Homology, Homotopy and Applications, vol. 21(2), 2019, pp.363–375

PERIODIC PROBLEM ON HOMOTOPY GROUPS OF CHANG COMPLEXES $C_r^{n+2,r}$

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(*communicated by John R. Klein*)

Abstract

In this paper, we find new examples, namely one class of Chang complexes $C_r^{n+2,r}$, such that their stable homotopy groups are direct summands of their unstable homotopy groups.

1. Introduction

The study of homotopy groups of finite complexes is one of the central problems in homotopy theory. The main tool for this problem is to decompose its loop space. Serre [**12**] showed that, p-locally for p an odd prime, ΩS^{2n} has the homotopy type of the product of S^{2n-1} and ΩS^{4n-1}. This decomposition reduces the computation of the odd p-primary part of the homotopy groups of spheres to that of odd dimensional spheres. Later Cohen-Moore-Neisendorfer [**6**] and Cohen [**5**] found a decomposition of the loop space $\Omega M(\mathbb{Z}/p^r, n)$, where $M(\mathbb{Z}/p^r, n)$ is the Moore space with the only non-trivial homological group $H_n(M(\mathbb{Z}/p^r, n)) \cong \mathbb{Z}/p^r$ for some prime p and $p = 2$ if $r > 1$. One application of these decompositions is determining the exponents of homotopy groups of Moore spaces. Little is known about $\Omega M(\mathbb{Z}/2, n)$. Recently Selick and Wu [**9, 10, 11**] developed a functorial decomposition approach to $\Omega\Sigma X$ involving the modular representation theory of symmetric groups. Furthermore, Wu [**14**] gave another decomposition of $\Omega\Sigma X$ using the free Lie power functor L_n. One can apply the last decomposition result of Beben-Wu [**1**] to find p-local (for an odd prime p) spaces with at most $(p-1)$ cells whose stable homotopy groups are summands of the unstable homotopy groups. Later this result was generalized by Chen and Wu [**4**] to the $p = 2$ case for 2-local spaces with 2 cells.

Motivated partly by Selick and Wu's functorial decomposition, the first and third authors [**16**] studied the decomposability of smash products of indecomposable complexes in $\mathbf{A}_n^2 (n \geqslant 3)$, where $\mathbf{A}_n^k$ is the homotopy category consisting of $(n-1)$-connected finite CW-complexes with dimension less than or equal to $n+k$ $(n \geqslant k+1)$. Our next goal is to see whether the stable homotopy groups of indecomposable $\mathbf{A}_n^2$-complexes (with 3 or 4 cells) are summands of their unstable homotopy groups or not.

The first author was partially supported by National Natural Science Foundation of China (Grant No. 11701430); the second and third authors were partially supported by National Natural Science Foundation of China (Grant No. 11661131004, 11131008).
Received December 11, 2018, revised March 17, 2019, April 14, 2019; published on June 5, 2019.
2010 Mathematics Subject Classification: 55P10, 55P35, 55Q05.
Key words and phrases: Chang-complex, smash product, loop space, decomposition, homotopy group.
Article available at http://dx.doi.org/10.4310/HHA.2019.v21.n2.a20

Since the decomposition of the 3-fold self smash product of any $\mathbf{A}_n^2$-complex has been completely determined and $\Sigma^* C_r^{n+2,r}$ is a wedge summand of $L_3(C_r^{n+2,r})$ (while other $\mathbf{A}_n^2$-complexes with 3 or 4 cells have no such good properties), we concentrate on the study of $C_r^{n+2,r}$ in this article. It is helpful to note the difference between Chen and Wu's argument and ours. Their results depend strongly on the assumption that the complexes considered have at most 2 cells when $p = 2$. They determined the iterated self smash product of the spaces by using modular representation theory. We have been unable to do the same for Chang complexes by using modular representation theory. Homotopy methods were used successfully to decompose iterated self smash products of Chang complexes and algebraic information from modular representation theory is essential to extract homotopical information on $\Omega\Sigma C_r^{n+2,r}$ from the decomposition.

$C_r^{n+2,r}(n \geqslant 3)$, one class of elementary Chang complexes, were found by Chang when he classified indecomposable homotopy types in $\mathbf{A}_n^2(n \geqslant 3)$ [**3**].

For stable CW-complexes X_i and $Y_j, (i = 1, \dots, t, j = 1, \dots, s)$, we have the identification:

$$[\bigvee_{i=1}^{t} X_i, \bigvee_{j=1}^{s} Y_j] = \left\{ f = (f_{kl}) = \begin{pmatrix} f_{11} & \cdots & f_{1t} \\ \vdots & \dots & \vdots \\ f_{s1} & \cdots & f_{st} \end{pmatrix} : f_{kl} \in [X_l, Y_k] \right\}.$$

The cell structures of **Elementary Chang complexes** are described as follows:

- $C_\eta^{n+2} = S^n \bigcup_\eta \mathbf{C}S^{n+1}$,
- $C^{n+2,s} = (S^n \vee S^{n+1}) \bigcup_{\binom{\eta}{2^s}} \mathbf{C}S^{n+1} = S^n \bigcup_{\eta q} \mathbf{C}M_{2^s}^n$;
- $C_r^{n+2} = S^n \bigcup_{(2^r,\eta)} \mathbf{C}(S^n \vee S^{n+1}) = M_{2^r}^n \bigcup_{i\eta} \mathbf{C}S^{n+1}$;
- $C_r^{n+2,s} = (S^n \vee S^{n+1}) \bigcup_{\begin{pmatrix} 2^r, \eta \\ 0\ , 2^s \end{pmatrix}} \mathbf{C}(S^n \vee S^{n+1}) = (M_{2^r}^n \vee S^n) \bigcup_{\binom{i\eta}{2^s}} \mathbf{C}S^{n+1}$

$$= S^n \bigcup_{(2^r,\eta q)} \mathbf{C}(S^n \vee M_{2^s}^n) = M_{2^r}^{n+1} \bigcup_{i\eta q} \mathbf{C}M_{2^s}^{n+1};$$

where η is the suspension of the Hopf map, i and q are the canonical inclusion and projection respectively, $n, r, s \in \mathbb{Z}^+$, $n \geqslant 3$ and $M_{p^r}^n$ denotes $M(\mathbb{Z}/p^r, n)$ for prime p.

Chang showed that all indecomposable homotopy types in $\mathbf{A}_n^2(n \geqslant 3)$ are spheres S^n, S^{n+1}, S^{n+2}, elementary Moore spaces $M_{p^r}^n$, $M_{p^r}^{n+1}$ (p is a prime, $r \in \mathbb{Z}^+$) and elementary Chang complexes listed above.

Our main results are given as follows.

Theorem 1.1 (Main theorem)**.** *Let $n \geqslant 3$.*

$$\Omega\Sigma C_r^{n+2,r} \simeq \prod_j \Omega\Sigma C_r^{k_j(n+1)+1,r} \times \textit{(some other space)},$$

where $2 < k_1 < k_2 < \cdots$ is a sequence of odd integers such that k_j is not a multiple of any k_i else for each j.

