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THE 3×3 LEMMA IN THE Σ-MAL'TSEV AND Σ-PROTOMODULAR SETTINGS. APPLICATIONS TO MONOIDS AND QUANDLES

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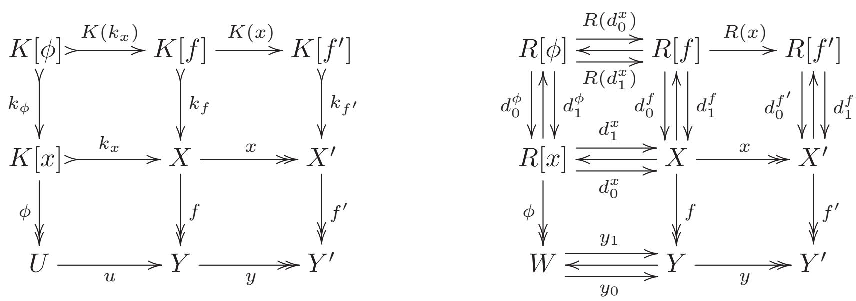
(*communicated by George Janelidze*)

Abstract

We investigate what is remaining of the 3×3 lemma and of the denormalized 3×3 lemma, valid in a pointed protomodular and in a Mal'tsev category, respectively, in the context of partial pointed protomodular and partial Mal'tsev categories, relatively to a class Σ of points (i.e. of split epimorphisms with a fixed section). The results apply, among other structures, to monoids, semirings, and quandles.

1. Introduction

The 3×3 lemma is a classical tool in homological algebra, with several applications. It holds for many algebraic structures, including groups. As shown in [**3**], the lemma is valid in any pointed regular protomodular [**1**] category: given any commutative diagram as the one on the left-hand side here below, where the three columns and the middle row are exact sequences, the upper row is exact if and only if the lower one is.



(1)

A *denormalized* version of the 3×3 lemma was proved in [**4**] to be valid in any (not necessarily pointed) regular Mal'tsev category: given any commutative diagram as the one on the right-hand side, where the three columns and the middle row are

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exact forks, the upper row is exact if and only if the lower one is. We recall that an exact fork is a diagram

$$R[x] \underset{d_0^x}{\overset{d_1^x}{\rightleftarrows}} X \xrightarrow{x} X'$$

in which x is a regular epimorphism and $(R[x], d_0^x, d_1^x)$ is its kernel pair.

Later, the denormalized 3×3 lemma was extended to the context of regular Goursat categories in [**20**]. A common generalization of the normalized and the denormalized versions of the 3×3 lemma was described in [**15**].

In the recent paper [**25**] it was shown that a particular version of the 3×3 lemma holds also in the category Mon of monoids, which is neither protomodular nor Mal'tsev: the 3×3 lemma holds for monoids when we replace exact sequences by *special Schreier* exact sequences. This notion, introduced in [**11**], originated from the notion of *Schreier split epimorphism* [**27, 22**]: these are the split epimorphisms that correspond to classical monoid actions. An action of a monoid B on a monoid X is a monoid homomorphism $B \to End(X)$. The 3×3 lemma for special Schreier exact sequences allowed to give in [**25**] a description of a Baer sum construction of special Schreier exact extensions with abelian kernel, obtained thanks to a *push forward* construction, analogous to the classical one for group extensions.

In order to understand and to describe categorically the (co)homological features of Schreier monoid extensions, the notions of pointed Σ-protomodular [**13**] and Σ-Mal'tsev category [**8**] have been introduced, with respect to a class Σ of *points*, i.e. split epimorphisms with a fixed section. The main examples of the first notion are: the categories of monoids and of semirings (see [**13**]), and, more generally, any Jónsson-Tarski variety [**21**]. An interesting example of a Σ-Mal'tsev category (which is not Σ-protomodular) is the category of quandles [**7**]. As for the case of monoids, in all these contexts the class Σl of Σ-special maps (see Definition 2.2) appeared to be very discriminating.

The aim of the present paper is to investigate what is remaining of the normalized and of the denormalized 3×3 lemma, respectively, in the abstract context of pointed Σ-protomodular and of Σ-Mal'tsev categories. Namely, we are interested in the description of the conditions under which, given any (normalized or denormalized) diagram as (1) above, where the three columns and the middle row are exact (or Σ-special), the upper row is exact (or Σ-special) when the lower one is (the so-called *upper* 3×3 *lemma*), and conversely (the so-called *lower* 3×3 *lemma*).

