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COUNIVERSAL SPACES WHICH ARE EQUIVARIANTLY COMMUTATIVE RING SPECTRA

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Abstract

The paper identifies which couniversal spaces have suspension spectra equivalent to commutative orthogonal ring G-spectra for a compact Lie group G. Equivalently these are the couniversal spaces admitting an action of an E_∞^G-operad. We show that they are precisely the couniversal spaces associated to a cofamily whose minimal subgroups are connected.

1. Introduction

For a compact Lie group G, our main result (Theorem 4.7 below) shows that a number of simple G-equivariant homotopy types have suspension spectra which are commutative orthogonal ring G-spectra.

To explain, we first suppose given a symmetric monoidal category G-Sp of G-spectra. A ring spectrum R is a monoid in G-Sp, but there are a number of different ways of describing commutativity. The simplest thing is to consider commutative monoids in G-Sp, but the implications of this structure for the homotopy type of R depend on the ambient category G-Sp. Instead we can consider that commutativity is a structure given by the action of a suitable operad on R; this depends on the operad but is then largely independent of the choice of G-Sp.

When G is trivial, we are used to working in models of spectra where commutative monoids in spectra are essentially the same as E_∞-rings, and these commutative ring spectra have a rich structure on their homotopy groups. Equivariantly there is a 'naive' operadic notion of commutativity for equivariant ring spectra that is precisely like the non-equivariant one, giving a rich structure on the H-equivariant homotopy groups for each subgroup H separately. However, the operadic notion of *equivariant* commutativity implies a great deal more structure, including multiplicative norm maps relating H-equivariant homotopy groups for different subgroups H. Commutative orthogonal ring G-spectra (i.e., commutative monoids in the category of orthogonal G-spectra) are *equivariant* commutative rings in this sense. At the very least, for every finite subgroup H of G an equivariantly commutative ring G-spectrum R has a norm map $\pi_0(R) \longrightarrow \pi_0^H(R)$ which preserves products and takes the unit to the unit. Accordingly if $\pi_0(R) = 0$ then $\pi_0^H(R) = 0$ for all finite subgroups H: this means, in

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particular, that when G is finite, couniversal spaces other than the sphere and the point do not admit an equivariantly commutative multiplication. Our main result shows that for couniversal spaces this is the only obstruction: if $\mathcal{C}$ is any cofamily whose minimal subgroups are connected then the suspension spectrum of the associated couniversal space admits an equivariantly commutative multiplication. Because of the additional structure this implies, the result has significant implications.

One reason for the importance of couniversal spaces is that they are naturally used for isotropic decompositions of the sphere. As such, they play a significant role in understanding the structure of rational G-equivariant spectra where G is a torus in [**4**]. That analysis involves constructing the model category of rational G-spectra for a torus G from a diagram of much simpler model categories. The simplest way to do this is to show these simpler model categories are categories of modules over commutative ring G-spectra, and that the diagram comes from a diagram of commutative ring G-spectra.

It is not hard to construct a suitable diagram of ring G-spectra, and their homotopy types are apparent from the construction. It remains to show that they are indeed *commutative* monoids so that their categories of modules are symmetric monoidal. If the ambient category of G-spectra is the category of orthogonal spectra, the commutative monoids are *equivariantly* commutative ring G-spectra, and as described above this is a substantial restriction on the homotopy type. When [**4**] was first written, it was expected that these couniversal spaces did not admit an equivariantly commutative multiplication, so [**4**] worked instead with the Blumberg-Hill category of orthogonal $\mathcal{L}$-spectra [**2**], where there are many more commutative monoids (they are naive commutative ring G-spectra in the sense above).

The motivating application of the present note is to show that, in fact, the ring G-spectra required in the construction of [**4**] can be represented by commutative monoids in the category of orthogonal G-spectra. It follows that the argument of [**4**] can be conducted directly in the category of orthogonal G-spectra rather than in the more elaborate category of spectra with an $\mathcal{L}$-action.

