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K_1-GROUPS VIA BINARY COMPLEXES OF FIXED LENGTH

DANIEL KASPROWSKI, BERNHARD KÖCK AND CHRISTOPH WINGES

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Abstract

We modify Grayson's model of K_1 of an exact category to give a presentation whose generators are binary acyclic complexes of length at most k for any given $k \geqslant 2$. As a corollary, we obtain another, very short proof of the identification of Nenashev's and Grayson's presentations.

1. Introduction

Let $\mathcal{N}$ be an exact category. Algebraic descriptions of Quillen's K_1-group of $\mathcal{N}$ in terms of explicit generators and relations have been given by Nenashev [**Nen98**] and Grayson [**Gra12**]. The generators in both descriptions are so-called binary acyclic complexes in $\mathcal{N}$. While Nenashev uses complexes of length at most 2, Grayson's generators are of arbitrary (finite) length. An algebraic proof of the fact that these two descriptions agree has been given in [**KW**]. In this paper, we will give another presentation of $K_1(\mathcal{N})$; this time the generators are binary acyclic complexes of length at most k for any $k \geqslant 2$, see Definition 2.2 and Theorem 2.4. A motivating question behind this new description is to determine the precise relations that in addition to Grayson's relations need to be divided out when restricting the generators in Grayson's description to complexes of length at most k. If $k = 2$, our relations are special cases of Nenashev's relations. In this sense, our presentation simplifies Nenashev's presentation. All this leads to a new, natural and sleek algebraic proof of the fact that Nenashev's and Grayson's descriptions agree, see Section 4.

The proof of our main result, see Section 3, basically proceeds by induction on k. The crucial ingredient in the inductive step is a shortening procedure for binary acyclic complexes, see Definition 3.5, as discovered by Grayson and also used in [**KW**]. The main new idea in this paper is the comparatively short and simple way of showing how this shortening procedure yields an inverse to passing from complexes of length k to complexes of length $k + 1$, see Proposition 3.9.

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2. Background and statement of main theorem

We recall a *binary acyclic complex* $\mathbb{P} = (P_*, d, d')$ in $\mathcal{N}$ is a graded object P_* in $\mathcal{N}$ supported on a finite subset of $[0, \infty]$ together with two maps $d, d' \colon P_* \to P_*$ of degree -1 such that both (P, d) and (P_*, d') are acyclic chain complexes in $\mathcal{N}$. Here, *acyclic* means that each differential $d_n \colon P_n \to P_{n-1}$ admits a factorisation into an admissible epimorphism followed by an admissible monomorphism

$$P_n \twoheadrightarrow J_{n-1} \rightarrowtail P_{n-1}$$

such that $J_n \rightarrowtail P_n \twoheadrightarrow J_{n-1}$ is a short exact sequence in $\mathcal{N}$ for every n. The differentials d and d' are called the *top* and *bottom* differential. We also write $\mathbb{P}^\top$ and $\mathbb{P}^\perp$ for the complexes (P_*, d) and (P_*, d'). If $d = d'$, we call $\mathbb{P}$ a *diagonal* binary acyclic complex.

A *morphism between binary acyclic complexes* $\mathbb{P}$ and $\mathbb{Q}$ is a degree 0 map between the underlying graded objects which is a chain map with respect to both differentials. According to [**Gra12**, Section 3], the obvious definition of short exact sequences turns the category of binary acyclic complexes into an exact category. We denote its Grothendieck group by $B_1(\mathcal{N})$.

Definition 2.1. The quotient of $B_1(\mathcal{N})$ obtained by declaring the classes of diagonal binary acyclic complexes to be zero is called *Grayson's K_1-group of $\mathcal{N}$* and denoted by $K_1(\mathcal{N})$.

Grayson proves in [**Gra12**, Corollary 7.2] that $K_1(\mathcal{N})$ is naturally isomorphic to Quillen's K_1-group of $\mathcal{N}$. This justifies our notation. Note that Grayson uses complexes supported on $[-\infty, \infty]$. By [**HKT17**, Proposition 1.4], defining $K_1(\mathcal{N})$ with complexes supported on $[0, \infty]$ as above yields the same K_1-group.

If we restrict the generators to be complexes supported on $[0, k]$ (for $k \geqslant 0$), we write $B_1^k(\mathcal{N})$ and $K_1^k(\mathcal{N})$ for the resulting abelian groups.

Definition 2.2.

(a) A *binary ladder* is a quadruple $(\mathbb{P}, \mathbb{Q}, \sigma, \tau)$ consisting of two binary complexes $\mathbb{P}$ and $\mathbb{Q}$ together with isomorphisms $\sigma \colon \mathbb{P}^\top \xrightarrow{\sim} \mathbb{Q}^\top$ and $\tau \colon \mathbb{P}^\perp \xrightarrow{\sim} \mathbb{Q}^\perp$.

(b) Any two isomorphisms $\alpha, \beta \colon P \xrightarrow{\sim} Q$ in $\mathcal{N}$ define a binary acyclic complex

$$P \underset{\beta}{\overset{\alpha}{\rightrightarrows}} Q$$

in $\mathcal{N}$ supported on $[0, 1]$. The corresponding element in any of the groups $B_1(\mathcal{N})$, $K_1(\mathcal{N})$ or $B_1^k(\mathcal{N})$, $K_1^k(\mathcal{N})$ for $k \geqslant 1$ is denoted by $\langle \alpha, \beta \rangle$.

