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HOMOTOPY CARTESIAN DIAGRAMS IN n-ANGULATED CATEGORIES

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Abstract

It has been proved by Bergh and Thaule that the higher mapping cone axiom is equivalent to the higher octahedral axiom for n-angulated categories. In this paper we use homotopy cartesian diagrams to give several new equivalent statements of the higher mapping cone axiom. As an application we give a new and elementary proof of the fact that the stable category of a Frobenius $(n-2)$-exact category is an n-angulated category, which was first proved by Jasso.

1. Introduction

Let n be an integer greater than or equal to three. The notion of n-angulated category was introduced by Geiss, Keller and Oppermann in [**5**] as the axiomatization of certain $(n-2)$-cluster tilting subcategories of triangulated categories. In particular, a 3-angulated category is a classical triangulated category. Examples of n-angulated categories can be found in [**5, 4, 8**]. Bergh and Thaule discussed the axioms of n-angulated categories in [**3**]. They showed that for n-angulated categories the higher mapping cone axiom is equivalent to the higher octahedral axiom.

The first aim and motivation of this paper is to understand the higher octahedral axiom. The n-angle induced by the higher octahedral axiom is very mysterious because it involves a lot of objects and morphisms. How do these objects and morphisms behave together? What are the morphisms of n-angles hidden in the higher octahedral axiom?

The second motivation is to discuss other equivalent statements of the higher mapping cone axiom. As is well known, there are quite a few equivalent statements of octahedral axiom such as the homotopy cartesian axiom, the base change axiom, the cobase change axiom and so on, which are used to construct triangles under varied conditions. Do their higher versions exist?

It turns out that homotopy cartesian diagrams provide a useful method to achieve our two goals. The notion of homotopy cartesian diagrams in n-angulated categories was first introduced in [**3**]. For triangulated case, homotopy cartesian square can be found in [**10, 9, 7**]. Since a homotopy cartesian square is the triangulated analogue

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of pushout and pullback square in an abelian category, inspired by the definition of $(n-2)$-pushout and $(n-2)$-pullback diagrams in $(n-2)$-abelian categories [**6**], we give an equivalent definition of homotopy cartesian diagrams to avoid dealing with the signs $(-1)^i$ in the n-angle (see Remark 2.4 for details). Then we prove that the higher mapping cone axiom is equivalent to the higher homotopy cartesian axiom, which is implied but not state explicitly in [**3**]. Using homotopy cartesian diagrams, we will give several other equivalent statements of the higher mapping cone axiom.

It has been proved by Jasso that the stable category $\underline{\mathcal{M}}$ of a Frobenius $(n-2)$-exact category $(\mathcal{M}, \mathcal{X})$ is an n-angulated category [**6**, Theorem 5.11]. We will give an alternative proof of [**6**, Theorem 5.11] by showing that the stable category $\underline{\mathcal{M}}$ satisfies the higher homotopy cartesian axiom. Our proof seems more elementary since the standard n-angles in $\underline{\mathcal{M}}$ are naturally induced by the $(n-2)$-exact sequences in $\mathcal{X}$ and the homotopy cartesian diagrams are naturally induced by the $(n-2)$-pushout and $(n-2)$-pullback diagrams.

This paper is organized as follows. In Section 2 we first recall the definition of n-angulated categories, then introduce the notion of homotopy cartesian diagrams and provide some needed facts. In Section 3 we use homotopy cartesian diagrams to give several new equivalent statements of the higher mapping cone axiom; see Theorem 3.1, Corollary 3.4 and Corollary 3.6. In Section 4 we apply the new characterizations of the higher mapping cone axiom to give a new and elementary proof of the fact that the stable category of a Frobenius $(n-2)$-exact category is an n-angulated category; see Theorem 4.7.

2. n-angulated categories and homotopy cartesian diagrams

Throughout this paper, we always assume that n is an integer greater than or equal to three. For convenience we recall the definition of n-angulated category from [**5**]. Let $\mathcal{C}$ be an additive category equipped with an automorphism $\Sigma\colon \mathcal{C} \to \mathcal{C}$. An n-Σ-*sequence* in $\mathcal{C}$ is a sequence of morphisms

$$X_\bullet = (X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \xrightarrow{f_3} \cdots \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} \Sigma X_1).$$

Its *left rotation* is the n-Σ-sequence

$$X_2 \xrightarrow{f_2} X_3 \xrightarrow{f_3} X_4 \xrightarrow{f_4} \cdots \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} \Sigma X_1 \xrightarrow{(-1)^n \Sigma f_1} \Sigma X_2.$$

An n-Σ-sequence $X_\bullet$ is *exact* if the induced sequence

$$\cdots \to \mathcal{C}(-, X_1) \to \mathcal{C}(-, X_2) \to \cdots \to \mathcal{C}(-, X_n) \to \mathcal{C}(-, \Sigma X_1) \to \cdots$$

is exact. A *morphism* of n-Σ-sequences is a sequence of morphisms $\varphi_\bullet = (\varphi_1, \varphi_2, \ldots, \varphi_n)$ such that the following diagram

$$\begin{array}{ccccccccccc}
X_1 & \xrightarrow{f_1} & X_2 & \xrightarrow{f_2} & X_3 & \xrightarrow{f_3} & \cdots & \xrightarrow{f_{n-1}} & X_n & \xrightarrow{f_n} & \Sigma X_1 \\
\downarrow{\scriptstyle \varphi_1} & & \downarrow{\scriptstyle \varphi_2} & & \downarrow{\scriptstyle \varphi_3} & & & & \downarrow{\scriptstyle \varphi_n} & & \downarrow{\scriptstyle \Sigma\varphi_1} \\
Y_1 & \xrightarrow{g_1} & Y_2 & \xrightarrow{g_2} & Y_3 & \xrightarrow{g_3} & \cdots & \xrightarrow{g_{n-1}} & Y_n & \xrightarrow{g_n} & \Sigma Y_1
\end{array}$$

commutes, where each row is an n-Σ-sequence. In this situation, we call $\varphi_\bullet$ a *weak isomorphism* if for some $1 \leqslant i \leqslant n$ both φ_i and φ_{i+1} (with $\varphi_{n+1} = \Sigma\varphi_1$) are isomorphisms. We call $\varphi_\bullet$ an *isomorphism* if $\varphi_1, \varphi_2, \ldots, \varphi_n$ are all isomorphisms.

Definition 2.1 ([**5**])**.** Let $\mathcal{C}$ be an additive category, Σ an automorphism of $\mathcal{C}$ and Θ a collection of n-Σ-sequences. We call $(\mathcal{C}, \Sigma, \Theta)$ a *pre-n-angulated category* and call the elements of Θ *n-angles* if Θ satisfies the following axioms:

(N1) (a) Θ is closed under isomorphisms, direct sums and direct summands.

(b) For each object $X \in \mathcal{C}$ the trivial sequence

$$(TX)_\bullet = (X \xrightarrow{1} X \to 0 \to \cdots \to 0 \to \Sigma X)$$

belongs to Θ.

(c) For each morphism $f_1\colon X_1 \to X_2$ in $\mathcal{C}$, there exists an n-Σ-sequence in Θ whose first morphism is f_1.

(N2) An n-Σ-sequence belongs to Θ if and only if its left rotation belongs to Θ.

(N3) Each commutative diagram

$$\begin{array}{ccccccccccc}
X_1 & \xrightarrow{f_1} & X_2 & \xrightarrow{f_2} & X_3 & \xrightarrow{f_3} & \cdots & \xrightarrow{f_{n-1}} & X_n & \xrightarrow{f_n} & \Sigma X_1 \\
\downarrow{\scriptstyle \varphi_1} & & \downarrow{\scriptstyle \varphi_2} & & \downarrow{\scriptstyle \varphi_3} & & & & \downarrow{\scriptstyle \varphi_n} & & \downarrow{\scriptstyle \Sigma\varphi_1} \\
Y_1 & \xrightarrow{g_1} & Y_2 & \xrightarrow{g_2} & Y_3 & \xrightarrow{g_3} & \cdots & \xrightarrow{g_{n-1}} & Y_n & \xrightarrow{g_n} & \Sigma Y_1
\end{array}$$

with rows in Θ can be completed to a morphism of n-Σ-sequences.

If Θ, moreover, satisfies the following axiom, then $(\mathcal{C}, \Sigma, \Theta)$ is called an *n-angulated category*:

(N4) In the situation of (N3), the morphisms $\varphi_3, \varphi_4, \ldots, \varphi_n$ can be chosen such that the mapping cone $C(\varphi_\bullet)$:

$$X_2 \oplus Y_1 \xrightarrow{\left(\begin{smallmatrix} -f_2 & 0 \\ \varphi_2 & g_1 \end{smallmatrix}\right)} X_3 \oplus Y_2 \xrightarrow{\left(\begin{smallmatrix} -f_3 & 0 \\ \varphi_3 & g_2 \end{smallmatrix}\right)} \cdots$$
$$\xrightarrow{\left(\begin{smallmatrix} -f_n & 0 \\ \varphi_n & g_{n-1} \end{smallmatrix}\right)} \Sigma X_1 \oplus Y_n \xrightarrow{\left(\begin{smallmatrix} -\Sigma f_1 & 0 \\ \Sigma\varphi_1 & g_n \end{smallmatrix}\right)} \Sigma X_2 \oplus \Sigma Y_1$$

belongs to Θ.

The following theorem shows that for n-angulated categories the higher mapping cone axiom is equivalent to the higher octahedral axiom.

Theorem 2.2 ([**3**, Theorem 4.4])**.** *Let $(\mathcal{C}, \Sigma, \Theta)$ be a pre-n-angulated category. Then Θ satisfies (N4) if and only if Θ satisfies (N4*):*

Given the following commutative diagram

$$\begin{array}{ccccccccccccccc}
X_1 & \xrightarrow{f_1} & X_2 & \xrightarrow{f_2} & X_3 & \xrightarrow{f_3} & X_4 & \xrightarrow{f_4} & \cdots & \xrightarrow{f_{n-2}} & X_{n-1} & \xrightarrow{f_{n-1}} & X_n & \xrightarrow{f_n} & \Sigma X_1 \\
\| & & \downarrow{\scriptstyle \varphi_2} & & \downarrow{\scriptstyle \varphi_3} & \swarrow & \downarrow{\scriptstyle \varphi_4} & & & & \downarrow{\scriptstyle \varphi_{n-1}} & \swarrow & \downarrow{\scriptstyle \varphi_n} & & \| \\
X_1 & \xrightarrow{g_1} & Y_2 & \xrightarrow{g_2} & Y_3 & \xrightarrow{g_3} & Y_4 & \xrightarrow{g_4} & \cdots & \xrightarrow{g_{n-2}} & Y_{n-1} & \xrightarrow{g_{n-1}} & Y_n & \xrightarrow{g_n} & \Sigma X_1 \\
\downarrow{\scriptstyle f_1} & & \| & & \downarrow{\scriptstyle \psi_3} & \swarrow{\scriptstyle \phi_4} & \downarrow{\scriptstyle \psi_4} & & & & \downarrow{\scriptstyle \psi_{n-1}} & \swarrow{\scriptstyle \phi_n} & \downarrow{\scriptstyle \psi_n} & & \downarrow{\scriptstyle \Sigma f_1} \\
X_2 & \xrightarrow{\varphi_2} & Y_2 & \xrightarrow{h_2} & Z_3 & \xrightarrow{h_3} & Z_4 & \xrightarrow{h_4} & \cdots & \xrightarrow{h_{n-2}} & Z_{n-1} & \xrightarrow{h_{n-1}} & Z_n & \xrightarrow{h_n} & \Sigma X_2
\end{array}$$

