Homology, Homotopy and Applications, vol. 22(1), 2020, pp.215–232

ON TRIVIALITIES OF EULER CLASSES OF ORIENTED VECTOR BUNDLES OVER MANIFOLDS

ANIRUDDHA C. NAOLEKAR, B. SUBHASH AND AJAY SINGH THAKUR

(*communicated by Donald M. Davis*)

Abstract

We study manifolds on which all oriented bundles have trivial Euler class. In this note, we give a complete a characterization of such manifolds in dimension less than six, in terms of their cohomology groups, and obtain some partial results for manifolds of dimension 6.

1. Introduction

One of the earliest known results on trivialities of characteristic classes of vector bundles was obtained by Atiyah-Hirzebruch by showing that the Stiefel-Whitney classes of any vector bundle over a 9-fold suspension $\Sigma^9 X$ of a CW-complex X are trivial [**1**, Theorem 2]. Further results on trivialities of Stiefel-Whitney classes of vector bundles over suspensions of CW-complexes have been studied in [**12, 16, 17**] and [**18**]. Similar results on trivialities of Chern classes of complex vector bundles over CW-complexes were studied in [**13**].

In recent times there has been some interest in trying to understand the set of Euler classes of oriented bundles over CW-complexes (see, for example, [**8, 9, 11, 14, 21**]). In this note we are mainly interested in trivialities of Euler class of oriented vector bundles over manifolds.

We say that a closed connected smooth n-manifold X has property $(\mathcal{E})$ (or satisfies $(\mathcal{E})$) if for every oriented vector bundle α over X, the Euler class $e(\alpha) = 0$.

Let $\mathcal{E}$ denote the set of diffeomorphism classes of closed connected smooth manifolds X that have property $(\mathcal{E})$. Let $\mathcal{E}_k$ denote the subset of $\mathcal{E}$ consisting of manifolds X with $\dim X = k$. In this note we shall completely describe the set $\mathcal{E}_i$ for $i \leqslant 5$ and obtain some necessary conditions for a closed connected smooth 6-manifold X to satisfy $(\mathcal{E})$.

Recall that a smooth homology n-sphere is a closed connected smooth n-manifold X such that $H_i(X;\mathbb{Z}) \cong H_i(S^n;\mathbb{Z})$. If X is a smooth homology n-sphere, where n is odd and if α is a oriented vector bundle over X, then $e(\alpha) = 0$ if $\operatorname{rank}\alpha \neq \dim X$. In the case when $\operatorname{rank}\alpha = \dim X$, we have $2e(\alpha) = 0$ which implies $e(\alpha) = 0$. It follows that every smooth homology n-sphere where n is odd has property $(\mathcal{E})$. The main results of this note are the following.

The research of third author is partially supported by DST-Inspire Faculty research grant (IFA-13-MA-26).

Received December 28, 2018, revised July 8, 2019; published on November 20, 2019.

2010 Mathematics Subject Classification: 57R20.

Key words and phrases: Euler class.

Article available at `http://dx.doi.org/10.4310/HHA.2020.v22.n1.a13`

Theorem 1.1. *Let X be a closed connected smooth 3-manifold. Then $X \in \mathcal{E}_3$ if and only if X is a smooth homology 3-sphere.* □

It is easy to see (see Proposition 2.1 below) that an even dimensional closed connected orientable smooth manifold cannot have property ($\mathcal{E}$). In dimension 4, the non-orientable closed connected smooth manifolds do not have property ($\mathcal{E}$) either. We prove the following.

Theorem 1.2. $\mathcal{E}_4 = \emptyset$. □

This is an immediate consequence of Theorem 3.2 below. The picture in dimension 5 is a bit more complicated. The following theorem completely describes the set $\mathcal{E}_5$.

Theorem 1.3. *Let X be a closed connected smooth 5-manifold. Then $X \in \mathcal{E}_5$ if and only if the following conditions are satisfied:*

1. $H^2(X;\mathbb{Z}) = 0$, $H^4(X;\mathbb{Z}) = 0$, *and*
2. *the 2-primary component of $H_2(X;\mathbb{Z})$ is trivial.* □

Thus, if X is a closed connected orientable smooth 5-manifold, then $X \in \mathcal{E}_5$ if and only if $\pi_1(X)$ is perfect and X is a $\mathbb{Z}_2$-homology sphere (see Remark 4.1 below). In particular, the projective space $\mathbb{RP}^5$ and the Lens spaces in dimension 5, do not satisfy ($\mathcal{E}$).

Using the Smale-Barden classification (see, [**15, 2**]) of simply connected closed smooth 5-manifolds, we construct examples of simply connected closed smooth 5-manifolds X that satisfy ($\mathcal{E}$). We also construct examples of non-simply connected closed smooth orientable 5-manifolds X that satisfy ($\mathcal{E}$) and closed smooth non-orientable manifolds X that satisfy ($\mathcal{E}$).

In dimension 6 we have not been able to derive a clear picture. We show, among other things, that $\mathcal{E}_6$ does not contain products and that any $X \in \mathcal{E}_6$ must have non-positive Euler characteristic.

Finally, we also compute the real and complex K-theory of $X \in \mathcal{E}_5$. As a consequence it follows that every orientable $X \in \mathcal{E}_5$ is stably parallelizable but not parallelizable since every such manifold has non-zero Kervaire semi-characteristic (see Remark 6.3(2) below).

This note is organized as follows. In Section 2 we collect some preliminary observations about the set $\mathcal{E}$, set up notations and state some results that we use for easy reference. In Section 3 we prove Theorem 1.1, Theorem 1.2 and state necessary conditions that any $X \in \mathcal{E}_6$ must satisfy. The proof of Theorem 1.3 is split into Sections 4 and 5 dealing with the orientable and non-orientable cases respectively. In Section 6 we compute the real and complex K-theory of $X \in \mathcal{E}_5$.

Conventions

Throughout, F (respectively, $F', F'', \ldots$) will denote a finite abelian group. The integer s (respectively, $s', s'', \ldots$) will denote the number of primes p_i that are equal to 2 in a direct sum decomposition of F (respectively, $F', F'', \ldots$)

$$F = \oplus_i \mathbb{Z}/p_i^{k_i}$$

with the p_i not necessarily distinct. Given s (respectively, $s', s'', \ldots$), the pair of integers u, v (respectively, $u', v'; u'', v''; \ldots$) will denote the number of integers k_i

with $k_i = 1$ and $k_i > 1$ respectively. Thus $s = u + v$ (respectively, $s' = u' + v'; \ldots$). ε^k will denote the trivial k-plane bundle. $\beta_{\mathbb{R}}$ will denote the underlying real bundle of a complex bundle β. In the sequel, a manifold will mean a closed, connected, smooth manifold.

2. Preliminaries

In this section we make some preliminary observations about the set $\mathcal{E}$. We begin with the following observation.