(c) Define $L_1^k(\mathcal{N})$ to be the quotient of $K_1^k(\mathcal{N})$ obtained by additionally imposing the relation

$$\mathbb{Q} - \mathbb{P} = \sum_{i=0}^{k} (-1)^i \langle \sigma_i, \tau_i \rangle$$

for every binary ladder $(\mathbb{P}, \mathbb{Q}, \sigma, \tau)$ in $\mathcal{N}$ such that $\mathbb{P}$ and $\mathbb{Q}$ are supported on $[0, k]$, $P_i = Q_i$ and all σ_i and τ_i are involutions, i.e. $\sigma_i^2 = \mathrm{id} = \tau_i^2$.

Remark 2.3. The object $L_1^k(\mathcal{N})$ has of course nothing to do with L-Theory; the L here is rather meant to refer to 'ladder'. It will follow from Proposition 3.9, that imposing the ladder relation in Definition 2.2(c) not just for σ_i, τ_i involutions but for all automorphisms yields an isomorphic group.

The following theorem is the precise formulation of our main result. Note that Lemma 3.3 below implies that we have a natural map $L_1^k(\mathcal{N}) \to K_1(\mathcal{N})$.

Theorem 2.4. *The canonical map*

$$L_1^k(\mathcal{N}) \to K_1(\mathcal{N})$$

is an isomorphism for every $k \geqslant 2$.

Section 3 contains the proof of Theorem 2.4.

3. Proof of the main theorem

Definition 3.1. For any object P in $\mathcal{N}$ let

$$\tau_P := \begin{pmatrix} 0 & \mathrm{id} \\ \mathrm{id} & 0 \end{pmatrix} : P \oplus P \xrightarrow{\sim} P \oplus P$$

denote the automorphism of $P \oplus P$ which switches the two summands.

Note that assigning $\langle \mathrm{id}_{P\oplus P}, \tau_P \rangle$ with any object $P \in \mathcal{N}$ defines a homomorphism $K_0(\mathcal{N}) \to B_1^1(\mathcal{N})$. In particular, $\langle \mathrm{id}_{P\oplus P}, \tau_P \rangle = \langle \mathrm{id}_{Q\oplus Q}, \tau_Q \rangle$ if P and Q represent the same element in K_0. As an aside, we remark that $\langle \mathrm{id}_{P\oplus P}, \tau_P \rangle$ is equal to $\langle \mathrm{id}_P, -\mathrm{id}_P \rangle$ in $B_1^1(\mathcal{N})$ and of order at most two in $L_1^1(\mathcal{N})$; both of these two facts are easy to prove but won't be used in this paper.

Lemma 3.2. *Let* $\mathbb{P}$ *be a binary acyclic complex in* $\mathcal{N}$ *supported on* $[0, k]$ *and let* $\mathrm{sw}(\mathbb{P})$ *denote the binary complex obtained from* $\mathbb{P}$ *by switching top and bottom differential. Then we have*

$$\mathrm{sw}(\mathbb{P}) = -\mathbb{P} \qquad \textit{in} \qquad L_1^k(\mathcal{N}).$$

Proof. We have $\mathbb{P} + \mathrm{sw}(\mathbb{P}) = \mathbb{P} \oplus \mathrm{sw}(\mathbb{P})$ in $B_1^k(\mathcal{N})$. The latter complex represents 0 in $L_1^k(\mathcal{N})$. To see this, consider the binary ladder $(\mathbb{P} \oplus \mathrm{sw}(\mathbb{P}), \mathbb{D}, \sigma, \tau)$ where $\mathbb{D}$ is the diagonal complex with $\mathbb{D}^\top = \mathbb{D}^\perp = (\mathbb{P} \oplus \mathrm{sw}(\mathbb{P}))^\top$, $\sigma = \mathrm{id}$ and τ switches the two summands $\mathbb{P}$ and $\mathrm{sw}(\mathbb{P})$, and note that $\sum_{i=0}^{k} (-1)^i P_i = 0$ in $K_0(\mathcal{N})$. $\square$

Regarding binary acyclic complexes supported on $[0, k]$ as complexes supported on $[0, k+1]$ defines a homomorphism

$$i_k \colon L_1^k(\mathcal{N}) \to L_1^{k+1}(\mathcal{N}).$$

The following lemma is basically a special case of the generalised Nenashev relation, see Definition 4.1 below and [**Har15**, Proposition 2.12]. We include a short proof to convince the reader that complexes supported on $[0, k+1]$ suffice to prove the desired relation. As usual, we write $\mathbb{P}[1]$ for the complex shifted by 1 (without changing the sign of the differentials d and d') so that $\mathbb{P}[1]_0 = 0$.

Lemma 3.3. *The homomorphism i_k naturally factorises as*

$$i_k \colon L_1^k(\mathcal{N}) \to K_1^{k+1}(\mathcal{N}) \twoheadrightarrow L_1^{k+1}(\mathcal{N}),$$

where the second map is the canonical epimorphism.

