathcal{C}$  and the relation  $X \leq Y$  if and only if  $\text{Map}_{\mathcal{C}}(X, Y) \neq \emptyset$ . Then we define  $h_0\mathcal{C}$  to be the nerve of the poset obtained from  $\tilde{h}_0\mathcal{C}$  by identifying isomorphic objects. There is a canonical map  $\theta_0: \mathcal{C} \rightarrow h_0\mathcal{C}$  defined as the composition of  $\theta_1: \mathcal{C} \rightarrow h_1\mathcal{C}$  with the nerve of the functor that takes each object in the homotopy category  $h_1\mathcal{C}$  to its class in  $h_0\mathcal{C}$  (with the unique definition on morphisms).

**Warning 2.10.** Note that an  $\infty$ -category  $\mathcal{C}$  is an essentially  $d$ -category if and only if all objects of  $\mathcal{C}$  are  $(d - 1)$ -truncated in the sense of T.5.5.6.1. Hence, another way to associate an essentially  $d$ -category with an  $\infty$ -category  $\mathcal{C}$  is to consider the full subcategory spanned by the  $(d - 1)$ -truncated objects. For a presentable  $\infty$ -category, this is denoted by  $\tau_{\leq d-1}\mathcal{C}$  in T.5.5.6.1 and called the  $(d - 1)$ -truncation of  $\mathcal{C}$ . We warn the reader that the two essentially  $d$ -categories  $h_d\mathcal{C}$  and  $\tau_{\leq d-1}\mathcal{C}$  are usually very different. For example, when  $\mathcal{C} = \mathcal{S}$  is the  $\infty$ -category of spaces,  $h_1\mathcal{S}$  is the ordinary homotopy category of spaces, while  $\tau_{\leq 0}\mathcal{S}$  is equivalent to the ordinary category of sets.

The map  $\theta_d$  has the following universal property.

**Lemma 2.11.** *Let  $d \geq -1$  and let  $\mathcal{C}$  be an  $\infty$ -category.*

- (1) *The simplicial set  $h_d\mathcal{C}$  is a  $d$ -category.*
- (2) *The canonical map  $\mathcal{C} \rightarrow h_d\mathcal{C}$  is an isomorphism if and only if  $\mathcal{C}$  is a  $d$ -category.*
- (3) *For every  $d$ -category  $\mathcal{D}$ , composition with the canonical map  $\mathcal{C} \rightarrow h_d\mathcal{C}$  induces an isomorphism of simplicial sets*

$$\text{Fun}(h_d\mathcal{C}, \mathcal{D}) \xrightarrow{\sim} \text{Fun}(\mathcal{C}, \mathcal{D}).$$

*Proof.* For  $d \geq 1$  this is the content of T.2.3.4.12. For  $d = -1$  this is trivial. For  $d = 0$ , (1) and (2) are obvious from the definition. For (3) observe that we have a factorization of the map in question:

$$\text{Fun}(h_0\mathcal{C}, \mathcal{D}) \rightarrow \text{Fun}(h_1\mathcal{C}, \mathcal{D}) \xrightarrow{\sim} \text{Fun}(\mathcal{C}, \mathcal{D}),$$

where the second map is an isomorphism (from the claim for  $d = 1$ ). Therefore, we can assume that  $\mathcal{C}$  is an ordinary category and  $\mathcal{D}$  is a poset and hence both simplicial sets are discrete. The result now follows from the observation that every functor  $\mathcal{C} \rightarrow \mathcal{D}$  factors uniquely through  $h_0\mathcal{C}$ .  $\square$

Using the above results, we get the following:

**Proposition 2.12.** *The inclusion  $\mathbf{Cat}_d \hookrightarrow \mathbf{Cat}_\infty$  admits a left adjoint  $h_d: \mathbf{Cat}_\infty \rightarrow \mathbf{Cat}_d$  with unit map given by  $\theta_d: \mathcal{C} \rightarrow h_d\mathcal{C}$ .*

*Proof.* By T.2.3.4.18, every essentially  $d$ -category is equivalent to a  $d$ -category and for every  $d$ -category  $\mathcal{D}$ , the map

$$\text{Fun}(h_d\mathcal{C}, \mathcal{D}) \rightarrow \text{Fun}(\mathcal{C}, \mathcal{D})$$

is an isomorphism by Lemma 2.11. Restricting to the maximal Kan sub-complexes, the map of simplicial sets

$$\theta_d^*: \text{Map}_{\mathbf{Cat}_d}(h_d\mathcal{C}, \mathcal{D}) \rightarrow \text{Map}_{\mathbf{Cat}_\infty}(\mathcal{C}, \mathcal{D})$$

is a homotopy equivalence. It now follows that  $\theta_d$  exhibits  $h_d\mathcal{C}$  as the  $\mathbf{Cat}_d$ -localization of  $\mathcal{C}$  in the sense of T.5.2.7.6. Thus, the claim about the existence of a left adjoint follows from T.5.2.7.8 and the claim about the unit follows from the proof of T.5.2.7.8.  $\square$

The main goal of this section is to show that for every  $\infty$ -category  $\mathcal{C}$ , the  $d$ -category  $h_d\mathcal{C}$  is obtained (as one would expect) by  $(d - 1)$ -truncation of the mapping spaces. The main ingredient is the following explicit description of the right mapping space in the  $d$ -homotopy category (we recall the construction of the right mapping space in the last paragraph of the next page).