Proposition 2.1. *Let X be a manifold.*

1. *If X is orientable and even dimensional, then $X \notin \mathcal{E}$.*
2. *If $X \in \mathcal{E}$, then $H^2(X;\mathbb{Z}) = 0$.*
3. *If $X \in \mathcal{E}$, then the Steenrod square $Sq^1\colon H^1(X;\mathbb{Z}_2) \longrightarrow H^2(X;\mathbb{Z}_2)$ is the zero homomorphism.*

Proof. The first claim follows from [**11**, Theorem 2.4, (1)]. If $x \in H^2(X;\mathbb{Z})$, then there exists a complex line bundle β with $c_1(\beta) = e(\beta_{\mathbb{R}}) = x$. This proves (ii). To prove (iii), let $x \in H^1(X;\mathbb{Z}_2)$ be non-zero and α a line bundle over X with $w_1(\alpha) = x$. Then as

$$Sq^1(x) = x^2 = w_2(\alpha \oplus \alpha)$$

and $w_2(\alpha \oplus \alpha)$ is the mod-2 reduction of $e(\alpha \oplus \alpha) = 0$, it follows that $Sq^1(x) = 0$. This proves (iii) and completes the proof of the proposition. □

Lemma 2.2. *Let X, Y be orientable manifolds of positive dimension. Then $X \times Y \notin \mathcal{E}$.*

Proof. If $X \times Y$ is even dimensional, then the previous proposition implies $X \times Y \notin \mathcal{E}$. If $X \times Y$ is odd dimensional, we assume that $\dim X = 2n$ is even. Then there exists a $2n$-plane bundle α over X with $e(\alpha) \neq 0$. The projection $X \times Y \longrightarrow X$ induces a monomorphism in cohomology with integer coefficients in degree $2n$. Thus the pullback of α to $X \times Y$ (by the projection) has non-zero Euler class. □

Thus $\mathcal{E}$ does not contain orientable products. It is clear that $\mathcal{E}_1$ consists of the circle S^1. Since $H^2(X;\mathbb{Z}) \neq 0$ for any 2-manifold, it follows that $\mathcal{E}_2 = \emptyset$.

We now set up some notations. We shall make use of the following maps. Throughout, $\rho_k, \rho_{4,2}$ will denote the homomorphisms

$$\rho_k\colon H^*(X;\mathbb{Z}) \longrightarrow H^*(X;\mathbb{Z}_k),$$
$$\rho_{4,2}\colon H^*(X;\mathbb{Z}_4) \longrightarrow H^*(X;\mathbb{Z}_2)$$

induced by the coefficient surjections. The homomorphism

$$i_*\colon H^*(X;\mathbb{Z}_2) \longrightarrow H^*(X;\mathbb{Z}_4)$$

is induced by the coefficient inclusion. Finally, recall that the Pontryagin square $\mathfrak{P}$ is a cohomology operation

$$\mathfrak{P}\colon H^{2k}(X;\mathbb{Z}_2) \longrightarrow H^{4k}(X;\mathbb{Z}_4)$$

with the property that

$$\rho_{4,2}\mathfrak{P}(x) = x^2.$$

It is well known (see, for example, [**19**, Theorem C]) that for any vector bundle α we have

$$\mathfrak{P}(w_2(\alpha)) = \rho_4(p_1(\alpha)) + i_*(w_1(\alpha)Sq^1(w_2(\alpha)) + w_4(\alpha)).$$

In particular, if either α is orientable or $w_2(\alpha) = 0$ we have

$$\mathfrak{P}(w_2(\alpha)) = \rho_4(p_1(\alpha)) + i_*(w_4(\alpha)).$$

The following observation will be crucially used in the proof of Theorem 1.3.

Proposition 2.3. *Let Y be a CW-complex. Assume that*

$$H_i(Y;\mathbb{Z}) = \begin{cases} \mathbb{Z} & i = 0, \\ 0 & i = 1, \\ F & i = 2. \end{cases}$$

Further assume that $\mathbb{Z}_2$ is a direct summand in $H_2(Y;\mathbb{Z}) = F$. Then the Steenrod square $Sq^1 \colon H^2(Y;\mathbb{Z}_2) \longrightarrow H^3(Y;\mathbb{Z}_2)$ is non-zero.

Proof. Observe that

$$H^i(Y;\mathbb{Z}_2) = \begin{cases} 0 & i = 1, \\ \mathbb{Z}_2^s & i = 2, \end{cases} \qquad H^i(Y;\mathbb{Z}_4) = \begin{cases} 0 & i = 1, \\ \mathbb{Z}_2^u \oplus \mathbb{Z}_4^v & i = 2. \end{cases}$$

The conclusion now follows from the fact that Sq^1 equals the Bockstein homomorphism associated to the exact sequence

$$0 \longrightarrow Z_2 \longrightarrow \mathbb{Z}_4 \longrightarrow \mathbb{Z}_2 \longrightarrow 0.$$

Indeed, we look at the following exact sequence

$$0 \to H^2(Y;\mathbb{Z}_2) \xrightarrow{i_*} H^2(Y;\mathbb{Z}_4) \xrightarrow{\rho_{4,2}} H^2(Y;\mathbb{Z}_2) \xrightarrow{Sq^1} H^3(Y;\mathbb{Z}_2) \longrightarrow \cdots .$$

Identifying the individual groups in the above sequence, we get the exact sequence

$$0 \to \mathbb{Z}_2^u \oplus \mathbb{Z}_2^v \xrightarrow{i_*} \mathbb{Z}_2^u \oplus \mathbb{Z}_4^v \xrightarrow{\rho_{4,2}} \mathbb{Z}_2^u \oplus \mathbb{Z}_2^v \xrightarrow{Sq^1} H^3(Y;\mathbb{Z}_2) \to \cdots .$$

Notice that as $s = u + v$ and $u \neq 0$, it follows that Sq^1 must be non-zero. This completes the proof. ☐

Remark 2.4. We remark that the above proposition remains true even in the case that

$$H_i(Y;\mathbb{Z}) = \begin{cases} \mathbb{Z} & i = 0, \\ \mathbb{Z} & i = 1, \\ F & i = 2. \end{cases}$$

The proof is similar to the proof of the above proposition.

We end this section by stating some results for easy reference in a form that we shall use.

Theorem 2.5 ([**5**, Theorem 1])**.** *Let X be a connected CW-complex of dimension $\leqslant 5$. Let $a \in H^2(X;\mathbb{Z}_2)$, $b \in H^4(X;\mathbb{Z}_2)$, $c \in H^4(X;\mathbb{Z})$ satisfy*

$$\rho_4(c) = \mathfrak{P}(a) + i_*(b).$$

Then there exists an oriented 5-plane bundle α such that $w_2(\alpha) = a$, $w_4(\alpha) = b$ and $p_1(\alpha) = c$. ☐

Theorem 2.6 ([**5**, Theorem 2])**.** *Let X be a connected CW-complex of dimension $\leqslant 5$. Let $a \in H^2(X;\mathbb{Z}_2)$ and $p \in H^4(X;\mathbb{Z})$ satisfy*

$$\rho_4(p) = \mathfrak{P}(a).$$

Then there exists an oriented 3-plane bundle α with $w_2(\alpha) = a$ and $p_1(\alpha) = p$. □

Theorem 2.7 ([**6**, Theorem 2])**.** *Let X be a connected CW-complex of dimension $\leqslant 7$. Let $P, E \in H^4(X;\mathbb{Z})$. Then there exists an oriented 4-plane bundle α with*

$$w_2(\alpha) = 0; \quad p_1(\alpha) = P; \quad e(\alpha) = E$$

if and only if there exists $U, V \in H^4(X;\mathbb{Z})$ such that

1. *$P = 2U$ and $E = 2V - U$,*
2. *$Sq^2\rho_2(U) = 0$ and $Sq^2\rho_2(V) = 0$,*
3. *$0 \in \Phi(U)$ and $0 \in \Phi(V)$,*

where Φ is the secondary cohomology operation from $H^4(X;\mathbb{Z})$ to $H^7(X;\mathbb{Z}_2)$ associated to the relation $Sq^2 \circ Sq^2\rho_2 = 0$. □

3. Dimensions 3, 4 and 6

In this section we shall prove Theorem 1.1, Theorem 1.2 and obtain some necessary conditions on 6-manifolds that satisfy ($\mathcal{E}$). We begin with the following observation that identifies the orientable manifolds in $\mathcal{E}_3$.