Proof. Let $(\mathbb{P}, \mathbb{Q}, \sigma, \tau)$ be a binary ladder with $\mathbb{P}, \mathbb{Q}$ supported on $[0, k]$. Then all rows and columns of the diagram

$$\begin{array}{ccccccc} \cdots & \rightrightarrows & P_2 & \rightrightarrows & P_1 & \rightrightarrows & P_0 \\ & & {\scriptstyle\sigma_2}\downdownarrows{\scriptstyle\tau_2} & & {\scriptstyle\sigma_1}\downdownarrows{\scriptstyle\tau_1} & & {\scriptstyle\sigma_0}\downdownarrows{\scriptstyle\tau_0} \\ \cdots & \rightrightarrows & Q_2 & \rightrightarrows & Q_1 & \rightrightarrows & Q_0 \end{array}$$

are binary acyclic complexes, top differentials commute with top differentials and bottom differentials commute with bottom differentials. Filtering the associated total complex $\mathbb{T}$ (which is a binary acyclic complex supported on $[0, k+1]$) "horizontally and vertically" then yields the relation

$$\mathbb{Q} + \mathbb{P}[1] = \mathbb{T} = \sum_{i=0}^{k} \langle \sigma_i, \tau_i \rangle [i]$$

in $B_1^{k+1}(\mathcal{N})$. If $\mathbb{P} = \mathbb{Q}$ and $\sigma = \tau = \mathrm{id}$, this shows that

$$\mathbb{P}[1] = -\mathbb{P} \tag{3.4}$$

in $K_1^{k+1}(\mathcal{N})$. The two equalities above finally show that

$$\mathbb{Q} - \mathbb{P} = \sum_{i=0}^{k} (-1)^i \langle \sigma_i, \tau_i \rangle$$

in $K_1^{k+1}(\mathcal{N})$, as desired. □

Definition 3.5. Let $k \geqslant 2$ and let $\mathbb{P} = (P_*, d, d')$ be a binary acyclic complex supported on $[0, k+1]$, and choose factorisations

$$d_2 \colon P_2 \twoheadrightarrow J \rightarrowtail P_1 \quad \text{and} \quad d_2' \colon P_2 \twoheadrightarrow K \rightarrowtail P_1$$

witnessing that (P_*, d) and (P_*, d') are acyclic. In the following, we denote the maps $P_2 \to J$ and $P_2 \to K$ again by d_2 and d_2'.

The *Grayson shortening of* $\mathbb{P}$ is the binary acyclic complex $\mathrm{sh}(\mathbb{P})$ supported on $[0, k]$ whose top component is given by

$$\begin{array}{ccccccc} \cdots & \longrightarrow & P_3 & \xrightarrow{d_3} & P_2 & \xrightarrow{d_2} & J \\ & & & & \oplus & & \oplus \\ & & \oplus & & K & \xrightarrow{\mathrm{id}} & K \\ & & & & \oplus & & \\ & & J & \xrightarrow{\mathrm{id}} & J & & \oplus \\ & & \oplus & & \oplus & & \\ & & K & \rightarrowtail & P_1 & \xrightarrow{d_1'} & P_0 \end{array}$$

and whose bottom component is

$$\begin{array}{ccccccc} \cdots & \longrightarrow & P_3 & \xrightarrow{d_3'} & P_2 & \xrightarrow{d_2'} & K \\ & & & & \oplus & & \oplus \\ & & \oplus & & J & \xrightarrow{\mathrm{id}} & J \\ & & & & \oplus & & \\ & & K & \xrightarrow{\mathrm{id}} & K & & \oplus \\ & & \oplus & & \oplus & & \\ & & J & \rightarrowtail & P_1 & \xrightarrow{d_1} & P_0 \end{array}$$

Note that we have permuted the summands in the bottom component for better legibility, but that we consider the summation order in the top component to be the definitive one.

Remark 3.6. Note that if $\mathbb{P}$ is supported on $[0, 1]$ then $\mathrm{sh}(\mathbb{P}) = \mathrm{sw}(\mathbb{P})$ and thus $\mathrm{sh}(\mathbb{P}) = -\mathbb{P} \in L_1^1(\mathcal{N})$ in this case by Lemma 3.2.

Remark 3.7. The complex $\mathrm{sh}(\mathbb{P})$ appears in handwritten notes by Grayson and has also been used in [**KW**, Section 5]. Note that our definition of $\mathrm{sh}(\mathbb{P})$ includes a shift by -1 so $\mathrm{sh}(\mathbb{P})$ is supported on $[0, k]$ rather than on $[1, k+1]$. This avoids bulky notations later.

For $\mathbb{P}$, J and K as in Definition 3.5, we have $\langle \mathrm{id}, \tau_J \rangle = \langle \mathrm{id}, \tau_K \rangle$ because $J = K$ in $K_0(\mathcal{N})$. We denote the latter element by $\tau_{\mathbb{P}}$. If, in fact, $J \cong K$, we replace the morphisms $P_2 \twoheadrightarrow K$ and $K \rightarrowtail P_1$ with $P_2 \twoheadrightarrow J$ and $J \rightarrowtail P_1$ by composing them with a fixed isomorphism between J and K. Then the ordinary (non-naive) truncations

$$\mathrm{t}_{\geqslant 1}(\mathbb{P}) := (\ \cdots \rightrightarrows P_3 \rightrightarrows P_2 \rightrightarrows J\ )$$

and

$$\mathrm{t}_{\leqslant 2}(\mathbb{P}) := (\ J \rightrightarrows P_1 \rightrightarrows P_0\ )$$

are binary acyclic complexes again. In this case, the following crucial lemma computes $\mathrm{sh}(\mathbb{P}) \in L_1^k(\mathcal{N})$ in terms of these truncations and $\tau_{\mathbb{P}}$.