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2. Operadic preliminaries

There are rare examples of spectra which are obviously strictly commutative rings, but it is much more usual to proceed indirectly. First we note that commutative ring spectra are those with an action of the operad Comm with one point in each degree, and then use the fact that the categories of algebras over equivalent operads are equivalent (of course the meaning of the word ‘equivalent’ needs to be elucidated). To see that a spectrum has the homotopy type of a commutative ring spectrum, we show that the spectrum admits the action of a convenient operad, and then show that the operad is equivalent in the appropriate sense to Comm.

2.1. N_∞-operads

In the equivariant world there are a range of essentially different operads governing commutative ring spectra: these are the N_∞-operads of Blumberg-Hill [**3**]. These are operads $\mathcal{O}$ in G-spaces whose n-th term $\mathcal{O}(n)$ is a universal space for a family $\mathcal{F}\mathcal{O}(n)$ of subgroups of $G \times \Sigma_n$: it is essential that $\mathcal{O}(n)$ is G-fixed and Σ_n-free, but within that class there is a wide range of options. We need only discuss the two extreme types of N_∞-operads.

At one extreme we have the non-equivariant E_∞-operads, which are as free as possible whilst being G-fixed. Equivalently, the n-th term is the universal space for the family

$$\mathcal{F}(n) = \{H \times 1 \subseteq G \times \Sigma_n \mid H \subseteq G\}.$$

There are of course many E_∞-operads, and we write $\mathbb{E}_\infty$ for a chosen one. For example we might use the linear isometries operad on a G-fixed universe, but we will use no special properties of the operad.

At the other extreme we have the E_∞^G-operads which are as fixed as possible whilst their nth term is Σ_n-free, so their n-th term is a universal space for the family

$$\mathcal{F}_G(n) = \{\Gamma \mid \Gamma \cap \Sigma_n = 1\}.$$

There are of course many E_∞^G-operads, and we write $\mathbb{E}_\infty^G$ for a chosen one. For example we might use the linear isometries operad on a complete G-universe, but we will use no special properties of the operad. We pause to recall that if $\Gamma \cap \Sigma_n = 1$ then Γ is a 'graph subgroup' in the sense that we have $\Gamma = \Gamma(L, \alpha)$ for some subgroup L of G and some homomorphism $\alpha\colon L \longrightarrow \Sigma_n$, where $\Gamma(L, \alpha) = \{(x, \alpha(x)) \mid x \in L\}$.

2.2. Commutative monoids and E_∞^G-operads

The relevance of E_∞^G-operads is the connection to the standard symmetric monoidal product of spectra. We will be satisfied with the statement for a single orthogonal G-spectrum.

Lemma 2.1. *An orthogonal G-spectrum is equivalent to a commutative monoid if and only if it is equivalent to an E_∞^G-algebra.*

Proof. We use the analogue of the traditional nonequivariant argument, where the main ingredient is [**7**, 15.5]. In the equivariant setting this is replaced by [**5**, B.117], which in turn corrects [**6**, III.8.4]. We note that the statement in [**5**] is only given for finite groups, but the argument applies as written to arbitrary compact Lie groups, giving the full replacement for the statement in [**6**]. □

2.3. Endomorphism operads

The other piece of standard material is to consider the endomorphism operad $\mathcal{E}_Y$ on a based space Y, defined by

$$\mathcal{E}_Y(n) = Map_*(Y^{\wedge n}, Y).$$

We automatically find Y is an $\mathcal{E}_Y$-algebra. Equally, if Y is a based G-space $\mathcal{E}_Y$ is an operad in G-spaces and Y is an algebra over it.

3. McClure's argument

McClure [**8**] argued as follows to construct an E_∞-operad acting on $\tilde{E}G$.

First we consider the endomorphism operad $\mathcal{E}_{\tilde{E}G}$, and then note that passage to fixed points gives a map

$$\phi(n)\colon \mathcal{E}_{\tilde{E}G}(n)^G = Map_*^G(\tilde{E}G^{\wedge n}, \tilde{E}G) \longrightarrow Map_*(S^0, S^0).$$

We write

$$D_{McC}(n) = \phi(n)^{-1}(id),$$

and note that this is also an operad acting on $\tilde{E}G$. Because $\phi(n)$ is a weak equivalence $D_{McC}(n)$ is contractible, so that $\mathbb{E}_\infty \times D_{McC}$ is an E_∞-operad acting on $\tilde{E}G$ as required.

