

THE OMEGA SPECTRUM FOR PENGELLEY'S BoP

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Abstract

We compute the homology of the spaces in the Omega spectrum for BoP . There is no torsion in $H_*(\underline{BoP}_i)$ for $i \geq 2$, and things are only slightly more complicated for $i < 2$. We find the complete homotopy type of \underline{BoP}_i for $i \leq 6$ and conjecture the homotopy type for $i > 6$. This completes the computation of all $H_*(\underline{MSU}_*)$.

1. Context

There are several standard (co)bordism theories and related spectra. We have, for example, unoriented (co)bordism, MO , oriented (co)bordism, MSO , Spin (co)bordism, $MSpin$, complex (co)bordism, MU , and special unitary cobordism, MSU . These theories are associated with Omega spectra classifying them, i.e. $M^k(X) = [X, \underline{M}_k]$ with $\Omega \underline{M}_{k+1} = \underline{M}_k$. We are only interested in the $p = 2$ versions so all spectra should be considered localized at 2. We will suppress the notation and let $M = M_{(2)}$.

Thom, [Tho54, 1954], computed the coefficients MO_* and gave the stable homotopy type of the spectrum MO . It is just the product of a lot of mod 2 Eilenberg-MacLane spectra. This gives the complete homotopy type of the \underline{MO}_k as well. From Serre's computation, [Ser53, 1953], of the cohomology of the Eilenberg-MacLane spaces, we also know $H_*(\underline{MO}_k) \pmod{2}$. Wall, [Wal60, 1960], showed MSO (at 2) is the stable product of integral and mod 2 Eilenberg-MacLane spectra. Again, this gives the complete homotopy type of the \underline{MSO}_k and, by Serre, the homology of these spaces. Anderson, Brown, and Peterson, [AEBP67, 1967], computed $MSpin_*$ and gave the stable homotopy type, and consequently, the unstable homotopy type of \underline{MSpin}_k . Stably, this is copies of the mod 2 Eilenberg-MacLane spectra and connected covers of the spectrum bo . Stong, [Sto63, 1963], computed the cohomology of (most of) the unstable connected covers for bo , giving (most of) $H_*(\underline{MSpin}_k)$. The gaps were filled in by Cowen Morton, [CM07, 2007]. At $p = 2$, Milnor, [Mil60, 1960], and Novikov, [Nov62, 1962], both computed MU_* . With the construction of the Brown-Peterson spectrum, [BP66, 1966], the stable homotopy of MU was described as a product of BP spectra. In [Wil73, 1973], the homology of \underline{BP}_k was computed and in [Wil75, 1975] a complete description of the unstable homotopy type was given.

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MSU has taken more time. Conner and Floyd finished the computation of MSU_* in [CF66b, 1966], but it wasn't until Pengelley constructed the spectrum BoP (the subject of this paper) in [Pen82, 1982], that the stable homotopy type of MSU (at 2) was given as the product of copies of BP and BoP . The purpose of this paper is to compute the homology of the \overline{BoP}_k and to conjecture the unstable homotopy type. This completes the computation of all $H_*(\overline{MSU}_*)$.

2. Introduction

In [Pen82], Pengelley constructed a 2-local spectrum, BoP , such that the special unitary cobordism spectrum localized at two, $MSU_{(2)}$, splits as many copies of various suspensions of BoP and BP , the Brown-Peterson spectrum. To simplify notation, we use \overline{BP} in place of $\prod_{a \geq 0} \Sigma^{8a} BP$. Andy Baker gives a stable cofibration

$$BoP \longrightarrow \overline{BP} \longrightarrow \Sigma^2 BoP. \quad (1)$$

We use homology with $\mathbb{Z}/(2)$ -coefficients. Our first theorem is about the homology of the spaces in the Omega spectrum for BoP :

Theorem 2.1. *For $i \geq 2$, we have a short exact sequence of Hopf algebras*

$$\mathbb{Z}/(2) \rightarrow H_*(\overline{BoP}_i) \longrightarrow H_*(\overline{BP}_i) \longrightarrow H_*(\overline{BoP}_{i+2}) \rightarrow \mathbb{Z}/(2).$$

The homology of all three terms is polynomial on even degree generators when i is even and exterior on odd degree generators when i is odd. There is no torsion in the $\mathbb{Z}_{(2)}$ -homology.

Remark 2.2. Of course the homology of the middle term, \overline{BP}_i , is well known already, [Wil73]. Because all are either exterior or polynomial, the short exact sequence is split as algebras. In addition, when $i > 2$ and even, the homology is bipolynomial (i.e. the cohomology is also polynomial). There is a temptation to believe that since the middle term is well-known, there must be some sort of degree-by-degree induction algorithm that allows one to bootstrap the computation of the homology of the other terms from that. Regrettably, this is not the case. The induction starts with the short exact sequence for $i = 2$, but that is a hard won exact sequence that depends on the degree by degree computations for all of the negative spaces, with an added exotic transition from negative to positive spaces.

By splicing these short exact sequences together there is a novel corollary with no obvious use. Using the composite maps $\overline{BP}_j \rightarrow \overline{BoP}_{j+2} \rightarrow \overline{BP}_{j+2}$, we get

Corollary 2.3. *For $i \geq 2$ there is a long exact sequence of Hopf algebras*

$$\begin{aligned} \mathbb{Z}/(2) \rightarrow H_*(\overline{BoP}_i) \longrightarrow H_*(\overline{BP}_i) \longrightarrow H_*(\overline{BP}_{i+2}) \longrightarrow \\ H_*(\overline{BP}_{i+4}) \longrightarrow H_*(\overline{BP}_{i+6}) \longrightarrow \cdots \end{aligned}$$

Remark 2.4. The object of interest to us is thus the zeroth homology of a chain complex of well understood Hopf algebras. Unfortunately, the unstable maps here are not at all understood. Stably, in (co)-homology, they are easy to see as everything is given by sums of cyclic modules over the Steenrod algebra. The actual stable map

$r: \overline{BP} \rightarrow \Sigma^2 \overline{BP}$ has the property that it covers Sq^2 on each copy of BP and has $r^2 = 0$. This might well be enough to determine r , but it is not to be messed with lightly. Furthermore, it is unlikely to give insight into the computation of the unstable maps with the usual generators.