whose rows are n-angles, there exist morphisms $\varphi_i\colon X_i \to Y_i$ for $3 \leqslant i \leqslant n$, $\psi_j\colon Y_j \to Z_j$ for $3 \leqslant j \leqslant n$ and $\phi_k\colon X_k \to Z_{k-1}$ for $4 \leqslant k \leqslant n$ such that each square in the above diagram commutes and the following n-Σ-sequence

$$X_3 \xrightarrow{\begin{pmatrix} f_3\\ \varphi_3\end{pmatrix}} X_4\oplus Y_3 \xrightarrow{\begin{pmatrix}-f_4&0\\ \varphi_4&-g_3\\ \phi_4&\psi_3\end{pmatrix}} X_5\oplus Y_4\oplus Z_3 \xrightarrow{\begin{pmatrix}-f_5&0&0\\ -\varphi_5&-g_4&0\\ \phi_5&\psi_4&h_3\end{pmatrix}} X_6\oplus Y_5\oplus Z_4$$
$$\xrightarrow{\begin{pmatrix}-f_6&0&0\\ \varphi_6&-g_5&0\\ \phi_6&\psi_5&h_4\end{pmatrix}} \cdots \xrightarrow{\begin{pmatrix}-f_{n-1}&0&0\\ (-1)^{n-1}\varphi_{n-1}&-g_{n-2}&0\\ \phi_{n-1}&\psi_{n-2}&h_{n-3}\end{pmatrix}} X_n\oplus Y_{n-1}\oplus Z_{n-2} \tag{1}$$
$$\xrightarrow{\begin{pmatrix}(-1)^n\varphi_n&-g_{n-1}&0\\ \phi_n&\psi_{n-1}&h_{n-2}\end{pmatrix}} Y_n\oplus Z_{n-1}\xrightarrow{(\psi_n\ h_{n-1})} Z_n \xrightarrow{\Sigma f_2\cdot h_n}\Sigma X_3$$

belongs to Θ.

In the rest of this section we assume that all n-angles are in a pre-n-angulated category $(\mathcal{C}, \Sigma, \Theta)$.

Definition 2.3 (cf. [**3**])**.** The following commutative diagram

$$\begin{array}{ccccccccc}
X_1 & \xrightarrow{f_1} & X_2 & \xrightarrow{f_2} & \cdots & \xrightarrow{f_{n-3}} & X_{n-2} & \xrightarrow{f_{n-2}} & X_{n-1}\\
\downarrow{\scriptstyle \varphi_1} & & \downarrow{\scriptstyle \varphi_2} & & & & \downarrow{\scriptstyle \varphi_{n-2}} & & \downarrow{\scriptstyle \varphi_{n-1}}\\
Y_1 & \xrightarrow{g_1} & Y_2 & \xrightarrow{g_2} & \cdots & \xrightarrow{g_{n-3}} & Y_{n-2} & \xrightarrow{g_{n-2}} & Y_{n-1}
\end{array}$$

in a pre-n-angulated category $\mathcal{C}$ is called a *homotopy cartesian diagram* if the following n-Σ-sequence

$$X_1 \xrightarrow{\begin{pmatrix}-f_1\\ \varphi_1\end{pmatrix}} X_2\oplus Y_1 \xrightarrow{\begin{pmatrix}-f_2&0\\ \varphi_2&g_1\end{pmatrix}} X_3\oplus Y_2 \xrightarrow{\begin{pmatrix}-f_3&0\\ \varphi_3&g_2\end{pmatrix}} \cdots$$
$$\cdots \xrightarrow{\begin{pmatrix}-f_{n-2}&0\\ \varphi_{n-2}&g_{n-3}\end{pmatrix}} X_{n-1}\oplus Y_{n-2} \xrightarrow{(\varphi_{n-1}\ g_{n-2})} Y_{n-1} \xrightarrow{\partial} \Sigma X_1 \tag{2}$$

is an n-angle for some morphism $\partial\colon Y_{n-1} \to \Sigma X_1$, where ∂ is called a *differential*.

Remark 2.4. (a) The n-angle (2) in the definition of homotopy cartesian diagram is slightly different from the one in the definition given in [**3**]. But the two n-angles are isomorphic:

$$\begin{array}{ccccccc}
X_1 & \xrightarrow{\begin{pmatrix}-f_1\\ \varphi_1\end{pmatrix}} & X_2\oplus Y_1 & \xrightarrow{\begin{pmatrix}-f_2&0\\ \varphi_2&g_1\end{pmatrix}} & X_3\oplus Y_2 & \xrightarrow{\begin{pmatrix}-f_3&0\\ \varphi_3&g_2\end{pmatrix}} & \cdots\\
\| & & \| & & \downarrow{\scriptstyle \begin{pmatrix}-1&0\\ 0&1\end{pmatrix}} & & \\
X_1 & \xrightarrow{\begin{pmatrix}-f_1\\ \varphi_1\end{pmatrix}} & X_2\oplus Y_1 & \xrightarrow{\begin{pmatrix}f_2&0\\ \varphi_2&g_1\end{pmatrix}} & X_3\oplus Y_2 & \xrightarrow{\begin{pmatrix}f_3&0\\ -\varphi_3&g_2\end{pmatrix}} & \cdots
\end{array}$$

$$\begin{array}{ccccccc}
\cdots & \xrightarrow{\begin{pmatrix}-f_{n-2}&0\\ \varphi_{n-2}&g_{n-3}\end{pmatrix}} & X_{n-1}\oplus Y_{n-2} & \xrightarrow{(\varphi_{n-1}\ g_{n-2})} & Y_{n-1} & \xrightarrow{\partial} & \Sigma X_1\\
 & & \downarrow{\scriptstyle \begin{pmatrix}(-1)^{n+1}&0\\ 0&1\end{pmatrix}} & & \| & & \|\\
\cdots & \xrightarrow{\begin{pmatrix}f_{n-2}&0\\ (-1)^n\varphi_{n-2}&g_{n-3}\end{pmatrix}} & X_{n-1}\oplus Y_{n-2} & \xrightarrow{((-1)^{n+1}\varphi_{n-1}\ g_{n-2})} & Y_{n-1} & \xrightarrow{\partial} & \Sigma X_1
\end{array}$$

(b) Since a homotopy cartesian square is the triangulated analogue of a pullback and pushout square in an abelian category, we can compare our definition with the notion of an $(n-2)$-pushout and $(n-2)$-pullback diagram in an $(n-2)$-abelian category [**6**].

Lemma 2.5 ([**5**, Lemma 2.4, Lemma 2.5])**.** *Let $(\mathcal{C}, \Sigma, \Theta)$ be a pre-n-angulated category, then the following hold:*

(a) All n-angles are exact.

(b) Let $\varphi_\bullet \colon X_\bullet \to Y_\bullet$ be a weak isomorphism of exact n-Σ-sequences. Then $X_\bullet$ is an n-angle if and only if $Y_\bullet$ is an n-angle.

Lemma 2.6. *Let*

$$\begin{array}{ccccccccccccc}
X_\bullet & & X_1 & \xrightarrow{f_1} & X_2 & \xrightarrow{f_2} & X_3 & \xrightarrow{f_3} & \cdots & \xrightarrow{f_{n-1}} & X_n & \xrightarrow{f_n} & \Sigma X_1 \\
\downarrow{\scriptstyle \varphi_\bullet} & & \| & & \downarrow{\scriptstyle \varphi_2} & & \downarrow{\scriptstyle \varphi_3} & & & & \downarrow{\scriptstyle \varphi_n} & & \| \\
Y_\bullet & & X_1 & \xrightarrow{g_1} & Y_2 & \xrightarrow{g_2} & Y_3 & \xrightarrow{g_3} & \cdots & \xrightarrow{g_{n-1}} & Y_n & \xrightarrow{g_n} & \Sigma X_1
\end{array}$$

be a morphism of n-angles with $X_1 = Y_1$. Then as an n-Σ-sequence, the mapping cone $C(\varphi_\bullet)$ is isomorphic to $M(\varphi_\bullet) \oplus (X_1)_\bullet$, where

$$M(\varphi_\bullet) = (X_2 \xrightarrow{\begin{pmatrix} -f_2 \\ \varphi_2 \end{pmatrix}} X_3 \oplus Y_2 \xrightarrow{\begin{pmatrix} -f_3 & 0 \\ \varphi_3 & g_2 \end{pmatrix}} \cdots \xrightarrow{\begin{pmatrix} -f_{n-1} & 0 \\ \varphi_{n-1} & g_{n-2} \end{pmatrix}} X_n \oplus Y_{n-1}$$
$$\xrightarrow{(\varphi_n \ g_{n-1})} Y_n \xrightarrow{\Sigma f_1 \cdot g_n} \Sigma X_2), \quad (X_1)_\bullet = (X_1 \to 0 \to \cdots \to 0 \to \Sigma X_1 \xrightarrow{1} \Sigma X_1).$$

In particular, $C(\varphi_\bullet)$ is exact if and only if $M(\varphi_\bullet)$ is exact.

Proof. It is easy to check the following isomorphism of n-Σ-sequences:

$$\begin{array}{lccccc}
C(\varphi_\bullet): & X_2 \oplus X_1 & \xrightarrow{\begin{pmatrix} -f_2 & 0 \\ \varphi_2 & g_1 \end{pmatrix}} & X_3 \oplus Y_2 & \xrightarrow{\begin{pmatrix} -f_3 & 0 \\ \varphi_3 & g_2 \end{pmatrix}} & \cdots \\
 & \downarrow{\scriptstyle \begin{pmatrix} 1 & f_1 \\ 0 & 1 \end{pmatrix}} & & \| & & \\
M(\varphi_\bullet) \oplus (X_1)_\bullet: & X_2 \oplus X_1 & \xrightarrow{\begin{pmatrix} -f_2 & 0 \\ \varphi_2 & 0 \end{pmatrix}} & X_3 \oplus Y_2 & \xrightarrow{\begin{pmatrix} -f_3 & 0 \\ \varphi_3 & g_2 \end{pmatrix}} & \cdots
\end{array}$$

$$\begin{array}{ccccccc}
\cdots \xrightarrow{\begin{pmatrix} -f_{n-1} & 0 \\ \varphi_{n-1} & g_{n-2} \end{pmatrix}} & X_n \oplus Y_{n-1} & \xrightarrow{\begin{pmatrix} -f_n & 0 \\ \varphi_n & g_{n-1} \end{pmatrix}} & \Sigma X_1 \oplus Y_n & \xrightarrow{\begin{pmatrix} -\Sigma f_1 & 0 \\ 1 & g_n \end{pmatrix}} & \Sigma X_2 \oplus \Sigma X_1 \\
 & \| & & \downarrow{\scriptstyle \begin{pmatrix} 0 & 1 \\ 1 & g_n \end{pmatrix}} & & \downarrow{\scriptstyle \begin{pmatrix} 1 & \Sigma f_1 \\ 0 & 1 \end{pmatrix}} \\
\cdots \xrightarrow{\begin{pmatrix} -f_{n-1} & 0 \\ \varphi_{n-1} & g_{n-2} \end{pmatrix}} & X_n \oplus Y_{n-1} & \xrightarrow{\begin{pmatrix} \varphi_n & g_{n-1} \\ 0 & 0 \end{pmatrix}} & Y_n \oplus \Sigma X_1 & \xrightarrow{\begin{pmatrix} \Sigma f_1 \cdot g_n & 0 \\ 0 & 1 \end{pmatrix}} & \Sigma X_2 \oplus \Sigma X_1
\end{array}$$