In these relative contexts, a curious phenomenon appears, in contrast to the "absolute" case of protomodular and Mal'tsev categories (that are Σ-protomodular and Σ-Mal'tsev, respectively, for the class Σ of all points). In fact, in the absolute contexts, the upper and the lower 3×3 lemmas are equivalent, both in the normalized and in the denormalized case (see [**18**] and [**16**]). In the relative contexts, this equivalence is no longer true (see Theorem 4.10, Proposition 6.2, and Theorem 6.7 below). This shows an unexpected asymmetry between the two parts of the 3×3 lemma.

The paper is organized as follows. In Section 2 we recall the notion of Σ-Mal'tsev category and give several examples. In Section 3 we obtain some properties of regular Σ-Mal'tsev categories that are used in Section 4, where we describe the versions of the denormalized 3×3 lemma that are valid in regular Σ-Mal'tsev categories. In Section 5 we recall the definition and the main properties of Σ-protomodular categories. In Section 6 we describe the versions of the (normalized) 3×3 lemma that hold in a regular Σ-protomodular category. In Section 7 we give an interpretation of the Baer sum construction in Barr-exact Σ-Mal'tsev categories.

2. Σ-Mal'tsev categories

2.1. The fibration of points

Throughout the paper, all the categories we consider will be finitely complete. A (generalized) *point* in a category $\mathbb{E}$ is a pair (f, s) of morphisms such that $fs = 1$; in other terms, f is a split epimorphism with a fixed section s. The category $Pt\mathbb{E}$ is the category whose objects are the points in $\mathbb{E}$ and whose morphisms are the pairs (y, x) of morphisms which make a square as below commutative, both downward and upward:

$$\begin{array}{ccc} X' & \xrightarrow{x} & X \\ {\scriptstyle f'}\downarrow\uparrow{\scriptstyle s'} & & {\scriptstyle f}\downarrow\uparrow{\scriptstyle s} \\ Y' & \xrightarrow[y]{} & Y. \end{array}$$

The codomain functor $¶_{\mathbb{E}} \colon Pt\mathbb{E} \to \mathbb{E}$ is a fibration whose cartesian maps are the pullbacks of split epimorphisms; it is called *the fibration of points* [**1**].

Let Σ be a pullback stable class of points in a category $\mathbb{E}$. We denote by $\Sigma Pt\mathbb{E}$ the full subcategory of $Pt\mathbb{E}$ whose objects are the points in Σ; the restriction of $¶_{\mathbb{E}}$ to the class Σ determines a subfibration of the fibration of points:

$$\begin{array}{ccc} \Sigma Pt\mathbb{E} & \overset{j}{\rightarrowtail} & Pt\mathbb{E} \\ {\scriptstyle ¶^{\Sigma}_{\mathbb{E}}}\downarrow & & \downarrow{\scriptstyle ¶_{\mathbb{E}}} \\ \mathbb{E} & = & \mathbb{E}. \end{array}$$

Recall from [**13**] the following:

Definition 2.2. *A reflexive relation on an object X:*

$$R \underset{d_1^R}{\overset{d_0^R}{\rightrightarrows}} X \quad (s_0^R \colon X \to R)$$

such that the pair (d_0^R, s_0^R) is a point in Σ is called a Σ-relation. *A morphism $f \colon X \to Y$ is said to be* Σ-special *when its kernel equivalence relation $R[f]$ is a Σ-equivalence relation.*

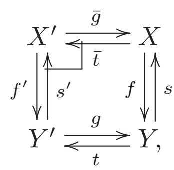
We denote by Σl the class of Σ-special morphisms. An object X is said to be Σ-*special* when the terminal map $\tau_X \colon X \to 1$ is Σ-special. Observe that, if a point (f, s) is in Σ, the morphism f is not necessarily Σ-special. However, when a Σ-special morphism f is split by s, the pair (f, s) is in Σ (see [**13**] for more details). As usual, we denote by $\mathbb{E}^2$ the category whose objects are the arrows in $\mathbb{E}$ and whose morphisms are the commutative squares. Finally, we denote by $\Sigma l\mathbb{E}^2$ the full subcategory of

$\mathbb{E}^2$ whose objects are the Σ-special morphisms, and by $\Sigma l_Y \mathbb{E}$ its subcategory whose morphisms have 1_Y as lower horizontal map.