Corollary 1.2. *Let $p \geqslant 3$ be an odd integer with $p \geqslant \frac{k-n+1}{n+1}$, then the group $\pi_k^s(C_r^{n+3,r})$ is a direct summand of $\pi_{k+(p-1)(n+1)}(C_r^{n+3,r})$ for $n \geqslant 3$.*

Remark 1.3. From [**16**], $\pi^s_{n+2}(C_r^{n+3,r}) \cong \mathbb{Z}/2^{r+1}$. Thus from Corollary 1.2, we get that there is a $\mathbb{Z}/2^{r+1}$-summand in $\pi_m(C_r^{n+3,r})$ for $m > n+2$ and $m \equiv n+2 \pmod{2n+2}$.

To make the calculations easier to follow, we prove Theorem 1.1 and Corollary 1.2 only for the case $n = 3$ in the following sections.

Acknowledgments

The authors would like to thank Fred Cohen, Jie Wu and Ruizhi Huang for reading an earlier version of this paper and for their many helpful suggestions. We are grateful to the anonymous referee for their careful reading of this paper and, in particular, for pointing out a mistake in a previous proof of Lemma 3.4.

2. Preliminaries

Let $X = \Sigma X'$ where X' is a path-connected, 2-local finite CW-complex, let nX and $X^{\wedge n}$ be the n-fold self wedge product and the smash product of X, respectively. Let S_n denote the symmetric group on n letters and let $\mathbb{Z}_{(2)}[S_n]$ denote the group ring over the 2-local integers $\mathbb{Z}_{(2)}$ generated by S_n. There is an action of $\mathbb{Z}_{(2)}[S_n]$ on $X^{\wedge n}$ by permuting coordinates and taking the summations. Let $V = \bar{H}_*(X;\mathbb{Z}/2)$, which is a graded $\mathbb{Z}/2$-module. For any $\delta \in \mathbb{Z}_{(2)}[S_n]$, there is a map $\delta\colon X^{\wedge n} \to X^{\wedge n}$, which induces a map by permuting factors of homologies:

$$\delta_*\colon V^{\otimes n} \to V^{\otimes n}.$$

Start with $\beta_2 = 1 - (1,2) \in \mathbb{Z}_{(2)}[S_2]$ and let

$$\beta_n = \beta_{n-1} \wedge id - (1,2,3,\ldots,n)(\beta_{n-1} \wedge id).$$

Then for any $x_1 \otimes x_2 \otimes \cdots \otimes x_n \in V^{\otimes n}$,

$$\beta_{n*}(x_1 \otimes x_2 \otimes \cdots \otimes x_n) = [[\cdots[[x_1,x_2],\ldots,x_{n-1}],x_n]].$$

From [**7**], for odd n, the elements $\frac{1}{n}\beta_n$ and $id - \frac{1}{n}\beta_n$ are orthogonal idempotents. Let $hocolim_f X^{\wedge n}$ be the mapping telescope of the sequence of maps $X^{\wedge n} \xrightarrow{f} X^{\wedge n} \xrightarrow{f} \cdots$, then $L_n(X) := hocolim_{\frac{1}{n}\beta_n} X^{\wedge n}$ is a wedge summand of $X^{\wedge n}$. Let $p_n\colon X^{\wedge n} \to L_n(X)$ be the projection and let $i_n\colon L_n(X) \hookrightarrow X^{\wedge n}$ be the canonical inclusion. We have

$$(\frac{1}{n}\beta_n)_* = i_{n*}p_{n*}\colon H_*(X^{\wedge n};\mathbb{Z}/2) \to H_*(X^{\wedge n};\mathbb{Z}/2).$$

Moreover, from the notation 1.5 of [**14**]

$$\bar{H}_*(L_n(X);\mathbb{Z}/2) \cong L_n(V),$$

where $L_n(V)$ is sub-vector space spanned by the Lie elements of tensor length n in the tensor algebra $T(V)$.

Let $1 = \Sigma_\alpha e_\alpha$ be an orthogonal primitive decomposition of the identity element in $\mathbb{Z}_{(2)}[S_n]$ in terms of primitive idempotents. For each α, we take $e_\alpha(X) =$

$hocolim_{e_\alpha} X^{\wedge n}$. From [**11, 15**], we have

$$X^{\wedge n} \simeq \bigvee_{\alpha} e_\alpha(X).$$

For $n = 3$, from [**8**], there is an orthogonal primitive decomposition $1 = e'_1 + e'_2 + e'_3$ of the identity element in $\mathbb{Z}/2[S_3]$, where $e'_1 = 1 + \sigma + \sigma^2$, $e'_2 = (1+\tau)(1+\tau\sigma)$, $e'_3 = (1+\tau\sigma)(1+\tau)$ are primitive elements and $\sigma = (1,2,3)$, $\tau = (1,2) \in S_3$. It is known that $e'_2(\mathbb{Z}/2[S_3]) \cong e'_3(\mathbb{Z}/2[S_3])$ and thus e'_2, e'_3 are related by conjugation in $\mathbb{Z}/2[S_3]$. By Theorem 1.9.4 of [**2**], we can lift this decomposition and conjugated relation up to $\mathbb{Z}_{(2)}[S_3]$, i.e., $1 = e_1 + e_2 + e_3$ and there is an element $e \in \mathbb{Z}_{(2)}[S_3]$, such that $e_3 = e^{-1}e_2e$. Hence

$$X^{\wedge 3} \simeq e_1(X) \vee e_2(X) \vee e_3(X).$$

Consider the following maps

$$e_2(X) \xrightarrow{e_3e^{-1}} e_3(X) \xrightarrow{e_2e} e_2(X) \xrightarrow{e_3e^{-1}} e_3(X).$$

By $H_*(e_i(X);\mathbb{Z}_{(2)}) = e_{i*}H_*(X^{\wedge 3};\mathbb{Z}_{(2)})$ for $i = 1,2,3$, we get that the composite of two adjacent maps above induces the identity map on $\mathbb{Z}_{(2)}$-coefficient homology groups. Thus

$$e_2(X) \simeq e_3(X).$$

We state the key proposition in the proof of the main theorem, whose proof is given in Section 4.

Proposition 2.1. *Let $n \geqslant 3$ be any odd integer, then $L_n(C_r^{5,r})$ has a wedge summand $C_r^{4n+1,r}$.*

In order to simplify the writing, let $H_*(-) = H(-;\mathbb{Z})$ in the following sections.

3. Some lemmas on Chang complexes

Let $v_n^*, v_{n+1}^*, \bar{v}_{n+1}^*, v_{n+2}^*$ be generators of $\bar{H}^*(C_r^{n+2,r};\mathbb{Z}/2)$, which are given in Lemma 3.2 of [**16**] and denote their dual elements in $H_*(C_r^{n+2,r};\mathbb{Z}/2)$ by v_n, v_{n+1}, $\bar{v}_{n+1}$, v_{n+2}. Note that $Sq^2(v_n^*) = v_{n+2}^*$.