4. Generalizing McClure's argument

4.1. Couniversal spaces

Given a group G and a family $\mathcal{F}$ of subgroups of G, we say that the join $\tilde{E}\mathcal{F} = S^0 * E\mathcal{F}$ is *the couniversal space* for the complementary cofamily $All \setminus \mathcal{F}$. Simplifying notation, for a cofamily $\mathcal{C}$, we write simply

$$E\mathcal{C} = \tilde{E}(\mathcal{C}^c).$$

This has two essential features: it has geometric isotropy $\mathcal{C}$, and $(E\mathcal{C})^H = S^0$ whenever $H \in \mathcal{C}$.

4.2. The endomorphism operad of a cofamily

We consider the endomorphism operad of $E\mathcal{C}$:

$$\mathcal{E}_{E\mathcal{C}}(n) = Map_*(E\mathcal{C}^{\wedge n}, E\mathcal{C}).$$

The following partial information about the homotopy type of this space will be useful later.

Lemma 4.1. *Given cofamilies $\mathcal{D}_1$ and $\mathcal{D}_2$ the space*

$$\mathrm{map}_*(E\mathcal{D}_1, E\mathcal{D}_2)$$

has the following properties

- *It is H-contractible if $H \notin \mathcal{D}_1 \cap \mathcal{D}_2$*
- *It is H-couniversal for the cofamily $\mathcal{D}_1 \cap \mathcal{D}_2$ if no subgroup of H lies in $\mathcal{D}_2 \setminus \mathcal{D}_1$*

Proof. It is clear that if H is not in $\mathcal{D}_1 \cap \mathcal{D}_2$ then $\mathrm{map}_*(E\mathcal{D}_1, E\mathcal{D}_2)$ is H-contractible, since one or other of the spaces is.

If $H \in \mathcal{D}_1 \cap \mathcal{D}_2$ we wish to argue that the map

$$\mathrm{map}_*(E\mathcal{D}_1, E\mathcal{D}_2)^H \longrightarrow \mathrm{map}_*(S^0, S^0) = S^0$$

is an equivalence. In other words, that any H-map $f\colon E\mathcal{D}_1 \longrightarrow E\mathcal{D}_2$ is determined by its restriction $S^0 \longrightarrow E\mathcal{D}_2$ to S^0. The obstruction to extension and uniqueness lie in $[E(\mathcal{D}_1)^c_+ \wedge S^k, E\mathcal{D}_2]^H$, which vanishes unless H has a subgroup $K \in \mathcal{D}_2 \setminus \mathcal{D}_1$. □

4.3. The couniversal operad of a cofamily

There is a $G \times \Sigma_n$-map $i_n\colon S^0 = (S^0)^{\wedge n} \longrightarrow (E\mathcal{C})^{\wedge n}$ inducing a $G \times \Sigma_n$-map

$$i_n^*\colon \mathcal{E}_{E\mathcal{C}}(n) = Map_*(E\mathcal{C}^{\wedge n}, E\mathcal{C}) \longrightarrow Map_*(S^0, E\mathcal{C}) = E\mathcal{C}.$$

We take

$$D\mathcal{C}(n) = (i_n^*)^{-1}(i_1).$$

We note that when $\mathcal{C} = \mathcal{NT}$ consists of the non-trivial subgroups the fixed point set $D\mathcal{NT}^G = D_{McC}$ is McClure's operad.

Lemma 4.2. *$D\mathcal{C}$ is an operad acting on $E\mathcal{C}$.*

Using this, we will show that for suitable cofamilies $\mathcal{C}$, the space $E\mathcal{C}$ is an algebra over an N_∞-operad with more highly structured algebras than $\mathbb{E}_\infty$.

4.4. Permutation powers and cofamilies

Let us think of the symmetric group Σ_n as the permutations of $\{1, 2, \ldots, n\}$. We consider the group $G \times \Sigma_n$ and let $p\colon G \times \Sigma_n \longrightarrow \Sigma_n$ and $\pi\colon G \times \Sigma_n \longrightarrow G$ be the projections.

If $\mathcal{C}$ is a cofamily of subgroups of G, we view $E\mathcal{C}$ as a trivial Σ_{n-1}-space and form the nth smash power $(E\mathcal{C})^{\wedge n}$ and view it as a $G \times \Sigma_n$-space.