We note that $(X_1)_\bullet$ is an n-angle since it is isomorphic to

$$X_1 \to 0 \to \cdots \to 0 \to \Sigma X_1 \xrightarrow{(-1)^n} \Sigma X_1.$$

Thus $C(\varphi_\bullet)$ is exact if and only if $M(\varphi_\bullet)$ is exact. □

Lemma 2.7. *Let*

$$\begin{array}{ccccccccccccc}
X_\bullet & & X_1 & \xrightarrow{f_1} & X_2 & \xrightarrow{f_2} & X_3 & \xrightarrow{f_3} \cdots \xrightarrow{f_{n-1}} & X_n & \xrightarrow{f_n} & \Sigma X_1 \\
\downarrow{\scriptstyle \varphi_\bullet} & & \| & & \downarrow{\scriptstyle \varphi_2} & & \downarrow{\scriptstyle \varphi_3} & & \downarrow{\scriptstyle \varphi_n} & & \| \\
Y_\bullet & & X_1 & \xrightarrow{g_1} & Y_2 & \xrightarrow{g_2} & Y_3 & \xrightarrow{g_3} \cdots \xrightarrow{g_{n-1}} & Y_n & \xrightarrow{g_n} & \Sigma X_1 \\
\downarrow{\scriptstyle \psi_\bullet} & & \| & & \| & & \downarrow{\scriptstyle \psi_3} & & \downarrow{\scriptstyle \psi_n} & & \| \\
Z_\bullet & & X_1 & \xrightarrow{g_1} & Y_2 & \xrightarrow{h_2} & Z_3 & \xrightarrow{h_3} \cdots \xrightarrow{h_{n-1}} & Z_n & \xrightarrow{h_n} & \Sigma X_1
\end{array}$$

be a commutative diagram whose rows are n-angles. Then

$$\begin{array}{ccccc}
X_2 & \xrightarrow{f_2} & X_3 & \xrightarrow{f_3} \cdots \xrightarrow{f_{n-1}} & X_n \\
\downarrow{\scriptstyle \varphi_2} & & \downarrow{\scriptstyle \varphi_3} & & \downarrow{\scriptstyle \varphi_n} \\
Y_2 & \xrightarrow{g_2} & Y_3 & \xrightarrow{g_3} \cdots \xrightarrow{g_{n-1}} & Y_n
\end{array}$$

is a homotopy cartesian diagram with the differential $\Sigma f_1 \cdot g_n$ if and only if

$$\begin{array}{ccccc}
X_2 & \xrightarrow{f_2} & X_3 & \xrightarrow{f_3} \cdots \xrightarrow{f_{n-1}} & X_n \\
\downarrow{\scriptstyle \varphi_2} & & \downarrow{\scriptstyle \psi_3\varphi_3} & & \downarrow{\scriptstyle \psi_n\varphi_n} \\
Y_2 & \xrightarrow{h_2} & Z_3 & \xrightarrow{h_3} \cdots \xrightarrow{h_{n-1}} & Z_n
\end{array}$$

is a homotopy cartesian diagram with the differential $\Sigma f_1 \cdot h_n$.

Proof. By Lemma 2.5(a), the n-angles $X_\bullet, Y_\bullet$ and $Z_\bullet$ are exact. Since the class of exact n-Σ-sequences is closed under mapping cones, we obtain that the mapping cones $C(\varphi_\bullet)$ and $C(\psi_\bullet\varphi_\bullet)$ are also exact. It is easy to see that we have the following commutative diagram

$$\begin{array}{ccccccccc}
M(\varphi_\bullet): & X_2 & \xrightarrow{\left(\begin{smallmatrix} -f_2 \\ \varphi_2 \end{smallmatrix}\right)} & X_3 \oplus Y_2 & \xrightarrow{\left(\begin{smallmatrix} -f_3 & 0 \\ \varphi_3 & g_2 \end{smallmatrix}\right)} & X_4 \oplus Y_3 & \xrightarrow{\left(\begin{smallmatrix} -f_4 & 0 \\ \varphi_4 & g_3 \end{smallmatrix}\right)} & \cdots \\
 & \| & & \| & & \downarrow{\scriptstyle \left(\begin{smallmatrix} 1 & 0 \\ 0 & \psi_3 \end{smallmatrix}\right)} & & \\
M(\psi_\bullet\varphi_\bullet): & X_2 & \xrightarrow{\left(\begin{smallmatrix} -f_2 \\ \varphi_2 \end{smallmatrix}\right)} & X_3 \oplus Y_2 & \xrightarrow{\left(\begin{smallmatrix} -f_3 & 0 \\ \psi_3\varphi_3 & h_2 \end{smallmatrix}\right)} & X_4 \oplus Z_3 & \xrightarrow{\left(\begin{smallmatrix} -f_4 & 0 \\ \psi_4\varphi_4 & h_3 \end{smallmatrix}\right)} & \cdots
\end{array}$$

$$\begin{array}{ccccccc}
\cdots \xrightarrow{\left(\begin{smallmatrix} -f_{n-1} & 0 \\ \varphi_{n-1} & g_{n-2} \end{smallmatrix}\right)} & X_n \oplus Y_{n-1} & \xrightarrow{(\varphi_n, g_{n-1})} & Y_n & \xrightarrow{\Sigma f_1 \cdot g_n} & \Sigma X_2 \\
 & \downarrow{\scriptstyle \left(\begin{smallmatrix} 1 & 0 \\ 0 & \psi_{n-1} \end{smallmatrix}\right)} & & \downarrow{\scriptstyle \psi_n} & & \| \\
\cdots \xrightarrow{\left(\begin{smallmatrix} -f_{n-1} & 0 \\ \psi_{n-1}\varphi_{n-1} & h_{n-2} \end{smallmatrix}\right)} & X_n \oplus Z_{n-1} & \xrightarrow{(\psi_n\varphi_n, h_{n-1})} & Z_n & \xrightarrow{\Sigma f_1 \cdot h_n} & \Sigma X_2
\end{array}$$

whose rows are exact n-Σ-sequences by Lemma 2.6. The result holds by definition and Lemma 2.5(b). □

3. Equivalent statements of higher mapping cone axiom

In this section we provide some equivalent statements of higher mapping cone axiom to explain the higher octahedral axiom. We leave the dual statements to the reader.

Theorem 3.1. *Let $(\mathcal{C}, \Sigma, \Theta)$ be a pre-n-angulated category. Then Θ satisfies (N4) if and only if Θ satisfies (N4-1):*

Given a commutative diagram

$$\begin{array}{ccccccccccc}
X_1 & \xrightarrow{f_1} & X_2 & \xrightarrow{f_2} & X_3 & \xrightarrow{f_3} & \cdots & \xrightarrow{f_{n-1}} & X_n & \xrightarrow{f_n} & \Sigma X_1 \\
\| & & \downarrow{\scriptstyle \varphi_2} & & \downarrow{\scriptstyle \varphi_3} & & & & \downarrow{\scriptstyle \varphi_n} & & \| \\
X_1 & \xrightarrow{g_1} & Y_2 & \xrightarrow{g_2} & Y_3 & \xrightarrow{g_3} & \cdots & \xrightarrow{g_{n-1}} & Y_n & \xrightarrow{g_n} & \Sigma X_1
\end{array}$$

whose rows are n-angles, there exist morphisms $\varphi_i\colon X_i \to Y_i$ for $3 \leqslant i \leqslant n$ such that the above diagram is commutative and the following

$$\begin{array}{ccccccc}
X_2 & \xrightarrow{f_2} & X_3 & \xrightarrow{f_3} & \cdots & \xrightarrow{f_{n-1}} & X_n \\
\downarrow{\scriptstyle \varphi_2} & & \downarrow{\scriptstyle \varphi_3} & & & & \downarrow{\scriptstyle \varphi_n} \\
Y_2 & \xrightarrow{g_2} & Y_3 & \xrightarrow{g_3} & \cdots & \xrightarrow{g_{n-1}} & Y_n
\end{array}$$

is a homotopy cartesian diagram whose differential is $\Sigma f_1 \cdot g_n$.

Proof. Assume that Θ satisfies (N4), then there exist morphisms $\varphi_i\colon X_i \to Y_i$ for $3 \leqslant i \leqslant n$ such that the mapping cone $C(\varphi_\bullet)$ is an n-angle. Since the class of n-angles is closed under direct summands, the remaining part of (N4-1) follows from Lemma 2.6.

Conversely, we assume that Θ satisfies (N4-1). Given a commutative diagram

$$\begin{array}{cccccccccccc}
X_\bullet\colon & X_1 & \xrightarrow{f_1} & X_2 & \xrightarrow{f_2} & X_3 & \xrightarrow{f_3} & \cdots & \xrightarrow{f_{n-1}} & X_n & \xrightarrow{f_n} & \Sigma X_1 \\
& \downarrow{\scriptstyle \varphi_1} & & \downarrow{\scriptstyle \varphi_2} & & & & & & & & \downarrow{\scriptstyle \Sigma\varphi_1} \\
Y_\bullet\colon & Y_1 & \xrightarrow{g_1} & Y_2 & \xrightarrow{g_2} & Y_3 & \xrightarrow{g_3} & \cdots & \xrightarrow{g_{n-1}} & Y_n & \xrightarrow{g_n} & \Sigma Y_1
\end{array}$$

whose rows are n-angles. The following commutative diagram

$$\begin{array}{cccccccc}
Y'_\bullet\colon & X_1 \oplus Y_1 & \xrightarrow{(\varphi_2 f_1\ g_1)} & Y_2 & \xrightarrow{g_2} & Y_3 & \xrightarrow{g_3} & \cdots \\
& \downarrow{\scriptstyle \left(\begin{smallmatrix} 1 & 0 \\ \varphi_1 & 1 \end{smallmatrix}\right)} & & \| & & \| & & \\
(X_1)_\bullet \oplus Y_\bullet\colon & X_1 \oplus Y_1 & \xrightarrow{(0\ g_1)} & Y_2 & \xrightarrow{g_2} & Y_3 & \xrightarrow{g_3} & \cdots
\end{array}$$

$$\begin{array}{ccccccc}
\cdots & \xrightarrow{g_{n-2}} & Y_{n-1} & \xrightarrow{\left(\begin{smallmatrix} 0 \\ g_{n-1} \end{smallmatrix}\right)} & \Sigma X_1 \oplus Y_n & \xrightarrow{\left(\begin{smallmatrix} -1 & 0 \\ \Sigma\varphi_1 & g_n \end{smallmatrix}\right)} & \Sigma X_1 \oplus \Sigma Y_1 \\
& & \| & & \| & & \downarrow{\scriptstyle \left(\begin{smallmatrix} 1 & 0 \\ \Sigma\varphi_1 & 1 \end{smallmatrix}\right)} \\
\cdots & \xrightarrow{g_{n-2}} & Y_{n-1} & \xrightarrow{\left(\begin{smallmatrix} 0 \\ g_{n-1} \end{smallmatrix}\right)} & \Sigma X_1 \oplus Y_n & \xrightarrow{\left(\begin{smallmatrix} -1 & 0 \\ 0 & g_n \end{smallmatrix}\right)} & \Sigma X_1 \oplus \Sigma Y_1
\end{array}$$