Lemma 3.1. *If a self-map $f\colon C_r^{n+2,r} \to C_r^{n+2,r}$ induces an automorphism on $H_n(C_r^{n+2,r};\mathbb{Z}/2)$ or on $H_{n+1}(C_r^{n+2,r};\mathbb{Z}/2)$, then f is a homotopy equivalence.*

Proof. If f induces an automorphism on $H_n(C_r^{n+2,r};\mathbb{Z}/2)$, then it induces an automorphism on $H_n(C_r^{n+2,r})$ and $H^n(C_r^{n+2,r};\mathbb{Z}/2)$.

Since $H^n(C_r^{n+2,r};\mathbb{Z}/2) \xrightarrow{Sq^2} H^{n+2}(C_r^{n+2,r};\mathbb{Z}/2)$ is an isomorphism, f induces an automorphism on $H^{n+2}(C_r^{n+2,r};\mathbb{Z}/2)$. From the following commutative diagram

$$\begin{array}{ccccc}
Ext(H_{n+1}(C_r^{n+2,r});\mathbb{Z}/2) & \hookrightarrow & H^{n+2}(C_r^{n+2,r};\mathbb{Z}/2) & \twoheadrightarrow & Hom(H_{n+2}(C_r^{n+2,r}),\mathbb{Z}/2) = 0 \\
\downarrow{\scriptstyle Ext(f_*;\mathbb{Z}/2)} & & \downarrow{\scriptstyle f^*(\cong)} & & \downarrow{\scriptstyle Hom(f_*,\mathbb{Z}/2)} \\
Ext(H_{n+1}(C_r^{n+2,r});\mathbb{Z}/2) & \hookrightarrow & H^{n+2}(C_r^{n+2,r};\mathbb{Z}/2) & \twoheadrightarrow & Hom(H_{n+2}(C_r^{n+2,r}),\mathbb{Z}/2) = 0
\end{array}$$

we have $Ext(f_*;\mathbb{Z}/2)\neq 0$, which implies that

$$f_*\colon H_{n+1}(C_r^{n+2,r})=\mathbb{Z}/2^r\xrightarrow{\cong}H_{n+1}(C_r^{n+2,r})=\mathbb{Z}/2^r.$$

So f induces automorphisms on all nontrivial homology groups of $C_r^{n+2,r}$; by the Whitehead theorem, f is a self-homotopy equivalence.

If f induces an automorphism on $H_{n+1}(C_r^{n+2,r};\mathbb{Z}/2)$, then it also does on $H^{n+1}(C_r^{n+2,r};\mathbb{Z}/2)$. The conclusion is easily obtained from the following commutative diagram

$$\begin{array}{ccccc}
Ext(H_n(C_r^{n+2,r});\mathbb{Z}/2) & \hookrightarrow & H^{n+1}(C_r^{n+2,r};\mathbb{Z}/2) & \twoheadrightarrow & Hom(H_{n+1}(C_r^{n+2,r}),\mathbb{Z}/2)\\
\downarrow{\scriptstyle Ext(f_*;\mathbb{Z}/2)} & & \downarrow{\scriptstyle f^*(\cong)} & & \downarrow{\scriptstyle Hom(f_*,\mathbb{Z}/2)}\\
Ext(H_n(C_r^{n+2,r});\mathbb{Z}/2) & \hookrightarrow & H^{n+1}(C_r^{n+2,r};\mathbb{Z}/2) & \twoheadrightarrow & Hom(H_{n+1}(C_r^{n+2,r}),\mathbb{Z}/2).
\end{array}$$ □

Lemma 3.2. *For any map $f\colon C_r^{n+2,r}\to C_\eta^5\wedge C_r^{n-2,r}$ $(n\geqslant 7)$, f induces a trivial homomorphism*

$$f_*=0\colon H_n(C_r^{n+2,r};\mathbb{Z}/2)\to H_n(C_\eta^5\wedge C_r^{n-2,r};\mathbb{Z}/2).$$

Proof. In order to simplify the writing, we prove this lemma for $n=7$.

Suppose that $H_7(C_r^{9,r};\mathbb{Z}/2)\xrightarrow{f_*}H_7(C_\eta^5\wedge C_r^{5,r};\mathbb{Z}/2)$ is non-trivial, then so is $f^*\colon H^7(C_\eta^5\wedge C_r^{5,r};\mathbb{Z}/2)\to H^7(C_r^{9,r};\mathbb{Z}/2)$. From the following commutative diagram

$$\begin{array}{ccccc}
H_7(C_r^{9,r})\otimes\mathbb{Z}/2 & \hookrightarrow & H_7(C_r^{9,r};\mathbb{Z}/2) & \twoheadrightarrow & Tor(H_6(C_r^{9,r}),\mathbb{Z}/2)=0\\
\downarrow{\scriptstyle f_*\otimes\mathbb{Z}/2} & & \downarrow{\scriptstyle f_*(\neq 0)} & & \downarrow{\scriptstyle Tor(f_*,\mathbb{Z}/2)}\\
H_7(C_\eta^5\wedge C_r^{5,r})\otimes\mathbb{Z}/2 & \hookrightarrow & H_7(C_\eta^5\wedge C_r^{5,r};\mathbb{Z}/2) & \twoheadrightarrow & Tor(H_6(C_\eta^5\wedge C_r^{5,r}),\mathbb{Z}/2)
\end{array}$$

we have $f_*\otimes\mathbb{Z}/2\neq 0$, which implies that

$$f_*\colon H_7(C_r^{9,r})=\mathbb{Z}/2^r\xrightarrow{\cong}H_7(C_\eta^5\wedge C_r^{5,r})=\mathbb{Z}/2^r.$$

Since $Sq^2\colon H^7(X;\mathbb{Z}/2)\to H^9(X;\mathbb{Z}/2)$ is an isomorphism for $X=C_r^{9,r}$, $C_\eta^5\wedge C_r^{5,r}$, we get that $f^*\colon H^9(C_\eta^5\wedge C_r^{5,r};\mathbb{Z}/2)\to H^9(C_r^{9,r};\mathbb{Z}/2)$ is nontrivial.

By the following commutative diagram

$$\begin{array}{ccccc}
Ext(H_8(C_\eta^5\wedge C_r^{5,r});\mathbb{Z}/2) & \hookrightarrow & H^9(C_\eta^5\wedge C_r^{5,r};\mathbb{Z}/2) & \twoheadrightarrow & Hom(H_9(C_\eta^5\wedge C_r^{5,r}),\mathbb{Z}/2)=\mathbb{Z}/2\\
\downarrow{\scriptstyle Ext(f_*;\mathbb{Z}/2)} & & \downarrow{\scriptstyle f^*(\neq 0)} & & \downarrow{\scriptstyle Hom(f_*,\mathbb{Z}/2)}\\
Ext(H_8(C_r^{9,r});\mathbb{Z}/2) & \hookrightarrow & H^9(C_r^{9,r};\mathbb{Z}/2) & \twoheadrightarrow & Hom(H_9(C_r^{9,r}),\mathbb{Z}/2)=0
\end{array}$$

we have $Ext(f_*;\mathbb{Z}/2)\neq 0$, which implies that

$$f_*\colon H_8(C_r^{9,r})=\mathbb{Z}/2^r\xrightarrow{\cong}H_8(C_\eta^5\wedge C_r^{5,r})=\mathbb{Z}/2^r.$$

It follows that $f^*\colon H^8(C_\eta^5\wedge C_r^{5,r};\mathbb{Z}/2)\to H^8(C_r^{9,r};\mathbb{Z}/2)$ is an isomorphism.