**Proposition 2.13.** *Let  $d \geq -1$  and let  $\mathcal{C}$  be an  $\infty$ -category. For every  $X, Y \in \mathcal{C}$ , there is a canonical isomorphism  $\alpha$  of simplicial sets rendering the following diagram commutative:*

$$\begin{array}{ccc} & \text{hom}_{\mathcal{C}}^R(X, Y) & \\ \beta \swarrow & & \searrow \gamma \\ \text{hom}_{h_d\mathcal{C}}^R(\theta_d(X), \theta_d(Y)) & \xrightarrow[\alpha]{\sim} & h_{d-1} \text{hom}_{\mathcal{C}}^R(X, Y), \end{array}$$

where  $\beta$  and  $\gamma$  are the obvious maps.

We defer the rather technical proof of Proposition 2.13 to the end of the section. Assuming Proposition 2.13, we get

**Corollary 2.14.** *Let  $d \geq -1$  and let  $\mathcal{C}$  be an  $\infty$ -category. The canonical map  $\theta_d: \mathcal{C} \rightarrow h_d\mathcal{C}$  is essentially surjective and for every  $X, Y \in \mathcal{C}$ , the induced map*

$$\text{Map}_{\mathcal{C}}(X, Y) \rightarrow \text{Map}_{h_d\mathcal{C}}(\theta_d(X), \theta_d(Y))$$

is a  $(d - 1)$ -truncation map.

*Proof.* It is clear that  $\theta_d$  is essentially surjective since it is surjective on objects. Let  $X, Y \in \mathcal{C}$  be two objects. Since the map

$$\text{Map}_{\mathcal{C}}(X, Y) \rightarrow \text{Map}_{h_d\mathcal{C}}(\theta_d(X), \theta_d(Y))$$

is represented by the map

$$\theta: \text{hom}_{\mathcal{C}}^R(X, Y) \rightarrow h_{d-1} \text{hom}_{\mathcal{C}}^R(X, Y),$$

it will be enough to show that for every Kan complex  $X$ , the map  $X \rightarrow h_{d-1}X$  is a  $(d - 1)$ -truncation map. We prove this by induction. For  $d \leq 0$  it is clear. For  $d \geq 1$ , recall that  $\text{hom}_X^R(p, q)$  has the homotopy type of the path space  $P_{p,q}X$  between  $p$  and  $q$  in  $X$  when viewed as a space. Thus, by induction,  $\theta$  is a map of spaces that is surjective on  $\pi_0$  and induces the  $(d - 2)$ -truncation map on path spaces

$$P_{p,q}X \rightarrow P_{p,q}(h_{d-1}X) \simeq h_{d-2}(P_{p,q}X).$$

It follows that  $\theta$  is a  $(d - 1)$ -truncation map. □

**Theorem 2.15.** *The inclusion functor  $\mathbf{Cat}_d \hookrightarrow \mathbf{Cat}_{\infty}$  admits a left adjoint  $h_d$  such that for every  $\infty$ -category  $\mathcal{C}$ , the value of  $h_d$  on  $\mathcal{C}$  is the  $d$ -homotopy category of  $\mathcal{C}$ , the unit transformation  $\theta_d: \mathcal{C} \rightarrow h_d\mathcal{C}$  is essentially surjective, and for all  $X, Y \in \mathcal{C}$ , the map of spaces*

$$\text{Map}_{\mathcal{C}}(X, Y) \rightarrow \text{Map}_{h_d\mathcal{C}}(\theta_d(X), \theta_d(Y))$$

is the  $(d - 1)$ -truncation map.

*Proof.* Combine Proposition 2.12 and Corollary 2.14. □

To prove Proposition 2.13, we begin by recalling the definitions of the “right” and “middle” mapping spaces. Let  $J: \mathbf{sSet} \rightarrow \mathbf{sSet}_{\partial\Delta^1/}$  be the functor given by  $J(K) = K \star \Delta^0/K$ , with the natural map  $\partial\Delta^1 \rightarrow J(K)$  taking 0 to the image of  $K$  and 1 to the cone point. Recall that by the definition of the right mapping space (right before T.1.2.2.3), we have

$$\text{hom}(\Delta^n, \text{hom}_{\mathcal{C}}^R(X, Y)) = \text{hom}_{(X,Y)}(J(\Delta^n), \mathcal{C}),$$

where the subscript  $(X, Y)$  in the right hand side means we take the subset of maps that restrict to  $(X, Y)$  on  $\partial\Delta^1$ . Since  $J$  preserves colimits, it follows that for every simplicial set  $K$ , we have a canonical isomorphism

$$\text{hom}(K, \text{hom}_{\mathcal{C}}^R(X, Y)) = \text{hom}_{(X,Y)}(J(K), \mathcal{C}).$$

Similarly, we can construct the “middle mapping space”. Let  $\Sigma: \mathbf{sSet} \rightarrow \mathbf{sSet}$  be the functor given by  $\Sigma(K) = K \diamond \Delta^0/K$ , where the alternative join operation  $\diamond$  is

that of definition T.4.2.1.1. This also comes with a canonical map  $\partial\Delta^1 \rightarrow \Sigma(K)$ , and similarly, from the definition of the middle mapping space (right after remark T.1.2.2.5), we have

$$\text{hom}(K, \text{hom}_{\mathcal{C}}^M(X, Y)) = \text{hom}_{(X, Y)}(\Sigma(K), \mathcal{C}).$$

There is a canonical categorical equivalence  $K \diamond \Delta^0 \xrightarrow{\sim} K \star \Delta^0$  that induces a categorical equivalence  $\Sigma K \rightarrow J(K)$  that induces a Kan equivalence

$$\Phi: \text{hom}_{\mathcal{C}}^R(X, Y) \xrightarrow{\sim} \text{hom}_{\mathcal{C}}^M(X, Y)$$

of Kan complexes.