2.3. Σ-Mal'tsev categories

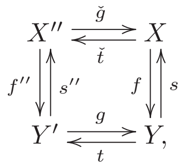
Recall from [**8**] the following:

Definition 2.4. *A finitely complete category* $\mathbb{E}$ *is said to be a* Σ-Mal'tsev *category when, given any pullback of points with* $(f,s) \in \Sigma$*:*



the pair $(s', \bar{t})$ *is jointly extremal epimorphic.*

The previous definition is equivalent to the following one: given any commutative square of points, with $(f,s) \in \Sigma$:



the unique induced morphism $\phi \colon X'' \to X'$ to the pullback of f and g is an extremal epimorphism.

Examples 2.5. 1. In the category Mon of monoids, a point $(f,s)\colon A \rightleftarrows B$ is *weakly Schreier* [**8**] if, for any $b \in B$, the map $\mu_b \colon \mathrm{Ker}(f) \to f^{-1}(b)$ defined by $\mu_b(x) = x \cdot s(b)$ is surjective. The point (f,s) is a *Schreier point* [**27, 22, 12**] if, for any $b \in B$, the map μ_b is bijective. It was observed in [**12**] that a point (f,s) is a Schreier point if and only if there exists a unique set-theoretical map $q_f \colon A \to \mathrm{Ker}(f)$ such that $a = q_f(a) \cdot sf(a)$ for all $a \in A$. The map q_f is called the *Schreier retraction* of (f,s). As shown in [**11, 8**], Mon is a Σ-protomodular category, and consequently, a Σ-Mal'tsev one (see Definition 5.3 and Theorem 5.4 below) for Σ either the class of Schreier or weakly Schreier points.

2. More generally, let $\mathbb{C}$ be a Jónsson-Tarski variety, i.e. a variety (in the sense of universal algebra) whose corresponding theory has a unique constant 0 and a binary term + satisfying the following axiom:

$$x + 0 = 0 + x = x.$$

The notion of a (weakly) Schreier point, given as above for monoids, actually makes sense in every Jónsson-Tarski variety. As shown in [**21**], every Jónsson-Tarski variety $\mathbb{C}$ is a Σ-protomodular (and hence Σ-Mal'tsev) category for the class Σ of Schreier points. Analogously, it is easy to see that $\mathbb{C}$ is Σ-protomodular for the class Σ of weakly Schreier points, too. In particular, the category $SRng$ of semirings is Σ-protomodular for both classes.

3. Both the definition of a Schreier point of monoids and the proof that Mon is a Σ-protomodular category, when Σ is the class of Schreier points, only make use of finite limits. Hence they are invariant under the Yoneda embedding. This means that it makes sense to consider *internal Schreier points* in every category $Mon\mathbb{E}$ of internal monoids in any category $\mathbb{E}$, and that $Mon\mathbb{E}$ is Σ-protomodular w.r.t. the class of such points. The same is true if we replace Mon with every Jónsson-Tarski variety.
4. A quandle [**19, 26**] is a set A equipped with two binary operations $\lhd$ and $\lhd^{-1}$ such that the following identities hold (for all $a,\ b,\ c \in A$):
   (A1) $a \lhd a = a = a \lhd^{-1} a$ (idempotency);
   (A2) $(a \lhd b) \lhd^{-1} b = a = (a \lhd^{-1} b) \lhd b$ (right invertibility);
   (A3) $(a \lhd b) \lhd c = (a \lhd c) \lhd (b \lhd c)$ and
   $(a \lhd^{-1} b) \lhd^{-1} c = (a \lhd^{-1} c) \lhd^{-1} (b \lhd^{-1} c)$ (self-distributivity).