Pengelley constructed a fibration $F \rightarrow BoP \rightarrow bo$ that gives a short exact sequence in homotopy. It also has the property that bo carries all of the torsion homotopy of BoP leaving F with no torsion in homotopy.

There is an old theorem of Conner and Floyd, [CF66a, Corollary 9.6, page 58], that says (really for MSU , but essentially):

$$\underline{BoP}_0 \simeq \underline{F}_0 \times \underline{bo}_0. \quad (2)$$

Notation is much simpler if we just assume that we are taking the 2-local version of bo and \underline{bo}_i throughout this paper.

We use this to get the following theorem covering the homology of the negative spaces:

Theorem 2.5. *For $i < 6$, we have*

$$H_*(\underline{BoP}_i) \simeq H_*(\underline{F}_i) \otimes H_*(\underline{bo}_i),$$

where $H_*(\underline{bo}_i)$ is well known and $H_*(\underline{F}_i)$ is polynomial on even degree elements when i is even and exterior on odd degree elements when i is odd. There is no torsion in the $\mathbb{Z}_{(2)}$ -homology of \underline{F}_i .

We need some background in order to state our final theorem about homotopy type, but first some notation. Let \mathcal{A} be the mod 2 Steenrod algebra and Q_k the Milnor primitives. Define $\mathcal{A}(k) = \mathcal{A}/\mathcal{A}(Q_0, Q_1, \dots, Q_k)$ and $\mathcal{A}(2, k) = \mathcal{A}/\mathcal{A}(Q_0, Sq^2, Q_1, \dots, Q_k)$.

Theorem 2.6 ([Wil75], rephrased from cohomology to homology).

1. *There exists a unique, up to homotopy, irreducible $(k-1)$ -connected H -space Y_k which has $H_*(Y_k; \mathbb{Z}_{(2)})$ and $\pi_*(Y_k)$ free over $\mathbb{Z}_{(2)}$.*
2. *If Z is an H -space with $H_*(Z; \mathbb{Z}_{(2)})$ and $\pi_*(Z)$ free over $\mathbb{Z}_{(2)}$, then $Z \simeq \prod_i Y_{k_i}$.*
3. *There are spectra $BP\langle n \rangle$ with $H^*(BP\langle n \rangle) = \mathcal{A}(n)$, $BP\langle n \rangle_* \simeq \mathbb{Z}_{(2)}[v_1, v_2, \dots, v_n]$ and $|v_i| = 2(2^i - 1)$.*
4. *For $2^{j+1} - 2 < k \leq 2^{j+2} - 2$, $Y_k \simeq \underline{BP}\langle j \rangle_k$.*
5. *$\underline{BP}\langle j \rangle_{2^{j+1}-2} \simeq \underline{BP}\langle j-1 \rangle_{2^{j+1}-2} \times \underline{BP}\langle j \rangle_{2^{j+2}-4}$.*

Theorem 2.7. *There is an irreducible splitting, not as H -spaces:*

$$\begin{aligned} \underline{BoP}_6 &\simeq \underline{bo}_6 \times \prod_{u \geq 0} Y_{8u+12} \simeq \underline{bo}_6 \times \prod_{2^{k-2} > u \geq 0} Y_{2^{k+1}+8u+4} \\ &\simeq \underline{bo}_6 \times \prod_{2^{k-2} > u \geq 0} \underline{BP}\langle k \rangle_{2^{k+1}+8u+4}. \end{aligned}$$

Remark 2.8. Looping down this splitting, we still have a splitting. After a few loops, it ceases to be irreducible. However, using Theorem 2.6(5), the irreducible splittings can be computed for as many loops as you want. This is a bad habit to get into though. The awkward notation is a result of there being one $BP\langle 2 \rangle$, two $BP\langle 3 \rangle$, four $BP\langle 4 \rangle$, eight $BP\langle 5 \rangle$, etc. All as a result of Theorem 2.6(4).

Now to the speculative part of the paper.

Conjecture 2.9. *There are spectra, $BoP\langle n \rangle$, with $BoP\langle 1 \rangle = bo$, with irreducible splittings*

$$\begin{aligned} \underline{BoP}_{8n-2} &\simeq \underline{BoP}\langle n \rangle_{8n-2} \times \prod_{2^{k-2}-n+1 > u \geq 0} \underline{BP}\langle k \rangle_{2^{k+1}+8(n+u)-4} \\ \underline{BoP}\langle n+1 \rangle_{8n-2} &\simeq \underline{BoP}\langle n \rangle_{8n-2} \times \prod_{2^{k-2} \geq n} \underline{BP}\langle k \rangle_{2^{k+2}-4}. \end{aligned}$$

There are spectra, $\overline{BP}\langle n \rangle$, with $\overline{BP}\langle 1 \rangle = bu$, such that we have a stable cofibration

$$BoP\langle n \rangle \longrightarrow \overline{BP}\langle n \rangle \longrightarrow \Sigma^2 BoP\langle n \rangle$$

inducing a short exact sequence on (co)homology.

Let $n > 2$. Write $n = 2^K + a + 1$ with $0 \leq a < 2^K$. Then

$$H^*(BoP\langle n \rangle) \simeq \bigoplus_{s=1}^{n-1} \sum_{K' \geq K}^{2^{K'+3+\epsilon}-8s} A(2, K' + 2 + \epsilon)$$

and

$$\overline{BP}\langle n \rangle \simeq \bigvee_{s=1}^{n-1} \bigvee_{K' \geq K}^{2^{K'+3+\epsilon}-8s} BP\langle K' + 2 + \epsilon \rangle,$$

where $\epsilon = 1$ if $0 < s \leq a$ or $2^K + 1 = n - a \leq s < n$ and $\epsilon = 0$ if $a < s < n - a = 2^K + 1$.