Lemma 3.8. *Let $\mathbb{P}$ be a binary acyclic complex supported on $[0, k+1]$ and suppose that $J \cong K$. Then we have*

$$\mathrm{sh}(\mathbb{P}) = \mathrm{t}_{\geqslant 1}(\mathbb{P})[-1] - \mathrm{t}_{\leqslant 2}(\mathbb{P}) - \tau_{\mathbb{P}} \qquad \text{in} \qquad L_1^k(\mathcal{N}).$$

In particular:

(a) If $\mathbb{P}$ is a diagonal complex, then $\mathrm{sh}(\mathbb{P}) = -\tau_{\mathbb{P}}$ *in $L_1^k(\mathcal{N})$.*

(b) If $P_0 = 0$, then $\mathrm{sh}(\mathbb{P}) = \mathbb{P}[-1] - \tau_{\mathbb{P}}$ *in $L_1^k(\mathcal{N})$.*

Proof. Without permuting any summands in the bottom component, $\mathrm{sh}(\mathbb{P})$ has top component

$$\begin{array}{ccccccc} \cdots & \longrightarrow & P_3 & \xrightarrow{d_3} & P_2 & \xrightarrow{d_2} & J \\ & & & & \oplus & & \oplus \\ & & \oplus & & J & \xrightarrow{\mathrm{id}} & J \\ & & & & \oplus & & \\ & & J & \xrightarrow{\mathrm{id}} & J & & \oplus \\ & & \oplus & & \oplus & & \\ & & J & \rightarrowtail & P_1 & \xrightarrow{d_1'} & P_0 \end{array}$$

and bottom component

$$
\begin{tikzcd}
\cdots \arrow[r] & P_3 \arrow[r, "d_3'"] & P_2 \arrow[ddr, "d_2'"] & J \\
 & & \oplus & \oplus \\
 & \oplus & J & J \\
 & & \oplus & \\
 & J \arrow[ddr, hook] & J \arrow[uuuur, "\mathrm{id}"] & \oplus \\
 & \oplus & \oplus & \\
 & J \arrow[uuuur, "\mathrm{id}"] & P_1 \arrow[r, "d_1"] & P_0
\end{tikzcd}
$$

Now consider the binary ladder

$$(\mathrm{sh}(\mathbb{P}), \mathrm{t}_{\geqslant 1}(\mathbb{P})[-1] \oplus \mathbb{J} \oplus \mathbb{J}[1] \oplus \mathrm{sw}(\mathrm{t}_{\leqslant 2}(\mathbb{P})), \sigma, \tau),$$

where $\mathbb{J}$ is the diagonal binary complex supported on $[0,1]$ given by $\mathbb{J}^{\top} = (J \xrightarrow{\mathrm{id}} J) = \mathbb{J}^{\perp}$, $\sigma = \mathrm{id}$ and τ is the automorphism switching the two copies of J in degrees 0, 1 and 2 and the identity in all higher degrees. From this binary ladder we obtain the following equality in $L_1^k(\mathcal{N})$:

$$\mathrm{t}_{\geqslant 1}(\mathbb{P})[-1] + \mathrm{sw}(\mathrm{t}_{\leqslant 2}(\mathbb{P})) = \mathrm{sh}(\mathbb{P}) + \tau_{\mathbb{P}}.$$

Using Lemma 3.2, we finally obtain the desired equality in $L_1^k(\mathcal{N})$:

$$\mathrm{sh}(\mathbb{P}) = \mathrm{t}_{\geqslant 1}(\mathbb{P})[-1] - \mathrm{t}_{\leqslant 2}(\mathbb{P}) - \tau_{\mathbb{P}}.$$

If $\mathbb{P}$ is a diagonal complex, both truncations are diagonal again and part (a) follows. If $P_0 = 0$, then $\mathrm{t}_{\geqslant 1}(\mathbb{P}) = \mathbb{P}$, $J = P_1$ and $\mathrm{t}_{\leqslant 2}(\mathbb{P}) = \langle \mathrm{id}_{P_1}, \mathrm{id}_{P_1} \rangle [1] = 0$ in $L_1^2(\mathcal{N})$; this shows part (b). ☐

Proposition 3.9. *The homomorphism*

$$i_k \colon L_1^k(\mathcal{N}) \to L_1^{k+1}(\mathcal{N})$$

is an isomorphism for all $k \geqslant 2$. Its inverse is induced by the assignment

$$\mathbb{P} \mapsto -\,\mathrm{sh}(\mathbb{P}) - \tau_{\mathbb{P}}.$$

Proof. We begin by showing that the assignment $\mathbb{P} \mapsto -\,\mathrm{sh}(\mathbb{P}) - \tau_{\mathbb{P}}$ induces a well-defined homomorphism $p_k \colon L_1^{k+1}(\mathcal{N}) \to L_1^k(\mathcal{N})$.