   The structure of quandle is of interest in knot theory, since the three axioms above correspond to the Reidemeister moves on oriented link diagrams. A quandle homomorphism is a map which preserves both $\lhd$ and $\lhd^{-1}$. In the category Qnd of quandles and quandle homomorphisms, a point $(f, s)\colon A \rightleftarrows B$ is called *puncturing* (resp. *acupuncturing*) if, for every $b \in B$, the map $\mu_b\colon f^{-1}(b) \to f^{-1}(b)$, defined by $\mu_b(a) = s(b) \lhd a$, is surjective (resp. bijective). In [**7**] it was shown that Qnd is a Σ-Mal'tsev category (which is not a Σ-protomodular one) with respect to both classes of puncturing and acupuncturing points.
5. Let Cat_B be the fiber above a set B of the fibration $(\ )_0\colon Cat \to Set$ which associates with every category its set of objects. A point (F, S) in Cat_B has *fibrant splittings* [**6**] if, for every arrow φ in the codomain of F, the arrow $S(\varphi)$ is cartesian. Cat_B is Σ-protomodular (and hence Σ-Mal'tsev) for the class Σ of points with fibrant splittings. We observe that Mon is Cat_1, where 1 is the one-element set, and that in this case the notion of point with fibrant splittings reduces to the notion of Schreier point.

We say that the class Σ is *point-congruous* when $\Sigma Pt\mathbb{E}$ is closed under finite limits in $Pt\mathbb{E}$ (which implies that it contains all the isomorphisms); in this case $\Sigma l\mathbb{E}^2$ is closed under finite limits in $\mathbb{E}^2$. If $\mathbb{C}$ is a Jónsson-Tarski variety, and Σ is the class of Schreier points, then Σ is a point-congruous class. This is the case, in particular, for the categories Mon of monoids and $SRng$ of semirings. Similarly, if $\mathbb{C} = Cat_B$, the class of points with fibrant splitting is point-congruous. If $\mathbb{C} = Qnd$, the class of acupuncturing points is point-congruous. Recall from [**8**] the following remarkable:

Theorem 2.6. *Suppose Σ is point-congruous and $\mathbb{E}$ is a Σ-Mal'tsev category, then:*

1. *when gf and g are Σ-special, so is f;*
2. *for any object Y, the subcategory $\Sigma l/Y$ of the slice category $\mathbb{E}/Y$ is a Mal'tsev category.*

In particular, if we denote by $\Sigma\mathbb{E}_\sharp = \Sigma l_1\mathbb{E}$ the full subcategory of $\mathbb{E}$ whose objects are the Σ-special objects, it is a Mal'tsev category [**13, 8**], called the *Mal'tsev core* of the point-congruous Σ-Mal'tsev category $\mathbb{E}$; any of its morphisms is Σ-special. If $\mathbb{C} = Mon$, $\Sigma\mathbb{C}_\sharp$ is the category Gp of groups; if $\mathbb{C} = SRng$, $\Sigma\mathbb{C}_\sharp$ is the category Rng of (not necessarily unitary) rings (in both cases, we are considering the class Σ of

Schreier points). If $\mathbb{C} = Qnd$, and Σ is the class of acupuncturing points, $\Sigma\mathbb{C}_\sharp$ is the category of *Latin quandles*, i.e. those quandles whose Cayley table for the operation $\lhd$ is a Latin square.

3. The regular context

In this section we suppose that the ground category $\mathbb{E}$ is regular. Then the categories $\mathbb{E}^2$ and $Pt\mathbb{E}$ are regular as well, and their regular epimorphisms are the levelwise ones.

We recall from [**4**] the following definition (see also [**17**], where the notion was first considered, under the name of *double extension*, in the category of groups):

Definition 3.1. *Let $\mathbb{E}$ be a regular category. Consider the following commutative square of regular epimorphisms:*

$$\begin{array}{ccc} X & \overset{x}{\twoheadrightarrow} & X' \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle f'} \\ Y & \underset{y}{\twoheadrightarrow} & Y'. \end{array}$$

It is said to be a regular pushout *if the induced morphism $(f, x)\colon X \to Y \times_{Y'} X'$ into the pullback of y and f' is a regular epimorphism.*

Any regular pushout is, in particular, a pushout. Moreover, if we consider the commutative diagram

$$\begin{array}{ccccc} & & R[f] & \overset{R(x)}{\twoheadrightarrow} & R[f'] \\ & & {\scriptstyle d_1^f}\downarrow\uparrow\downarrow{\scriptstyle d_0^f} & & {\scriptstyle d_1^{f'}}\downarrow\uparrow\downarrow{\scriptstyle d_0^{f'}} \\ R[x] & \underset{d_1^x}{\overset{d_0^x}{\leftleftarrows\!\!\rightarrow}} & X & \overset{x}{\twoheadrightarrow} & X' \\ {\scriptstyle R(f)}\big\downarrow & & {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle f'} \\ R[y] & \underset{d_1^y}{\overset{d_0^y}{\leftleftarrows\!\!\rightarrow}} & Y & \underset{y}{\twoheadrightarrow} & Y', \end{array}$$

where we denote by $R[x]$ the kernel pair of x, and by $R(f)$ the morphism between the kernel pairs induced by f, we have that the induced arrows $R(f)$ and $R(x)$ are regular epimorphisms.