Let w_3^*,w_5^* be the generators of $H^*(C_\eta^5;\mathbb{Z}/2)$.

$Sq^2\colon H^6(C_\eta^5\wedge C_r^{5,r};\mathbb{Z}/2)\to H^8(C_\eta^5\wedge C_r^{5,r};\mathbb{Z}/2)$ is non-trivial because $Sq^2(w_3^*\otimes v_3^*)=w_3^*\otimes v_5^*+w_5^*\otimes v_3^*\neq 0$.

Therefore, we get a contradiction by the following commutative diagram

$$\begin{array}{ccc} H^6(C_\eta^5 \wedge C_r^{5,r};\mathbb{Z}/2) & \xrightarrow{f^*} & H^6(C_r^{9,r};\mathbb{Z}/2)=0 \\ {\scriptstyle Sq^2\neq 0}\downarrow & & \downarrow{\scriptstyle Sq^2} \\ H^8(C_\eta^5 \wedge C_r^{5,r};\mathbb{Z}/2) & \xrightarrow{f^*\cong} & H^8(C_r^{9,r};\mathbb{Z}/2). \end{array}$$

□

We will use the following cofiber sequences
C1: $C_r^4 \xrightarrow{2^r q_4} S^4 \xrightarrow{i_{\underline{C}}} C_r^{5,r} \xrightarrow{q_{\underline{C}}} C_r^5 \xrightarrow{2^r q_5} S^5$;
C2: $S^3 \xrightarrow{2^r j_3} C_\eta^5 \xrightarrow{i_C} C_r^5 \xrightarrow{q_C} S^4 \xrightarrow{2^r j_4} C_\eta^6$,
which come from the cofiber sequences **Cof 5** of $C_r^{k,s}$ and **Cof 3** of C_r^k in Section 3.2 of [**16**], respectively.

Lemma 3.3.
(1) *In the following cofiber sequence induced by cofiber sequence **C1**,*

$$S^4 \wedge C_r^{5,r} \xrightarrow{i_{\underline{C}}\wedge id} C_r^{5,r}\wedge C_r^{5,r} \xrightarrow{q_{\underline{C}}\wedge id} C_r^5 \wedge C_r^{5,r} \xrightarrow{2^r q_5\wedge id} S^5 \wedge C_r^{5,r}$$

there is a section $s_1\colon C_r^5 \wedge C_r^{5,r} \to C_r^{5,r}\wedge C_r^{5,r}$ *such that* $(q_{\underline{C}}\wedge id)s_1 \simeq id$.
(2) *In the following cofiber sequence induced by cofiber sequence **C2**,*

$$S^3 \wedge C_r^{5,r} \xrightarrow{2^r j_3\wedge id} C_\eta^5 \wedge C_r^{5,r} \xrightarrow{i_C\wedge id} C_r^5 \wedge C_r^{5,r} \xrightarrow{q_C\wedge id} S^4 \wedge C_r^{5,r}$$

there is a section $s_2\colon S^4 \wedge C_r^{5,r} \to C_r^5 \wedge C_r^{5,r}$ *such that* $(q_C\wedge id)s_2 \simeq id$.

Proof. We only prove (1), the proof of (2) is exactly the same as that of (1).

Let $f = i_{\underline{C}} \wedge id$ and $g = q_{\underline{C}}\wedge id$. Since $(i_{\underline{C}})_*$ (resp. $(q_{\underline{C}})_*$) is an injection (resp. surjection) in degree 4 homology, so f_* (resp. g_*) is an injection (surjection) in degree 7 homology. We have a short exact sequence

$$0 \to H_*(S^4\wedge C_r^{5,r};\mathbb{Z}/2) \xrightarrow{f_*} H_*(C_r^{5,r}\wedge C_r^{5,r};\mathbb{Z}/2) \xrightarrow{g_*} H_*(C_r^5\wedge C_r^{5,r};\mathbb{Z}/2) \to 0.$$

From Theorem 1.1 of [**16**], $C_r^{5,r}\wedge C_r^{5,r} \simeq 2C_r^{9,r} \vee (C_\eta^5 \wedge C_r^{5,r})$. Let

$$C_r^{5,r}\wedge C_r^{5,r} \xrightarrow{(P_1,P_2,P_3)^T} C_r^{9,r}\vee C_r^{9,r} \vee (C_\eta^5\wedge C_r^{5,r}),$$

where $P_i (i = 1,2,3)$ are the canonical projections, to the first, second and third wedge summands, respectively; A^T is the transpose of a matrix A.

By Lemma 3.2, $H_7(P_3 f;\mathbb{Z}/2) = 0$. Hence there is a map $P = P_1$ or $P = P_2$, such that $H_7(Pf;\mathbb{Z}/2) \neq 0$. By Lemma 3.1,

$$Pf\colon S^4 \wedge C_r^{5,r} \xrightarrow{f} C_r^{5,r}\wedge C_r^{5,r} \xrightarrow{P} C_r^{9,r}$$

is a homotopy equivalence.

Let $h := (Pf)^{-1}P\colon C_r^{5,r}\wedge C_r^{5,r} \xrightarrow{P} C_r^{9,r} \xrightarrow{(Pf)^{-1}} S^4\wedge C_r^{5,r}$, where $(Pf)^{-1}$ is the homotopy inverse of Pf. Then $hf \simeq id$, h is a retraction for f; equivalently, there is a section $s_1\colon C_r^5\wedge C_r^{5,r} \to C_r^{5,r}\wedge C_r^{5,r}$ such that $(q_{\underline{C}}\wedge id)s_1 \simeq id$. □

Next we study the homotopy type of $L_3(C_r^{5,r})$.

Lemma 3.4. $L_3(C_r^{5,r}) \simeq C_r^{13,r} \vee 2(C_\eta^5 \wedge C_r^{9,r})$.

Proof. $L_3(C_r^{5,r})$ is a wedge summand of $(C_r^{5,r})^{\wedge 3}$. From Theorem 1.1 of [**16**], $C_r^{5,r} \wedge C_r^{5,r} \simeq 2C_r^{9,r} \vee (C_\eta^5 \wedge C_r^{5,r})$, hence

$$(C_r^{5,r})^{\wedge 3} \simeq 4C_r^{13,r} \vee 4(C_\eta^5 \wedge C_r^{9,r}) \vee (C_\eta^5 \wedge C_\eta^5 \wedge C_r^{5,r}),$$

where $C_r^{13,r}$ and $C_\eta^5 \wedge C_r^{9,r}$ are indecomposable [**16**].

From (2), $dim\bar{H}_*(L_3(C_r^{5,r}); \mathbb{Z}/2) = dimL_3(V) = 20$, where $V = \mathbb{Z}/2\,\langle v_3, v_4, \bar{v}_4, v_5\rangle$. So we easily get the homology groups of $L_3(C_r^{5,r})$

$\bar{H}_k(L_3(C_r^{5,r}))$

$k =$	10	11	12	13
	$\bigoplus_2 \mathbb{Z}/2^r$	$\bigoplus_3 \mathbb{Z}/2^r$	$\bigoplus_3 \mathbb{Z}/2^r$	$\bigoplus_2 \mathbb{Z}/2^r$

Let $Z := C_\eta^5 \wedge C_\eta^5 \wedge C_r^{5,r}$.