shows that $Y'_\bullet$ is an n-angle since $(X_1)_\bullet \oplus Y_\bullet$ is an n-angle. By (N4-1), the following diagram

$$
\begin{array}{ccccccc}
X_1 \oplus Y_1 & \xrightarrow{\left(\begin{smallmatrix} f_1 & 0 \\ 0 & 1 \end{smallmatrix}\right)} & X_2 \oplus Y_1 & \xrightarrow{(\, f_2 \;\; 0\,)} & X_3 & \xrightarrow{f_3} & \cdots \\
\| & & \downarrow{\scriptstyle (\,\varphi_2 \;\; g_1\,)} & & \downarrow{\scriptstyle \varphi_3} & & \\
X_1 \oplus Y_1 & \xrightarrow{(\,\varphi_2 f_1 \;\; g_1\,)} & Y_2 & \xrightarrow{g_2} & Y_3 & \xrightarrow{g_3} & \cdots
\end{array}
$$

$$
\begin{array}{ccccccc}
\cdots & \xrightarrow{f_{n-2}} & X_{n-1} & \xrightarrow{f_{n-1}} & X_n & \xrightarrow{\left(\begin{smallmatrix} f_n \\ 0 \end{smallmatrix}\right)} & \Sigma X_1 \oplus \Sigma Y_1 \\
 & & \downarrow{\scriptstyle \varphi_{n-1}} & & \downarrow{\scriptstyle \left(\begin{smallmatrix} \varphi'_n \\ \varphi_n \end{smallmatrix}\right)} & & \| \\
\cdots & \xrightarrow{g_{n-2}} & Y_{n-1} & \xrightarrow{\left(\begin{smallmatrix} 0 \\ g_{n-1} \end{smallmatrix}\right)} & \Sigma X_1 \oplus Y_n & \xrightarrow{\left(\begin{smallmatrix} -1 & 0 \\ \Sigma\varphi_1 & g_n \end{smallmatrix}\right)} & \Sigma X_1 \oplus \Sigma Y_1
\end{array}
$$

can be completed to a morphism of n-angles such that the following sequence

$$
X_2 \oplus Y_1 \xrightarrow{\left(\begin{smallmatrix} -f_2 & 0 \\ \varphi_2 & g_1 \end{smallmatrix}\right)} X_3 \oplus Y_2 \xrightarrow{\left(\begin{smallmatrix} -f_3 & 0 \\ \varphi_3 & g_2 \end{smallmatrix}\right)} X_4 \oplus Y_3 \xrightarrow{\left(\begin{smallmatrix} -f_4 & 0 \\ \varphi_4 & g_3 \end{smallmatrix}\right)} \cdots
$$

$$
\cdots \xrightarrow{\left(\begin{smallmatrix} -f_{n-1} & 0 \\ \varphi_{n-1} & g_{n-2} \end{smallmatrix}\right)} X_n \oplus Y_{n-1} \xrightarrow{\left(\begin{smallmatrix} \varphi'_n & 0 \\ \varphi_n & g_{n-1} \end{smallmatrix}\right)} \Sigma X_1 \oplus Y_n \xrightarrow{\left(\begin{smallmatrix} -\Sigma f_1 & 0 \\ \Sigma\varphi_1 & g_n \end{smallmatrix}\right)} \Sigma X_2 \oplus \Sigma Y_1
$$

is an n-angle, where $\varphi'_n = -f_n$ by the commutativity of the above rightmost square. □

Since the proof of (N4-1) ⇒ (N4) does not use (N3), we immediately obtain the following corollary.

Corollary 3.2. *Let $\mathcal{C}$ be an additive category with an automorphism Σ and Θ be a class of n-Σ-sequences. Then $(\mathcal{C}, \Sigma, \Theta)$ is an n-angulated category if and only if Θ satisfies (N1), (N2) and (N4-1).*

Remark 3.3. We can compare Corollary 3.2 with [**1**, Theorem 3.1], where the author proved that $(\mathcal{C}, \Sigma, \Theta)$ is an n-angulated category if and only if Θ satisfies (N1), (N2) and (N4*).

Corollary 3.4. *Let $(\mathcal{C}, \Sigma, \Theta)$ be a pre-n-angulated category. Then the following statements are equivalent:*

(a) Θ satisfies (N4).

(b) Θ satisfies (N4-2):

Given an n-angle $X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} \Sigma X_1$ and a morphism $\varphi_1 \colon X_1 \to Y_1$, there exists a commutative diagram

$$
\begin{array}{ccccccccccc}
X_1 & \xrightarrow{f_1} & X_2 & \xrightarrow{f_2} & \cdots & \xrightarrow{f_{n-2}} & X_{n-1} & \xrightarrow{f_{n-1}} & X_n & \xrightarrow{f_n} & \Sigma X_1 \\
\downarrow{\scriptstyle \varphi_1} & & \downarrow{\scriptstyle \varphi_2} & & & & \downarrow{\scriptstyle \varphi_{n-1}} & & \| & & \downarrow{\scriptstyle \Sigma\varphi_1} \\
Y_1 & \xrightarrow{g_1} & Y_2 & \xrightarrow{g_2} & \cdots & \xrightarrow{g_{n-2}} & Y_{n-1} & \xrightarrow{g_{n-1}} & X_n & \xrightarrow{\Sigma\varphi_1 \cdot f_n} & \Sigma Y_1
\end{array}
$$

such that the second row is an n-angle and

$$\begin{array}{ccccccc}
X_1 & \xrightarrow{f_1} & X_2 & \xrightarrow{f_2} & \cdots & \xrightarrow{f_{n-2}} & X_{n-1} \\
\downarrow{\scriptstyle \varphi_1} & & \downarrow{\scriptstyle \varphi_2} & & & & \downarrow{\scriptstyle \varphi_{n-1}} \\
Y_1 & \xrightarrow{g_1} & Y_2 & \xrightarrow{g_2} & \cdots & \xrightarrow{g_{n-2}} & Y_{n-1}
\end{array}$$

is a homotopy cartesian diagram where $(-1)^n f_n \cdot g_{n-1}$ is the differential.

(c) Θ satisfies (N4-3):

Given two morphisms $f_1\colon X_1 \to X_2$ and $\varphi_2\colon X_2 \to Y_2$, there exists a commutative diagram

$$\begin{array}{ccccccccccc}
X_1 & \xrightarrow{f_1} & X_2 & \dashrightarrow^{f_2} & X_3 & \dashrightarrow^{f_3} & \cdots & \dashrightarrow^{f_{n-1}} & X_n & \dashrightarrow^{f_n} & \Sigma X_1 \\
\| & & \downarrow{\scriptstyle \varphi_2} & & \downarrow{\scriptstyle \varphi_3} & & & & \downarrow{\scriptstyle \varphi_n} & & \| \\
X_1 & \xrightarrow{\varphi_2 f_1} & Y_2 & \dashrightarrow^{g_2} & Y_3 & \dashrightarrow^{g_3} & \cdots & \dashrightarrow^{g_{n-1}} & Y_n & \dashrightarrow^{g_n} & \Sigma X_1
\end{array}$$

such that each row is an n-angle and

$$\begin{array}{ccccccc}
X_2 & \xrightarrow{f_2} & X_3 & \xrightarrow{f_3} & \cdots & \xrightarrow{f_{n-1}} & X_n \\
\downarrow{\scriptstyle \varphi_2} & & \downarrow{\scriptstyle \varphi_3} & & & & \downarrow{\scriptstyle \varphi_n} \\
Y_2 & \xrightarrow{g_2} & Y_3 & \xrightarrow{g_3} & \cdots & \xrightarrow{g_{n-1}} & Y_n
\end{array}$$

is a homotopy cartesian diagram where $\Sigma f_1 \cdot g_n$ is the differential.

Proof. (a) $\Rightarrow$ (b). By (N2) and (N1)(c), we have the following commutative diagram

$$\begin{array}{ccccccccccc}
\Sigma^{-1}X_n & \xrightarrow{(-1)^n\Sigma^{-1}f_n} & X_1 & \xrightarrow{f_1} & X_2 & \xrightarrow{f_2} & \cdots & \xrightarrow{f_{n-2}} & X_{n-1} & \xrightarrow{f_{n-1}} & X_n \\
\| & & \downarrow{\scriptstyle \varphi_1} & & & & & & & & \| \\
\Sigma^{-1}X_n & \xrightarrow{(-1)^n\varphi_1\cdot\Sigma^{-1}f_n} & Y_1 & \xrightarrow{g_1} & Y_2 & \xrightarrow{g_2} & \cdots & \xrightarrow{g_{n-2}} & Y_{n-1} & \xrightarrow{g_{n-1}} & X_n
\end{array}$$

whose rows are n-angles. Now (b) follows from (N4-1) and Theorem 3.1.

(b) $\Rightarrow$ (c). For the morphism $f_1\colon X_1 \to X_2$, by (N1)(c) and (N2) we assume that $X_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} \Sigma X_1 \xrightarrow{(-1)^n\Sigma f_1} \Sigma X_2$ is an n-angle. Then (c) follows from (N4-2).

(c) $\Rightarrow$ (a). Given a commutative diagram

$$\begin{array}{ccccccccccc}
X_1 & \xrightarrow{f_1} & X_2 & \xrightarrow{f_2} & X_3 & \xrightarrow{f_3} & \cdots & \xrightarrow{f_{n-1}} & X_n & \xrightarrow{f_n} & \Sigma X_1 \\
\| & & \downarrow{\scriptstyle \varphi_2} & & & & & & & & \| \\
X_1 & \xrightarrow{g_1} & Y_2 & \xrightarrow{g_2} & Y_3 & \xrightarrow{g_3} & \cdots & \xrightarrow{g_{n-1}} & Y_n & \xrightarrow{g_n} & \Sigma X_1
\end{array}$$

whose rows are n-angles. By (c), there exists a commutative diagram

$$\begin{array}{ccccccccccc}
X_1 & \xrightarrow{f_1} & X_2 & \dashrightarrow^{f'_2} & X'_3 & \dashrightarrow^{f'_3} & \cdots & \dashrightarrow^{f'_{n-1}} & X'_n & \dashrightarrow^{f'_n} & \Sigma X_1 \\
\| & & \downarrow{\scriptstyle \varphi_2} & & \downarrow{\scriptstyle \varphi'_3} & & & & \downarrow{\scriptstyle \varphi'_n} & & \| \\
X_1 & \xrightarrow{g_1} & Y_2 & \dashrightarrow^{g'_2} & Y'_3 & \dashrightarrow^{g'_3} & \cdots & \dashrightarrow^{g'_{n-1}} & Y'_n & \dashrightarrow^{g'_n} & \Sigma X_1
\end{array}$$

such that each row is an n-angle and

$$\begin{array}{ccccccc} X_2 & \xrightarrow{f_2'} & X_3' & \xrightarrow{f_3'} & \cdots & \xrightarrow{f_{n-1}'} & X_n' \\ \downarrow{\scriptstyle\varphi_2} & & \downarrow{\scriptstyle\varphi_3'} & & & & \downarrow{\scriptstyle\varphi_n'} \\ Y_2 & \xrightarrow{g_2'} & Y_3' & \xrightarrow{g_3'} & \cdots & \xrightarrow{g_{n-1}'} & Y_n' \end{array}$$