Remark 2.10. Particular thanks to Andy Baker for the cofibration (1). Thanks also to Dave Johnson, David Pengelley, Vitaly Lorman, Vladimir Verchinine, and the referee. Because of my interest in cobordism, I have always wanted to study Pengelley's BoP but never found the time. I was inspired by a vague recollection of a 2001 email from Mike Slack suggesting that there is no torsion in spaces in the Omega spectrum. After proving these results, I went back and reviewed my emails from Mike and found much more than that. I discovered a 1995 email that used the same notation $BoP\langle n \rangle$. The email does not specify the detail in Conjecture 2.9, but went further in one sense. In [Sla98], Slack proved that if an infinite loop space had no odd torsion in its integral homology, then it also had no odd torsion in homotopy. This did not hold for $p = 2$ as is illustrated by the spaces like \underline{bo}_2 , and now \underline{BoP}_k , $k \geq 2$. However, Michael Slack conjectured that you could classify all infinite loop spaces with two-primary torsion-free homology in terms of products of the spaces $\underline{BoP}\langle n \rangle_k$ that are irreducible and torsion-free in homology. He left mathematics before he completed his proofs and wrote things up. If he hadn't left mathematics, I'm sure everything in this paper, and more, would have been done by Michael Slack 15 years ago.

In Section 3, we do a quick review of what we need about Hopf algebras and the bar spectral sequence. We review what we need about BoP in Section 4. In Section 5 we do our computation for the spaces \underline{F}_i , $i < 6$, and prove Theorem 2.5. In Section 6, we move on to \underline{BoP}_i , $i \geq 2$ and prove Theorem 2.1. We prove the splitting, Theorem 2.7, in Section 7 and discuss Conjecture 2.9 in the last two sections.

3. Hopf algebras and the bar spectral sequence

All of the spaces we deal with are spaces in Omega spectra, i.e., $Z = \{\underline{Z}_i\}$ with $\Omega \underline{Z}_{i+1} \simeq \underline{Z}_i$. The spectra are all connective, i.e., $\pi_*^S(Z) = 0$ for $* < 0$.

As a result, the mod 2 homology of all our spaces, $H_*(\underline{Z}_i)$, are all bicommutative, biassociative graded (and sometimes bigraded) Hopf algebras. When we use the zero component, \underline{Z}'_i , the Hopf algebra is connected. The standard reference for Hopf algebras is, of course, [MM65].

Because we know the homotopy of our spaces, we know the zeroth homology of the spaces in the Omega spectra. We have $H_0(\underline{Z}_*) \simeq \mathbb{Z}/(2)[Z^*]$, the group ring on the coefficients. More precisely, $H_0(\underline{Z}_i) \simeq \mathbb{Z}/(2)[Z^i] \simeq \mathbb{Z}/(2)[\pi_{-i}^S(Z)]$. Then we have $H_*(\underline{Z}_i) \simeq \mathbb{Z}/(2)[Z^i] \otimes H_*(\underline{Z}'_i)$. Except for bo and BoP , all of our spaces have homotopy only in even degrees. As a result, the above isomorphism degenerates when i is odd because $\underline{Z}_i = \underline{Z}'_i$.

Our main tool is the bar spectral sequence. We state what we need here.

Theorem 3.1 ([RS65]). *There is a first quadrant homology spectral sequence of Hopf algebras going from $H_*(\underline{Z}_{i-1})$ to $H_*(\underline{Z}_i)$ with*

$$E_{*,*}^2 \simeq \text{Tor}_{*,*}^{H_*(\underline{Z}_{i-1})}(\mathbb{Z}/(2), \mathbb{Z}/(2)) \Rightarrow H_*(\underline{Z}_i),$$

$$d_r : E_{u,v}^r \longrightarrow E_{u-r, v+r-1}^r.$$

Remark 3.2. Because we are dealing with infinite loop spaces we do not run into weird convergence problems associated with problematic fundamental groups. In fact, we will do away with the use of the zero component, \underline{Z}'_i , and, instead, use the convention that we think of the group ring $\mathbb{Z}/(2)[\mathbb{Z}_{(2)}]$ as a polynomial algebra on one generator. When there is no torsion in the homotopy, the use of this convention gives $\text{Tor}_{*,*}^{H_0(\underline{Z}_i)}(\mathbb{Z}/(2), \mathbb{Z}/(2))$ is an exterior algebra on the suspensions of the $\mathbb{Z}/(2)[\mathbb{Z}_{(2)}]$ generators in $\mathbb{Z}/(2)[Z^i]$, all located in $(1, 0)$. This follows from the Hurewicz isomorphism in degree 1 if not the algebra.

We actually only need the spectral sequence in very limited situations. Keep in mind that Tor commutes with tensor products.

Proposition 3.3. *If $H_*(\underline{Z}_{i-1})$ is polynomial on even degree generators for degrees $* < j$, then $H_*(\underline{Z}_i)$ is an exterior algebra on the suspensions of the generators in degrees $* \leq j$. If $H_*(\underline{Z}_{i-1})$ is exterior on odd degree generators for degrees $* < j$, then $E^2 = E^\infty H_*(\underline{Z}_i)$ is an even degree divided power algebra on the suspensions of the generators in degrees $* \leq j$.*

Proof. To compute Tor you need a resolution. Since we are only using degrees $* < j$, it follows that our input gives all $E_{u,v}^2$ with $v < j$. Also, ignoring the $\mathbb{Z}/(2)$ in degree zero, there is nothing in the zero filtration. Since $u > 0$, this gives all $E_{u,v}^2$ with total degree $u + v \leq j$.