All non-trivial reduced homology groups of Z are given as follows:

$\bar{H}_k(C_\eta^5 \wedge C_\eta^5 \wedge C_r^{5,r})$

$k =$	9	10	11	12	13	14
	$\mathbb{Z}/2^r$	$\mathbb{Z}/2^r$	$\mathbb{Z}/2^r \oplus \mathbb{Z}/2^r$	$\mathbb{Z}/2^r \oplus \mathbb{Z}/2^r$	$\mathbb{Z}/2^r$	$\mathbb{Z}/2^r$

Firstly, we show that if $Z \simeq U \vee V$, with $H_9U = \mathbb{Z}/2^r$, then $H_{10}U = \mathbb{Z}/2^r$.

From [**15**], we have a 2-local homotopy equivalence

$$C_\eta^5 \wedge C_\eta^5 \wedge C_\eta^5 \simeq 2C_\eta^{13} \vee (C_\eta^5 \wedge C_\nu^{10}),$$

where $C_\nu^{10} = \Sigma^2\mathbb{H}P^2$, with $Sq^4\colon H^6(C_\nu^{10}; \mathbb{Z}/2) \xrightarrow{\cong} H^{10}(C_\nu^{10}; \mathbb{Z}/2)$.

Thus

$$C_\eta^5 \wedge C_\eta^5 \wedge C_\eta^5 \wedge C_r^{5,r} \simeq 2(C_\eta^{13} \wedge C_r^{5,r}) \vee (C_\eta^5 \wedge C_\nu^{10} \wedge C_r^{5,r}).$$

The nontrivial reduced homology groups of $L = C_\eta^5 \wedge C_\nu^{10} \wedge C_r^{5,r}$ are given as follows

$\bar{H}_k(C_\eta^5 \wedge C_\nu^{10} \wedge C_r^{5,r})$

$k =$	12	13	14	15	16	17	18	19
	$\mathbb{Z}/2^r$	$\mathbb{Z}/2^r$	$\mathbb{Z}/2^r$	$\mathbb{Z}/2^r$	$\mathbb{Z}/2^r$	$\mathbb{Z}/2^r$	$\mathbb{Z}/2^r$	$\mathbb{Z}/2^r$

(Sq1) $0 \to H^{12}(L; \mathbb{Z}/2) \xrightarrow{Sq^6} H^{18}(L; \mathbb{Z}/2) \xrightarrow{Sq^2} H^{20}(L; \mathbb{Z}/2) \to 0$ is exact;

(Sq2) $H^{12}(L; \mathbb{Z}/2) \xrightarrow{Sq^4} H^{16}(L; \mathbb{Z}/2)$ is injective;

(Sq3) $H^{12}(L; \mathbb{Z}/2) \xrightarrow{Sq^8} H^{20}(L; \mathbb{Z}/2)$ is an isomorphism.

$C_\eta^5 \wedge C_\eta^5 \wedge C_\eta^5 \wedge C_r^{5,r}$ and $C_\eta^{13} \wedge C_r^{5,r}$ are self dual under Spanier-Whitehead-Duality $D_{32}\colon \mathbf{A}_{12}^8 \to \mathbf{A}_{12}^8$, so is L.

Suppose $L \simeq X \vee Y$ with X indecomposable and $H_{12}X = \mathbb{Z}/2^r$, then $H^{12}(X; \mathbb{Z}/2) \cong H^{12}(L; \mathbb{Z}/2) = \mathbb{Z}/2$. By (Sq3), $H^{20}(X; \mathbb{Z}/2) \cong H^{20}(L; \mathbb{Z}/2) = \mathbb{Z}/2$, and thus $H_{19}X = \mathbb{Z}/2^r$. From (Sq1), $H^{18}(X; \mathbb{Z}/2) \cong H^{18}(L; \mathbb{Z}/2) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$. Hence $H_{17}X \cong H_{18}X = \mathbb{Z}/2^r$. Since L is self dual and X contains both the bottom and top cells of L, X is also self dual. We have $H_{13}X \cong H_{14}X = \mathbb{Z}/2^r$. From (Sq2), $H_{15}X \oplus H_{16}X \neq 0$, hence $H_{15}X \cong H_{16}X = \mathbb{Z}/2^r$ by self duality of X. Thus $\bar{H}_m(Y) = 0$ for any m, which implies that Y is contractible. Thus we get L is indecomposable.

$C_\eta^5 \wedge C_\eta^5 \wedge C_\eta^5 \wedge C_r^{5,r} \simeq (C_\eta^5 \wedge U) \vee (C_\eta^5 \wedge V) \simeq 2(C_\eta^{13} \wedge C_r^{5,r}) \vee (C_\eta^5 \wedge C_\nu^{10} \wedge C_r^{5,r})$.

Since $C_\eta^{13} \wedge C_r^{5,r}$ and $C_\eta^5 \wedge C_\nu^{10} \wedge C_r^{5,r}$ are indecomposable and by observing the bottom cells, we get $C_\eta^5 \wedge C_\nu^{10} \wedge C_r^{5,r}$ is a wedge summand of $C_\eta^5 \wedge U$, hence $C_\eta^5 \wedge V$ is homotopy equivalent to $C_\eta^{13} \wedge C_r^{5,r}$ or $2C_\eta^{13} \wedge C_r^{5,r}$. By the Künneth formula, $H_{10}V = 0$. Hence $H_{10}U = \mathbb{Z}/2^r$.

From the analysis above, the nontrivial 10-dimensional homology groups of $L_3(C_r^{5,r})$ must come from the wedge summand $4(C_\eta^5 \wedge C_r^{9,r})$ of $(C_r^{5,r})^{\wedge 3}$. So $L_3(C_r^{5,r}) \simeq 2(C_\eta^5 \wedge C_r^{9,r}) \vee Q$, where Q is a space whose only nontrivial homology groups are $H_{11}Q \cong H_{12}Q \cong \mathbb{Z}/2^r$. Since the sequence $H^{10}(Z;\mathbb{Z}/2) \xrightarrow{Sq^2} H^{12}(Z;\mathbb{Z}/2) \xrightarrow{Sq^2} H^{14}(Z;\mathbb{Z}/2)$ is exact, the Moore spaces $M_{2^r}^{11}$ and $M_{2^r}^{12}$ cannot be wedge summands of Z. Hence Q is not equivalent to $M_{2^r}^{11} \vee M_{2^r}^{12}$. By the classification of indecomposable complexes in $\mathbf{A}_n^2(n>3)$, $Q \simeq C_r^{13,r}$. This finishes the proof of Lemma 3.4. □

Define a map

$$\phi\colon C_r^{13,r} \simeq S^4 \wedge S^4 \wedge C_r^{5,r} \to C_r^{5,r} \wedge C_r^{5,r} \wedge C_r^{5,r}$$

by $\phi = (id \wedge s_1 s_2)(T \wedge id)(id \wedge i_{\underline{C}} \wedge id)$, where $T(X \wedge Y) = Y \wedge X$. Then $\phi_*(\iota_4 \otimes \iota_4 \otimes u) = v_4 \otimes \bar{v}_4 \otimes u$ for $u \in \{v_3, v_4, \bar{v}_4, v_5\}$, where ϕ_* is the induced map of ϕ on $\mathbb{Z}/2$-homology groups and ι_n is the generator of $\bar{H}_*(S^n;\mathbb{Z}/2)$.