Lemma 4.3. *The $G \times \Sigma_n$-space $E\mathcal{C}^{\wedge n}$ is couniversal.*

Proof. Consider any G-space X and form the $G \times \Sigma_n$-space $X^{\wedge n}$. We will consider fixed points under a subgroup $\Delta \subseteq G \times \Sigma_n$.

Consider the orbits $o_1, \ldots, o_s$ of $\{1, \ldots, n\}$ under $p(\Delta)$, and choose orbit representatives $d_i \in o_i$. Now write $\Delta_i = p^{-1}((\Sigma_n)_{d_i}) \cap \Delta$ for the subgroup of Δ fixing d_i.

We then see that there is a homeomorphism

$$h\colon \bigwedge_{i=1}^{s} X^{\pi(\Delta_i)} \xrightarrow{\cong} (X^{\wedge n})^{\Delta}.$$

The ith factor in the domain gives the d_ith coordinate in $X^{\wedge n}$ and hence determines the coordinates in o_i. More precisely, if $m \in o_i$ we may choose $\delta \in \Delta$ with $p(\delta)(d_i) = m$, and then

$$h(x_1 \wedge \ldots \wedge x_s)_m = \pi(\delta) x_i.$$

Since x_i is fixed by Δ_i this is independent of the choice of δ. The verification that h is a homeomorphism is straightforward.

Applying this to $X = E\mathcal{C}$ we see that X^Δ is always either S^0 or contractible. The collection of subgroups for which it is S^0 is obviously a cofamily. □

If we write $C(\mathcal{C}, n)$ for the geometric isotropy of $E\mathcal{C}^{\wedge n}$, then by the lemma $E\mathcal{C}^{\wedge n} \simeq EC(\mathcal{C}, n)$.

Lemma 4.4.

$$C(\mathcal{C}, n) \subseteq \pi^* \mathcal{C}.$$

Proof. We show that if Δ is not in the right hand side it is not in the left hand side.

If $\pi(\Delta)$ does not lie in $\mathcal{C}$, then $(E\mathcal{C})^{\pi(\Delta)} \neq S^0$. Suppose then that $x = x(1) \in E\mathcal{C} \setminus S^0$ is a non-trivial element of $\mathcal{C}$ fixed by $\pi(\Delta)$. Now write $x(i) = \sigma^i x(1)$ where $\sigma = (123\cdots n)$. We then have $x = x(1) \wedge x(2) \wedge \cdots \wedge x(n)$ fixed by $\pi(\Delta) \times \Sigma_n$ and hence by its subgroup Δ. Hence $\Delta \notin C(\mathcal{C}, n)$. □

Lemma 4.5. *If $\mathcal{C}$ is closed under passage to finite index subgroups then*

$$\pi^*\mathcal{C} \cap \mathcal{F}_G(n) \subseteq C(\mathcal{C}, n).$$

Proof. Suppose $\Delta \subseteq G \times \Sigma_n$ lies in the intersection, which is to say $L := \pi(\Delta) \in \mathcal{C}$, and $\Delta = \Gamma(L, \alpha)$ is a graph subgroup. We will show that $\Delta \in C(\mathcal{C}, n)$. Since $C(\mathcal{C}, n)$ is a cofamily, it suffices to show that the subgroup $\Delta' = \Gamma(L_e, \alpha|_{L_e})$ lies in $C(\mathcal{C}, n)$, where L_e is the identity component of L.

However, since Σ_n is discrete $\alpha|_{L_e}$ is trivial, so that $\Delta' = \Gamma(L_e, const) = L_e$. However, $L = \pi(\Delta)$ lies in $\mathcal{C}$, so its finite index subgroup L_e also lies in $\mathcal{C}$ by hypothesis:

$$(E\mathcal{C}^{\wedge n})^{\Delta} \subseteq (E\mathcal{C}^{\wedge n})^{\Delta'} = (E\mathcal{C}^{L_e})^{\wedge n} = (S^0)^{\wedge n} = S^0.$$

Hence $(E\mathcal{C}^{\wedge n})^{\Delta} = S^0$ as required. □

Lemma 4.6. *The map*

$$i_n^* \colon \mathcal{E}_{E\mathcal{C}}(n) = \mathrm{map}_*(E\mathcal{C}^n, E\mathcal{C}) \longrightarrow \mathrm{map}_*(S^0, E\mathcal{C}) = E\mathcal{C}$$

is an $\mathcal{F}_G(n)$-equivalence of $G \times \Sigma_n$-spaces.