If $i - 1$ is even, we take Tor of a polynomial algebra with even degree generators x_{2i} to get an exterior algebra on the elements that are suspensions of the generators in bidegree $(1, 2i)$. Since they are in filtration 1, there are no differentials on them. Because of this, there can be no differentials from degree (u, v) when $v < j$. Any differential that hits something in total degree $* \leq j$, must come from (u, v) with

$u > 2$ and $u + v \leq j + 1$, but this requires $v < j$, so there can be no differentials interfering. We have E^∞ for $* \leq j$ is an exterior algebra on odd degree generators. There can be no extensions because all that can happen is an odd exterior generator squares to an even generator, but there are none in degrees $* \leq j$.

If $i - 1$ is odd, Tor of an exterior algebra with odd degree generators x_{2i+1} is a divided power algebra, $\Gamma[\sigma x_{2i+1}]$, on the elements σx_{2i+1} in $(1, 2i + 1)$. Thus Tor, up through degrees $* \leq j$ is even degree so there can be no differentials on them. Just as above, they cannot be hit by differentials. There can still be extension problems with respect to squaring elements. \square

Remark 3.4. In the case of the spectral sequence above for \overline{BP}_i , we know that the homology for i odd is exterior and for i even polynomial. That means that all of the possible extension problems in the spectral sequence for i even must be solved.

4. Review of BoP

We need to collect a few facts. Recall our notation, $\overline{BP} = \prod_{a \geq 0} \Sigma^{8a} BP$.

Proposition 4.1 (from A. Baker). *There is a stable cofibration*

$$BoP \longrightarrow \overline{BP} \longrightarrow \Sigma^2 BoP.$$

This is torsion free and short exact on (co)homology.

Note that there is no torsion in homology for these spectra because they are even degree.

Proof. We smash BoP with the cofibration

$$S^1 \longrightarrow S^0 \longrightarrow C(\eta)$$

to get

$$\Sigma BoP \longrightarrow BoP \longrightarrow BoP \wedge C(\eta)$$

or

$$BoP \longrightarrow BoP \wedge C(\eta) \longrightarrow \Sigma^2 BoP.$$

All we need to do is show that

$$BoP \wedge C(\eta) \simeq \overline{BP}.$$

Let A be the mod 2 Steenrod algebra and Q_i the Milnor primitives, [Mil58]. From [Pen82] we know the mod 2 cohomology

$$H^*(BoP) \simeq \bigoplus_{a \geq 0} \Sigma^{8a} A/A(Q_0, Sq^2, Q_1, Q_2, Q_3, \dots).$$

The usual way to write this is to take out Q_0 on both the left and right, but that is equivalent to taking out all of the Q_i on the right. Then Sq^2 is taken out on the right as well. For future use we prefer this notation. We know $H^*(C(\eta))$ just has cells in degree 0 and 2 connected by Sq^2 . This gives

$$H^*(BoP \wedge C(\eta)) \simeq H^*(BoP) \otimes H^*(C(\eta)) \simeq \bigoplus_{a \geq 0} \Sigma^{8a} A/A(Q_0, Q_1, Q_2, Q_3, \dots).$$

This last is the cohomology of \overline{BP} , and anything with such cohomology is a product of BP spectra, [BP66]. The short exact sequence follows. \square

We also make use of the old result (of R. Wood, see [KLW04, 2.3.1] for a discussion of references) that comes about in a similar way.

Proposition 4.2. *There is a stable cofibration*

$$bo \longrightarrow bu \longrightarrow \Sigma^2 bo.$$

This is short exact on $\mathbb{Z}/(2)$ -(co)homology.

We always assume our bo and bu to be the 2-local versions and will suppress using the notation $bo_{(2)}$ and $bu_{(2)}$.

We need to define two spectra, F and X . X is easy.

Lemma 4.3 (for BoP , [Pen82]). *There are stable cofibrations that define spectra F and X with torsion-free even degree homotopy. They give short exact sequences in their homotopy groups.*

$$\begin{aligned} F &\longrightarrow BoP \longrightarrow bo, \\ X &\longrightarrow \overline{BP} \longrightarrow bu. \end{aligned}$$

Proof. The spectrum bu at $p = 2$ is sometimes called $BP\langle 1 \rangle$ and there is a map $BP \rightarrow BP\langle 1 \rangle$, see [JW73]. The other copies of BP do not enter in. \square

This all gives rise to horizontal and vertical cofibrations:

$$\begin{array}{ccccc} F & \longrightarrow & BoP & \longrightarrow & bo \\ \downarrow & & \downarrow & & \downarrow \\ X & \longrightarrow & \overline{BP} & \longrightarrow & bu \\ \downarrow & & \downarrow & & \downarrow \\ \Sigma^2 F & \longrightarrow & \Sigma^2 BoP & \longrightarrow & \Sigma^2 bo \end{array} \quad (3)$$

We need some unstable information to make our computations work.

Theorem 4.4 ($i = 0$, Corollary 9.6, page 58, and following comments [CF66a]).

$$\begin{aligned} \underline{BoP}_i &\simeq \underline{F}_i \times \underline{bo}_i & i \leq 6, \\ \underline{\overline{BP}}_i &\simeq \underline{X}_i \times \underline{bu}_i & i \leq 6. \end{aligned}$$

Proof. Of course Conner and Floyd did not use BoP and BP in their work because they did not exist at the time. The above proposition is the modern interpretation. They only proved these cases for $i = 0$ and that is all we will use. Of course, the $i < 0$ cases follow. Only in the end will we improve on this and give the \underline{BoP}_i splittings for $0 < i \leq 6$. The BP - bu case is already known from [Wil75] which can be used to give the entire homotopy type of \underline{X}_i for $i \leq 8$. \square

We need just a few more things.