If $\mathbb{P}' \rightarrowtail \mathbb{P} \twoheadrightarrow \mathbb{P}''$ is a short exact sequence of binary acyclic complexes supported on $[0, k+1]$, then we have induced short exact sequences $J' \rightarrowtail J \twoheadrightarrow J''$ and $K' \rightarrowtail K \twoheadrightarrow K''$ by [**Büh10**, Corollary 3.6]. In particular, we obtain a short exact sequence $\mathrm{sh}(\mathbb{P}') \rightarrowtail \mathrm{sh}(\mathbb{P}) \twoheadrightarrow \mathrm{sh}(\mathbb{P}'')$. Since we also have $\tau_{\mathbb{P}} = \tau_{\mathbb{P}'} + \tau_{\mathbb{P}''}$, we obtain an induced homomorphism

$$p_k'' \colon B_1^{k+1}(\mathcal{N}) \to L_1^k(\mathcal{N}).$$

Let $\mathbb{P}$ be a diagonal binary acyclic complex supported on $[0, k+1]$, then we have $p_k''(\mathbb{P}) = 0$ in $L_1^k(\mathcal{N})$ by Lemma 3.8(a). In particular, p_k'' induces a homomorphism

$$p_k' \colon K_1^{k+1}(\mathcal{N}) \to L_1^k(\mathcal{N}).$$

If $(\mathbb{P}, \mathbb{Q}, \sigma, \tau)$ is a binary ladder in $\mathcal{N}$ with $\mathbb{P}$ and $\mathbb{Q}$ supported on $[0, k+1]$, then σ_1 and τ_1 induce isomorphisms $\sigma^J \colon J_{\mathbb{P}} \xrightarrow{\sim} J_{\mathbb{Q}}$ and $\tau^K \colon K_{\mathbb{P}} \xrightarrow{\sim} K_{\mathbb{Q}}$, respectively. If σ_1 and

τ_2 are involutions, so are σ^J and τ^K. These define a binary ladder

$$(\mathrm{sh}(\mathbb{P}), \mathrm{sh}(\mathbb{Q}), \mathrm{sh}(\sigma), \mathrm{sh}(\tau))$$

with

$$\begin{aligned} \mathrm{sh}(\sigma)_0 &= \sigma^J \oplus \tau^K \oplus \tau_0, \quad \mathrm{sh}(\sigma)_1 = \sigma_2 \oplus \tau^K \oplus \sigma^J \oplus \tau_1, \quad \mathrm{sh}(\sigma)_2 = \sigma_3 \oplus \sigma^J \oplus \tau^K, \\ \mathrm{sh}(\tau)_0 &= \sigma^J \oplus \tau^K \oplus \sigma_0, \quad \mathrm{sh}(\tau)_1 = \tau_2 \oplus \tau^K \oplus \sigma^J \oplus \sigma_1, \quad \mathrm{sh}(\tau)_2 = \tau_3 \oplus \sigma^J \oplus \tau^K, \end{aligned}$$

and $\mathrm{sh}(\sigma)_i = \sigma_{i+1}$ and $\mathrm{sh}(\tau)_i = \tau_{i+1}$ for $i \geqslant 3$. Hence we have

$$\begin{aligned} \sum_{i=0}^{k} (-1)^i \langle \mathrm{sh}(\sigma)_i, \mathrm{sh}(\tau)_i \rangle &= \langle \tau_0, \sigma_0 \rangle - \langle \sigma_2, \tau_2 \rangle - \langle \tau_1, \sigma_1 \rangle + \langle \sigma_3, \tau_3 \rangle + \sum_{i=3}^{\infty} (-1)^i \langle \sigma_{i+1}, \tau_{i+1} \rangle \\ &= -\sum_{i=0}^{k+1} (-1)^i \langle \sigma_i, \tau_i \rangle \end{aligned}$$

in $L_1^1(\mathcal{N})$ by Lemma 3.2. If the given ladder is as in Definition 2.2(c), it follows that

$$\begin{aligned} p'_k(\mathbb{Q} - \mathbb{P}) &= -\,\mathrm{sh}(\mathbb{Q}) - \tau_{\mathbb{Q}} + \mathrm{sh}(\mathbb{P}) + \tau_{\mathbb{P}} = -\sum_{i \geqslant 0} (-1)^i \langle \mathrm{sh}(\sigma)_i, \mathrm{sh}(\tau)_i \rangle \\ &= p'_k \left(\sum_{i \geqslant 0} (-1)^i \langle \sigma_i, \tau_i \rangle \right) \end{aligned}$$

in $L_1^k(\mathcal{N})$ since $J_{\mathbb{P}} \cong J_{\mathbb{Q}}$, hence $\tau_{\mathbb{P}} = \tau_{\mathbb{Q}}$, and since, by Remark 3.6, p'_k maps $\langle \sigma_i, \tau_i \rangle \in K_1^{k+1}(\mathcal{N})$ to $\langle \sigma_i, \tau_i \rangle \in L_1^k(\mathcal{N})$ for each i. Consequently, p'_k induces the desired homomorphism

$$p_k \colon L_1^{k+1}(\mathcal{N}) \to L_1^k(\mathcal{N}).$$

We now show that $p_k \circ i_k = \mathrm{id}$ for all $k \geqslant 2$. Let $\mathbb{P}$ be a binary acyclic complex in $\mathcal{N}$ supported on $[0, k]$. By Equation (3.4) and Lemma 3.8(b), we have

$$p_k(i_k(\mathbb{P})) = -p_k(\mathbb{P}[1]) = \mathrm{sh}(\mathbb{P}[1]) + \tau_{\mathbb{P}[1]} = (\mathbb{P} - \tau_{\mathbb{P}[1]}) + \tau_{\mathbb{P}[1]} = \mathbb{P}$$

in $L_1^k(\mathcal{N})$, as was to be shown.