Proposition 3.1. *Let X be an orientable 3-manifold. Then $X \in \mathcal{E}_3$ if and only if X is a homology 3-sphere.*

Proof. Assume that $X \in \mathcal{E}_3$. By Proposition 2.1, (ii), we have $H^2(X;\mathbb{Z}) = 0$. This implies that $H_i(X;\mathbb{Z}) = 0$, $i = 1, 2$. Conversely, if X is a homology 3-sphere it follows from the comments in Section 1 that $X \in \mathcal{E}_3$. □

To complete the proof of Theorem 1.1 we shall show that $\mathcal{E}_3$ does not contain any non-orientable manifold.

Proof of Theorem 1.1. Assume that there exists $X \in \mathcal{E}_3$ with X non-orientable. Then, as $H^2(X;\mathbb{Z}) = 0$, the integral and mod-2 cohomology groups of X can be seen to be

$$H^i(X;\mathbb{Z}) = \begin{cases} \mathbb{Z} & i = 0, \\ \mathbb{Z} & i = 1, \\ 0 & i = 2, \\ \mathbb{Z}_2 & i = 3, \end{cases} \qquad H^i(X;\mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2 & i = 0, \\ \mathbb{Z}_2 & i = 1, \\ \mathbb{Z}_2 & i = 2, \\ \mathbb{Z}_2 & i = 3. \end{cases}$$

By Remark 2.4, there exists $a \in H^2(X;\mathbb{Z}_2)$ with $Sq^1(a) \neq 0$. Clearly,

$$\rho_4(p) = \mathfrak{P}(a) = 0,$$

where $0 = p \in H^4(X;\mathbb{Z})$. Hence, by Theorem 2.6, there exists an orientable 3-plane bundle α over X with $w_2(\alpha) = a \neq 0$. Now,

$$w_3(\alpha) = Sq^1(w_2(\alpha)) = Sq^1(a) \neq 0.$$

This implies $e(\alpha) \neq 0$ which is a contradiction. This completes the proof. □

We now turn to the proof of Theorem 1.2. We have already seen that $H^2(X;\mathbb{Z}) = 0$ is a necessary condition for a manifold to satisfy $(\mathcal{E})$. In low dimensions, an additional necessary condition is the vanishing of the fourth integral cohomology. The precise statement is the following.

Theorem 3.2. *Let X be a n-manifold with $n = 4, 5$. If X satisfies $(\mathcal{E})$, then $H^4(X;\mathbb{Z}) = 0$.*

Proof. Assume, if possible, that $H^4(X;\mathbb{Z}) \neq 0$. Let $U \in H^4(X;\mathbb{Z})$ with $U \neq 0$. Let $V = U$, $P = 2U$ and $E = 2U - U = U$. Then as all the three conditions of Theorem 2.7 are satisfied, there exists an oriented 4-plane bundle α with $e(\alpha) = U \neq 0$. This contradiction proves the theorem. □

Proof of Theorem 1.2. If X is a 4-manifold, then $H^4(X;\mathbb{Z})$ equals either $\mathbb{Z}$ or $\mathbb{Z}_2$. Hence $\mathcal{E}_4 = \emptyset$. This completes the proof of Theorem 1.2. □

As a consequence of the above theorem we obtain the following observation.

Corollary 3.3. *Suppose $X = X_1 \times X_2$ is a non-orientable manifold, where X_1, X_2 are manifolds of positive dimension and* $\dim(X) = 5$. *Then X does not satisfy $(\mathcal{E})$.*

Proof. If $\dim X_1 = 2$, then $H^2(X;\mathbb{Z}) \neq 0$. If $\dim X_1 = 4$, then $H^4(X;\mathbb{Z}) \neq 0$. In either case X does not satisfy $(\mathcal{E})$. □

We end this section by deriving some necessary conditions on manifolds $X \in \mathcal{E}_6$. We do not know if the conclusion of Theorem 3.2 remains true for 6-manifolds. We can, however, make the following observation.

Theorem 3.4. *Suppose $X \in \mathcal{E}_6$. Then*

1. *$H^4(X;\mathbb{Z})$ is a finite elementary abelian 2-group, and*
2. *the Euler characteristic $\chi(X) \leqslant 0$.*

Proof. Since $H^4(X;\mathbb{Z}_2)$ is a finite elementary abelian 2-group the conclusion (1) follows if

$$\rho_2 \colon H^4(X;\mathbb{Z}) \longrightarrow H^4(X;\mathbb{Z}_2)$$

is a monomorphism. If $U \neq 0$ and $U \in \ker(\rho_2)$ we set

$$V = U, \quad E = U, \quad P = 2U.$$

As $Sq^2\rho_2(U) = 0$ we have, by Theorem 2.7, an oriented 4-plane bundle α with

$$e(\alpha) = U \neq 0.$$

This contradiction forces ρ_2 to be a monomorphism and (1) follows. In particular, $H^4(X;\mathbb{Z})$ is finite.

To prove (2), we note that, since X is necessarily non-orientable, the integral homology of X must now be of the form

$$H_i(X;\mathbb{Z}) = \begin{cases} \mathbb{Z} & i=0, \\ \mathbb{Z}^{\ell} & i=1, \\ F & i=2, \\ \mathbb{Z}^t \oplus F' & i=3, \\ F'' & i=4, \\ \mathbb{Z}^{\ell'} \oplus \mathbb{Z}_2 & i=5, \\ 0 & i=6, \end{cases}$$

where F' is a finite elementary abelian 2-group. The Euler characteristic can now be computed to be

$$\chi(X) = 1 - \ell - t - \ell'.$$

Evidently $\ell \geqslant 1$ and so we must have $\chi(X) \leqslant 0$. This completes the proof. □

We end this section by showing that a non-orientable 6-dimensional product cannot satisfy $(\mathcal{E})$. Note that if a 6-manifold X satisfies $(\mathcal{E})$, then X is necessarily non-orientable.

Proposition 3.5. *Suppose $X = X_1 \times X_2$ is a non-orientable manifold, where X_1, X_2 are manifolds of positive dimension and* $\dim(X) = 6$*. Then X does not satisfy $(\mathcal{E})$.*

Proof. Assume that X_1 is non-orientable. We look at several cases. If $\dim(X_1) = 2$, then $H^2(X;\mathbb{Z}) \neq 0$ and hence X does not satisfy $(\mathcal{E})$. If $\dim(X_1) = 3$, then as X_1 is non-orientable there exists (by Theorem 1.1) an oriented bundle α over X_1 with non-zero Euler class. Pulling back α to X via the projection $X \longrightarrow X_1$ gives an oriented bundle over X with non-zero Euler class. Thus in this case too X does not satisfy $(\mathcal{E})$. If $\dim(X_1) = 4$, then arguing as in the previous case (and using Theorem 1.2) we see that X does not satisfy $(\mathcal{E})$. Finally, let $\dim(X_1) = 5$. Then first observe that $X_2 = S^1$ and hence $H^1(X_2;\mathbb{Z}) = \mathbb{Z}$. We now have two cases. In the case that X_1 does not satisfy $(\mathcal{E})$, then arguing as in the previous two cases we see that X does not satisfy $(\mathcal{E})$. In the case that X_1 satisfies $(\mathcal{E})$, we have $H^1(X_1;\mathbb{Z}) = \mathbb{Z}$ (see Section 5) and hence $H^2(X;\mathbb{Z}) \neq 0$. Thus in this case too X does not satisfy $(\mathcal{E})$. This completes the proof. □

4. Dimension 5: the orientable case

In this section we shall prove Theorem 1.3 in the case X is orientable.