Lemma 4.5. *The homotopy groups give a split short exact sequence of free $\mathbb{Z}_{(2)}$ -modules for the fibration*

$$\underline{F}_i \longrightarrow \underline{X}_i \longrightarrow \underline{F}_{i+2}.$$

Proof. In the diagram (3), all of the $\mathbb{Z}/(2)$ -groups are in BoP and bo . All of the $\mathbb{Z}_{(2)}$ -free groups in the other spectra are in even degrees. The result follows. \square

Remark 4.6. It is perhaps unfair to assume too much pre-existing knowledge. To correct that oversight, we give the homotopy groups of the well-known objects in diagram (3).

$$\begin{aligned} BP_* &\simeq \mathbb{Z}/(2)[v_1, v_2, \dots], & |v_n| &= 2(2^n - 1), & bu_* &\simeq \mathbb{Z}/(2)[v_1], \\ bo_{4i} &\simeq \mathbb{Z}/(2), & & & bo_{8i+1} &\simeq \mathbb{Z}/(2) \simeq bo_{8i+2}. \end{aligned}$$

In addition, we need some information about homology.

Proposition 4.7. *The homology, $H_*(\overline{BP}_i)$, is torsion free and is polynomial on even degree generators for i even and exterior on odd degree generators for i odd. The homology, $H_*(\underline{X}_i)$, $i < 6$, is torsion free and is polynomial on even degree generators for i even and exterior on odd degree generators for i odd.*

There is a short exact sequence of polynomial Hopf algebras

$$\mathbb{Z}/(2) \longrightarrow H_*(bo_2) \longrightarrow H_*(bu_2) \longrightarrow H_*(bo_4) \longrightarrow \mathbb{Z}/(2).$$

Proof. The homology of \overline{BP}_i is known from [Wil73] and that for \underline{X}_i follows from the splitting, Theorem 4.4. The given short exact sequence is well-known, but see Remark 4.9 below as well. \square

Remark 4.8. This can be refined to see that \underline{X}_i has no torsion for $i \leq 8$, but there the homology is not polynomial.

Remark 4.9. Again, it seems only fair to recall some of the well known results. $bu_2 = BU$, and, as such, the homology is just $H_*(BU) \simeq \mathbb{Z}/(2)[x_{2i}]$. The spectrum bo is more complicated with $bo_0 = Z \times BO = \underline{KO}_0$. By Bott periodicity we always have $bo_i = \underline{bo}_{i-8} = \underline{KO}_i$ for $i < 4$ (in particular, for all negative i). Let P denote a polynomial algebra and E an exterior algebra. Also let z_i denote an i -th degree element. We know that

$$\begin{aligned} H_*(bo_0) &\simeq \mathbb{Z}/(2)[Z] \otimes P[x_i] \quad i > 0, & H_*(bo_1) &\simeq P[x_{2i+1}], \\ H_*(bo_2) &\simeq P[x_{4i+2}], & H_*(bo_3) &\simeq E[x_{4i+3}], \\ H_*(\underline{KO}_4) &\simeq \mathbb{Z}/(2)[Z] \otimes H_*(bo_4) \simeq \mathbb{Z}/(2)[Z] \otimes P[x_{4i}] \quad i > 0, \\ H_*(\underline{KO}_5) &\simeq E(x_1) \otimes H_*(bo_5) \simeq E(x_1) \otimes E[x_{4i+1}] \quad i > 0, \\ H_*(\underline{KO}_6) &\simeq E(x_{2^k}) \otimes H_*(bo_6) \simeq E(x_{2^k}) \otimes E[x_{2i}] \quad 2i \neq 2^k, \\ H_*(\underline{KO}_7) &\simeq E(x_i). \end{aligned}$$

This is all conveniently written down in [CS02] and [KW07, Theorem 25.2].

5. $H_*(\underline{F}_i)$, $i < 6$

The goal of this section is to prove the following theorem.

Theorem 5.1. *For $i \leq 6$, the fibration of Lemma 4.5 gives a short exact sequence of Hopf algebras*

$$\mathbb{Z}/(2) \rightarrow H_*(\underline{F}_i) \longrightarrow H_*(\underline{X}_i) \longrightarrow H_*(\underline{F}_{i+2}) \rightarrow \mathbb{Z}/(2).$$

When i is odd, all three are exterior algebras on odd degree generators. When $i < 6$ is even, all three are polynomial algebras on even degree generators.

Remark 5.2. This, plus the Conner-Floyd result, equation (2) and Theorem 4.4, gives Theorem 2.5 for $i \leq 0$. We will have to return to Theorem 2.5 later for the $0 < i < 6$.

Proof. We already have the homology of \underline{X}_i by Proposition 4.7. We do our proof by induction on degree. We know that on the zero-degree homology we have such an exact sequence because there the groups are just given by the homotopy groups where we have exactness from Lemma 4.5. Exactness for $H_1(-)$ follows from our convention on Tor or just using the Hurewicz isomorphisms. $H_2(-)$ and exactness follows in the same way. One can even go one step further and get $H_3(-)$. This starts our induction.

For our induction, we assume the result for degrees $* < j$ and we will show it holds for degree j as well. By induction, we have the result for

$$\mathbb{Z}/(2) \rightarrow H_*(\underline{F}_{i-1}) \longrightarrow H_*(\underline{X}_{i-1}) \longrightarrow H_*(\underline{F}_{i+1}) \rightarrow \mathbb{Z}/(2), \quad * < j.$$

Since these are either exterior or polynomial, this exact sequence is split as algebras. Computing Tor only depends on the algebra structure, so in our range $* \leq j$, we know from Proposition 3.3 that we get a short exact sequence of Hopf algebras on the $E^2 = E^\infty$ terms.

When i is odd, we are done because the results are exterior. When i is even, we get the inclusion $H_*(\underline{F}_i) \rightarrow H_*(\underline{X}_i)$. Since the second one is polynomial, the first must also be polynomial (Hopf algebra structure requires this).