Proposition 3.2. *Let $\mathbb{E}$ be a regular Σ-Mal'tsev category. Consider any regular epimorphism of points as on the right-hand side of the following diagram:*

$$\begin{array}{ccccc} R[x] & \underset{d_1^x}{\overset{d_0^x}{\leftleftarrows\!\!\rightarrow}} & X & \overset{x}{\twoheadrightarrow} & X' \\ {\scriptstyle R(f)}\downarrow\uparrow{\scriptstyle R(s)} & & {\scriptstyle f}\downarrow\uparrow{\scriptstyle s} & & {\scriptstyle f'}\downarrow\uparrow{\scriptstyle s'} \\ R[y] & \underset{d_1^y}{\overset{d_0^y}{\leftleftarrows\!\!\rightarrow}} & Y & \underset{y}{\twoheadrightarrow} & Y'. \end{array}$$

When its domain (f, s) is in Σ, or when the regular epimorphism y is Σ-special, the *downward square on the right is a regular pushout, and the arrow $R(x)\colon R[f] \to R[f']$ is a regular epimorphism.*

Proof. Consider the diagram

$$\begin{array}{ccccccccc}
R[x] & \xrightarrow{\psi} & \bar{X} & \xrightarrow{\bar{d}_0^y} & X & \xrightarrow{\phi} & \bar{X}' & \xrightarrow{\bar{y}} & X' \\
{\scriptstyle R(f)}\downarrow\uparrow{\scriptstyle R(s)} & & {\scriptstyle \bar{f}}\downarrow\uparrow{\scriptstyle \bar{s}} & & {\scriptstyle f}\downarrow\uparrow{\scriptstyle s} & & {\scriptstyle \bar{f}'}\downarrow\uparrow{\scriptstyle \bar{s}'} & & {\scriptstyle f'}\downarrow\uparrow{\scriptstyle s'} \\
R[y] & = & R[y] & \xrightarrow[d_0^y]{} & Y & = & Y & \xrightarrow[y]{} & Y',
\end{array}$$

where $(\bar{f}', \bar{s}')$ is the pullback of (f', s') along y, $(\bar{f}, \bar{s})$ is the pullback of (f, s) along d_0^y, and ψ and ϕ are induced by the universal property of the pullback. Since $\mathbb{E}$ is a Σ-Mal'tsev category, ψ is a regular epimorphism whenever (f, s) is in Σ, or whenever (d_0^y, s_0^y) is in Σ, namely when y is Σ-special. Denote by $\delta_1 = (x\bar{d}_0^y, d_1^y \bar{f})\colon \bar{X} \to \bar{X}'$ the induced morphism into the pullback. The morphism δ_1 is the pullback of x along $\bar{y}$, and so it is a regular epimorphism. Moreover, we get $\phi\, d_1^x = \delta_1\, \psi$, which is a regular epimorphism, being the composite of two regular epimorphisms. Accordingly, ϕ itself is a regular epimorphism, and the square in question is a regular pushout. □

Whence an important consequence about the direct images of equivalence relations along regular epimorphisms; in general they are no longer equivalence relations, but only reflexive and symmetric ones. However, we get:

Corollary 3.3. *Let $\mathbb{E}$ be a regular Σ-Mal'tsev category, $f\colon X \twoheadrightarrow Y$ a regular epimorphism, and R an equivalence relation on X. When R is a Σ-equivalence relation, or when f is Σ-special, the direct image $f(R)$ is itself an equivalence relation on Y.*

Proof. Consider the following diagram, where $f(R)$ is given by the canonical decomposition $R \twoheadrightarrow f(R) \rightarrowtail Y \times Y$ of the map $(f\, d_0^R, f\, d_1^R)\colon R \to Y \times Y$:

$$\begin{array}{ccc}
R & \overset{\check{f}}{\twoheadrightarrow} & f(R) \\
{\scriptstyle d_0^R}\downarrow\ \uparrow{\scriptstyle s_0^R}\ \downarrow{\scriptstyle d_1^R} & & {\scriptstyle d_0}\downarrow\ \uparrow{\scriptstyle s_0}\ \downarrow{\scriptstyle d_1} \\
X & \underset{f}{\twoheadrightarrow} & Y.
\end{array}$$