Proof of Theorem 1.3. We assume that $X \in \mathcal{E}_5$ is orientable. We shall show that the conditions (1) and (2) of Theorem 1.3 are satisfied.

By Proposition 2.1 (ii) and Theorem 3.2 we have $H^2(X;\mathbb{Z}) = 0$ and $H^4(X;\mathbb{Z}) = 0$. Thus (1) is satisfied. To prove (2), we first observe that the integral homology and cohomology groups of X can be seen to be

$$H_i(X;\mathbb{Z}) = \begin{cases} \mathbb{Z} & i=0, \\ 0 & i=1, \\ F & i=2, \\ 0 & i=3, \\ 0 & i=4, \\ \mathbb{Z} & i=5, \end{cases} \qquad H^i(X;\mathbb{Z}) = \begin{cases} \mathbb{Z} & i=0, \\ 0 & i=1, \\ 0 & i=2, \\ F & i=3, \\ 0 & i=4, \\ \mathbb{Z} & i=5. \end{cases}$$

where we recall that F denotes a finite abelian group. Thus the mod-2 cohomology groups of X are

$$H^i(X;\mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2 & i=0, \\ 0 & i=1, \\ \mathbb{Z}_2^s & i=2, \\ \mathbb{Z}_2^s & i=3, \\ 0 & i=4, \\ \mathbb{Z}_2 & i=5. \end{cases}$$

To complete the proof we shall show that $s = u + v = 0$, where $u,\ v$ are as in the proof of Proposition 2.3. We first show that $u = 0$.

Assume that $u > 0$. We shall see that this leads to a contradiction. Since $u > 0$, we conclude by Proposition 2.3 that

$$Sq^1\colon H^2(X;\mathbb{Z}_2) \longrightarrow H^3(X;\mathbb{Z}_2)$$

is non-zero. Let $a \in H^2(X;\mathbb{Z}_2)$ with $Sq^1(a) \neq 0$. Since the Pontryagin square

$$\mathfrak{P}\colon H^2(X;\mathbb{Z}_2) \longrightarrow H^4(X;\mathbb{Z}_4) = 0$$

is the zero homomorphism, we see that

$$\rho_4(0) = \mathfrak{P}(a).$$

Thus, by Theorem 2.6, there exists an oriented 3-plane bundle α with $w_2(\alpha) = a \neq 0$. By Wu's formula we have

$$w_3(\alpha) = Sq^1(w_2(\alpha)) = Sq^1(a) \neq 0.$$

This forces $e(\alpha) \neq 0$. This contradiction implies $u = 0$.

Next we show that $v = 0$. Assuming the contrary, that is, $v > 0$ we shall construct a 5-dimensional CW-complex X_2 and a map $h\colon X \longrightarrow X_2$ such that

1. the homology groups of X_2 in degrees $\leqslant 2$ are of the form of the homology groups of Y in Proposition 2.3,
2. $H_2(X_2;\mathbb{Z}) = \mathbb{Z}_2$, and
3. $h^*\colon H^3(X_2;\mathbb{Z}) = \mathbb{Z}^\ell \oplus \mathbb{Z}_2 \longrightarrow H^3(X;\mathbb{Z}) = F$ is injective on the torsion subgroup.

We shall construct the space X_2 in two steps.

Step I. Let X_1 be the CW-complex obtained from X by killing the fundamental group $\pi_1(X)$ of X. Let $f\colon X \longrightarrow X_1$ denote the inclusion. Since X_1 is obtained from X by attaching 2-cells, we have $H_i(X_1) = H_i(X)$ for $i \neq 1, 2$. As X_1 is simply connected,

$H_1(X_1;\mathbb{Z}) = 0$. To compute $H_2(X_1;\mathbb{Z})$ we look at the exact sequence

$$0 \longrightarrow H_2(X;\mathbb{Z}) \xrightarrow{f_*} H_2(X_1) \longrightarrow H_2(X_1, X;\mathbb{Z}) \longrightarrow 0.$$

As $H_2(X_1, X;\mathbb{Z}) = \mathbb{Z}^t$ (say) is free abelian, we have

$$H_2(X_1;\mathbb{Z}) = F \oplus \mathbb{Z}^t.$$

Thus the integral homology groups of X_1 are of the form

$$H_i(X_1;\mathbb{Z}) = \begin{cases} \mathbb{Z} & i = 0, \\ 0 & i = 1, \\ \mathbb{Z}^t \oplus F & i = 2, \\ 0 & i = 3, \\ 0 & i = 4, \\ \mathbb{Z} & i = 5. \end{cases}$$

In the exact sequence

$$H^3(X_1, X;\mathbb{Z}) \longrightarrow H^3(X_1;\mathbb{Z}) \xrightarrow{f^*} H^3(X;\mathbb{Z}) \longrightarrow H^4(X_1, X;\mathbb{Z}),$$

the first and the fourth groups are zero and hence

$$f^*\colon H^3(X_1;\mathbb{Z}) = F \longrightarrow H^3(X;\mathbb{Z}) = F$$

is an isomorphism.

Step II. From the above step, we have that X_1 is simply connected and $\pi_2(X_1) = H_2(X_1;\mathbb{Z}) = F \oplus \mathbb{Z}^t$. Since $v > 0$, there exists an index two subgroup, H say, of F. Let X_2 be obtained from X_1 by killing the index two subgroup $\mathbb{Z}^t \oplus H$ of $\pi_2(X_1) = \mathbb{Z}^t \oplus F$. Let $g\colon X_1 \longrightarrow X_2$ denote the inclusion and $h = g \circ f$. Note that

$$g_*\colon H_2(X_1;\mathbb{Z}) = F \oplus \mathbb{Z}^t \longrightarrow H_2(X_2;\mathbb{Z}) = \mathbb{Z}_2$$

is onto. Since we have an exact sequence

$$0 \longrightarrow H_3(X_2;\mathbb{Z}) \longrightarrow H_3(X_2, X_1;\mathbb{Z})$$

and the last group is free abelian we conclude that $H_3(X_2;\mathbb{Z}) = \mathbb{Z}^\ell$ (say) is free abelian. The integral homology groups of X_2 are now of the form

$$H_i(X_2;\mathbb{Z}) = \begin{cases} \mathbb{Z} & i = 0, \\ 0 & i = 1, \\ \mathbb{Z}_2 & i = 2, \\ \mathbb{Z}^\ell & i = 3, \\ 0 & i = 4, \\ \mathbb{Z} & i = 5. \end{cases}$$

Hence, by Proposition 2.3

$$Sq^1\colon H^2(X_2;\mathbb{Z}_2) = \mathbb{Z}_2 \longrightarrow H^3(X;\mathbb{Z}_2) = \mathbb{Z}_2^\ell \oplus \mathbb{Z}_2$$

is non-zero. We now argue that

$$g^*\colon H^3(X_2;\mathbb{Z}) = \mathbb{Z}^\ell \oplus \mathbb{Z}_2 \longrightarrow H^3(X_1;\mathbb{Z}) = F$$

is injective on the torsion subgroup. This follows readily from the commutative diagram

$$\begin{array}{ccccccccc} 0 \to & \mathrm{Ext}(H_2(X_1;\mathbb{Z}),\mathbb{Z}) = F & \longrightarrow & H^3(X_1;\mathbb{Z}) = F & \longrightarrow & \mathrm{Hom}(H_3(X_1;\mathbb{Z}),\mathbb{Z}) & \to 0 \\ & \uparrow & & \uparrow & & \uparrow & \\ 0 \to & \mathrm{Ext}(H_2(X_2;\mathbb{Z}),\mathbb{Z}) = \mathbb{Z}_2 & \to & H^3(X_2;\mathbb{Z}) = \mathbb{Z}^\ell \oplus \mathbb{Z}_2 & \to & \mathrm{Hom}(H_3(X_2;\mathbb{Z}),\mathbb{Z}) & \to 0. \end{array}$$

Observe that the first vertical map is a monomorphism, since $g_*\colon H_2(X_1;\mathbb{Z}) = F \oplus \mathbb{Z}^t \longrightarrow H_2(X_2;\mathbb{Z}) = \mathbb{Z}_2$ is onto. The horizontal maps in the first square are monomorphisms. This forces

$$g^*\colon H^3(X_2;\mathbb{Z}) = \mathbb{Z}_2 \oplus \mathbb{Z}^\ell \longrightarrow H^3(X_1;\mathbb{Z}) = F$$

to be injective on the torsion subgroup $\mathbb{Z}_2$ of $H^3(X_2;\mathbb{Z})$.