We are left with showing that p_k is also a right-inverse to i_k. Let $\mathbb{P}$ be a binary acyclic complex in $\mathcal{N}$ supported on $[0, k+1]$. Since the definition of $\mathrm{sh}(\mathbb{P})$ is independent of whether we regard $\mathbb{P}$ as a complex supported on $[0, k+1]$ or $[0, k+2]$, we have

$$i_k(p_k(\mathbb{P})) = p_{k+1}(i_{k+1}(\mathbb{P})) = \mathbb{P}$$

in $L_1^{k+1}(\mathcal{N})$, as desired. □

Proof of Theorem 2.4. From Lemma 3.3 we obtain the directed system

$$K_1^2(\mathcal{N}) \twoheadrightarrow L_1^2(\mathcal{N}) \to K_1^3(\mathcal{N}) \twoheadrightarrow L_1^3(\mathcal{N}) \to \cdots.$$

Since the colimit of the cofinal sub-system $K_1^2(\mathcal{N}) \to K_1^3(\mathcal{N}) \to \cdots$ is $K_1(\mathcal{N})$, the colimit of the displayed system is $K_1(\mathcal{N})$ as well. Hence, the colimit of the cofinal sub-system $L_1^2(\mathcal{N}) \to L_1^3(\mathcal{N}) \to \cdots$ is also $K_1(\mathcal{N})$. Furthermore, all the connecting maps in this sub-system are isomorphisms by Proposition 3.9. The claim follows. □

Remark 3.10. Grayson shows in the handwritten notes mentioned in Remark 3.7 that $\mathrm{sh}(\mathbb{P}) \in K_1(\mathcal{N})$ differs from $\mathbb{P}$ by classes of binary acyclic complexes of length at most 2. By induction, this proves that the canonical map $K_1^2(\mathcal{N}) \to K_1(\mathcal{N})$ is surjective. While Grayson uses slightly involved double complex arguments, we use simpler and at the same time more potent arguments and also prove the simple relation $\mathbb{P} + \mathrm{sh}(\mathbb{P}) = -\tau_{\mathbb{P}}$ in $K_1(\mathcal{N})$.

Corollary 3.11 ([**KW**, Theorem 1.4])**.** *The canonical map $K_1^3(\mathcal{N}) \to K_1(\mathcal{N})$ is onto and admits a canonical section.*

Proof. The right inverse is given by the inverse of the bijection $L_1^2(\mathcal{N}) \to K_1(\mathcal{N})$ from Theorem 2.4 composed with the map $L_1^2(\mathcal{N}) \to K_1^3(\mathcal{N})$ from Lemma 3.3. □

Remark 3.12. The inverse of the isomorphism $L_1^2(\mathcal{N}) \to K_1(\mathcal{N})$ from Theorem 2.4 admits an explicit description. This agrees with the map Ψ appearing in the proof of [**KW**, Theorem 1.1].

4. The relation to Nenashev's description

In this section, we compare Nenashev's and Grayson's descriptions of K_1.

Definition 4.1. *Nenashev's K_1-group $K_1^{\mathrm{N}}(\mathcal{N})$ of $\mathcal{N}$* is defined as the abelian group generated by binary acyclic complexes $\mathbb{P}$ of length 2 subject to the following relations:

(1) If $\mathbb{P}$ is a diagonal complex, then $\mathbb{P} = 0$.

(2) If

$$\begin{tikzcd} P_2' \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift left] \arrow[d, shift right] & P_1' \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift left] \arrow[d, shift right] & P_0' \arrow[d, shift left] \arrow[d, shift right] \\ P_2 \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift left] \arrow[d, shift right] & P_1 \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift left] \arrow[d, shift right] & P_0 \arrow[d, shift left] \arrow[d, shift right] \\ P_2'' \arrow[r, shift left] \arrow[r, shift right] & P_1'' \arrow[r, shift left] \arrow[r, shift right] & P_0'' \end{tikzcd}$$

is a diagram in $\mathcal{N}$ such that all rows and columns are binary acyclic complexes, top differentials commute with top differentials and bottom differentials commute with bottom differentials, then

$$\mathbb{P}_0 - \mathbb{P}_1 + \mathbb{P}_2 = \mathbb{P}' - \mathbb{P} + \mathbb{P}''.$$

Nenashev proves in [**Nen98**] that $K_1^{\mathrm{N}}(\mathcal{N})$ is canonically isomorphic to Quillen's K_1-group of $\mathcal{N}$. The following corollary purely algebraically proves that $K_1^{\mathrm{N}}(\mathcal{N})$ is isomorphic to $K_1(\mathcal{N})$, i.e., to Grayson's K_1-group of $\mathcal{N}$. By [**Gra12**, Remark 8.1], regarding a binary acyclic complex of length 2 as a class in $K_1(\mathcal{N})$ defines a map $K_1^{\mathrm{N}}(\mathcal{N}) \to K_1(\mathcal{N})$.

Corollary 4.2 ([**KW**, Theorem 1.1])**.** *The canonical map*

$$K_1^{\mathrm{N}}(\mathcal{N}) \to K_1(\mathcal{N})$$

is an isomorphism.