Lemma 3.5. *The composition map*

$$h := \bar{p}p_3\phi\colon C_r^{13,r} \xrightarrow{\phi} (C_r^{5,r})^{\wedge 3} \xrightarrow{p_3} L_3(C_r^{5,r}) \simeq C_r^{13,r} \vee 2(C_\eta^5 \wedge C_r^{9,r}) \xrightarrow{\bar{p}} C_r^{13,r}$$

is a homotopy equivalence, where $\bar{p}$ is the canonical projection.

Proof. Note that $H_*(C_r^{13,r};\mathbb{Z}/2) \xrightarrow{(p_3\phi)_*} H_*(L_3(C_r^{5,r});\mathbb{Z}/2) \xrightarrow{i_{3*}} H_*((C_r^{5,r})^{\wedge 3};\mathbb{Z}/2)$ takes $\iota_4 \otimes \iota_4 \otimes u$ to $[[v_4, \bar{v}_4], u]$ for $u \in \{v_3, v_4, \bar{v}_4, v_5\}$, hence $i_{3*}(p_3\phi)_*$ is a monomorphism, which implies that $(p_3\phi)_*$ is a monomorphism. From Lemma 3.2, for any wedge summand $C_\eta^5 \wedge C_r^{9,r}$ of $L_3(C_r^{5,r})$, the map $f = \bar{p}_2 p_3 \phi$ induces the trivial homomorphism

$$f_*\colon H_{11}(C_r^{13,r};\mathbb{Z}/2) \to H_{11}(C_\eta^5 \wedge C_r^{9,r};\mathbb{Z}/2),$$

where $L_3(C_r^{5,r}) \simeq C_r^{13,r} \vee (C_\eta^5 \wedge C_r^{9,r}) \xrightarrow{\bar{p}_2} C_\eta^5 \wedge C_r^{9,r}$ is the canonical projection to this wedge summand.

Then the lemma is easily obtained from Lemma 3.1. □

4. Proof of Proposition 2.1

Let

$$\phi_1 := i_3 p_3 \phi\colon C_r^{13,r} \simeq S^4 \wedge S^4 \wedge C_r^{5,r} \xrightarrow{\phi} (C_r^{5,r})^{\wedge 3} \xrightarrow{p_3} L_3(C_r^{5,r}) \xrightarrow{i_3} (C_r^{5,r})^{\wedge 3};$$
$$\varphi_1 := h^{-1}\bar{p}p_3\colon (C_r^{5,r})^{\wedge 3} \xrightarrow{p_3} L_3(C_r^{5,r}) \xrightarrow{\bar{p}} C_r^{13,r} \xrightarrow{h^{-1}} C_r^{13,r},$$

where h^{-1} is the homotopy inverse of h in Lemma 3.5. So

$$\varphi_1\phi_1 = h^{-1}\bar{p}p_3 i_3 p_3 \phi = h^{-1}\bar{p}p_3\phi = h^{-1}h \simeq id. \tag{1}$$

Let $\phi_k = i_{2k+1}p_{2k+1}(\phi_1 \wedge id)(id \wedge \phi_{k-1})$, i.e., the composite of the following maps

$$C_r^{8k+5,r} \simeq (S^4)^{\wedge 2} \wedge C_r^{8k-3,r} \xrightarrow{id\wedge\phi_{k-1}} (S^4)^{\wedge 2} \wedge (C_r^{5,r})^{\wedge 2k-1} = C_r^{13,r} \wedge (C_r^{5,r})^{\wedge 2k-2}$$
$$\xrightarrow{\phi_1\wedge id} (C_r^{5,r})^{\wedge 2k+1} \xrightarrow{p_{2k+1}} L_{2k+1}(C_r^{5,r}) \xrightarrow{i_{2k+1}} (C_r^{5,r})^{\wedge 2k+1},$$

and $\varphi_k = (id \wedge \varphi_1)(\varphi_{k-1} \wedge id)$, i.e., the composite of the following maps

$$(C_r^{5,r})^{\wedge 2k+1} = (C_r^{5,r})^{\wedge 2k-1} \wedge (C_r^{5,r})^{\wedge 2} \xrightarrow{\varphi_{k-1}\wedge id} (S^4)^{\wedge 2k-2} \wedge (C_r^{5,r})^{\wedge 3}$$
$$\xrightarrow{id\wedge\varphi_1} (S^4)^{\wedge 2k} \wedge C_r^{5,r} \simeq C_r^{8k+5}.$$

Proof of Proposition 2.1. Since $i_{2k+1}p_{2k+1}$ factors through $L_{2k+1}(C_r^{5,r})$, $C_r^{8k+5,r}$ is a wedge summand of $L_{2k+1}(C_r^{5,r})$ is a direct corollary of Lemma 4.1. □

Lemma 4.1. *$\varphi_k\phi_k$ is a homotopy equivalence.*

Proof. By Lemma 3.1, it suffices to show that $\varphi_{k*}\phi_{k*}$ on $H_{8k+3}(C_r^{8k+5,r};\mathbb{Z}/2)$ is an automorphism:

$$C_r^{8k+5,r} \simeq (S^4)^{\wedge 2k} \wedge C_r^{5,r} \xrightarrow{\varphi_k\phi_k} C_r^{8k+5,r} \simeq (S^4)^{\wedge 2k} \wedge C_r^{5,r}.$$

Actually, we shall prove that $\varphi_{k*}\phi_{k*}(\iota_4^{\otimes 2k} \otimes v_3) = \iota_4^{\otimes 2k} \otimes v_3$ by induction.

From (1), $\varphi_1\phi_1 \simeq id$.

For $C_r^{8k-3,r} \simeq (S^4)^{\wedge 2k-2} \wedge C_r^{5,r} \xrightarrow{\varphi_{k-1}\phi_{k-1}} C_r^{8k-3,r} \simeq (S^4)^{\wedge 2k-2} \wedge C_r^{5,r}$, assume that the induced endomorphism $\varphi_{k-1*}\phi_{k-1*}$ on $H_*(C_r^{8k-3,r};\mathbb{Z}/2)$ is the identity on generators $\iota_4^{\otimes 2k-2} \otimes v_3 \in H_{8k-5}((S^4)^{\wedge 2k-2} \wedge C_r^{5,r};\mathbb{Z}/2)$.

Denote by $ad(x)(y) = [y,x]$ and $ad^{i+1}(x)(y) = [ad^i(x)(y), x]$ for $i \geqslant 1$. The coefficients are taken mod-2, hence $[u,v] = [v,u]$ and $ad([u,v])(u) = [[u,v],u] = [u,[u,v]]$.

Claim 1.