Now observe that as $H^4(X_2;\mathbb{Z}_4) = 0$, the Pontryagin square

$$\mathfrak{P}\colon H^2(X_2;\mathbb{Z}_2) \longrightarrow H^4(X_2;\mathbb{Z}_4)$$

is the zero map. We may now conclude, as before, that there exists an orientable 3-plane bundle α over X_2 with $e(\alpha) \neq 0$. As $2e(\alpha) = 0$, we have that $e(\alpha)$ is of order two in $H^3(X_2;\mathbb{Z}) = \mathbb{Z}^\ell \oplus \mathbb{Z}_2$. Hence $e(h^*(\alpha)) \neq 0$. This contradiction proves $v = 0$ and hence (2) holds.

Conversely, if the two conditions of the theorem are satisfied and α is an oriented k-plane bundle over X with $k = 3$ or 5, then as the integral cohomology groups are as described above and $2e(\alpha) = 0$ it readily follows that $e(\alpha) = 0$. Hence $X \in \mathcal{E}_5$. This completes the proof of the theorem. □

Remark 4.1. We remark that if X is an orientable 5-manifold, then X satisfies the two conditions in Theorem 1.3 if and only if X satisfies either of the following two sets of equivalent conditions:

1. $\pi_1(X)$ is perfect,
2. X is a rational homology sphere, and
3. the 2-primary component of $H_2(X;\mathbb{Z})$ is trivial.

and

1. $\pi_1(X)$ is perfect, and
2. X is a $\mathbb{Z}_2$-homology sphere.

We now make the following observations.

Corollary 4.2. *If $X \in \mathcal{E}_5$ is orientable and α a vector bundle over X, then $w(\alpha) = 1$. Also, X is an oriented boundary.*

Proof. We may assume that α is a 5-plane bundle. By the remark above, X is a $\mathbb{Z}_2$-homology sphere and hence $w_i(\alpha) = 0$ for $1 \leqslant i < 5$. Since $e(\alpha) = 0$ we have $w_5(\alpha) = 0$. This completes the proof. □

In particular, X is oriented cobordant to the sphere S^5. This is clearly not a sufficient condition for an orientable 5-manifold to satisfy $(\mathcal{E})$. Indeed, the projective space $\mathbb{RP}^5$ is oriented cobordant to S^5 but does not satisfy $(\mathcal{E})$. It is well known that

$\Omega_5^{SO} \cong \mathbb{Z}_2$ is generated by the Wu manifold $W = SU(3)/SO(3)$. The Wu manifold W is a simply connected rational homology 5-sphere with $H_2(X;\mathbb{Z}) \cong \mathbb{Z}_2$. Hence W does not satisfy $(\mathcal{E})$.

Suppose $X, Y \in \mathcal{E}_5$ are both orientable. It is then straightforward to check that the connected sum $X\#Y$ satisfies the two conditions of Theorem 1.3. Thus we have the following.

Corollary 4.3. *Suppose $X, Y \in \mathcal{E}_5$ with both X and Y orientable. Then their connected sum $X\#Y \in \mathcal{E}_5$.* □

We now give examples of $X \in \mathcal{E}_5$ with X orientable and not simply connected.

Example 4.4. Let G be a finitely presented perfect group with trivial Schur multiplier. Then by [**10**, Theorem 1], there exists an integral homology 5-sphere K with $\pi_1(K) = G$. Let F be a finite abelian group with trivial 2-primary component. Then, by the Smale-Barden theory, [**15, 2**], there exists a simply connected 5-manifold M with $H_2(M;\mathbb{Z}) = F \oplus F$. It is clear that $K, M \in \mathcal{E}_5$. As observed above, we have that $K\#M \in \mathcal{E}_5$. In particular, $\mathcal{E}_5$ contains non-simply connected orientable manifolds.

Given a connected CW-complex Y, there exists an exact sequence

$$\pi_2(Y) \longrightarrow H_2(Y;\mathbb{Z}) \longrightarrow H_2(\pi_1(Y);\mathbb{Z}) \longrightarrow 0$$

(see, for example, [**4**, Theorem 5.2]) where, by definition, the last group is the Schur multiplier of $\pi_1(Y)$. Since $H_2(X;\mathbb{Z})$ is finite for every orientable $X \in \mathcal{E}_5$ it follows that the Schur multiplier of $\pi_1(X)$ is finite for every orientable $X \in \mathcal{E}_5$ and does not contain any element of order 2. It would be interesting to know if the fundamental group of any orientable $X \in \mathcal{E}_5$ has trivial Schur multiplier.

5. Dimension 5: the non-orientable case

In this section we prove Theorem 1.3 when X is non-orientable, thereby completing the proof of Theorem 1.3. We begin with some observations.

Let X be a non-orientable 5-manifold with $H^2(X;\mathbb{Z}) = 0$. Then the integral homology and cohomology groups of X are of the form below

$$H_i(X;\mathbb{Z}) = \begin{cases} \mathbb{Z} & i = 0, \\ \mathbb{Z}^\ell & i = 1, \\ F & i = 2, \\ \mathbb{Z}^t \oplus F' & i = 3, \\ \mathbb{Z}^{\ell'} \oplus \mathbb{Z}_2 & i = 4, \\ 0 & i = 5, \end{cases} \qquad H^i(X;\mathbb{Z}) = \begin{cases} \mathbb{Z} & i = 0, \\ \mathbb{Z}^\ell & i = 1, \\ 0 & i = 2, \\ \mathbb{Z}^t \oplus F & i = 3, \\ \mathbb{Z}^{\ell'} \oplus F' & i = 4, \\ \mathbb{Z}_2 & i = 5. \end{cases}$$

Thus we have,

$$H^i(X;\mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2 & i = 0, \\ \mathbb{Z}_2^\ell & i = 1, \\ \mathbb{Z}_2^s & i = 2, \\ \mathbb{Z}_2^t \oplus \mathbb{Z}_2^{s'} \oplus \mathbb{Z}_2^s & i = 3, \\ \mathbb{Z}_2^{\ell'} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2^{s'} & i = 4, \\ \mathbb{Z}_2 & i = 5. \end{cases}$$

By Poincaré duality we have

$$\ell = \ell' + 1 + s'$$

and

$$s = t + s' + s.$$

Hence $t + s' = 0$. This forces $t = 0 = s'$ and $\ell = \ell' + 1$. In particular, $\ell \geqslant 1$ (compare proof of Proposition 3.5).