For  $f: K \rightarrow \text{hom}_{\mathcal{C}}^R(X, Y)$ , we denote by  $\bar{f}: J(K) \rightarrow \mathcal{C}$  the corresponding map in the definition of  $\text{hom}_{\mathcal{C}}^R(X, Y)$ . We also denote by  $F = \Phi \circ f$  and  $\bar{F}: \Sigma(K) \rightarrow \mathcal{C}$  the corresponding map in the definition of  $\text{hom}_{\mathcal{C}}^M(X, Y)$ . We begin with the following technical lemma.

**Lemma 2.16.** *Given simplicial sets  $A \subseteq B$  and  $D$ , there is a canonical isomorphism*

$$\Sigma(B \rtimes_A D) \xrightarrow{\sim} \Sigma B \rtimes_{\Sigma A} D.$$

*Proof.* Consider the following diagram (with the obvious maps) and compute the colimit, starting once with the rows and once with the columns:

$$\begin{array}{ccc} \partial\Delta^1 & \longleftarrow \partial\Delta^1 \times D & \longrightarrow \partial\Delta^1 \times D & & \partial\Delta^1 \times (\Delta^0 \rtimes_{\Delta^0} D) \\ \uparrow & & \uparrow & & \uparrow \\ \partial\Delta^1 \times A & \longleftarrow \partial\Delta^1 \times A \times D & \longrightarrow \partial\Delta^1 \times B \times D & & \partial\Delta^1 \times (B \rtimes_A D) \\ \downarrow & & \downarrow & & \downarrow \\ \Delta^1 \times A & \longleftarrow \Delta^1 \times A \times D & \longrightarrow \Delta^1 \times B \times D & & \Delta^1 \times (B \rtimes_A D) \\ & & & & \\ \Sigma A & \longleftarrow \Sigma A \times D & \longrightarrow \Sigma B \times D & & \Sigma B \rtimes_{\Sigma A} D \simeq \Sigma(B \rtimes_A D). \quad \square \end{array}$$

The following lemma compares the different models of the mapping space.

**Lemma 2.17.** *Given simplicial sets  $A \subseteq B$  and two maps  $f, g: B \rightarrow \text{hom}_{\mathcal{C}}^R(X, Y)$ , the following are equivalent:*

- (1)  $f, g: B \rightarrow \text{hom}_{\mathcal{C}}^R(X, Y)$  agree on  $A$  (resp. homotopic rel.  $A$ ).
- (2)  $F, G: B \rightarrow \text{hom}_{\mathcal{C}}^M(X, Y)$  agree on  $A$  (resp. homotopic rel.  $A$ ).
- (3)  $\bar{f}, \bar{g}: J(B) \rightarrow \mathcal{C}$  agree on  $J(A)$  (resp. homotopic rel.  $J(A)$ ).
- (4)  $\bar{F}, \bar{G}: \Sigma(B) \rightarrow \mathcal{C}$  agree on  $\Sigma(A)$  (resp. homotopic rel.  $\Sigma(A)$ ).

*Proof.* In each of the statements (1)-(4) we refer to “agree on . . .” as the *first part* and to “homotopic rel . . .” as the *second part*. We start with the equivalence (1)  $\iff$  (2). The first part follows from the fact that

$$\Phi: \text{hom}_{\mathcal{C}}^R(X, Y) \xrightarrow{\sim} \text{hom}_{\mathcal{C}}^M(X, Y)$$

is a monomorphism and second part from the fact that it is a homotopy equivalence of Kan complexes. In the equivalence (3)  $\iff$  (4), the first part follows from the

fact that  $\Sigma A \rightarrow J(A)$  is an epimorphism and the second part can be seen as follows: the maps  $\bar{f}, \bar{g}: J(B) \rightarrow \mathcal{C}$  are homotopic rel  $J(A)$  if and only if they are equivalent as elements of the  $\infty$ -category that is the fiber over  $\bar{f}|_{J(A)} = \bar{g}|_{J(A)}$  (which is also a homotopy fiber) of the categorical fibration  $\mathcal{C}^{J(B)} \rightarrow \mathcal{C}^{J(A)}$ . Since we have functorial categorical equivalences  $\Sigma(A) \xrightarrow{\sim} J(A)$  and  $\Sigma(B) \xrightarrow{\sim} J(B)$ , this is the same as showing that the corresponding maps  $\bar{F}, \bar{G}: \Sigma(B) \rightarrow \mathcal{C}$  are equivalent in the fiber of  $\mathcal{C}^{\Sigma(B)} \rightarrow \mathcal{C}^{\Sigma(A)}$  (which is also the homotopy fiber). This in turn is the same as having  $\bar{F}, \bar{G}$  homotopic rel.  $\Sigma A$ . It is left to show the equivalence (2)  $\iff$  (4). The first part is clear. The second part amounts to showing the equivalence of two extension problems. If  $F|_A = G|_A$ , we get a map  $F \cup_A G$  from  $B \cup_A B \simeq B \rtimes_A \partial\Delta^1$  to  $\text{hom}_{\mathcal{C}}^M(X, Y)$  and  $F$  and  $G$  are homotopic rel.  $A$  if and only if  $F \cup_A G$  extends to the relative cylinder  $B \rtimes_A \Delta^1$ . In terms of maps to  $\mathcal{C}$ , this is equivalent to the extension problem

$$\begin{array}{ccc} \Sigma(B \rtimes_A \partial\Delta^1) & \longrightarrow & \mathcal{C} \\ \downarrow & \dashrightarrow & \\ \Sigma(B \rtimes_A \Delta^1) & & \end{array}$$