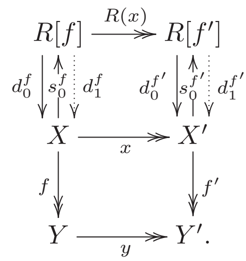
According to the previous proposition, under the assumption that R is a Σ-equivalence relation or that f is Σ-special, the split vertical square indexed by 0 is a regular pushout. Accordingly, the induced arrow $R(\check{f})\colon R[d_0^R] \to R[d_0]$ is a regular epimorphism. Thanks to Proposition 1.14 in [**9**], we obtain that $f(R)$ is an equivalence relation. □

Proposition 3.4. *Let $\mathbb{E}$ be a regular Σ-Mal'tsev category. Consider a commutative square of regular epimorphisms as below, where f is Σ-special:*

$$\begin{array}{ccc}
X & \overset{x}{\twoheadrightarrow} & X' \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\
Y & \underset{y}{\twoheadrightarrow} & Y'.
\end{array} \qquad (2)$$

Suppose, moreover, that the induced arrow $R(x)\colon R[f] \to R[f']$ *is a regular epimorphism. Then the square* (2) *is a regular pushout and the arrow* $R(f)\colon R[x] \to R[y]$ *is a regular epimorphism. In other words,* (2) *is a regular pushout if and only if* $R(x)$ *is a regular epimorphism.*

Proof. Complete Diagram (2) by the kernel equivalence relations of the vertical maps:



The previous proposition shows that the non-dotted upper part of this diagram is a regular pushout, and consequently, that the induced arrow $\psi\colon R[f] \to \overline{R[f']}$, where $\overline{R[f']}$ is the pullback of $d_0^{f'}$ and x, is a regular epimorphism. The same method as in the previous proposition allows us to show that the induced arrow $\phi\colon X \to \bar{X}'$, where $\bar{X}'$ is the pullback of f' and y, is a regular epimorphism, and consequently, that the square (2) is a regular pushout. □

4. The denormalized 3×3 lemma

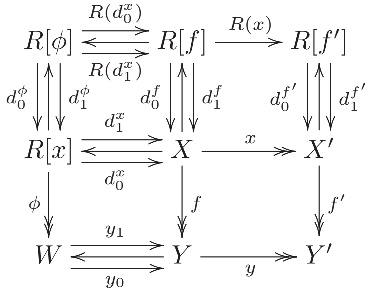
4.1. Preliminary observations

A diagram

$$G \underset{d_0}{\overset{d_1}{\rightleftarrows}} X \xrightarrow{f} Y, \qquad (3)$$

where the left-hand side part is a reflexive graph and f coequalizes the pair (d_0, d_1), is said to be *left exact* when G is the kernel equivalence relation $R[f]$ of f, while it is *right exact* when f is the coequalizer of the pair (d_0, d_1). It is said to be *exact* when it is both left and right exact.

A commutative diagram



(4)

is said to be a *weakly* 3×3 *diagram* when the middle row and the three columns are exact, and a 3×3 *diagram* when all the rows and columns are exact. In a weakly

3×3 diagram, the pair $(R(d_0^x), R(d_1^x))$ is necessarily jointly monomorphic, and the map y is necessarily an extremal epimorphism. Since ϕ is an epimorphism, the lower row is right exact if and only if the lower right-hand side square is a pushout. Now suppose $\mathbb{E}$ is regular.

Proposition 4.2 ([**4**])**.** *Let $\mathbb{E}$ be a regular category. In a weakly 3×3 diagram, the upper row is left exact if and only if the pair (y_0, y_1) is jointly monomorphic.*

We recall from [**4**] the "denormalized 3×3 lemma" for regular Mal'tsev categories:

Proposition 4.3. *Let $\mathbb{E}$ be a regular Mal'tsev category. Given any weakly 3×3 diagram (4), the following conditions are equivalent:*

(i) the upper row is exact;

(ii) the lower row is exact;

(iii) (4) is a 3×3 diagram.