If further $H^4(X;\mathbb{Z}) = 0$, then the integral homology and cohomology groups of X take the following form

$$H_i(X;\mathbb{Z}) = \begin{cases} \mathbb{Z} & i = 0, \\ \mathbb{Z} & i = 1, \\ F & i = 2, \\ 0 & i = 3, \\ \mathbb{Z}_2 & i = 4, \\ 0 & i = 5, \end{cases} \qquad H^i(X;\mathbb{Z}) = \begin{cases} \mathbb{Z} & i = 0, \\ \mathbb{Z} & i = 1, \\ 0 & i = 2, \\ F & i = 3, \\ 0 & i = 4, \\ \mathbb{Z}_2 & i = 5. \end{cases}$$

and hence we have

$$H^i(X;\mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2 & i = 0, \\ \mathbb{Z}_2 & i = 1, \\ \mathbb{Z}_2^s & i = 2, \\ \mathbb{Z}_2^s & i = 3, \\ \mathbb{Z}_2 & i = 4, \\ \mathbb{Z}_2 & i = 5. \end{cases}$$

Before starting the proof of Theorem 1.3 we make some observations.

Lemma 5.1. *Let X be a non-orientable 5-manifold such that $H^2(X;\mathbb{Z}) = 0$ and $H^4(X;\mathbb{Z}) = 0$. Then the homomorphisms $Sq^1\colon H^4(X;\mathbb{Z}_2) = \mathbb{Z}_2 \longrightarrow H^5(X;\mathbb{Z}_2) = \mathbb{Z}_2$ and $i_*\colon H^4(X;\mathbb{Z}_2) = \mathbb{Z}_2 \longrightarrow H^4(X;\mathbb{Z}_4) = \mathbb{Z}_2$ are isomorphisms.*

Proof. Consider the long exact sequence

$$\begin{aligned} \cdots \to H^4(X;\mathbb{Z}_2) &\xrightarrow{i^*} H^4(X;\mathbb{Z}_4) \to H^4(X;\mathbb{Z}_2) \xrightarrow{Sq^1} H^5(X;\mathbb{Z}_2) \\ &\to H^5(X;\mathbb{Z}_4) \to H^5(X;\mathbb{Z}_2) \to 0 \end{aligned}$$

corresponding to the short exact sequence

$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathbb{Z}_4 \longrightarrow \mathbb{Z}_2 \longrightarrow 0$$

of coefficient groups. Note that $H^i(X,\mathbb{Z}_2) = \mathbb{Z}_2$ and $H^i(X,\mathbb{Z}_4) = \mathbb{Z}_2$ for $i = 4,\ 5$, hence it follows that i^* and Sq^1 are isomorphisms. $\square$

Lemma 5.2. *Let X be a non-orientable 5-manifold with $H^2(X;\mathbb{Z}) = 0$, $H^4(X;\mathbb{Z}) = 0$ and $H^2(X;\mathbb{Z}_2) = 0$. Let α be an oriented 5-plane bundle over X. Then $e(\alpha) = 0$.*

Proof. We note that

$$Sq^1(w_4(\alpha)) = w_1(\alpha)w_4(\alpha) + w_5(\alpha) = w_5(\alpha)$$

as α is orientable. As $w_2(\alpha) = 0$ and $p_1(\alpha) = 0$ we have

$$0 = \mathfrak{P}(w_2(\alpha)) = \rho_4(p_1(\alpha)) + i_*(w_4(\alpha)) = i_*(w_4(\alpha)).$$

Thus $w_4(\alpha) = 0$ which implies that $w_5(\alpha) = 0$ and hence $e(\alpha) = 0$. □

Proof of Theorem 1.3. Assume that $X \in \mathcal{E}_5$ is non-orientable. Then, by Proposition 2.1 and Theorem 3.2, $H^i(X;\mathbb{Z}) = 0$ for $i = 2, 4$ and hence (1) is satisfied. Observe that the homology and cohomology groups of X are as noted above. To prove (2) we shall show, as in the orientable case, that $s = u + v = 0$. We first show that $u = 0$.

Assume that $u > 0$. We shall see that this leads to a contradiction. By the remark after Proposition 2.3 we have that

$$Sq^1\colon H^2(X;\mathbb{Z}_2) = \mathbb{Z}_2^s \longrightarrow H^3(X;\mathbb{Z}_2) = \mathbb{Z}_2^s$$

is non-zero. Hence there exists $a \in H^2(X;\mathbb{Z}_2)$ with $Sq^1(a) \neq 0$. Also note that $H^4(X;\mathbb{Z}) = 0$. We now consider the Pontryagin square

$$\mathfrak{P}\colon H^2(X;\mathbb{Z}_2) = \mathbb{Z}_2^s \longrightarrow H^4(X;\mathbb{Z}_4) = \mathbb{Z}_2.$$

If $\mathfrak{P}(a) = 0$, then as in the orientable case we get an oriented 3-plane bundle α over X with $w_2(\alpha) = a$. This implies as before that $w_3(\alpha) \neq 0$ and hence $e(\alpha) \neq 0$ which is a contradiction. In the case that $\mathfrak{P}(a) \neq 0$ we can find $b \in H^4(X;\mathbb{Z}_2) = \mathbb{Z}_2$, $b \neq 0$ with $i_*(b) = \mathfrak{P}(a)$. By Theorem 2.5, there exists an oriented 5-plane bundle α with $w_4(\alpha) = b \neq 0$. By Lemma 5.1,

$$w_5(\alpha) = Sq^1(w_4(\alpha)) \neq 0$$

and hence $e(\alpha) \neq 0$. This contradiction forces $u = 0$.

We now show that $v = 0$. We assume that $v > 0$ and obtain a contradiction. The proof proceeds as in the orientable case and the case above. As before we construct spaces X_1 and X_2 where X_1 is obtained from X by killing $\pi_1(X)$ and X_2 is obtained from X_1 by killing an index two subgroup of $\pi_2(X_1) = H_2(X_1;\mathbb{Z}) = \mathbb{Z}^t \oplus F$. Let $f\colon X \longrightarrow X_1$ and $g\colon X_1 \longrightarrow X_2$ denote the inclusion with $h = g \circ f$. As in the orientable case it can now be checked that

1. $h^*\colon H^3(X_2;\mathbb{Z}) = \mathbb{Z}^\ell \oplus \mathbb{Z}_2 \longrightarrow H^3(X;\mathbb{Z}) = F$ is injective on the torsion subgroup, and
2. $h^*\colon H^5(X_2;\mathbb{Z}) = \mathbb{Z}_2 \longrightarrow H^5(X;\mathbb{Z}) = \mathbb{Z}_2$ is an isomorphism.

Since X_2 is obtained from X by attaching 2-cells and 3-cells we see that

$$H^4(X_2;\mathbb{Z}) = H^4(X_1;\mathbb{Z}) = H^4(X;\mathbb{Z}) = 0.$$

Also as $H_2(X_2;\mathbb{Z}) = \mathbb{Z}_2$ we have

$$Sq^1\colon H^2(X_2;\mathbb{Z}_2) = \mathbb{Z}_2 \longrightarrow H^3(X_2;\mathbb{Z}_2)$$

is non-zero. As before we fix $a \in H^2(X_2;\mathbb{Z}_2)$ with $Sq^1(a) \neq 0$. If $\mathfrak{P}(a) = 0$, then arguing as above we have an oriented 3-plane bundle α with $e(\alpha) \neq 0$. This implies $h^*(e(\alpha)) \neq 0$. This is a contradiction. If $\mathfrak{P}(a) \neq 0$, then we have an oriented 5-plane bundle with $e(\alpha) \neq 0$. Hence $h^*(e(\alpha)) \neq 0$ and we have a contradiction. This forces $v = 0$. This completes the proof of the theorem in one direction.