(i) $\beta_{3*}([x,y] \otimes z) = 0$; $\beta_{3*}(z \otimes [x,y]) = [[x,y],z]$;
$\beta_{3*}([[x,y],z]) = [[x,y],z]$,

(ii) $\beta_{2k+1*}([[x,y],x] \otimes y \otimes w) + \beta_{2k+1*}([[x,y],y] \otimes x \otimes w) = 0$,

where $x,y,z \in \bar{H}_*(X;\mathbb{Z}/2)$ and $w \in \bar{H}_*(X^{\wedge 2k-3};\mathbb{Z}/2)$.

Proof of Claim 1. (i) of **Claim 1** is obvious. We prove (ii) of **Claim 1** in the following.

$$[[x,y],x] \otimes y + [[x,y],y] \otimes x$$
$$= x \otimes x \otimes y \otimes y + y \otimes x \otimes x \otimes y + y \otimes y \otimes x \otimes x + x \otimes y \otimes y \otimes x.$$

Note that $\beta_{2k+1*}(x \otimes x \otimes y \otimes y \otimes w)$ and $\beta_{2k+1*}(y \otimes y \otimes x \otimes x \otimes w)$ are zero. By the Jacobi Identity, $[[[y,x],x],y] + [[[x,y],y],x] = [[y,x],[y,x]] = 0$, hence $\beta_{2k+1*}(y \otimes x \otimes x \otimes y \otimes w) + \beta_{2k+1*}(x \otimes y \otimes y \otimes x \otimes w) = 0$. Thus we complete the proof (ii) of **Claim 1**. □

Claim 2. $\beta_{2k+1*}([[v_4,\bar{v}_4],v_3] \otimes [v_4,\bar{v}_4]^{\otimes k-1}) = ad^k([v_4,\bar{v}_4])(v_3)$ for $k \geqslant 1$.

Proof of Claim 2. We prove **Claim 2** by induction on k.

For $k = 1$, $\beta_{3*}([[v_4,\bar{v}_4],v_3]) = [[v_4,\bar{v}_4],v_3] = ad([v_4,\bar{v}_4])(v_3)$.

Assume that $\beta_{2k-1*}([[v_4,\bar{v}_4],v_3]\otimes[v_4,\bar{v}_4]^{\otimes k-2})=ad^{k-1}([v_4,\bar{v}_4])(v_3)$. Then

$$\begin{aligned}
&\beta_{2k+1*}([[v_4,\bar{v}_4],v_3]\otimes[v_4,\bar{v}_4]^{\otimes k-1})\\
&\quad=\beta_{2k+1*}([[v_4,\bar{v}_4],v_3]\otimes[v_4,\bar{v}_4]^{\otimes k-2}\otimes v_4\otimes\bar{v}_4)\\
&\qquad+\beta_{2k+1*}([[v_4,\bar{v}_4],v_3]\otimes[v_4,\bar{v}_4]^{\otimes k-2}\otimes\bar{v}_4\otimes v_4)\\
&\quad=[[\beta_{2k-1*}([[v_4,\bar{v}_4],v_3]\otimes[v_4,\bar{v}_4]^{\otimes k-2}),v_4],\bar{v}_4]\\
&\qquad+[[\beta_{2k-1*}([[v_4,\bar{v}_4],v_3]\otimes[v_4,\bar{v}_4]^{\otimes k-2}),\bar{v}_4],v_4]\\
&\quad=[[ad^{k-1}([v_4,\bar{v}_4])(v_3),v_4],\bar{v}_4]+[\,[ad^{k-1}([v_4,\bar{v}_4])(v_3),\bar{v}_4],v_4]\\
&\quad=[[ad^{k-1}([v_4,\bar{v}_4])(v_3),[v_4,\bar{v}_4]]]\qquad\text{(Jacobi's identity)}\\
&\quad=ad^{k}([v_4,\bar{v}_4])(v_3).
\end{aligned}$$

We complete the proof of **Claim 2**. □

Claim 3. $\phi_{k*}(\iota_4^{\otimes 2k}\otimes v_3)=ad^k([v_4,\bar{v}_4])(v_3)$.

Proof of Claim 3. We prove **Claim 3** by induction on k.

$$\phi_1(\iota_4\otimes\iota_4\otimes v_3)=[[v_4,\bar{v}_4],v_3]=ad([v_4,\bar{v}_4])(v_3).$$

Assume that $\phi_{k-1*}(\iota_4^{\otimes 2k-2}\otimes v_3)=ad^{k-1}([v_4,\bar{v}_4])(v_3)$.

$$\begin{aligned}
\phi_{k*}(\iota_4^{\otimes 2k}\otimes v_3)&=i_{2k+1_*}p_{2k+1_*}(\phi_1\wedge id)_*(id\otimes\phi_{k-1*})(\iota_4^{\otimes 2}\otimes\iota_4^{\otimes 2k-2}\otimes v_3)\\
&=\beta_{2k+1*}(\phi_1\wedge id)_*(\iota_4^{\otimes 2}\otimes ad^{k-1}([v_4,\bar{v}_4])(v_3))\\
&=\beta_{2k+1*}(\phi_1\wedge id)_*(\iota_4^{\otimes 2}\otimes[v_4,\bar{v}_4]\otimes ad^{k-2}([v_4,\bar{v}_4])(v_3))\\
&\quad+\beta_{2k+1*}(\phi_1\wedge id)_*(\iota_4^{\otimes 2}\otimes ad^{k-2}([v_4,\bar{v}_4])(v_3)\otimes[v_4,\bar{v}_4])
\end{aligned}$$

Observation A.

$$\begin{aligned}
&\beta_{2k+1*}(\phi_1\wedge id)_*(\iota_4^{\otimes 2}\otimes[v_4,\bar{v}_4]\otimes ad^{k-2}([v_4,\bar{v}_4])(v_3))\\
&\quad=\beta_{2k+1*}(\phi_1\wedge id)_*(\iota_4^{\otimes 2}\otimes v_4\otimes\bar{v}_4\otimes ad^{k-2}([v_4,\bar{v}_4])(v_3))\\
&\qquad+\beta_{2k+1*}(\phi_1\wedge id)_*(\iota_4^{\otimes 2}\otimes\bar{v}_4\otimes v_4\otimes ad^{k-2}([v_4,\bar{v}_4])(v_3))\\
&\quad=\beta_{2k+1*}([[v_4,\bar{v}_4],v_4]\otimes\bar{v}_4\otimes ad^{k-2}([v_4,\bar{v}_4])(v_3))\\
&\qquad+\beta_{2k+1*}([[v_4,\bar{v}_4],\bar{v}_4]\otimes v_4\otimes ad^{k-2}([v_4,\bar{v}_4])(v_3))\\
&\quad=0.\qquad\text{(by (ii) of }\textbf{Claim 1}\text{)}
\end{aligned}$$