Proof. Since the relations used to define $L_1^2(\mathcal{N})$ are special cases of Nenashev's relation, the canonical surjection $K_1^2(\mathcal{N}) \twoheadrightarrow K_1^{\mathrm{N}}(\mathcal{N})$ factors via $L_1^2(\mathcal{N})$, yielding a surjection $L_1^2(\mathcal{N}) \twoheadrightarrow K_1^{\mathrm{N}}(\mathcal{N})$. Since we have a commutative diagram

$$\begin{tikzcd} L_1^2(\mathcal{N}) \arrow[r] \arrow[d] & K_1(\mathcal{N}) \\ K_1^{\mathrm{N}}(\mathcal{N}) \arrow[ur] & \end{tikzcd}$$

it follows from Theorem 2.4 that $L_1^2(\mathcal{N}) \to K_1^{\mathrm{N}}(\mathcal{N})$ and $K_1^{\mathrm{N}}(\mathcal{N}) \to K_1(\mathcal{N})$ are isomorphisms. $\square$

Remark 4.3. The bijectivity of the map $L_1^2(\mathcal{N}) \to K_1^{\mathrm{N}}(\mathcal{N})$ in the proof of the previous corollary means that Nenashev's relation (Definition 4.1(2)) can be expressed in $K_1^2(\mathcal{N})$ as a linear combination of relations arising from binary ladders that we used to define $L_1^2(\mathcal{N})$ (Definition 2.2(c)). The object of this remark is to explicitly write down such a linear combination.

Let

$$\begin{tikzcd} M_2 \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift left] \arrow[d, shift right] & M_1 \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift left] \arrow[d, shift right] & M_0 \arrow[d, shift left] \arrow[d, shift right] \\ N_2 \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift left] \arrow[d, shift right] & N_1 \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift left] \arrow[d, shift right] & N_0 \arrow[d, shift left] \arrow[d, shift right] \\ P_2 \arrow[r, shift left] \arrow[r, shift right] & P_1 \arrow[r, shift left] \arrow[r, shift right] & P_0 \end{tikzcd}$$

be a diagram in $\mathcal{N}$ as in Definition 4.1(2). Let $\mathbb{T}$ denote the associated binary total complex (of length 4). Choose factorisations

$$M_1 \oplus N_2 \twoheadrightarrow J_3 \rightarrowtail M_0 \oplus N_1 \oplus P_2 \twoheadrightarrow J_2 \rightarrowtail N_0 \oplus P_1$$

of the second and third top differential of $\mathbb{T}$, and define K_3 and K_2 analogously in terms of the bottom differentials of $\mathbb{T}$. The next step, according to our earlier constructions, would be to apply the Grayson shortening twice to $\mathbb{T}$, but this results in a complicated complex with superfluous terms. We rather apply twice just the idea behind the Grayson shortening in order to obtain the following complex $\mathbb{T}'$ of length 2 with obvious differentials (top differentials on the left hand side, bottom differentials on the right hand side):

$$\begin{tikzcd}[row sep=tiny] M_2 \arrow[r] & M_1 \oplus N_2 \arrow[r] & J_3 \\ & \oplus & \oplus \\ \oplus & K_3 \arrow[r] & K_3 \\ & \oplus & \\ J_3 \arrow[r] & J_3 & \oplus \\ \oplus & \oplus & \\ K_3 \arrow[r] & M_0 \oplus N_1 \oplus P_2 \arrow[r] & K_2 \\ & \oplus & \oplus \\ \oplus & J_2 \arrow[r] & J_2 \\ & \oplus & \\ K_2 \arrow[r] & K_2 & \oplus \\ \oplus & \oplus & \\ J_2 \arrow[r] & N_0 \oplus P_1 \arrow[r] & P_0 \end{tikzcd} \qquad \begin{tikzcd}[row sep=tiny] M_2 \arrow[r] & M_1 \oplus N_2 \arrow[ddr] & J_3 \\ & \oplus & \oplus \\ \oplus & K_3 & K_3 \\ & \oplus & \\ J_3 \arrow[ddr] & J_3 \arrow[uuuur] & \oplus \\ \oplus & \oplus & \\ K_3 \arrow[uuuur] & M_0 \oplus N_1 \oplus P_2 \arrow[ddr] & K_2 \\ & \oplus & \oplus \\ \oplus & J_2 & J_2 \\ & \oplus & \\ K_2 \arrow[ddr] & K_2 \arrow[uuuur] & \oplus \\ \oplus & \oplus & \\ J_2 \arrow[uuuur] & N_0 \oplus P_1 \arrow[r] & P_0 \end{tikzcd}$$

The obvious admissible monomorphisms $P_2 \to J_2$ and $P_2 \to K_2$ (the cokernel of both is N_0) define an admissible monomorphism from the binary acyclic complex $T_b(\mathbb{P})$

$$\begin{tikzcd}
 & P_2 \arrow[r] & P_2 \\
 & \oplus & \oplus \\
 & P_2 \arrow[r] & P_2 \\
 & \oplus & \\
P_2 \arrow[r] & P_2 & \oplus \\
\oplus & \oplus & \\
P_2 \arrow[r] & P_1 \arrow[r] & P_0
\end{tikzcd}
\qquad
\begin{tikzcd}
 & P_2 \arrow[ddr] & P_2 \\
 & \oplus & \oplus \\
 & P_2 & P_2 \\
 & \oplus & \\
P_2 \arrow[ddr] & P_2 \arrow[uuuur] & \oplus \\
\oplus & \oplus & \\
P_2 \arrow[uuuur] & P_1 \arrow[r] & P_0
\end{tikzcd}$$