However, we have not yet shown that $H_*(\underline{F}_{i+2})$ is polynomial for i even, $i < 6$. To see this, we use the following short exact sequence we have by induction

$$\mathbb{Z}/(2) \rightarrow H_*(\underline{F}_{i+1}) \longrightarrow H_*(\underline{X}_{i+1}) \longrightarrow H_*(\underline{F}_{i+3}) \rightarrow \mathbb{Z}/(2), \quad * < j.$$

Since i is even and $i < 6$, $i + 1 < 6$ fits our induction hypothesis. Once again we apply Tor to this split short exact sequence of algebras to get a short exact sequence on the $E^2 = E^\infty$ terms. Now we have the inclusion $H_*(\underline{F}_{i+2}) \rightarrow H_*(\underline{X}_{i+2})$ forcing $H_*(\underline{F}_{i+2})$ to be polynomial. If $i = 4$, this gives

$$\mathbb{Z}/(2) \rightarrow H_*(\underline{F}_6) \longrightarrow H_*(\underline{X}_6) \longrightarrow H_*(\underline{F}_8) \rightarrow \mathbb{Z}/(2), \quad * \leq j.$$

In this case, the right hand term need not be polynomial, but is still even degree (and cofree). \square

Proof of Theorem 2.5. We use the Atiyah-Hirzebruch spectral sequence (AHSS) for the fibration $\underline{F}_i \rightarrow \underline{BoP}_i \rightarrow \underline{bo}_i$, $0 < i \leq 6$. The E^2 term is

$$E^2 \simeq H_*(\underline{F}_i) \otimes H_*(\underline{bo}_i) \Rightarrow H_*(\underline{BoP}_i).$$

These spectral sequences all collapse. For $i = 2, 4$, and 6 they collapse because they are even degree. For $i = 3$ and 5 , everything is exterior on odd degree generators so there is no place for a differential to go. The $i = 1$ case is a bit more complicated. The $H_*(\underline{F}_1)$ term is exterior on odd degree generators and $H_*(\underline{bo}_1)$ is polynomial on odd degree generators. Differentials must start on the odd degree polynomial generators of $H_*(\underline{bo}_1)$ and end up in an even degree, so, as with the exterior case, there is nothing to hit. In addition, there are no algebra extension problems for $0 < i < 6$. For $1 < i < 6$ this is because the E^2 term is either exterior on odd degree generators or polynomial on even degree generators. In the case of $i = 1$, we have to look at the maps $H_*(\underline{F}_1) \rightarrow H_*(\underline{BoP}_1) \rightarrow H_*(\underline{bo}_1)$ to see that the exterior elements from

$H_*(\underline{F}_1)$ cannot have extension problems and since $H_*(\underline{bo}_1)$ is polynomial, this splits as algebras. \square

Remark 5.3. Where this gets interesting is for the $i = 6$ case. We have the short exact sequence $H_*(\underline{F}_6) \rightarrow H_*(\underline{BoP}_6) \rightarrow H_*(\underline{bo}_6)$. We will soon see that $H_*(\underline{BoP}_6)$ is polynomial. However, $H_*(\underline{bo}_6)$ is exterior. Later we will even show that $\underline{BoP}_6 \simeq \underline{F}_6 \times \underline{bo}_6$, just not as H-spaces. So, the squares of the exterior generators in $H_*(\underline{bo}_6)$ must be non-zero in $H_*(\underline{F}_6)$.

6. $H_*(\underline{BoP}_i)$, $i \geq 2$

To begin our induction for the proof of Theorem 2.1, we need the following.

Theorem 6.1. *There is a short exact sequence of polynomial Hopf algebras:*

$$\mathbb{Z}/(2) \rightarrow H_*(\underline{BoP}_2) \longrightarrow H_*(\overline{BP}_2) \longrightarrow H_*(\underline{BoP}_4) \rightarrow \mathbb{Z}/(2).$$

Proof. Consider the diagram

$$\begin{array}{ccccc} \underline{F}_2 & \longrightarrow & \underline{BoP}_2 & \longrightarrow & \underline{bo}_2 \\ \downarrow & & \downarrow & & \downarrow \\ \underline{X}_2 & \longrightarrow & \overline{BP}_2 & \longrightarrow & \underline{bu}_2 \\ \downarrow & & \downarrow & & \downarrow \\ \underline{F}_4 & \longrightarrow & \underline{BoP}_4 & \longrightarrow & \underline{bo}_4 \end{array}$$

Take the E^2 -terms of the AHSS for the horizontal fibrations and maps to get

$$H_*(\underline{F}_2) \otimes H_*(\underline{bo}_2) \longrightarrow H_*(\underline{X}_2) \otimes H_*(\underline{bu}_2) \longrightarrow H_*(\underline{F}_4) \otimes H_*(\underline{bo}_4).$$

These spectral sequences are all even degree so collapse, are polynomial so there are no extension problems, and the maps form the required short exact sequence by Theorem 5.1 and Proposition 4.7. \square

Proof of Theorem 2.1. We proceed pretty much as we did with the short exact sequences for negative spaces. We inductively assume our result for all $* < j$. We can certainly start our induction because we know the result for $j = 2$ (the only groups here are $H_2(\underline{BoP}_2) \simeq H_2(\overline{BP}_2) \simeq \mathbb{Z}/(2)$. All others are zero.) So, we do our induction on j . If i is odd, $i > 2$, we know

$$\mathbb{Z}/(2) \rightarrow H_*(\underline{BoP}_{i-1}) \longrightarrow H_*(\overline{BP}_{i-1}) \longrightarrow H_*(\underline{BoP}_{i+1}) \rightarrow \mathbb{Z}/(2) \quad (4)$$

satisfies our inductive hypothesis for $* < j$.

Compute Tor on this split polynomial short exact sequence to get a split short exact (collapsing) bar spectral sequence of exterior algebras, by Proposition 3.3. This completes the i odd cases.