is a homotopy cartesian diagram where $\Sigma f_1 \cdot g_n'$ is the differential. It follows from (N3) that we have the following commutative diagram

$$\begin{array}{ccccccccccccc} X_\bullet & & X_1 & \xrightarrow{f_1} & X_2 & \xrightarrow{f_2} & X_3 & \xrightarrow{f_3} & \cdots & \xrightarrow{f_{n-1}} & X_n & \xrightarrow{f_n} & \Sigma X_1 \\ \downarrow{\scriptstyle\theta_\bullet} & & \| & & \| & & \downarrow{\scriptstyle\theta_3} & & & & \downarrow{\scriptstyle\theta_n} & & \| \\ X_\bullet' & & X_1 & \xrightarrow{f_1} & X_2 & \xrightarrow{f_2'} & X_3' & \xrightarrow{f_3'} & \cdots & \xrightarrow{f_{n-1}'} & X_n' & \xrightarrow{f_n'} & \Sigma X_1 \\ \downarrow{\scriptstyle\varphi_\bullet'} & & \| & & \downarrow{\scriptstyle\varphi_2} & & \downarrow{\scriptstyle\varphi_3'} & & & & \downarrow{\scriptstyle\varphi_n'} & & \| \\ Y_\bullet' & & X_1 & \xrightarrow{g_1} & Y_2 & \xrightarrow{g_2'} & Y_3' & \xrightarrow{g_3'} & \cdots & \xrightarrow{g_{n-1}'} & Y_n' & \xrightarrow{g_n'} & \Sigma X_1 \\ \downarrow{\scriptstyle\psi_\bullet} & & \| & & \| & & \downarrow{\scriptstyle\psi_3} & & & & \downarrow{\scriptstyle\psi_n} & & \| \\ Y_\bullet & & X_1 & \xrightarrow{g_1} & Y_2 & \xrightarrow{g_2} & Y_3 & \xrightarrow{g_3} & \cdots & \xrightarrow{g_{n-1}} & Y_n & \xrightarrow{g_n} & \Sigma X_1 \end{array}$$

whose rows are n-angles. Lemma 2.7 and its dual imply that

$$\begin{array}{ccccccc} X_2 & \xrightarrow{f_2} & X_3 & \xrightarrow{f_3} & \cdots & \xrightarrow{f_{n-1}} & X_n \\ \downarrow{\scriptstyle\varphi_2} & & \downarrow{\scriptstyle\psi_3\varphi_3'\theta_3} & & & & \downarrow{\scriptstyle\psi_n\varphi_n'\theta_n} \\ Y_2 & \xrightarrow{g_2} & Y_3 & \xrightarrow{g_3} & \cdots & \xrightarrow{g_{n-1}} & Y_n \end{array}$$

is a homotopy cartesian diagram and $\Sigma f_1 \cdot g_n$ is the differential. Thus (N4-1) holds and (a) follows from Theorem 3.1. □

Remark 3.5. The axiom (N4-1) and axiom (N4-2) are the higher analogues of homotopy cartesian axiom and cobase change axiom respectively. See [**9, 7, 2**] for reference.

Now we use the higher homotopy cartesian axiom (N4-1) to explain the higher octahedral axiom (N4*).

Corollary 3.6. *Let $(\mathcal{C}, \Sigma, \Theta)$ be a pre-n-angulated category. Then the following statements are equivalent:*

(a) Θ satisfies (N4).

(b) Θ satisfies (N4-4):

Given the following commutative diagram

$$\begin{array}{ccccccccccccccccc} X_1 & \xrightarrow{f_1} & X_2 & \xrightarrow{f_2} & X_3 & \xrightarrow{f_3} & X_4 & \xrightarrow{f_4} & \cdots & \xrightarrow{f_{n-2}} & X_{n-1} & \xrightarrow{f_{n-1}} & X_n & \xrightarrow{f_n} & \Sigma X_1 \\ \| & & \downarrow{\scriptstyle\varphi_2} & & \downarrow{\scriptstyle\varphi_3} & & \downarrow{\scriptstyle\varphi_4} & & & & \downarrow{\scriptstyle\varphi_{n-1}} & & \downarrow{\scriptstyle\varphi_n} & & \| \\ X_1 & \xrightarrow{g_1} & Y_2 & \xrightarrow{g_2} & Y_3 & \xrightarrow{g_3} & Y_4 & \xrightarrow{g_4} & \cdots & \xrightarrow{g_{n-2}} & Y_{n-1} & \xrightarrow{g_{n-1}} & Y_n & \xrightarrow{g_n} & \Sigma X_1 \\ \downarrow{\scriptstyle f_1} & & \| & & \downarrow{\scriptstyle\psi_3} & \swarrow{\scriptstyle\theta_4} & \downarrow{\scriptstyle\psi_4} & & & & \downarrow{\scriptstyle\psi_{n-1}} & \swarrow{\scriptstyle\theta_n} & \downarrow{\scriptstyle\psi_n} & & \downarrow{\scriptstyle\Sigma f_1} \\ X_2 & \xrightarrow{\varphi_2} & Y_2 & \xrightarrow{h_2} & Z_3 & \xrightarrow{h_3} & Z_4 & \xrightarrow{h_4} & \cdots & \xrightarrow{h_{n-2}} & Z_{n-1} & \xrightarrow{h_{n-1}} & Z_n & \xrightarrow{h_n} & \Sigma X_2 \end{array}$$

whose rows are n-angles, there exist morphisms $\varphi_i\colon X_i \to Y_i$ for $3 \leqslant i \leqslant n$, $\psi_j\colon Y_j \to Z_j$ for $3 \leqslant j \leqslant n$ and $\theta_k\colon X_k \to Z_{k-1}$ for $4 \leqslant k \leqslant n$ such that the following diagram

$$\begin{array}{ccccccccc}
X_1 & \xrightarrow{f_1} & X_2 & \xrightarrow{f_2} & X_3 & \xrightarrow{f_3} & X_4 & \xrightarrow{f_4} & \cdots \\
\| & & \downarrow{\scriptstyle \varphi_2} & & \downarrow{\scriptstyle \varphi_3} & & \downarrow{\scriptstyle \varphi_4} & & \\
X_1 & \xrightarrow{g_1} & Y_2 & \xrightarrow{g_2} & Y_3 & \xrightarrow{g_3} & Y_4 & \xrightarrow{g_4} & \cdots \\
\downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle \binom{0}{1}} & & \downarrow{\scriptstyle \binom{0}{1}} & & \downarrow{\scriptstyle \binom{0}{1}} & & \\
X_2 & \xrightarrow{\binom{-f_2}{\varphi_2}} & X_3 \oplus Y_2 & \xrightarrow{\left(\begin{smallmatrix} -f_3 & 0 \\ \varphi_3 & g_2 \end{smallmatrix}\right)} & X_4 \oplus Y_3 & \xrightarrow{\left(\begin{smallmatrix} -f_4 & 0 \\ \varphi_4 & g_3 \end{smallmatrix}\right)} & X_5 \oplus Y_4 & \xrightarrow{\left(\begin{smallmatrix} -f_5 & 0 \\ \varphi_5 & g_4 \end{smallmatrix}\right)} & \cdots \\
\| & & \downarrow{\scriptstyle (0\ 1)} & & \downarrow{\scriptstyle (\theta_4\ \psi_3)} & & \downarrow{\scriptstyle (\theta_5\ \psi_4)} & & \\
X_2 & \xrightarrow{\varphi_2} & Y_2 & \xrightarrow{h_2} & Z_3 & \xrightarrow{h_3} & Z_4 & \xrightarrow{h_4} & \cdots \\
\downarrow{\scriptstyle -f_2} & & \downarrow{\scriptstyle \binom{0}{g_2}} & & \downarrow{\scriptstyle \left(\begin{smallmatrix} 0 \\ 0 \\ 1 \end{smallmatrix}\right)} & & \downarrow{\scriptstyle \left(\begin{smallmatrix} 0 \\ 0 \\ 1 \end{smallmatrix}\right)} & & \\
X_3 & \xrightarrow{\binom{f_3}{-\varphi_3}} & X_4 \oplus Y_3 & \xrightarrow{\left(\begin{smallmatrix} f_4 & 0 \\ -\varphi_4 & -g_3 \\ \theta_4 & \psi_3 \end{smallmatrix}\right)} & X_5 \oplus Y_4 \oplus Z_3 & \xrightarrow{\left(\begin{smallmatrix} f_5 & 0 & 0 \\ -\varphi_5 & -g_4 & 0 \\ \theta_5 & \psi_4 & h_3 \end{smallmatrix}\right)} & X_6 \oplus Y_5 \oplus Z_4 & \xrightarrow{\left(\begin{smallmatrix} f_6 & 0 & 0 \\ -\varphi_6 & -g_5 & 0 \\ \theta_6 & \psi_5 & h_4 \end{smallmatrix}\right)} & \cdots
\end{array}$$

$$\begin{array}{ccccccccc}
\cdots & \xrightarrow{f_{n-3}} & X_{n-2} & \xrightarrow{f_{n-2}} & X_{n-1} & \xrightarrow{f_{n-1}} & X_n & \xrightarrow{f_n} & \Sigma X_1 \\
& & \downarrow{\scriptstyle \varphi_{n-2}} & & \downarrow{\scriptstyle \varphi_{n-1}} & & \downarrow{\scriptstyle \varphi_n} & & \| \\
\cdots & \xrightarrow{g_{n-3}} & Y_{n-2} & \xrightarrow{g_{n-2}} & Y_{n-1} & \xrightarrow{g_{n-1}} & Y_n & \xrightarrow{g_n} & \Sigma X_1 \\
& & \downarrow{\scriptstyle \binom{0}{1}} & & \downarrow{\scriptstyle \binom{0}{1}} & & \| & & \downarrow{\scriptstyle \Sigma f_1} \\
\cdots & \xrightarrow{\left(\begin{smallmatrix} -f_{n-2} & 0 \\ \varphi_{n-2} & g_{n-3} \end{smallmatrix}\right)} & X_{n-1} \oplus Y_{n-2} & \xrightarrow{\left(\begin{smallmatrix} -f_{n-1} & 0 \\ \varphi_{n-1} & g_{n-2} \end{smallmatrix}\right)} & X_n \oplus Y_{n-1} & \xrightarrow{(\varphi_n\ g_{n-1})} & Y_n & \xrightarrow{\Sigma f_1 \cdot g_n} & \Sigma X_2 \\
& & \downarrow{\scriptstyle (\theta_{n-1}\ \psi_{n-2})} & & \downarrow{\scriptstyle (\theta_n\ \psi_{n-1})} & & \downarrow{\scriptstyle \psi_n} & & \| \\
\cdots & \xrightarrow{h_{n-3}} & Z_{n-2} & \xrightarrow{h_{n-2}} & Z_{n-1} & \xrightarrow{h_{n-1}} & Z_n & \xrightarrow{h_n} & \Sigma X_2 \\
& & \downarrow{\scriptstyle \left(\begin{smallmatrix} 0 \\ 0 \\ 1 \end{smallmatrix}\right)} & & \downarrow{\scriptstyle \binom{0}{1}} & & \| & & \downarrow{\scriptstyle -\Sigma f_2} \\
\cdots & \xrightarrow{\left(\begin{smallmatrix} f_{n-1} & 0 & 0 \\ -\varphi_{n-1} & -g_{n-2} & 0 \\ \theta_{n-1} & \psi_{n-2} & h_{n-3} \end{smallmatrix}\right)} & X_n \oplus Y_{n-1} \oplus Z_{n-2} & \xrightarrow{\left(\begin{smallmatrix} -\varphi_n & -g_{n-1} & 0 \\ \theta_n & \psi_{n-1} & h_{n-2} \end{smallmatrix}\right)} & Y_n \oplus Z_{n-1} & \xrightarrow{(\psi_n\ h_{n-1})} & Z_n & \xrightarrow{-\Sigma f_2 \cdot h_n} & \Sigma X_3
\end{array} \tag{3}$$

is commutative where each row is an n-angle.