On the other hand, by the assumption that  $\bar{F}|_{\Sigma A} = \bar{G}|_{\Sigma A}$  we get a map  $\bar{F} \cup_{\Sigma A} \bar{G}$  from  $\Sigma B \rtimes_{\Sigma A} \partial\Delta^1$  to  $\mathcal{C}$  and  $\bar{F}$  and  $\bar{G}$  are homotopic rel.  $\Sigma A$  if and only if it extends to the relative cylinder  $\Sigma B \rtimes_{\Sigma A} \Delta^1$ . By Lemma 2.16 for  $D = \Delta^1, \partial\Delta^1$ , the two extension problems are isomorphic.  $\square$

We are now ready to prove Proposition 2.13.

*Proof (of Proposition 2.13).* For  $d \leq 0$  this follows directly from the definitions, and so we assume that  $d \geq 1$ . Let  $K$  be a simplicial set. On the one hand,

$$\begin{aligned} \text{hom}\left(K, \text{hom}_{h_d\mathcal{C}}^R(X, Y)\right) &= \text{hom}_{(X, Y)}(J(K), h_d\mathcal{C}) \\ &= \left[J(K)^{d-1}, J(K)^d, J(K)^{d+1}; \mathcal{C}\right]_{(X, Y)}, \end{aligned}$$

where the subscript  $(X, Y)$  indicates that we take only the subset of maps that restrict to  $(X, Y)$  on  $\partial\Delta^1 \hookrightarrow J(K)$  (observe that this is independent of the representative as  $\partial\Delta^1 \subseteq J(K)^{d-1}$ ). On the other hand,

$$\text{hom}\left(K, h_{d-1} \text{hom}_{\mathcal{C}}^R(X, Y)\right) = \left[K^{d-2}, K^{d-1}, K^d; \text{hom}_{\mathcal{C}}^R(X, Y)\right].$$

We will argue that this last set is in natural bijection with the set

$$\left[J(K^{d-2}), J(K^{d-1}), J(K^d), \mathcal{C}\right]_{(X, Y)}.$$

First, by definition of the right mapping space we have a natural bijection

$$\overline{(-)}: \text{hom}(K^{d-1}, \text{hom}_{\mathcal{C}}^R(X, Y)) \xrightarrow{\sim} \text{hom}_{(X, Y)}(J(K^{d-1}), \mathcal{C}).$$

Second,  $f$  extends to  $K^d$  if and only if  $\bar{f}$  extends to  $J(K^d)$ . Likewise, it is clear that two maps  $f, g: K^{d-1} \rightarrow \text{hom}_{\mathcal{C}}^R(X, Y)$  agree on  $K^{d-2}$  if and only if the corresponding maps  $\bar{f}, \bar{g}: J(K^{d-1}) \rightarrow \mathcal{C}$  agree on  $J(K^{d-2})$ . Hence, the only thing we need to show is that  $f$  and  $g$  are homotopic rel.  $K^{d-2}$  if and only if  $\bar{f}$  and  $\bar{g}$  are homotopic rel.



$J(K^{d-2})$  and this follows from (1)  $\iff$  (3) in Lemma 2.17. It remains to observe that for every simplicial set  $K$  and every  $d \geq 1$  we have a canonical isomorphism  $J(K^{d-1}) \xrightarrow{\sim} J(K)^d$ . Hence, we get a natural bijection

$$\text{hom}\left(K, \text{hom}_{h_d \mathcal{C}}^R(X, Y)\right) \simeq \text{hom}\left(K, h_{d-1} \text{hom}_{\mathcal{C}}^R(X, Y)\right)$$

and therefore an isomorphism  $\alpha: \text{hom}_{h_d \mathcal{C}}^R(X, Y) \simeq h_{d-1} \text{hom}_{\mathcal{C}}^R(X, Y)$ .

Finally, we need to show that the isomorphism we have constructed is compatible with the maps  $\theta: \text{hom}_{\mathcal{C}}^R(X, Y) \rightarrow h_{d-1} \text{hom}_{\mathcal{C}}^R(X, Y)$  and  $\beta: \text{hom}_{\mathcal{C}}^R(X, Y) \rightarrow \text{hom}_{h_d \mathcal{C}}^R(X, Y)$ . For this, consider a map  $f: K \rightarrow \text{hom}_{\mathcal{C}}^R(X, Y)$ . The composition  $\theta \circ f$  is represented by the restriction  $f|_{K^{d-1}}$ , which corresponds to the map  $f|_{K^{d-1}}: J(K^{d-1}) \rightarrow \mathcal{C}$ . On the other hand, the composition  $\beta \circ f$  corresponds to the restriction of  $\bar{f}: J(K) \rightarrow \mathcal{C}$  to  $J(K)^{d+1}$  and these are identified by  $\alpha$ .  $\square$

### 3. $d$ -Operads

We now develop the basic theory of (essentially)  $d$ -operads in analogy with (and by bootstrapping of) the theory of  $d$ -categories. First,

**Definition 3.1.** Let  $d \geq -1$ . An *essentially  $d$ -operad* is an  $\infty$ -operad  $\mathcal{O}$  such that for all  $X_1, \dots, X_n, Y \in \mathcal{O}$ , the multi-mapping space  $\text{Mul}_{\mathcal{O}}(\{X_1, \dots, X_n\}; Y)$  is  $(d - 1)$ -truncated. We denote by  $\mathbf{Op}_d$  the full subcategory of  $\mathbf{Op}_{\infty}$  spanned by essentially  $d$ -operads.