If i is even, we can assume $i > 2$ because we have done all of $i = 2$ already.

We have diagram (4), for $* < j$, where the $i - 1$ is now odd (and > 2). Taking Tor for the bar spectral sequence we get a short exact sequence of collapsing even degree divided power algebras (Proposition 3.3). The middle term is polynomial so

the left-hand side, $H_*(\underline{BoP}_j)$ is as well, for degrees $* \leq j$. The problem is to show that the right hand term, $H_*(\underline{BoP}_{i+2})$, is polynomial. However, we have

$$\mathbb{Z}/(2) \rightarrow H_*(\underline{BoP}_{i+1}) \longrightarrow H_*(\overline{BP}_{i+1}) \longrightarrow H_*(\underline{BoP}_{i+3}) \rightarrow \mathbb{Z}/(2)$$

satisfies our induction hypothesis for $* < j$. (This is why we can't just do an induction on i for all degrees at once.) These are all odd, so exterior. Taking Tor of this gives our usual short exact sequence of Hopf algebras, divided power algebras on even degrees (so collapse) through degrees $* \leq j$. Because the middle term is polynomial, the left hand term is too, i.e., $H_*(\underline{BoP}_{i+2})$.

This completes the proof. \square

7. Proof of the splitting

The proof of Theorem 2.7 is mostly about knowing how to write the homotopy groups. To this end we have:

Lemma 7.1. *As graded abelian groups*

$$BoP_* \simeq bo_* \oplus_{k \geq 2} \oplus_{u=0}^{2^{k-2}-1} \Sigma^{2^{k+1}+8u-2} BP\langle k \rangle_*.$$

Proof. BoP_* and bo_* have the same $\mathbb{Z}/(2)$ groups and everything else is torsion free. As a result, all we have to do is show both sides are the same rationally. The Poincare series for the rational homotopy of the \overline{BP}_* is:

$$\frac{1}{1-x^8} \times \prod_{j>0} \frac{1}{1-x^{2(2^j-1)}}.$$

From this and the rational short exact sequence of (1) and Lemma 4.5, we can read off the Poincare series for the homotopy of BoP :

$$\frac{1}{1-x^8} \times \frac{1}{1-x^4} \times \prod_{j>1} \frac{1}{1-x^{2(2^j-1)}}.$$

The rational Poincare series for the right-hand side (RHS) is

$$\frac{1}{1-x^4} + \sum_{k \geq 2} x^{2^{k+1}-2} \sum_{u=0}^{2^{k-2}-1} \frac{x^{8u}}{\prod_{j>0}^k (1-x^{2(2^j-1)}}.$$

We need to show these two are the same. Multiply both sides by the denominator on the left-hand side, i.e.

$$(1-x^8)(1-x^4) \prod_{j>1} (1-x^{2(2^j-1)}).$$

This would leave a $1-x^2$ in the denominator of the RHS, but we can divide that into $1-x^4$ to get $1+x^2$ in the numerator. We want to show:

$$1 = (1-x^8) \prod_{j>1} (1-x^{2(2^j-1)}) \\ + (1-x^8)(1+x^2) \sum_{k \geq 2} x^{2^{k+1}-2} \prod_{j>k} (1-x^{2(2^j-1)}) \left(\sum_{u=0}^{2^{k-2}-1} x^{8u} \right).$$

But

$$\sum_{u=0}^{2^{k-2}-1} x^{8u} = \frac{1 - x^{2^{k+1}}}{1 - x^8}.$$

The $(1 - x^8)$ cancels out and we have the RHS is

$$(1 - x^8) \prod_{j>1} (1 - x^{2(2^j-1)}) + \sum_{k \geq 2} x^{2^{k+1}-2} (1 + x^2) (1 - x^{2^{k+1}}) \prod_{j>k} (1 - x^{2(2^j-1)}).$$

We need three definitions to complete the proof. Let

$$\begin{aligned} A_s &= \sum_{k \geq s} x^{2^{k+1}-2} (1 + x^2) (1 - x^{2^{k+1}}) \prod_{j>k} (1 - x^{2(2^j-1)}), \\ B_s &= (1 - x^{2^{s+1}}) \prod_{j \geq s} (1 - x^{2(2^j-1)}), \\ C_s &= x^{2(2^s-1)} (1 + x^2) (1 - x^{2^{s+1}}) \prod_{j>s} (1 - x^{2(2^j-1)}). \end{aligned}$$

We have two straightforward identities:

$$RHS = B_2 + A_2, \quad A_s = C_s + A_{s+1}.$$

We plan to show that

$$B_{s+1} = B_s + C_s.$$

This would give us

$$B_s + A_s = B_s + C_s + A_{s+1} = B_{s+1} + A_{s+1}.$$

We note that A_s has higher and higher powers of x in it and that $B_s - 1$ also has higher and higher powers of x in it. Take the limit as s goes to infinity and we see that the RHS = 1. We still have to show the inductive step. So, we compute $B_s + C_s$. We have

$$(1 - x^{2^{s+1}}) \prod_{j \geq s} (1 - x^{2(2^j-1)}) + x^{2(2^s-1)} (1 + x^2) (1 - x^{2^{s+1}}) \prod_{j>s} (1 - x^{2(2^j-1)}).$$

Factor out

$$\prod_{j>s} (1 - x^{2(2^j-1)})$$

to get

$$\left[(1 - x^{2^{s+1}}) (1 - x^{2(2^s-1)}) + x^{2(2^s-1)} (1 + x^2) (1 - x^{2^{s+1}}) \right] \prod_{j>s} (1 - x^{2(2^j-1)}).$$

Looking at what is in the brackets, we have

$$(1 - x^{2^{s+1}}) \left[(1 - x^{2(2^s-1)}) + x^{2(2^s-1)} (1 + x^2) \right] = (1 - x^{2^{s+1}}) (1 + x^{2^{s+1}}) = 1 - x^{2^{s+2}}$$

which is exactly what we needed to finish the proof. \square

We need to show that

Claim 7.2. The $\underline{BP}\langle k \rangle_{2^{k+1}+8u+4}$ of Theorem 2.7 are irreducible.