(c) Θ satisfies (N4-5):

Given two morphisms $f_1\colon X_1 \to X_2$ and $\varphi_2\colon X_2 \to Y_2$, there exists a commutative diagram (3) such that each row is an n-angle.

Proof. (a) $\Rightarrow$ (b). By Theorem 3.1 and (N4-1), there exist morphisms $\varphi_i\colon X_i \to Y_i$ for $3 \leqslant i \leqslant n$ such that the diagram (3) involving the first two rows is commutative and the third row is the n-angle given by the homotopy cartesian diagram. By (N4-1) again, there exist morphisms $\psi_j\colon Y_j \to Z_j$ for $3 \leqslant j \leqslant n$ and $\theta_k\colon X_k \to Z_{k-1}$ for $4 \leqslant k \leqslant n$ such that the diagram involving the third row and the fourth row is commutative, and the n-angle given by the homotopy cartesian diagram is the direct sum of the fifth row and the trivial n-angle $(TY_2)_\bullet$. Other commutative squares are trivial.

(b) $\Rightarrow$ (c). It follows from (N1)(c).

(c) ⇒ (a). The first three rows in diagram (3) implies that Θ satisfies (N4-3). Thus (a) follows from Corollary 3.4. □

Remark 3.7. If we replace θ_k by $(-1)^{k+1}\phi_k$ for $4 \leqslant k \leqslant n$ in the last row of diagram (3), then we have the following isomorphism of n-angles

$$
\begin{array}{ccccccccc}
X_3 & \xrightarrow{\left(\begin{smallmatrix} f_3 \\ -\varphi_3 \end{smallmatrix}\right)} & X_4 \oplus Y_3 & \xrightarrow{\left(\begin{smallmatrix} f_4 & 0 \\ -\varphi_4 & -g_3 \\ -\phi_4 & \psi_3 \end{smallmatrix}\right)} & X_5 \oplus Y_4 \oplus Z_3 & \xrightarrow{\left(\begin{smallmatrix} f_5 & 0 & 0 \\ -\varphi_5 & -g_4 & 0 \\ \phi_5 & \psi_4 & h_3 \end{smallmatrix}\right)} & X_6 \oplus Y_5 \oplus Z_4 & \xrightarrow{\left(\begin{smallmatrix} f_6 & 0 & 0 \\ -\varphi_6 & -g_5 & 0 \\ -\phi_6 & \psi_5 & h_4 \end{smallmatrix}\right)} & \cdots \\
\Vert & & \downarrow{\scriptstyle \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)} & & \downarrow{\scriptstyle \left(\begin{smallmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{smallmatrix}\right)} & & \downarrow{\scriptstyle \left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{smallmatrix}\right)} & & \\
X_3 & \xrightarrow{\left(\begin{smallmatrix} f_3 \\ \varphi_3 \end{smallmatrix}\right)} & X_4 \oplus Y_3 & \xrightarrow{\left(\begin{smallmatrix} -f_4 & 0 \\ \varphi_4 & -g_3 \\ \phi_4 & \psi_3 \end{smallmatrix}\right)} & X_5 \oplus Y_4 \oplus Z_3 & \xrightarrow{\left(\begin{smallmatrix} -f_5 & 0 & 0 \\ -\varphi_5 & -g_4 & 0 \\ \phi_5 & \psi_4 & h_3 \end{smallmatrix}\right)} & X_6 \oplus Y_5 \oplus Z_4 & \xrightarrow{\left(\begin{smallmatrix} -f_6 & 0 & 0 \\ \varphi_6 & -g_5 & 0 \\ \phi_6 & \psi_5 & h_4 \end{smallmatrix}\right)} & \cdots
\end{array}
$$

$$
\begin{array}{ccccccccc}
\cdots & \xrightarrow{\left(\begin{smallmatrix} f_{n-1} & 0 & 0 \\ -\varphi_{n-1} & -g_{n-2} & 0 \\ (-1)^n\phi_{n-1} & \psi_{n-2} & h_{n-3} \end{smallmatrix}\right)} & X_n \oplus Y_{n-1} \oplus Z_{n-2} & \xrightarrow{\left(\begin{smallmatrix} -\varphi_n & -g_{n-1} & 0 \\ (-1)^{n+1}\phi_n & \psi_{n-1} & h_{n-2} \end{smallmatrix}\right)} & Y_n \oplus Z_{n-1} & \xrightarrow{(\psi_n \ h_{n-1})} & Z_n & \xrightarrow{-\Sigma f_2 \cdot h_n} & \Sigma X_3 \\
& & \downarrow{\scriptstyle \left(\begin{smallmatrix} (-1)^n & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{smallmatrix}\right)} & & \downarrow{\scriptstyle \left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)} & & \downarrow{\scriptstyle -1} & & \Vert \\
\cdots & \xrightarrow{\left(\begin{smallmatrix} -f_{n-1} & 0 & 0 \\ (-1)^{n-1}\varphi_{n-1} & -g_{n-2} & 0 \\ \phi_{n-1} & \psi_{n-2} & h_{n-3} \end{smallmatrix}\right)} & X_n \oplus Y_{n-1} \oplus Z_{n-2} & \xrightarrow{\left(\begin{smallmatrix} (-1)^n\varphi_n & -g_{n-1} & 0 \\ \phi_n & \psi_{n-1} & h_{n-2} \end{smallmatrix}\right)} & Y_n \oplus Z_{n-1} & \xrightarrow{(\psi_n \ h_{n-1})} & Z_n & \xrightarrow{\Sigma f_2 \cdot h_n} & \Sigma X_3
\end{array}
$$

where the second row is the n-angle (1) given by the higher octahedral axiom (N4*). Indeed, (N4-4) is nothing but the proof of (N4) implying (N4*). Moreover, by (N4-4) we easily obtain the morphisms of n-angles hidden in (N4*).

4. An application

In this section we apply the idea of homotopy cartesian diagram developed in Section 3 to give a new proof of [**6**, Theorem 5.11]. We first recall some basic definitions and facts on Frobenius n-exact categories from [**6**].

Definition 4.1. Let $\mathcal{C}$ be an additive category and $f_1 \colon X_1 \to X_2$ be a morphism in $\mathcal{C}$. An $(n-2)$-*cokernel* of f_1 is a sequence

$$(f_2, f_3, \ldots, f_{n-1}) \colon X_2 \xrightarrow{f_2} X_3 \xrightarrow{f_3} \cdots \xrightarrow{f_{n-1}} X_n$$

such that the induced sequence

$$0 \to \mathcal{C}(X_n, Y) \to \mathcal{C}(X_{n-1}, Y) \to \cdots \to \mathcal{C}(X_3, Y) \to \mathcal{C}(X_2, Y) \to \mathcal{C}(X_1, Y)$$

is exact for each $Y \in \mathcal{C}$. In this case, the sequence

$$X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-2}} X_{n-1} \xrightarrow{f_{n-1}} X_n \to 0$$

is called *right* $(n-2)$-*exact*. The notion of $(n-2)$-*kernel* and of *left* $(n-2)$-*exact sequence* are defined dually. A sequence

$$0 \to X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-2}} X_{n-1} \xrightarrow{f_{n-1}} X_n \to 0$$

is called $(n-2)$-*exact* if it is both right and left $(n-2)$-exact.

Definition 4.2. A commutative diagram

$$\begin{array}{ccccccccc}
X_1 & \xrightarrow{f_1} & X_2 & \xrightarrow{f_2} & \cdots & \xrightarrow{f_{n-3}} & X_{n-2} & \xrightarrow{f_{n-2}} & X_{n-1}\\
\downarrow{\scriptstyle\varphi_1} & & \downarrow{\scriptstyle\varphi_2} & & & & \downarrow{\scriptstyle\varphi_{n-2}} & & \downarrow{\scriptstyle\varphi_{n-1}}\\
Y_1 & \xrightarrow{g_1} & Y_2 & \xrightarrow{g_2} & \cdots & \xrightarrow{g_{n-3}} & Y_{n-2} & \xrightarrow{g_{n-2}} & Y_{n-1}
\end{array}$$

is an $(n-2)$-*pushout diagram* if the following sequence

$$X_1 \xrightarrow{\begin{pmatrix} -f_1 \\ \varphi_1 \end{pmatrix}} X_2 \oplus Y_1 \xrightarrow{\begin{pmatrix} -f_2 & 0 \\ \varphi_2 & g_1 \end{pmatrix}} X_3 \oplus Y_2 \xrightarrow{\begin{pmatrix} -f_3 & 0 \\ \varphi_3 & g_2 \end{pmatrix}} \cdots$$
$$\cdots \xrightarrow{\begin{pmatrix} -f_{n-2} & 0 \\ \varphi_{n-2} & g_{n-3} \end{pmatrix}} X_{n-1} \oplus Y_{n-2} \xrightarrow{(\varphi_{n-1}\ g_{n-2})} Y_{n-1} \to 0$$

is right $(n-2)$-exact.

Let $\mathcal{M}$ be an additive category and $\mathcal{X}$ be a class of $(n-2)$-exact sequences in $\mathcal{M}$. The pair $(\mathcal{M}, \mathcal{X})$ is called an $(n-2)$-*exact category* if it satisfies some axioms which are similar to those of exact categories (see [**6**, Definition 4.2] for details). We will frequently use the following lemma and its dual.

Lemma 4.3 ([**6**, Proposition 4.8])**.** *Let $(\mathcal{M}, \mathcal{X})$ be an $(n-2)$-exact category. If the sequence*

$$0 \to X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-2}} X_{n-1} \xrightarrow{f_{n-1}} X_n \to 0$$

is an $(n-2)$-exact sequence in $\mathcal{X}$, then the following statements are equivalent:

(a) The diagram

$$\begin{array}{ccccccccc}
X_1 & \xrightarrow{f_1} & X_2 & \xrightarrow{f_2} & \cdots & \xrightarrow{f_{n-3}} & X_{n-2} & \xrightarrow{f_{n-2}} & X_{n-1}\\
\downarrow{\scriptstyle\varphi_1} & & \downarrow{\scriptstyle\varphi_2} & & & & \downarrow{\scriptstyle\varphi_{n-2}} & & \downarrow{\scriptstyle\varphi_{n-1}}\\
Y_1 & \xrightarrow{g_1} & Y_2 & \xrightarrow{g_2} & \cdots & \xrightarrow{g_{n-3}} & Y_{n-2} & \xrightarrow{g_{n-2}} & Y_{n-1}
\end{array}$$

is both an $(n-2)$-pushout and an $(n-2)$-pullback diagram.