Proof. We must show that for any $\Delta \in \mathcal{F}_G(n)$, the map is an equivalence in Δ-fixed points.

We observe that by Lemmas 4.4 and 4.5, we have $C(\mathcal{C}, n) \cap \mathcal{F}_G(n) = \pi^*\mathcal{C} \cap \mathcal{F}_G(n)$. This shows that with $\mathcal{D}_1 = C(\mathcal{C}, n)$, $\mathcal{D}_2 = \pi^*\mathcal{C}$, and $H = \Delta \in \mathcal{F}_G(n)$ the condition $\Delta \notin C(\mathcal{C}, n) \setminus \pi^*\mathcal{C}$ of Lemma 4.1 is satisfied, so that $\mathrm{map}_*(E\mathcal{C}^n, E\mathcal{C})$ is Δ-universal for $C(\mathcal{C}, n)|_{\Delta} \cap (\pi^*\mathcal{C})|_{\Delta} = (\pi^*\mathcal{C})|_{\Delta}$. □

4.5. McClure's argument extended

We now apply the above to the operad $D\mathcal{C}$ of Subsection 4.2.

Theorem 4.7. *If $\mathcal{C}$ is a cofamily then the space $E\mathcal{C}$ is an E_∞^G-algebra if and only if $\mathcal{C}$ is closed under passage to finite index subgroups (i.e., the minimal elements of $\mathcal{C}$ are connected subgroups).*

Proof. If there is a finite index inclusion $K \subseteq H$ of subgroups with $H \in \mathcal{C}$ and $K \notin \mathcal{C}$, then the assumption that $E\mathcal{C}$ is E_∞^G leads to a contradiction. Indeed $\pi_0^K(E\mathcal{C}) = 0$ so that $1 = 0$ in that ring. On the other hand, by Segal-tom Dieck splitting, $\pi_0^H(E\mathcal{C}) \neq 0$ so that $1 \neq 0$ in $\pi_0^H(E\mathcal{C})$. The existence of a norm map then gives a contradiction since $\mathrm{norm}_K^H(1) = 1$.

Now suppose $\mathcal{C}$ is closed under passage to finite index subgroups. By Lemma 4.2 there is an action of $D\mathcal{C}$ on $E\mathcal{C}$, and hence also an action of $\mathbb{E}_\infty^G \times D\mathcal{C}$. It remains to show that the nth term in this operad is universal for $\mathcal{F}_G(n)$. In other words, we need to show that if $\Gamma \in \mathcal{F}_G(n)$ is a graph subgroup then $D\mathcal{C}(n)^{\Gamma} \simeq *$.

By Lemma 4.6, the map

$$i_n^*\colon \mathcal{E}_{E\mathcal{C}}(n) = \mathrm{map}_*(E\mathcal{C}^n, E\mathcal{C}) \longrightarrow \mathrm{map}_*(S^0, E\mathcal{C}) = E\mathcal{C}$$

is an $\mathcal{F}_G(n)$-equivalence, and hence $D\mathcal{C}(n)$ is $\mathcal{F}_G(n)$-contractible as required. □

The following special cases are used extensively in [**4, 1**].

Corollary 4.8. *If G is a torus and K is a connected subgroup then $S^{\infty V(K)} = \bigcup_{V^K=0} S^V$ is an E_∞^G-algebra.*

Proof. The space $S^{\infty V(K)}$ is couniversal for the cofamily $\mathsf{V}(K) = \{H \mid H \supseteq K\}$ of subgroups containing K. By hypothesis, the unique minimal subgroup K of $\mathsf{V}(K)$ is connected. □

It is often useful to argue by induction on the dimension of subgroups.

Corollary 4.9. *If G is any compact Lie group and $\mathcal{C}_t$ denotes the cofamily of subgroups of dimension $\geqslant t$ then $E\mathcal{C}_t$ is an E_∞^G-algebra.*

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