*Example 3.2.* Two important special cases are:

- (1) A symmetric monoidal  $\infty$ -category is an essentially  $d$ -operad if and only if its underlying  $\infty$ -category is an essentially  $d$ -category.
- (2) A reduced  $\infty$ -operad  $\mathcal{P}$  is an essentially  $d$ -operad if and only if the corresponding symmetric sequence of  $n$ -ary operations  $\{\mathcal{P}(n)\}_{n \geq 0}$  consists of  $(d - 1)$ -truncated spaces.

The corresponding strict notion is the following:

**Definition 3.3.** Let  $\mathcal{O}$  be an  $\infty$ -operad.

- (1) For  $d \geq 1$ , we say that  $\mathcal{O}$  is a  *$d$ -operad* if  $\mathcal{O}^{\otimes}$  is a  $d$ -category.
- (2) We say that  $\mathcal{O}$  is a *0-operad* if  $\mathcal{O}^{\otimes}$  is a skeletal 1-category and  $p$  is faithful.
- (3) We say that  $\mathcal{O}$  is a  *$(-1)$ -operad* if either  $\mathcal{O}^{\otimes} = \emptyset$  or  $p$  is an isomorphism.

*Remark 3.4.* A  $d$ -operad is intended to bear the same relation to an essentially  $d$ -operad as a  $d$ -category does to an essentially  $d$ -category; i.e. it is a strict model for an  $\infty$ -operad in which all multi-mapping spaces are  $(d - 1)$ -truncated.

Next, we define the notion of a  $d$ -homotopy operad of an  $\infty$ -operad, which is analogous to the notion of a  $d$ -homotopy category of an  $\infty$ -category. We begin by showing that the functor  $h_d$  behaves well with respect to inner and coCartesian edges.

**Proposition 3.5.** Let  $d \geq -1$  and let  $p: \mathcal{C} \rightarrow \mathcal{D}$  be a functor, where  $\mathcal{C}$  is an  $\infty$ -category and  $\mathcal{D}$  a  $d$ -category.

- (1) If the functor  $p: \mathcal{C} \rightarrow \mathcal{D}$  is an inner fibration, then so is  $h_d(p): h_d(\mathcal{C}) \rightarrow h_d(\mathcal{D}) = \mathcal{D}$ .
- (2) If in addition  $f$  is a  $p$ -coCartesian morphism in  $\mathcal{C}$ , then  $h_d(f)$  is  $h_d(p)$ -coCartesian in  $h_d\mathcal{C}$ .

*Proof.* For  $d = -1, 0$ , both assertions are trivial to check and so we assume that  $d \geq 1$ . The argument that  $h_d(p)$  is an inner fibration is similar to the argument that  $h_d(f)$  is coCartesian and so we shall prove them together. Using T.2.4.1.4, we need to consider the lifting problem

$$\begin{array}{ccc} \Lambda_i^m & \longrightarrow & h_d\mathcal{C} \\ \downarrow & \nearrow & \downarrow \\ \Delta^m & \longrightarrow & \mathcal{D} \end{array}$$

for some  $m \geq 2$  and either

- (1)  $0 < i < m$  or
- (2)  $i = 0$  and  $\Delta^{\{0,1\}} \subseteq \Lambda_0^m$  is mapped in  $h_d\mathcal{C}$  to  $h_d(f)$ .

For  $m \geq d + 3$ , we have  $\text{sk}^j \Lambda_i^m = \text{sk}^j \Delta^m$  for all  $j \leq d + 1$ , and so the map

$$\text{hom}(\Delta^m, h_d\mathcal{C}) \rightarrow \text{hom}(\Lambda_i^m, h_d\mathcal{C})$$

is a bijection and there is nothing to prove. For  $m \leq d + 2$ , we have  $\Lambda_i^m = \text{sk}^{d+1} \Lambda_i^m$ , and so the map

$$\text{hom}(\Lambda_i^m, \mathcal{C}) \twoheadrightarrow \text{hom}(\Lambda_i^m, h_d\mathcal{C})$$

is surjective, hence the map  $\Lambda_i^m \rightarrow h_d\mathcal{C}$  factors through  $\Lambda_i^m \rightarrow \mathcal{C}$ . Now, the functor  $\mathcal{C} \rightarrow h_d\mathcal{C}$  identifies only homotopic morphisms (for  $d \geq 1$ ); hence in (2) the image of  $\Delta^{\{0,1\}}$  in  $\mathcal{C}$  is coCartesian. Thus, in both cases we can solve the corresponding lifting problem in  $\mathcal{C}$ , which induces a lift in the original square.  $\square$

**Definition 3.6.** Given an  $\infty$ -operad  $p: \mathcal{O} \rightarrow \mathbf{Fin}_*$ , we define its  $d$ -homotopy operad  $h_d\mathcal{O}$  to be a map of simplicial sets  $p_d: (h_d\mathcal{O})^\otimes \rightarrow \mathbf{Fin}_*$  defined as follows:

- (1) For  $d \geq 1$ , we simply apply  $h_d$  to  $p$  as a functor between  $\infty$ -categories and use the fact that  $\mathbf{Fin}_*$  is a 1-category; hence there is a canonical isomorphism  $h_d(\mathbf{Fin}_*) \simeq \mathbf{Fin}_*$ .
- (2) For  $d = 0$ , we first construct the (ordinary) category  $\tilde{h}_0\mathcal{O}^\otimes$  whose objects are those of  $\mathcal{O}^\otimes$  and each mapping space is replaced by its image in  $\mathbf{Fin}_*$ . Then we identify isomorphic objects in  $\tilde{h}_0\mathcal{O}^\otimes$  (note that there is a unique induced composition, since isomorphic objects are mapped to the same object in  $\mathbf{Fin}_*$ ) and, finally, we define  $(h_0\mathcal{O})^\otimes$  to be the nerve of the resulting category, with  $p_0$  being the obvious map to  $\mathbf{Fin}_*$ .
- (3) For  $d = -1$ , we define  $p_d: \mathbf{Fin}_* \rightarrow \mathbf{Fin}_*$  to be the identity functor if  $\mathcal{O}^\otimes \neq \emptyset$  and the unique functor  $p_d: \emptyset \rightarrow \mathbf{Fin}_*$ , otherwise.

In all three cases we have a canonical map of simplicial sets  $\theta_d: \mathcal{O}^\otimes \rightarrow (h_d\mathcal{O})^\otimes$  over  $\mathbf{Fin}_*$ .

**Warning 3.7.** For every  $\infty$ -operad  $\mathcal{O}$  and  $d \geq 1$  we have  $(h_d\mathcal{O})^\otimes \simeq h_d(\mathcal{O}^\otimes)$ , but for  $d \leq 0$  we get something slightly different. The reason for this is that  $(h_d\mathcal{C})^\otimes$

corresponds to the application of  $h_d$  fiber-wise to the map  $p: \mathcal{O}^\otimes \rightarrow \mathbf{Fin}_*$ . Since  $\mathbf{Fin}_*$  is a 1-category, for  $d \geq 1$  this is the same as applying  $h_d$  to  $p$ , but for  $d \leq 0$  it is not.

**Lemma 3.8.** *Let  $p: \mathcal{O}^\otimes \rightarrow \mathbf{Fin}_*$  be an  $\infty$ -operad.*

- (1) *The map  $p_d: (h_d\mathcal{O})^\otimes \rightarrow \mathbf{Fin}_*$  is a  $d$ -operad.*
- (2) *The canonical map  $\theta_d: \mathcal{O} \rightarrow h_d\mathcal{O}$  is a map of  $\infty$ -operads.*
- (3) *Given an  $\infty$ -operad map  $F: \mathcal{O} \rightarrow \mathcal{U}$ , the induced map  $h_dF: h_d\mathcal{O} \rightarrow h_d\mathcal{U}$  on  $d$ -homotopy operads, is an  $\infty$ -operad map.*

*Proof.* For  $d = -1$ , there is nothing to prove in (1)–(3) and so we assume that  $d \geq 0$ .

(1) For  $d = 0$ , it is clear that  $(h_0\mathcal{O})^\otimes$  is a skeletal 1-category, with  $p_0$  faithful; and for  $d \geq 1$ , it is clear that  $(h_d\mathcal{O})^\otimes$  is a  $d$ -category. Hence, we only need to show that  $(h_d\mathcal{O})^\otimes$  is an  $\infty$ -operad. For this we need to check the three conditions of Definition A.2.1.1.10.

- Since  $p: \mathcal{O}^\otimes \rightarrow \mathbf{Fin}_*$  is an  $\infty$ -operad, for every inert morphism  $f: \langle m \rangle \rightarrow \langle n \rangle$  and an object  $\bar{X} \in h_d\mathcal{O}_{\langle m \rangle}^\otimes$ , we can lift  $\bar{X}$  to  $X \in \mathcal{O}_{\langle m \rangle}^\otimes$  and find a coCartesian lift  $g: X \rightarrow Y$  of  $f$  in  $\mathcal{O}^\otimes$ . For  $d \geq 1$ , the image  $\bar{g}$  of  $g$  in  $(h_d\mathcal{O})^\otimes$  is a coCartesian lift of  $f$  by Proposition 3.5. For  $d = 0$ , we use the dual of T.2.4.4.3 to show that  $\bar{g}$  is coCartesian. The functor  $(h_0\mathcal{O})^\otimes \rightarrow \mathbf{Fin}_*$  is an inner fibration (as the nerve of a functor of ordinary categories by T.2.3.1.5) and for every  $\bar{Z} \in (h_0\mathcal{O})_{\langle m \rangle}^\otimes$ , pre-composition with  $\bar{g}$  induces a diagram

$$\begin{array}{ccc} \mathrm{Map}_{(h_0\mathcal{O})^\otimes}(\bar{Y}, \bar{Z}) & \longrightarrow & \mathrm{Map}_{(h_0\mathcal{O})^\otimes}(\bar{X}, \bar{Z}), \\ \downarrow & & \downarrow \\ \mathrm{Map}_{\mathbf{Fin}_*}(\langle m \rangle, \langle k \rangle) & \longrightarrow & \mathrm{Map}_{\mathbf{Fin}_*}(\langle n \rangle, \langle k \rangle) \end{array}$$

and this is a homotopy pullback as  $p_0$  is faithful and it is a pullback of sets.