(b) The sequence

$$0 \to X_1 \xrightarrow{\begin{pmatrix} -f_1 \\ \varphi_1 \end{pmatrix}} X_2 \oplus Y_1 \xrightarrow{\begin{pmatrix} -f_2 & 0 \\ \varphi_2 & g_1 \end{pmatrix}} X_3 \oplus Y_2 \xrightarrow{\begin{pmatrix} -f_3 & 0 \\ \varphi_3 & g_2 \end{pmatrix}} \cdots$$
$$\cdots \xrightarrow{\begin{pmatrix} -f_n & 0 \\ \varphi_{n-2} & g_{n-3} \end{pmatrix}} X_{n-1} \oplus Y_{n-2} \xrightarrow{(\varphi_{n-1}\ g_{n-2})} Y_{n-1} \to 0$$

is an $(n-2)$-exact sequence in $\mathcal{X}$.

(c) There exists a commutative diagram

$$\begin{array}{ccccccccccccc}
0 & \longrightarrow & X_1 & \xrightarrow{f_1} & X_2 & \xrightarrow{f_2} & \cdots & \xrightarrow{f_{n-3}} & X_{n-2} & \xrightarrow{f_n} & X_{n-1} & \xrightarrow{f_{n-1}} & X_n & \longrightarrow & 0\\
 & & \downarrow{\scriptstyle\varphi_1} & & \downarrow{\scriptstyle\varphi_2} & & & & \downarrow{\scriptstyle\varphi_{n-2}} & & \downarrow{\scriptstyle\varphi_{n-1}} & & \| & & \\
0 & \longrightarrow & Y_1 & \xrightarrow{g_1} & Y_2 & \xrightarrow{g_2} & \cdots & \xrightarrow{g_{n-3}} & Y_{n-2} & \xrightarrow{g_{n-2}} & Y_{n-1} & \xrightarrow{g_{n-1}} & X_n & \longrightarrow & 0
\end{array}$$

whose rows are $(n-2)$-exact sequences in $\mathcal{X}$.

Let $(\mathcal{M}, \mathcal{X})$ be an $(n-2)$-exact category. Recall that an object $I \in \mathcal{M}$ is called *$\mathcal{X}$-injective* if for any $(n-2)$-exact sequence

$$0 \to X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-2}} X_{n-1} \xrightarrow{f_{n-1}} X_n \to 0$$

in $\mathcal{X}$ and any morphism $g \in \mathcal{M}(X_1, I)$, there is a morphism $h \in \mathcal{M}(X_2, I)$ such that $g = hf_1$. We denote by $\mathcal{I}$ the full subcategory of $\mathcal{M}$ consisting of $\mathcal{X}$-injectives. We say $(\mathcal{M}, \mathcal{X})$ *has enough $\mathcal{X}$-injectives* if for any object $X \in \mathcal{M}$, there is an $(n-2)$-exact sequence

$$0 \to X \xrightarrow{i_1} I_1(X) \xrightarrow{i_2} I_2(X) \to \cdots \xrightarrow{i_{n-2}} I_{n-2}(X) \xrightarrow{i_{n-1}} Y \to 0$$

in $\mathcal{X}$ such that $I_i(X) \in \mathcal{I}$. We say $(\mathcal{M}, \mathcal{X})$ is *Frobenius* if it has enough $\mathcal{X}$-injectives, enough $\mathcal{X}$-projectives and if $\mathcal{X}$-injective objects and $\mathcal{X}$-projective objects coincide.

From now on, we assume that $(\mathcal{M}, \mathcal{X})$ is a Frobenius $(n-2)$-exact category. We denote by $\underline{\mathcal{M}}$ the stable category $\mathcal{M}/[\mathcal{I}]$. Given a morphism $f\colon X \to Y$ in $\mathcal{M}$, we denote by $\underline{f}$ the image of f in $\underline{\mathcal{M}}$ under the canonical functor $\mathcal{M} \to \underline{\mathcal{M}}$.

The following lemma is clear. It is a variant of [**6**, Lemma 2.1].

Lemma 4.4. *Assume that we have the following two commutative diagrams*

$$\begin{array}{ccccccccccccc}
0 & \longrightarrow & X_1 & \xrightarrow{f_1} & X_2 & \xrightarrow{f_2} & \cdots & \xrightarrow{f_{n-2}} & X_{n-1} & \xrightarrow{f_{n-1}} & X_n & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \varphi_1} & & \downarrow{\scriptstyle \varphi_2} & & & & \downarrow{\scriptstyle \varphi_{n-1}} & & \downarrow{\scriptstyle \varphi_n} & & \\
0 & \longrightarrow & Y_1 & \xrightarrow{g_1} & I_1 & \xrightarrow{g_2} & \cdots & \xrightarrow{g_{n-2}} & I_{n-2} & \xrightarrow{g_{n-1}} & Y_n & \longrightarrow & 0
\end{array}$$

and

$$\begin{array}{ccccccccccccc}
0 & \longrightarrow & X_1 & \xrightarrow{f_1} & X_2 & \xrightarrow{f_2} & \cdots & \xrightarrow{f_{n-2}} & X_{n-1} & \xrightarrow{f_{n-1}} & X_n & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \varphi_1} & & \downarrow{\scriptstyle \varphi'_2} & & & & \downarrow{\scriptstyle \varphi'_{n-1}} & & \downarrow{\scriptstyle \varphi'_n} & & \\
0 & \longrightarrow & Y_1 & \xrightarrow{g_1} & I_1 & \xrightarrow{g_2} & \cdots & \xrightarrow{g_{n-2}} & I_{n-2} & \xrightarrow{g_{n-1}} & Y_n & \longrightarrow & 0
\end{array}$$

whose rows are $(n-2)$-exact sequences in $\mathcal{X}$ and $I_i \in \mathcal{I}$. Then $\underline{\varphi_n} = \underline{\varphi'_n}$ in $\underline{\mathcal{M}}$.

Following [**6**, Proposition 5.8], there exists an automorphism $\Sigma\colon \underline{\mathcal{M}} \to \underline{\mathcal{M}}$. For convenience we recall the definition of Σ. For each $X \in \mathcal{M}$, we choose an $(n-2)$-exact sequence

$$0 \to X \xrightarrow{i_1} I_1(X) \xrightarrow{i_2} I_2(X) \to \cdots \xrightarrow{i_{n-2}} I_{n-2}(X) \xrightarrow{i_{n-1}} \Sigma X \to 0$$

in $\mathcal{X}$ such that $I_i(X) \in \mathcal{I}$. For each morphism $f\colon X \to Y$, we have the following commutative diagram

$$\begin{array}{ccccccccccccccc}
0 & \longrightarrow & X & \xrightarrow{i_1} & I_1(X) & \xrightarrow{i_2} & I_2(X) & \xrightarrow{i_3} & \cdots & \xrightarrow{i_{n-2}} & I_{n-2}(X) & \xrightarrow{i_{n-1}} & \Sigma X & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle I_1(f)} & & \downarrow{\scriptstyle I_2(f)} & & & & \downarrow{\scriptstyle I_{n-2}(f)} & & \downarrow{\scriptstyle \Sigma f} & & \\
0 & \longrightarrow & Y & \xrightarrow{j_1} & I_1(Y) & \xrightarrow{j_2} & I_2(Y) & \xrightarrow{j_3} & \cdots & \xrightarrow{j_{n-2}} & I_{n-2}(Y) & \xrightarrow{j_{n-1}} & \Sigma Y & \longrightarrow & 0
\end{array}$$

with rows in $\mathcal{X}$. By Lemma 4.4 the morphism $\underline{\Sigma f}$ does not depend on the choice of $I_i(f)$. We define $\Sigma\underline{f} = \underline{\Sigma f}$. It is easily seen that the functor Σ is an automorphism of $\underline{\mathcal{M}}$.

Given an $(n-2)$-exact sequence

$$0 \to X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-2}} X_{n-1} \xrightarrow{f_{n-1}} X_n \to 0$$

in $\mathcal{X}$, we call the sequence

$$X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-2}} X_{n-1} \xrightarrow{f_{n-1}} X_n \xrightarrow{(-1)^n f_n} \Sigma X_1$$

a *standard n-angle* if there exists the following commutative diagram

$$\begin{array}{ccccccccccccc}
0 & \longrightarrow & X_1 & \xrightarrow{f_1} & X_2 & \xrightarrow{f_2} & \cdots & \xrightarrow{f_{n-2}} & X_{n-1} & \xrightarrow{f_{n-1}} & X_n & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle a_2} & & & & \downarrow{\scriptstyle a_{n-1}} & & \downarrow{\scriptstyle f_n} & & \\
0 & \longrightarrow & X_1 & \xrightarrow{i_1} & I_1(X_1) & \xrightarrow{i_2} & \cdots & \xrightarrow{i_{n-2}} & I_{n-2}(X_1) & \xrightarrow{i_{n-1}} & \Sigma X_1 & \longrightarrow & 0.
\end{array}$$

We denote by $\Theta(\mathcal{X})$ the class of n-Σ-sequences which are isomorphic to standard n-angles.

Lemma 4.5. *Let $(\mathcal{M}, \mathcal{X})$ be a Frobenius $(n-2)$-exact category. Assume that the following is a commutative diagram*

$$\begin{array}{ccccccccccccc}
0 & \longrightarrow & X_1 & \xrightarrow{i_1} & I_1(X_1) & \xrightarrow{i_2} & \cdots & \xrightarrow{i_{n-2}} & I_{n-2}(X_1) & \xrightarrow{i_{n-1}} & \Sigma X_1 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle g_1} & & & & \downarrow{\scriptstyle g_{n-2}} & & \| & & \\
0 & \longrightarrow & X_2 & \xrightarrow{f_2} & X_3 & \xrightarrow{f_3} & \cdots & \xrightarrow{f_{n-1}} & X_n & \xrightarrow{f_n} & \Sigma X_1 & \longrightarrow & 0
\end{array}$$

whose rows are $(n-2)$-exact sequences in $\mathcal{X}$ and $I_i(X_1) \in \mathcal{I}$. Then the sequence

$$X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \xrightarrow{f_3} \cdots \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} \Sigma X_1$$

belongs to $\Theta(\mathcal{X})$.

Proof. By Lemma 4.3 we have the following morphism of $(n-2)$-exact sequences

$$\begin{array}{cccccccc}
0 \to & X_1 & \xrightarrow{\left(\begin{smallmatrix} -i_1 \\ f_1 \end{smallmatrix}\right)} & I_1(X_1) \oplus X_2 & \xrightarrow{\left(\begin{smallmatrix} -i_2 & 0 \\ g_1 & f_2 \end{smallmatrix}\right)} & I_2(X_1) \oplus X_3 & \xrightarrow{\left(\begin{smallmatrix} -i_3 & 0 \\ g_2 & f_3 \end{smallmatrix}\right)} & \cdots \\
 & \| & & \downarrow{\scriptstyle (-1\ 0)} & & \downarrow{\scriptstyle (1\ 0)} & & \\
0 \to & X_1 & \xrightarrow{i_1} & I_1(X_1) & \xrightarrow{i_2} & I_2(X_1) & \xrightarrow{i_3} & \cdots
\end{array}$$

$$\begin{array}{cccccc}
\cdots & \xrightarrow{\left(\begin{smallmatrix} -i_{n-2} & 0 \\ g_{n-3} & f_{n-2} \end{smallmatrix}\right)} & I_{n-2}(X_1) \oplus X_{n-1} & \xrightarrow{(g_{n-2}, f_{n-1})} & X_n & \to 0 \\
 & & \downarrow{\scriptstyle ((-1)^{n-2}\ 0)} & & \downarrow{\scriptstyle (-1)^{n-2} f_n} & \\
\cdots & \xrightarrow{i_{n-2}} & I_{n-2}(X_1) & \xrightarrow{i_{n-1}} & \Sigma X_1 & \to 0.
\end{array}$$

Thus the sequence

$$X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \xrightarrow{f_3} \cdots \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} \Sigma X_1$$

belongs to $\Theta(\mathcal{X})$ by definition. □

Remark 4.6. We point out that our definition of standard n-angles is different from the definition given in [**6**]. They are the same up to isomorphisms by Lemma 4.5 and [**6**, Lemma 5.10].