Conversely, assume that X is non-orientable and satisfies the two conditions of the theorem. In particular $s = 0$. Now if α is an oriented 3-plane bundle then clearly $e(\alpha) = 0$. If α is an oriented 5-plane bundle, then by Lemma 5.2 we have $e(\alpha) = 0$. Hence $X \in \mathcal{E}_5$. This completes the proof of the theorem. □

As in the orientable case we have the following:

Corollary 5.3. *If $X \in \mathcal{E}_5$ is non-orientable and α a vector bundle over X, then*

1. *$w(\alpha) = 1$ if α is orientable, and*
2. *$w_i(\alpha) = 0$, $i \geqslant 2$, if α is non-orientable.*

In particular, X is an unoriented boundary.

Proof. We first prove (1). It is enough to prove this when α is an orientable 5-plane bundle. Since $e(\alpha) = 0$ we have $w_5(\alpha) = 0$. The equality

$$Sq^1(w_4(\alpha)) = w_5(\alpha) = 0,$$

together with the fact that Sq^1 is an isomorphism (Lemma 5.1) implies that $w_4(\alpha) = 0$. Since $H^2(X; \mathbb{Z}_2) = 0$ we have $w_2(\alpha) = 0$. That $w_1(\alpha) = 0$ follows from the assumption that α is orientable. This forces $w_3(\alpha) = 0$ as the first non-zero Stiefel-Whitney class must appear in a degree a power of 2. This proves (1).

Next we prove (2). Assume that α is non-orientable. We assume as before that α is a 5-plane bundle. As $H^2(X; \mathbb{Z}_2) = 0$ and

$$Sq^1(w_2(\alpha)) = w_3(\alpha),$$

we have $w_i(\alpha) = 0$, $i = 2, 3$. We next show that $w_5(\alpha) = 0$. Assume $w_5(\alpha) \neq 0$. Now

$$Sq^1(w_4(\alpha)) = w_1(\alpha)w_4(\alpha) + w_5(\alpha).$$

If $w_4(\alpha) = 0$, then $0 = Sq^1(w_4(\alpha)) = w_5(\alpha)$ is a contradiction. On the other hand assuming $w_4(\alpha) \neq 0$ we have

$$Sq^1(w_4(\alpha)) = w_1(\alpha)w_4(\alpha) + w_5(\alpha) = 0$$

which is a contradiction since Sq^1 is an isomorphism, by Lemma 5.1. Thus $w_5(\alpha) = 0$. Finally, we show that $w_4(\alpha) = 0$. This follows from the fact that

$$i_*(w_4(\alpha)) + \rho_4(p_1(\alpha)) = \mathfrak{P}(w_2(\alpha)) = 0$$

implying that $i_*(w_4(\alpha)) = 0$. This implies $w_4(\alpha) = 0$. This completes the proof of (2). □

Suppose $X, Y \in \mathcal{E}_5$ where at least one of X and Y is orientable. Then it is easy to check that the connected sum $X \# Y$ satisfies the two conditions of Theorem 1.3. Thus we have the following.

Corollary 5.4. *Suppose $X, Y \in \mathcal{E}_5$ where at least one of X and Y is orientable. Then $X \# Y \in \mathcal{E}_5$.* □

We mention that if $X, Y \in \mathcal{E}_5$ are both non-orientable, then as $H_1(X \# Y; \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}$, it follows that $X \# Y$ does not satisfy $(\mathcal{E})$. We end this section by giving examples of non-orientable 5-manifolds that satisfy $(\mathcal{E})$.

Example 5.5. Let $E = S^4 \widetilde{\times} S^1$ denote the twisted S^4 bundle over S^1. The space E is obtained as the quotient

$$E = S^4 \times [0,1]/\sim,$$

where $(x,0) \sim (-x,1)$. The integral homology groups of E are of the form

$$H_i(E;\mathbb{Z}) = \begin{cases} \mathbb{Z} & i=0, \\ \mathbb{Z} & i=1, \\ 0 & i=2, \\ 0 & i=3, \\ \mathbb{Z}_2 & i=4, \\ 0 & i=5. \end{cases}$$

Thus, by Theorem 1.3, $E \in \mathcal{E}_5$. Observe that E is non-orientable and $\pi_1(E) = \mathbb{Z}$. Let $M \in \mathcal{E}_5$ be an orientable manifold. Then $E\#M$ is non-orientable and $E\#M \in \mathcal{E}_5$.

Notice that if $X \in \mathcal{E}_5$ is non-orientable, then $\pi_1(X)$ is necessarily infinite. However, as $H_2(X;\mathbb{Z})$ is finite, it follows that the Schur multiplier of $\pi_1(X)$ is still finite and has trivial 2-primary component.

6. K-theory

In this section we describe the real and complex K-theory of manifolds $X \in \mathcal{E}_5$. Recall the Atiyah-Hirzebruch spectral sequence for $\widetilde{K}(X)$ is the spectral sequence with E_2 term given by

$$E_2^{p,q} = \widetilde{H}^p(X; K^q(\text{point})).$$

Similarly the Atiyah-Hirzebruch spectral sequence for $\widetilde{KO}(X)$ is the spectral sequence with E_2 term given by

$$E_2^{p,q} = \widetilde{H}^p(X; KO^q(\text{point})).$$

We have the following description of the real and complex K-theory for an orientable $X \in \mathcal{E}_5$.

Theorem 6.1. *Let $X \in \mathcal{E}_5$ be orientable with $H_2(X,\mathbb{Z}) = F$.*

1. *$\widetilde{KO}^{-i}(X)$ is given as follows:*

i	0	1	2	3	4	5	6	7
$\widetilde{KO}^{-i}(X)$	0	F	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2 \oplus F$	0	$\mathbb{Z}$

2. *$\widetilde{K}^0(X) = 0$ and $\widetilde{K}^{-1}(X) = \mathbb{Z} \oplus F$.*

Proof. The Atiyah-Hirzebruch spectral sequence for both $\widetilde{KO}(X)$ and $\widetilde{K}(X)$ collapses after the sixth page, since all the differentials are zero. The proof of the theorem now follows easily by analyzing these spectral sequences and using the cohomology description of X when $X \in \mathcal{E}_5$ and X orientable, except for the value of $\widetilde{K}^{-1}(X)$. For the computation of $\widetilde{K}^{-1}(X)$, we consider the following long exact sequence (see

p. 340 of [20]),

$$\cdots \to \widetilde{KO}^0(X) \to \widetilde{KO}^{-1}(X) \to \widetilde{K}^{-1}(X) \to \widetilde{KO}^1(X) \to \widetilde{KO}^0(X) \to \cdots.$$

As $\widetilde{KO}^0(X) = 0$, $\widetilde{KO}^{-1}(X) = F$ and $\widetilde{KO}^1(X) = \mathbb{Z}$, we conclude that $\widetilde{K}^{-1}(X) = \mathbb{Z} \oplus F$. □

When $X \in \mathcal{E}_5$ is non-orientable, we have the following description of real and complex K-theory.