- Let  $\bar{X} \in (h_d\mathcal{O})_{\langle m \rangle}^\otimes$  and  $\bar{Y} \in (h_d\mathcal{O})_{\langle n \rangle}^\otimes$  and let  $f: \langle m \rangle \rightarrow \langle n \rangle$  be a morphism in  $\mathbf{Fin}_*$ . We first observe that

$$\mathrm{Map}_{(h_d\mathcal{O})^\otimes}^f(X, Y) \simeq h_{d-1} \left( \mathrm{Map}_{\mathcal{O}^\otimes}^f(X, Y) \right).$$

For  $d \geq 1$  this follows from Proposition 2.13 and for  $d = 0$  it follows directly from the definition. Hence,

$$\begin{aligned} \mathrm{Map}_{(h_d\mathcal{O})^\otimes}^f(X, Y) &\simeq h_{d-1} \left( \mathrm{Map}_{\mathcal{O}^\otimes}^f(X, Y) \right) \simeq h_{d-1} \left( \prod_{1 \leq i \leq n} \mathrm{Map}_{\mathcal{O}^\otimes}^{\rho^i \circ f}(X, Y_i) \right) \\ &\simeq \prod_{1 \leq i \leq n} h_{d-1} \left( \mathrm{Map}_{\mathcal{O}^\otimes}^{\rho^i \circ f}(X, Y_i) \right) \simeq \prod_{1 \leq i \leq n} \mathrm{Map}_{(h_d\mathcal{O})^\otimes}^{\rho^i \circ f}(X, Y_i). \end{aligned}$$

Note that we use the fact that  $h_d$  preserves finite products of spaces.

- For every finite collection of objects  $\bar{X}_1, \dots, \bar{X}_n \in (h_d\mathcal{O})_{\langle 1 \rangle}^\otimes$  that are lifted to objects of  $\mathcal{O}_{\langle 1 \rangle}^\otimes$ , there is an object  $X \in \mathcal{O}_{\langle n \rangle}^\otimes$  and coCartesian morphisms  $f_i: X \rightarrow X_i$  covering  $\rho^i: \langle n \rangle \rightarrow \langle 1 \rangle$ . The images of those maps in  $(h_d\mathcal{O})^\otimes$  are coCartesian as well and satisfy the analogous property (Proposition 3.5(2)).

(2) From the proof of (1),  $\theta_d$  maps inert morphisms in  $\mathcal{O}^\otimes$  to inert morphisms in  $h_d\mathcal{O}^\otimes$ .

(3) We need to show that  $h_dF$  maps inert morphisms to inert morphisms. For  $d = 0$ , this is automatic. For  $d \geq 1$ , let  $\bar{f}: X \rightarrow Y$  be an inert morphism in  $(h_d\mathcal{O})^\otimes$ . There is a coCartesian morphism  $f: X \rightarrow Y'$  in  $\mathcal{O}^\otimes$  with the same image as  $\bar{f}$  in  $\mathbf{Fin}_*$ ; hence its image in  $(h_d\mathcal{O})^\otimes$  is equivalent to  $f$ . Since the composition  $\mathcal{O}^\otimes \rightarrow \mathcal{U}^\otimes \rightarrow (h_d\mathcal{U})^\otimes$  preserves inert morphisms, it follows that the image of  $f$  in  $(h_d\mathcal{U})^\otimes$  is inert and since the image of  $\bar{f}$  in  $(h_d\mathcal{U})^\otimes$  is equivalent to the image of  $f$ , it is inert as well.  $\square$

The following lemma provides the universal property of  $\theta_d$  by analogy with Lemma 2.11 for  $d$ -categories.

**Lemma 3.9.** *Let  $\mathcal{O}$  be an  $\infty$ -operad.*

- (1) *For every  $d$ -operad  $\mathcal{U}$ , pre-composition with  $\theta_d$  induces an isomorphism of simplicial sets*

$$\underline{\text{Alg}}_{h_d\mathcal{O}}(\mathcal{U}) \rightarrow \underline{\text{Alg}}_{\mathcal{O}}(\mathcal{U})$$

*and, in particular, a homotopy equivalence*

$$\text{Map}_{\mathbf{Op}_\infty}(h_d\mathcal{O}, \mathcal{U}) \rightarrow \text{Map}_{\mathbf{Op}_\infty}(\mathcal{O}, \mathcal{U}).$$

- (2)  *$\mathcal{O}$  is a  $d$ -operad if and only if  $\theta_d$  is an isomorphism.*

*Proof.* (1) Assume that  $d \geq 1$ . By the analogous fact for  $\infty$ -categories, the composition with  $\theta_d$  induces an isomorphism

$$\text{Fun}_{\mathbf{Fin}_*}((h_d\mathcal{O})^\otimes, \mathcal{U}^\otimes) \xrightarrow{\sim} \text{Fun}_{\mathbf{Fin}_*}(\mathcal{O}^\otimes, \mathcal{U}^\otimes).$$

The simplicial set  $\underline{\text{Alg}}_{\mathcal{O}}(\mathcal{U})$  is the full subcategory of  $\text{Fun}_{\mathbf{Fin}_*}(\mathcal{O}^\otimes, \mathcal{U}^\otimes)$  spanned by maps of  $\infty$ -operads (and similarly for  $h_d\mathcal{O}$  instead of  $\mathcal{O}$ ). The claim now follows from the fact that the image of a coCartesian edge in  $\mathcal{O}^\otimes$  is coCartesian in  $(h_d\mathcal{O})^\otimes$  (Proposition 3.5(2)) and, conversely, every inert morphism in  $(h_d\mathcal{O})^\otimes$  is up to equivalence the image of an inert morphism in  $\mathcal{O}^\otimes$  (lift the source to some object  $X \in \mathcal{O}^\otimes$  and choose any inert map with domain  $X$ ).