to (the bottom half of) $\mathbb{T}'$. Similarly, we have an admissible epimorphism from (the top half of) $\mathbb{T}'$ to the binary acyclic complex $T_f(\mathbb{M})$

$$\begin{tikzcd}
M_2 \arrow[r] & M_1 \arrow[r] & M_0 \\
 & \oplus & \oplus \\
\oplus & M_0 \arrow[r] & M_0 \\
 & \oplus & \\
M_0 \arrow[r] & M_0 & \\
\oplus & \oplus & \\
M_0 \arrow[r] & M_0 &
\end{tikzcd}
\qquad
\begin{tikzcd}
M_2 \arrow[r] & M_1 \arrow[ddr] & M_0 \\
 & \oplus & \oplus \\
\oplus & M_0 & M_0 \\
 & \oplus & \\
M_0 \arrow[ddr] & M_0 \arrow[uuuur] & \\
\oplus & \oplus & \\
M_0 \arrow[uuuur] & M_0 &
\end{tikzcd}$$

which obviously factors modulo $T_b(\mathbb{P})$. The resulting epimorphism $\mathbb{T}'/T_b(\mathbb{P}) \to T_f(\mathbb{M})$ has kernel $T_{b,f}(\mathrm{sw}(\mathbb{N}))$ which is obtained from $\mathbb{N}$ by first switching top and bottom differential and then, similarly to $T_b(\mathbb{P})$ and $T_f(\mathbb{M})$, by adding copies of N_2 above $\mathrm{sw}(\mathbb{N})$ and copies of N_0 below $\mathrm{sw}(\mathbb{N})$. Hence we have

$$\mathbb{T}' = T_b(\mathbb{P}) + T_{b,f}(\mathrm{sw}(\mathbb{N})) + T_f(\mathbb{M}) \quad \text{in} \quad B_1(\mathcal{N}).$$

Applying the switching automorphism τ_{P_2} at the appropriate place in all three degrees of $T_b(\mathbb{P})$ produces the direct sum of $\mathbb{P}$ and a diagonal complex, so we obtain the relation

$$T_b(\mathbb{P}) = \mathbb{P} + \tau_{P_2} \quad \text{in} \quad L_1^2(\mathcal{N}). \tag{4.4}$$

Similarly, we obtain $T_f(\mathbb{M}) = \mathbb{M} + \tau_{M_0}$ and, also using Lemma 3.2, $T_{b,f}(\mathrm{sw}(\mathbb{N})) = -\mathbb{N} + \tau_{N_0} + \tau_{N_2}$. Hence we have

$$\mathbb{T}' = \mathbb{P} - \mathbb{N} + \mathbb{M} + \tau_{P_2} + \tau_{N_0} + \tau_{N_2} + \tau_{M_0} \quad \text{in} \quad L_1^2(\mathcal{N}).$$

Let $\mathbb{C}_i$ denote the binary acyclic complex $M_i \rightrightarrows N_i \rightrightarrows P_i$. Filtering the total complex by columns, we similarly obtain

$$\mathbb{T}' = \mathbb{C}_0 - \mathbb{C}_1 + \mathbb{C}_2 + \tau_{M_0} + \tau_{P_1} + \tau_{M_1} + \tau_{P_2} \quad \text{in} \quad L_1^2(\mathcal{N}).$$

The exact sequences $\mathbb{N}$ and $\mathbb{C}_1$ furthermore imply the relations

$$\tau_{N_0} + \tau_{N_2} = \tau_{N_1} = \tau_{P_1} + \tau_{M_1} \quad \text{in} \quad B_1(\mathcal{N}).$$

Hence we finally obtain the Nenashev relation

$$\mathbb{P} - \mathbb{N} + \mathbb{M} = \mathbb{C}_0 - \mathbb{C}_1 + \mathbb{C}_2 \quad \text{in} \quad L_1^2(\mathcal{N}). \tag{4.5}$$

Put slightly differently, in $B_1(\mathcal{N})$, the difference of the two sides in (4.5) is equal to the sum of (the negative of) the relation given by (4.4) and the analogous relations for $T_f(\mathbb{M})$, $T_{b,f}(\mathrm{sw}(\mathbb{N}))$, $T_b(\mathbb{C}_0)$, $T_f(\mathbb{C}_2)$ and $T_{b,f}(\mathrm{sw}(\mathbb{C}_1))$. Each of these relations in turn is made up of a ladder and a diagonal relation apart from those for $T_{b,f}(\mathrm{sw}(\mathbb{N}))$ and $T_{b,f}(\mathrm{sw}(\mathbb{C}_1))$ which in addition involve the ladder and diagonal relation occurring in the proof of Lemma 3.2.

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Daniel Kasprowski `kasprowski@uni-bonn.de`

Rheinische Friedrich-Wilhelms-Universität Bonn, Mathematisches Institut, Endenicher Allee 60, 53115 Bonn, Germany

Bernhard Köck `b.koeck@soton.ac.uk`

School of Mathematical Sciences, University of Southampton, Highfield, Southampton SO17 1BJ, United Kingdom

Christoph Winges `winges@math.uni-bonn.de`

Rheinische Friedrich-Wilhelms-Universität Bonn, Mathematisches Institut, Endenicher Allee 60, 53115 Bonn, Germany