In any category $\mathbb{E}$, we shall say that a weakly 3×3 diagram (4) satisfies the denormalized 3×3 lemma if the three previous conditions are equivalent for the diagram.

Proposition 4.4. *Let $\mathbb{E}$ be a regular category. A weakly 3×3 diagram (4), such that the lower right-hand side square is a regular pushout, satisfies the 3×3 lemma.*

Proof. In such a diagram, the maps $R(x)$ and $R(f)$ are necessarily regular epimorphisms. Since the lower right-hand side square is a pushout, the lower row is right exact. Suppose now that the lower row is left exact; then the upper row is left exact by Proposition 4.2. Since $R(x)$ is a regular epimorphism, the upper row is exact. Conversely, when the upper row is exact, the pair (y_0, y_1) is jointly monomorphic, and the induced arrow $t\colon W \to R[y]$ is a monomorphism such that $t\,\phi = R(f)$. Since $R(f)$ is a regular epimorphism, t is a regular epimorphism as well, and consequently, an isomorphism. Accordingly the lower row is left exact and, since g is a regular epimorphism, right exact. □

4.5. The regular Σ-Mal'tsev context

We shall now investigate what is remaining of the denormalized 3×3 lemma in a regular Σ-Mal'tsev category $\mathbb{E}$.

Theorem 4.6. *Let $\mathbb{E}$ be a regular Σ-Mal'tsev category. Consider a weakly 3×3 diagram (4):*

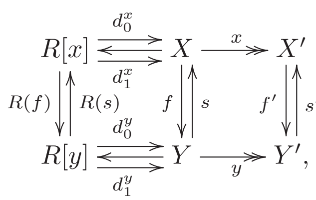
1. *if the morphism x is Σ-special and the lower row is exact, then the upper row is exact;*
2. *if the morphism f is Σ-special and the upper row is exact, then the lower row is exact;*
3. *if both f and x are Σ-special, then the diagram (4) satisfies the denormalized 3×3 lemma.*

Proof. 1. If the lower row is exact, we have $\phi = R(f)$. According to Proposition 3.4, the lower right-hand side square is a regular pushout, since x is Σ-special. Then the upper row is exact by Proposition 4.4.

2. When the upper row is exact, the map $R(x)$ is a regular epimorphim. When f is Σ-special, the lower right-hand side square is a regular pushout. Again by Proposition 4.4, the lower row is exact.
3. It is an immediate consequence of (1) and (2). □

We shall now investigate the conditions under which the rows are Σ-special. For that, let us recall from [**8**] the following definition:

Definition 4.7. *Let $\mathbb{E}$ be a regular category, and Σ a class of points. This class is said to be* 2-regular *whenever, given any regular epimorphism of points as on the right-hand side of the diagram*

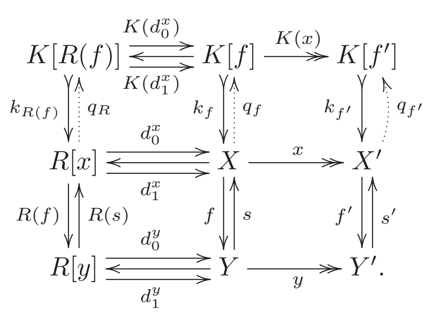


the point (f', s') is in Σ as soon as both (f, s) and $(R(f), R(s))$ belong to Σ.

The reason why a class Σ satisfying the condition above is called 2-regular is that in [**8**] three different levels of "regularity" (i.e. stability of Σ under regular epimorphisms) were considered. Furthermore, if $\mathbb{E}$ is a regular category and Σ is a 2-regular class of points in $\mathbb{E}$, then the full subcategory $\Sigma l/Y$ of the slice category $\mathbb{E}/Y$, whose objects are the Σ-special morphisms, is a regular category (see Proposition 4.9 below).