Theorem 6.2. *Let $X \in \mathcal{E}_5$ be non-orientable with $H_2(X, \mathbb{Z}) = F$. Then $\widetilde{K}^0(X) = 0$ and $\widetilde{K}^{-1}(X) = \mathbb{Z} \oplus \mathbb{Z}_2 \oplus F$. The KO-groups of X is as follows:*

i	0	1	2	3	4	5	6	7
$\widetilde{KO}^{-i}(X)$	$\mathbb{Z}_2$	$\mathbb{Z}_2 \oplus F$	0	$\mathbb{Z} \oplus \mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus F or \mathbb{Z}_4 \oplus F$	$\mathbb{Z}_2$	$\mathbb{Z} \oplus \mathbb{Z}_2$

Proof. As noted above, the Atiyah-Hirzebruch spectral sequences for $\widetilde{K}(X)$ and $\widetilde{KO}(X)$ collapses after the sixth page. The proof of the theorem now follows easily by further analyzing the spectral sequences, except for the value of $\widetilde{KO}^0(X)$ and $\widetilde{KO}^{-6}(X)$. From the analysis we observe that $\widetilde{KO}^0(X)$ is either 0 or $\mathbb{Z}_2$ and $\widetilde{KO}^{-6}(X)$ is either $\mathbb{Z}_2$ or 0.

Since X is non-orientable, the tangent bundle over X will represent the non-zero element in $\widetilde{KO}^0(X)$. Hence $\widetilde{KO}^0(X) \cong \mathbb{Z}_2$.

If $\widetilde{KO}^{-6}(X) = 0$ then we get the following short exact sequence (see p. 340 of [20])

$$0 \to \widetilde{KO}^{-7}(X) \xrightarrow{\epsilon} \widetilde{K}^{-7}(X) \to \widetilde{KO}^{-5}(X) \to 0,$$

where the first map ϵ is the complexification map. Then ϵ will map the $\mathbb{Z}_2$-summand of $\widetilde{KO}^{-7}(X)$ onto the $\mathbb{Z}_2$ summand of $\widetilde{K}^{-7}(X)$. If $\rho\colon \widetilde{K}^{-7}(X) \to \widetilde{KO}^{-7}(X)$ denotes the real restriction then using the fact that the composition $\rho \circ \epsilon$ is multiplication by 2, it is easy to see that ϵ will map the $\mathbb{Z}$-summand of $\widetilde{KO}^{-7}(X)$ into the $\mathbb{Z}$-summand of $\widetilde{K}^{-7}(X)$ either isomorphically or by multiplication of 2. Applying this to the above short exact sequence, we get the contradiction on the cardinality of the groups. Hence, $\widetilde{KO}^{-6}(X) = \mathbb{Z}_2$. □

Remark 6.3.
1. It follows from the KO-theory computations that if $X \in \mathcal{E}_5$ is orientable, then X is stably parallelizable. Thus, these X provide examples of stably parallelizable $\mathbb{Z}_2$-homology 5-spheres. On the other hand it is known that there exist $\mathbb{Z}_2$-homology 5-spheres, for example the Lens space $L(5; 1, 1, 1)$ in the notation of [7], that are not stably parallelizable.
2. As noted above, if $X \in \mathcal{E}_5$ is orientable, then X is stably parallelizable. The Kervaire semi-characteristic $\chi^*(X)$ of a $(2n+1)$-dimensional manifold is given by (see [3])

$$\chi^*(X) = \sum_{i=0}^{n} \dim H^i(X; \mathbb{Z}_2) \bmod 2.$$

If $X \in \mathcal{E}_5$ is orientable, then the computation of cohomology groups of $X \in \mathcal{E}_5$ in Theorem 1.3, shows that $\chi^*(X) = 1$. This implies that if $X \in \mathcal{E}_5$ is orientable, then X cannot be parallelizable by Theorem 1.2 from [**3**]. Moreover, in this case it follows that $\operatorname{span}(X) = 1$.

Acknowledgments

The authors thank the anonymous referee for his detailed and valuable comments which has helped in improving the general presentation of the paper and correcting a few inaccuracies.

References

[**1**] Atiyah, M., and Hirzebruch, F., *Bott periodicity and the parallelizability of the spheres*, Proc. Cambridge Philos. Soc., 57 (1961), 223–226.

[**2**] Barden, D., *Simply connected five-manifolds*, Ann. of Math. (2), 82 (3) (1965), 365–385.

[**3**] Bredon, G.E., and Kosinski, A., *Vector fields on π-manifolds*, Ann. of Math., 84 (1) (1966), 85–90.

[**4**] Brown, K.S., *Cohomology of Groups*, Springer, Grad. Texts in Math., vol. 87, 1982.

[**5**] Čadek, M., and Vanžura, J., *On the classification of oriented bundles over* 5*-complexes*, Czechoslovak Math. J., 43 (118), no. 4, (1993), 753–764.

[**6**] Čadek, M., and Vanžura, J., *On oriented vector bundles over CW-complexes of dimension* 6 *and* 7, Comment. Math. Univ. Carolin., 33 (4) (1992), 727–736.

[**7**] Ewing, J., Moolgavkar, S., and Smith, L., *Stable parallelizability of Lens spaces*, J. Pure Appl. Algebra, 10 (1977), 177–191.

[**8**] Guijarro L., Schick T., and Walschap G., *Bundles with spherical Euler class*, Pacific J. Math., 27 (2) (2002), 377–392.

[**9**] Guijarro, L., and Walschap, G., *Transitive holonomy group and rigidity in nonnegative curvature*, Math. Z., 237 (2001), 265–281.

[**10**] Kervaire, Michel, A., *Smooth homology spheres and their fundamental groups*, Trans. Amer. Math. Soc., 144 (1969), 67–72

[**11**] Naolekar, A.C., *Realizing cohomology classes as Euler classes*, Math. Slovaca, 62 (5) (2012), 949–966.

[**12**] Naolekar, A.C., and Thakur, A.S., *Vector bundles over iterated suspensions of stunted real projective spaces.*, Acta Math. Hungar., 142 (2014), 339–347.

[**13**] Naolekar, A.C., and Thakur, A.S., *On trivialities of Chern classes*, Acta Math. Hungar., 144 (1) (2014), 99–109.

[**14**] Naolekar, A.C., and Thakur, A.S., *Euler classes of vector bundles over iterated suspensions of real projective spaces*, Math. Slovaca, 68 (3) (2018), 677–684.

[**15**] Smale, S., *On the structure of* 5*-manifolds*, Ann. of Math., 75 (1) (1962), 38–46.

[**16**] Tanaka, R., *On trivialities of Stiefel-Whitney classes of vector bundles over highly connected complexes*, Topology Appl., 155 (2008), 1687–1693.

[17] Tanaka, R., *On trivialities of Stiefel-Whitney classes of vector bundles over iterated suspension spaces*, Homology Homotopy Appl., 12 (1) (2010), 357–366.

[18] Thakur, A.S., *On trivialities of Stiefel-Whitney classes of vector bundles over iterated suspensions of Dold manifolds*, Homology Homotopy Appl., 15 (2013), 223–233.

[19] Thomas, E., *On the cohomology of the real Grassmann complexes and the characteristic classes of n-plane bundles*, Trans. Amer. Math. Soc., 96 (1) (1960), 67–89.

[20] Toda, H., *Order of the identity class of a suspension space*, Ann. of Math., 78 (1963), 300–325.

[21] Walschap, G., *The Euler class as a cohomology generator*, Illinois J. Math., 46 (1) (2002), 165–169.

Aniruddha C. Naolekar `ani@isibang.ac.in`

Stat-Math Unit, Indian Statistical Institute, 8th Mile, Mysore Road, RVCE Post, Bangalore 560059, India

B. Subhash `subhash@iisertirupati.ac.in`

Indian Institute of Science Education and research (IISER) Tirupati, Tirupati 517507, India

Ajay Singh Thakur `asthakur@iitk.ac.in`

Department of Mathematics and Statistics, Indian Institute of Technology Kanpur, Kanpur 208016, India