The following theorem was first proved by Jasso. We will give a new proof.

Theorem 4.7 ([**6**, Theorem 5.11]). *Let $(\mathcal{M},\mathcal{X})$ be a Frobenius $(n-2)$-exact category, then $(\underline{\mathcal{M}},\Sigma,\Theta(\mathcal{X}))$ is an n-angulated category.*

Proof. By Corollary 3.2, we only need to show that $\Theta(\mathcal{X})$ satisfies (N1), (N2) and (N4-1). Here we only prove (N4-1).

Given a commutative diagram

$$\begin{array}{ccccccccccc} X_1 & \xrightarrow{\underline{f_1}} & X_2 & \xrightarrow{\underline{f_2}} & X_3 & \xrightarrow{\underline{f_3}} & \cdots & \xrightarrow{\underline{f_{n-1}}} & X_n & \xrightarrow{(-1)^n\underline{f_n}} & \Sigma X_1 \\ \| & & \downarrow{\scriptstyle \underline{\varphi_2}} & & \downarrow{\scriptstyle \underline{\varphi_3}} & & & & \downarrow{\scriptstyle \underline{\varphi_n}} & & \| \\ X_1 & \xrightarrow{\underline{g_1}} & Y_2 & \xrightarrow{\underline{g_2}} & Y_3 & \xrightarrow{\underline{g_3}} & \cdots & \xrightarrow{\underline{g_{n-1}}} & Y_n & \xrightarrow{(-1)^n\underline{g_n}} & \Sigma X_1 \end{array}$$

with rows standard n-angles, we need to show that there exist morphisms $\varphi_i\colon X_i\to Y_i$ for $3\leqslant i\leqslant n$ such that $\underline{\varphi_i f_{i-1}}=\underline{g_{i-1}\varphi_{i-1}}$ and $\underline{f_n}=\underline{g_n\varphi_n}$, and the following

$$\begin{array}{ccccccc} X_2 & \xrightarrow{\underline{f_2}} & X_3 & \xrightarrow{\underline{f_3}} & \cdots & \xrightarrow{\underline{f_{n-1}}} & X_n \\ \downarrow{\scriptstyle \underline{\varphi_2}} & & \downarrow{\scriptstyle \underline{\varphi_3}} & & & & \downarrow{\scriptstyle \underline{\varphi_n}} \\ Y_2 & \xrightarrow{\underline{g_2}} & Y_3 & \xrightarrow{\underline{g_3}} & \cdots & \xrightarrow{\underline{g_{n-1}}} & Y_n \end{array}$$

is a homotopy cartesian diagram whose differential is $(-1)^n\Sigma\underline{f_1}\cdot\underline{g_n}$.

Indeed, we assume that

$$0\to X_1\xrightarrow{f_1}X_2\xrightarrow{f_2}X_3\xrightarrow{f_3}\cdots\xrightarrow{f_{n-1}}X_n\to 0$$

and

$$0\to X_1\xrightarrow{g_1}Y_2\xrightarrow{g_2}Y_3\xrightarrow{g_3}\cdots\xrightarrow{g_{n-1}}Y_n\to 0$$

are $(n-2)$-exact sequences in $\mathcal{X}$. Since $\underline{g_1}=\underline{\varphi_2 f_1}$, there exist two morphisms $a\colon X_1\to I$ and $b\colon I\to Y_2$ such that $I\in\mathcal{I}$ and $g_1-\varphi_2 f_1=ba$. Since $I\in\mathcal{I}$, there is a morphism $c\colon X_2\to I$ such that $a=cf_1$. Thus, $g_1=(\varphi_2+bc)f_1$. Consequently, there exist morphisms $\varphi_i\colon X_i\to Y_i$ for $3\leqslant i\leqslant n$ such that the following diagram

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & X_1 & \xrightarrow{f_1} & X_2 & \xrightarrow{f_2} & X_3 & \xrightarrow{f_3} \cdots \xrightarrow{f_{n-1}} & X_n & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle \varphi_2+bc} & & \downarrow{\scriptstyle \varphi_3} & & \downarrow{\scriptstyle \varphi_n} & & \\ 0 & \longrightarrow & X_1 & \xrightarrow{g_1} & Y_2 & \xrightarrow{g_2} & Y_3 & \xrightarrow{g_3} \cdots \xrightarrow{g_{n-1}} & Y_n & \longrightarrow & 0 \end{array}$$

is commutative. Hence, the sequence

$$0\to X_2\xrightarrow{\begin{pmatrix}-f_2\\ \varphi_2+bc\end{pmatrix}}X_3\oplus Y_2\xrightarrow{\begin{pmatrix}-f_3 & 0\\ \varphi_3 & g_2\end{pmatrix}}\cdots\xrightarrow{\begin{pmatrix}-f_{n-1} & 0\\ \varphi_{n-1} & g_{n-2}\end{pmatrix}}X_n\oplus Y_{n-1}\xrightarrow{(\varphi_n,g_{n-1})}Y_n\to 0$$

is an $(n-2)$-exact sequence in $\mathcal{X}$ by the dual of Lemma 4.3, which induces the following standard n-angle

$$X_2\xrightarrow{\begin{pmatrix}-\underline{f_2}\\ \underline{\varphi_2}\end{pmatrix}}X_3\oplus Y_2\xrightarrow{\begin{pmatrix}-\underline{f_3} & 0\\ \underline{\varphi_3} & \underline{g_2}\end{pmatrix}}X_4\oplus Y_3\xrightarrow{\begin{pmatrix}-\underline{f_4} & 0\\ \underline{\varphi_4} & \underline{g_3}\end{pmatrix}}\cdots$$

$$\cdots\xrightarrow{\begin{pmatrix}-\underline{f_{n-1}} & 0\\ \underline{\varphi_{n-1}} & \underline{g_{n-2}}\end{pmatrix}}X_n\oplus Y_{n-1}\xrightarrow{(\underline{\varphi_n},\underline{g_{n-1}})}Y_n\xrightarrow{(-1)^n\underline{h}}\Sigma X_2$$

for some morphism $h\colon Y_n \to \Sigma X_2$. The following commutative diagram

$$\begin{array}{ccccccccccccccc}
0 & \to & X_1 & \xrightarrow{f_1} & X_2 & \xrightarrow{f_2} & X_3 & \xrightarrow{f_3} & \cdots & \xrightarrow{f_{n-2}} & X_{n-1} & \xrightarrow{f_{n-1}} & X_n & \to & 0\\
 & & \| & & \downarrow{\scriptstyle \varphi_2+bc} & & \downarrow{\scriptstyle \varphi_3} & & & & \downarrow{\scriptstyle \varphi_{n-1}} & & \downarrow{\scriptstyle \varphi_n} & & \\
0 & \to & X_1 & \xrightarrow{g_1} & Y_2 & \xrightarrow{g_2} & Y_3 & \xrightarrow{g_3} & \cdots & \xrightarrow{g_{n-2}} & Y_{n-1} & \xrightarrow{g_{n-1}} & Y_n & \to & 0\\
 & & \| & & \downarrow{\scriptstyle a_2} & & \downarrow{\scriptstyle a_3} & & & & \downarrow{\scriptstyle a_{n-1}} & & \downarrow{\scriptstyle g_n} & & \\
0 & \to & X_1 & \xrightarrow{i_1} & I_1(X_1) & \xrightarrow{i_2} & I_2(X_1) & \xrightarrow{i_3} & \cdots & \xrightarrow{i_{n-2}} & I_{n-2}(X_1) & \xrightarrow{i_{n-1}} & \Sigma X_1 & \to & 0
\end{array}$$

and Lemma 4.4 implies that $\underline{f_n} = \underline{g_n\varphi_n}$. Comparing the following two diagrams

$$\begin{array}{ccccccccccccccc}
0 & \to & X_1 & \xrightarrow{g_1} & Y_2 & \xrightarrow{g_2} & Y_3 & \xrightarrow{g_3} & \cdots & \xrightarrow{g_{n-2}} & Y_{n-1} & \xrightarrow{g_{n-1}} & Y_n & \to & 0\\
 & & \| & & \downarrow{\scriptstyle a_2} & & \downarrow{\scriptstyle a_3} & & & & \downarrow{\scriptstyle a_{n-1}} & & \downarrow{\scriptstyle g_n} & & \\
0 & \to & X_1 & \xrightarrow{i_1} & I_1(X_1) & \xrightarrow{i_2} & I_2(X_1) & \xrightarrow{i_3} & \cdots & \xrightarrow{i_{n-2}} & I_{n-2}(X_1) & \xrightarrow{i_{n-1}} & \Sigma X_1 & \to & 0\\
 & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle I_1(f_1)} & & \downarrow{\scriptstyle I_2(f_1)} & & & & \downarrow{\scriptstyle I_{n-2}(f_1)} & & \downarrow{\scriptstyle \Sigma f_1} & & \\
0 & \to & X_2 & \xrightarrow{j_1} & I_1(X_2) & \xrightarrow{j_2} & I_2(X_2) & \xrightarrow{j_3} & \cdots & \xrightarrow{j_{n-2}} & I_{n-2}(X_2) & \xrightarrow{j_{n-1}} & \Sigma X_2 & \to & 0
\end{array}$$

and

$$\begin{array}{ccccccccccccc}
0 & \to & X_1 & \xrightarrow{g_1} & Y_2 & \xrightarrow{g_2} & \cdots & \xrightarrow{g_{n-2}} & Y_{n-1} & \xrightarrow{g_{n-1}} & Y_n & \to & 0\\
 & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle \binom{0}{1}} & & & & \downarrow{\scriptstyle \binom{0}{1}} & & \| & & \\
0 & \to & X_2 & \xrightarrow{\left(\begin{smallmatrix}-f_2\\ \varphi_2+bc\end{smallmatrix}\right)} & X_3\oplus Y_2 & \xrightarrow{\left(\begin{smallmatrix}-f_3 & 0\\ \varphi_3 & g_2\end{smallmatrix}\right)} & \cdots & \xrightarrow{\left(\begin{smallmatrix}-f_{n-1} & 0\\ \varphi_{n-1} & g_{n-2}\end{smallmatrix}\right)} & X_n\oplus Y_{n-1} & \xrightarrow{(\varphi_n, g_{n-1})} & Y_n & \to & 0\\
 & & \| & & \downarrow{\scriptstyle (b_2,c_2)} & & & & \downarrow{\scriptstyle (b_{n-1},c_{n-1})} & & \downarrow{\scriptstyle h} & & \\
0 & \to & X_2 & \xrightarrow{j_1} & I_1(X_2) & \xrightarrow{j_2} & \cdots & \xrightarrow{j_{n-2}} & I_{n-2}(X_2) & \xrightarrow{j_{n-1}} & \Sigma X_2 & \to & 0,
\end{array}$$

we have $\underline{h} = \Sigma\underline{f_1}\cdot\underline{g_n}$ by Lemma 4.4. We are done. □

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