Proof. By Theorem 2.6(4), all we need to do is show that $2^{k+1} - 2 < 2^{k+1} + 8u + 4 \leq 2^{k+2} - 2$ when $2^{k-2} > u \geq 0$. This is a simple exercise. \square

Proof of Theorem 2.7. From Remark 5.3, we have the short exact sequence

$$\mathbb{Z}/(2) \rightarrow H_*(\underline{F}_6) \rightarrow H_*(\underline{BoP}_6) \rightarrow H_*(\underline{bo}_6) \rightarrow \mathbb{Z}/(2),$$

We have shown that \underline{F}_6 has no torsion in homology or homotopy, so Theorem 2.6(2) gives us the homotopy type of \underline{F}_6 as the $\underline{BP}\langle k \rangle_i$ part of Theorem 2.7. We need a map

$$\underline{BoP}_6 \rightarrow \underline{bo}_6 \times \prod_{2^{k-2} > u \geq 0} \underline{BP}\langle k \rangle_{2^{k+1}+8u+4}$$

that gives an isomorphism on homotopy. We have the map to \underline{bo}_6 . Using the proofs of Theorem 2.6 in [Wil75] we can construct our other maps. If we have a double no-torsion H-space, Z , to get a map to Y_k , it is enough to start with a map $Z \rightarrow K(\mathbb{Z}/(2), k)$. All the k -invariants are torsion (trivially true for any H-space) and so zero, so we can lift the map to $Z \rightarrow Y_k$. Picking our k_i appropriately, this gives the required map $Z \rightarrow \prod Y_{k_i}$. In our case, we have the maps from \underline{F}_6 , but by the short exact sequence, we can lift cohomology classes of \underline{F}_6 to \underline{BoP}_6 . \underline{BoP}_6 has no torsion in cohomology, so we can get our lifts to Y_k even though \underline{BoP}_6 has 2-torsion in homotopy. This now gives us our maps and our homotopy equivalence. \square

8. Discussion of the conjecture, Part 1

The easiest way to illustrate our evidence for our Conjecture 2.9 is to look at the $n = 1$ case

$$\underline{BoP}\langle 2 \rangle_6 \simeq \underline{bo}_6 \times \prod_{2^{k-2} \geq 1} \underline{BP}\langle k \rangle_{2^{k+2}-4}.$$

Since all the terms on the right-hand side are even degree and have no torsion in homology, we would want the property to hold if we delooped twice to get our $\underline{BoP}\langle 2 \rangle_8$. We know that the splitting is not as H-spaces though because the homology of \underline{bo}_6 is an exterior algebra. If our $\underline{BoP}\langle 2 \rangle_8$ exists, we need the squares of the elements in $H_*(\underline{bo}_6)$ to lie in the other terms. We know that they must be somewhere in our splitting for \underline{BoP}_6 , we just want them in these specific spaces.

The homology of $\underline{BP}\langle k \rangle_{2^{k+2}-4}$ is known to be polynomial from [Wil73], but we are going to need to get very technical and use the results of [RW77] where we write down generators for this homology. Since this is a speculative part of the paper, we will not go into the necessary lengthy review of [RW77] needed, but the interested reader can pursue this on their own.

We have, from Remark 4.9, $H_*(\underline{bo}_6) \simeq E[x_{2^j}], 2^j \neq 2^k$. If we write

$$j = 2^{s_1} + 2^{s_2} + \cdots + 2^{s_k}, \quad 0 \leq s_1 < s_2 < \cdots < s_k, \quad k > 1.$$

We conjecture that

$$x_{2j}^{*2} \simeq b_{(s_1)}^2 b_{(s_2-1)}^4 b_{(s_3-2)}^8 \cdots b_{(s_k-k+1)}^{2^k} \in H_*(\underline{BP}\langle k \rangle_{2^{k+2}-4}).$$

If we have this relation and double suspend to the homology of $\underline{BP}\langle k \rangle_{2^{k+2}-2}$, because suspension kills star products, the elements

$$b_{(s_0)} b_{(s_1)}^2 b_{(s_2-1)}^4 b_{(s_3-2)}^8 \cdots b_{(s_k-k+1)}^{2^k} \in \underline{BP}\langle k \rangle_{2^{k+2}-2} \quad \text{with } s_0 \leq s_1$$

would be zero. The reason this looks good is because the homology of $\underline{BP}\langle k \rangle_{2^{k+2}-2}$ is not polynomial. Better, using [RW77], we can see that the above elements are exactly the elements whose squares are zero, and we cannot have such elements in the homology of \underline{BoP}_8 because it is polynomial.

Every 8 deloopings we find ourselves in the same position and we have to incorporate more and more of the $\underline{BP}\langle k \rangle_i$ into our $\underline{BoP}\langle n \rangle_i$ to maintain polynomial algebras with no torsion.

9. Discussion of the conjecture, Part 2

The conjecture for $\overline{BP}\langle n \rangle$ follows from the conjecture for $H^*(BoP\langle n \rangle)$. This later conjecture comes from a hypothetical inductive (on n) computation based on the conjectured unstable splitting of $BoP\langle n \rangle$ in terms of $BoP\langle n-1 \rangle$ and the $BP\langle k \rangle$. It is what delayed the paper so long.

There are a few observations worth noting. First, as n goes to infinity, $H^*(BoP\langle n \rangle)$ goes to $H^*(BoP)$.

Next, if we look at the summand on degree $8q$, the first n it appears with is $n = 2^J + 1 - u$ if $q = 2^J + u$, $0 \leq u < 2^J$.

In conclusion, and enterprising individual might compute the homology of all our spaces giving names to the generators and then find the splitting using this information. It would seem to be a lot of work.

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