Hence

$$\begin{aligned}
&\phi_{k*}(\iota_4^{\otimes 2k}\otimes v_3)\\
&\quad=\beta_{2k+1*}(\phi_1\wedge id)_*(\iota_4^{\otimes 2}\otimes ad^{k-2}([v_4,\bar{v}_4])(v_3)\otimes[v_4,\bar{v}_4])\\
&\quad=\beta_{2k+1*}(\phi_1\wedge id)_*(\iota_4^{\otimes 2}\otimes[v_4,\bar{v}_4]\otimes ad^{k-3}([v_4,\bar{v}_4])(v_3)\otimes[v_4,\bar{v}_4])\\
&\qquad+\beta_{2k+1*}(\phi_1\wedge id)_*(\iota_4^{\otimes 2}\otimes ad^{k-3}([v_4,\bar{v}_4])(v_3)\otimes[v_4,\bar{v}_4]^{\otimes 2})\\
&\quad=\beta_{2k+1*}(\phi_1\wedge id)_*(\iota_4^{\otimes 2}\otimes ad^{k-3}([v_4,\bar{v}_4])(v_3)\otimes[v_4,\bar{v}_4]^{\otimes 2})\quad\text{(as }\textbf{Observation A}\text{)}\\
&\quad=\cdots
\end{aligned}$$

$$
\begin{aligned}
&= \beta_{2k+1*}(\phi_1 \wedge id)_*(\iota_4^{\otimes 2} \otimes ad([v_4, \bar{v}_4])(v_3) \otimes [v_4, \bar{v}_4]^{\otimes k-2}) \\
&= \beta_{2k+1*}(\phi_1 \wedge id)_*(\iota_4^{\otimes 2} \otimes [v_4, \bar{v}_4] \otimes v_3 \otimes [v_4, \bar{v}_4]^{\otimes k-2}) \\
&\quad +\beta_{2k+1*}(\phi_1 \wedge id)_*(\iota_4^{\otimes 2} \otimes v_3 \otimes [v_4, \bar{v}_4]^{\otimes k-1}) \\
&= \beta_{2k+1*}([[v_4, \bar{v}_4], v_3] \otimes [v_4, \bar{v}_4]^{\otimes k-1}) \quad (\text{as } \textbf{Observation A}) \\
&= ad^k([v_4, \bar{v}_4])(v_3). \qquad (\text{by } \textbf{Claim 2})
\end{aligned}
$$

This finishes the proof of **Claim 3**. □

Claim 4. $\varphi_{k*}([v_4, \bar{v}_4] \otimes ad^{k-1}([v_4, \bar{v}_4])(v_3)) = 0$.

Proof of Claim 4. Note that

$$
\begin{aligned}
\varphi_k &= (id \wedge \varphi_1)(\varphi_{k-1} \wedge id) \\
&= (id \wedge \varphi_1)(id \wedge \varphi_1 \wedge id)(\varphi_{k-2} \wedge id) \\
&= \cdots \\
&= (id \wedge \varphi_1) \cdots (\varphi_1 \wedge id) \\
&= (id \wedge \varphi_1) \cdots ((\varphi_1 \beta_3) \wedge id) \quad (\text{note that } \varphi_1 \beta_3 = \varphi_1).
\end{aligned}
$$

By (i) of **Claim 1**, we have $\varphi_{k*}([x, y] \otimes \cdots) = 0$.
This finishes the proof of **Claim 4**. □

Now we have

$$
\begin{aligned}
&\varphi_{k*}\phi_{k*}(\iota_4^{\otimes 2k} \otimes v_3) \\
&= \varphi_{k*}(ad^k([v_4, \bar{v}_4])(v_3)) \\
&= \varphi_{k*}([v_4, \bar{v}_4] \otimes ad^{k-1}([v_4, \bar{v}_4])(v_3)) + \varphi_{k*}(ad^{k-1}([v_4, \bar{v}_4])(v_3) \otimes [v_4, \bar{v}_4]) \\
&= \varphi_{k*}(ad^{k-1}([v_4, \bar{v}_4])(v_3) \otimes [v_4, \bar{v}_4]) \quad (\text{by } \textbf{Claim 4}) \\
&= (id \wedge \varphi_1)_*(\varphi_{k-1} \wedge id)_*(ad^{k-1}([v_4, \bar{v}_4])(v_3) \otimes [v_4, \bar{v}_4]) \\
&= (id \wedge \varphi_1)_*(\varphi_{k-1*}(ad^{k-1}([v_4, \bar{v}_4])(v_3)) \otimes [v_4, \bar{v}_4]) \\
&= (id \otimes \varphi_{1*})(\varphi_{k-1*}(\phi_{k-1*}(\iota_4^{\otimes 2k-2} \otimes v_3)) \otimes [v_4, \bar{v}_4]) \quad (\text{by } \textbf{Claim 3}) \\
&= (id \otimes \varphi_{1*})(\iota_4^{\otimes 2k-2} \otimes v_3 \otimes [v_4, \bar{v}_4]) \quad (\text{by inductive condition }) \\
&= \iota_4^{\otimes 2k-2} \otimes \varphi_{1*}(v_3 \otimes [v_4, \bar{v}_4]) \\
&= \iota_4^{\otimes 2k-2} \otimes \varphi_{1*}(\beta_{3*}(v_3 \otimes [v_4, \bar{v}_4])) \quad (\text{note that } \varphi_1 \beta_3 = \varphi_1) \\
&= \iota_4^{\otimes 2k-2} \otimes \varphi_{1*}([[v_4, \bar{v}_4], v_3]) \quad (\text{by (i) of } \textbf{Claim 1}) \\
&= \iota_4^{\otimes 2k-2} \otimes \varphi_{1*}(\phi_{1*}(\iota_4^{\otimes 2} \otimes v_3)) \\
&= \iota_4^{\otimes 2k} \otimes v_3.
\end{aligned}
$$

Therefore, we complete the proof of Lemma 4.1. □

5. Proof of Theorem 1.1 and Corollary 1.2

The following Proposition 5.1 is easily obtained from Theorem 1.6 of [**14**].

Proposition 5.1. *Let $X = \Sigma X'$ be a path-connected 2-local finite CW-complex. Let $2 < k_1 < k_2 < \cdots$ be a sequence of odd integers such that no k_j is a multiple of any* *other. Then there exists a topological space A such that*

$$\Omega\Sigma X \simeq \prod_j \Omega\Sigma L_{k_j}(X) \times A$$

localized at 2.

Proof of Theorem 1.1 for $n = 3$. By Proposition 2.1 and the Hilton-Milnor Theorem [**13**], we get

$$\Omega\Sigma L_{k_j}(C_r^{5,r}) \simeq \Omega\Sigma C_r^{4k_j+1,r} \times (\text{some other spaces}).$$

Hence Theorem 1.1 is easily obtained from Proposition 5.1. □

Proof of Corollary 1.2 for $n = 3$. $\pi_m(\Sigma C_r^{4p+1,r})$ is a summand of $\pi_m(C_r^{6,r})$ for any odd integer $p \geqslant 3$ from Theorem 1.1.

Let p be large enough such that $p \geqslant \frac{k-2}{4}$. By the Freudenthal suspension theorem,

$$\pi_{k+4p-4}(\Sigma C_r^{4p+1,r}) \cong \pi^s_{k+4p-4}(\Sigma C_r^{4p+1,r}) \cong \pi^s_k(C_r^{6,r}).$$

Therefore, $\pi^s_k(C_r^{6,r})$ is a summand of $\pi_{k+4p-4}(C_r^{6,r})$. □

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