For  $d = 0$ , essentially the same argument works, only now the inert maps of  $(h_0\mathcal{O})^\otimes$  are precisely those whose image in  $\mathbf{Fin}_*$  is inert and therefore the inert maps of  $(h_0\mathcal{O})^\otimes$  are again precisely the images of inert maps in  $\mathcal{O}^\otimes$ . For  $d = -1$ , the claim is obvious.

(2) Follows from (1) and the Yoneda lemma in the 1-category  $\mathbf{POp}_\infty$  of  $\infty$ -preoperads (see A.2.1.4.2).  $\square$

**Lemma 3.10.** *Let  $d \geq -1$  and let  $\mathcal{O}$  be an  $\infty$ -operad. The canonical map  $\theta_d: \mathcal{O} \rightarrow h_d\mathcal{O}$  is essentially surjective and for all  $X_1, \dots, X_n, Y \in \underline{\mathcal{O}}$ , the map*

$$\text{Mul}_{\mathcal{O}}(\{X_1, \dots, X_n\}; Y) \rightarrow \text{Mul}_{h_d\mathcal{O}}(\{\theta_d(X_1), \dots, \theta_d(X_n)\}; \theta_d(Y))$$

*is a  $(d - 1)$ -truncation map.*

*Proof.* The map  $\theta_d: \mathcal{O} \rightarrow h_d\mathcal{O}$  is surjective on objects and hence is essentially surjective. For  $d \geq 1$ , the second assertion follows from the corresponding fact for  $\infty$ -categories; and for  $d = -1, 0$ , it follows directly from the definition.  $\square$

**Corollary 3.11.** *An  $\infty$ -operad is an essentially  $d$ -operad if and only if it is equivalent to a  $d$ -operad.*

The following is the analogue of Theorem 2.15 for  $\infty$ -operads.

**Theorem 3.12.** *The inclusion  $\mathbf{Op}_d \hookrightarrow \mathbf{Op}_\infty$  admits a left adjoint  $h_d$ , such that for every  $\infty$ -operad  $\mathcal{O}$  the value of  $h_d$  on  $\mathcal{O}$  is the  $d$ -homotopy operad of  $\mathcal{O}$ , the unit transformation  $\theta_d: \mathcal{O} \rightarrow h_d\mathcal{O}$  is essentially surjective, and for all objects  $X_1, \dots, X_n, Y \in \mathcal{O}$ , the map of spaces*

$$\mathrm{Mul}_{\mathcal{O}}(\{X_1, \dots, X_n\}; Y) \rightarrow \mathrm{Mul}_{h_d\mathcal{O}}(\{\theta_d(X_1), \dots, \theta_d(X_n)\}; \theta_d(Y))$$

is the  $(d-1)$ -truncation map.

*Proof.* Follows from Lemma 3.10, Lemma 3.9 (the universal property of  $\theta_d$ ) and Corollary 3.11 analogously to the proof for  $d$ -categories.  $\square$

We conclude with a simple consequence of the theory of  $d$ -operads, that showcases the effectiveness of the strict model.

**Proposition 3.13.** *Let  $\mathcal{O}$  be an  $\infty$ -operad and let  $\mathcal{U}$  be an (essentially)  $d$ -operad. The  $\infty$ -category  $\mathrm{Alg}_{\mathcal{O}}(\mathcal{U})$  is an (essentially)  $d$ -category.*

*Proof.* Since an  $\infty$ -operad  $\mathcal{U}$  is an essentially  $d$ -operad if and only if it is equivalent to a (strict)  $d$ -operad, it is enough to prove the strict version. By definition, the  $\infty$ -category  $\mathrm{Alg}_{\mathcal{O}}(\mathcal{U})$  is a full subcategory of  $\mathrm{Fun}(\mathcal{O}^{\otimes}, \mathcal{U}^{\otimes})$ . For  $d \geq 1$ , the  $\infty$ -category  $\mathcal{U}^{\otimes}$  is a  $d$ -category and, therefore, by T.2.3.4.8, the  $\infty$ -category  $\mathrm{Fun}(\mathcal{O}^{\otimes}, \mathcal{U}^{\otimes})$  is a  $d$ -category as well. Hence, every full subcategory of it is a  $d$ -category. For  $d = 0$ , by Lemma 3.9 we can assume that  $\mathcal{O}^{\otimes}$  is a 0-operad as well and therefore both  $\mathcal{O}^{\otimes}$  and  $\mathcal{U}^{\otimes}$  are skeletal 1-categories with faithful projection to  $\mathbf{Fin}_*$ . Observing that  $\mathrm{Alg}_{\mathcal{O}}(\mathcal{U})$  is a full subcategory of  $\mathrm{Fun}_{\mathbf{Fin}_*}(\mathcal{O}^{\otimes}, \mathcal{U}^{\otimes})$  and using the faithfulness of the projections to  $\mathbf{Fin}_*$ , we see that the mapping spaces are either empty or singletons. For  $d = -1$ , the claim is obvious.  $\square$

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Tomer M. Schlank [tomer.schlank@mail.huji.ac.il](mailto:tomer.schlank@mail.huji.ac.il)

Mathematics Department, Hebrew University of Jerusalem, Edmond j. Safra Campus Givat Ram, Jerusalem, 9190401, Israel

Lior Yanovski [lior.yanovski@mail.huji.ac.il](mailto:lior.yanovski@mail.huji.ac.il)

Mathematics Department, Hebrew University of Jerusalem, Edmond j. Safra Campus Givat Ram, Jerusalem, 9190401, Israel