Proposition 4.8. *In the category Mon of monoids, the class Σ of Schreier points is* 2*-regular (see [**6**], where it is asserted, but not proved). More generally, given any finitely complete regular category $\mathbb{E}$, the class of internal Schreier points in the category $Mon\mathbb{E}$ of internal monoids is* 2*-regular.*

Proof. Consider a horizontal morphism of split epimorphisms, as in the lower right-hand side part of the diagram



Complete the diagram with the horizontal kernel pairs and the vertical kernels. Commutation of limits makes the upper row left exact. Since y and x are regular epimorphisms, and since (f, s) is a Schreier point, then the lower right-hand side square is a regular pushout by Proposition 3.2. Accordingly the map $K(x)$ is a regular epimorphism and the upper row is right exact as well. The maps x and $K(x)$ are still regular epimorphisms (= surjective maps) in the category Set of sets. Accordingly the Schreier retractions q_f and q_R produce the desired retraction $q_{f'}$. As for the internal case, it is easy to check that the category $Mon\mathbb{E}$ of internal monoids in $\mathbb{E}$ is regular when $\mathbb{E}$ is regular. Then the proof is just an internal version of the one we did for Mon. □

The class of acupuncturing points in the category Qnd of quandles is 2-regular, see [**7**].

Proposition 4.9. *Let Σ be a point-congruous and 2-regular class of points in a regular category $\mathbb{E}$. Then the full subcategory $\Sigma l/Y$ of the slice category $\mathbb{E}/Y$ is a regular category. Moreover, if f and h are as in diagram* (5) *below, if h a regular epimorphism, and there exists a morphism g such that f factors as $f = g\,h$, then g is Σ-special if and only if f and ϕ are. In other words, a map h in $\mathbb{E}/Y$ is a regular epimorphism in $\Sigma l/Y$ if and only if it is a regular epimorphism in $\mathbb{E}$ such that the kernel equivalence relation $R[f]$ belongs to $\Sigma l/Y$. Accordingly, when $\mathbb{E}$ is Barr-exact, the category $\Sigma l/Y$ is Barr-exact as well.*

Proof. Let us first show that $\Sigma l/Y$ admits quotients of effective equivalence relations. Consider a morphism h in $\Sigma l/Y$ as in the lower right-hand side square in the diagram below (which means that both f and f' are Σ-special):

$$\begin{array}{ccccc}
R[\phi] & \underset{R(d_1^h)}{\overset{R(d_0^h)}{\rightleftarrows}} & R[f] & \xrightarrow{R(h)} & R[f'] \\
{\scriptstyle d_0^\phi}\downarrow\uparrow\downarrow{\scriptstyle d_1^\phi} & & {\scriptstyle d_0^f}\downarrow\uparrow\downarrow{\scriptstyle d_1^f} & & {\scriptstyle d_0^{f'}}\downarrow\uparrow\downarrow{\scriptstyle d_1^{f'}} \\
R[h] & \underset{d_0^h}{\overset{d_1^h}{\rightleftarrows}} & X & \xrightarrow{h} & X' \\
{\scriptstyle \phi}\downarrow & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f'} \\
Y & = & Y & = & Y.
\end{array} \tag{5}$$

By commutation of limits, the upper row is left exact, and so the left-hand side of the two-level upper part of the diagram produces the kernel equivalence relation of the right-hand side morphism $(h, R(h))$ in the category $Eq\mathbb{E}$ of equivalence relations in $\mathbb{E}$. Since Σ is point-congruous and both $R[f]$ and $R[f']$ are Σ-equivalence relations, so is $R[\phi]$, and hence ϕ is Σ-special. Now let $h = m\,x$ be the decomposition of h with m a monomorphism and x a regular epimorphism, so that we have $R[h] = R[x]$ and $R[\phi] = R[R(x)]$. In this way, we get the following regular epimorphism $(x, R(x))$ between equivalence relations on the right-hand side:

$$\begin{array}{ccccc}
R[\phi] = R[R(x)] & \underset{R(d_1^h)}{\overset{R(d_0^h)}{\rightleftarrows}} & R[f] & \overset{R(x)}{\twoheadrightarrow} & R[f'\,m] \\
{\scriptstyle d_0^\phi}\downarrow\uparrow\downarrow{\scriptstyle d_1^\phi} & & {\scriptstyle d_0^f}\downarrow\uparrow\downarrow{\scriptstyle d_1^f} & & {\scriptstyle d_0^{f'}}\downarrow\uparrow\downarrow{\scriptstyle d_1^{f'}} \\
R[h] = R[x] & \underset{d_0^h}{\overset{d_1^h}{\rightleftarrows}} & X & \underset{x}{\twoheadrightarrow} & X'',
\end{array}$$

because the morphism $R(x)$ is a regular epimorphism: indeed, by $(f'\,m)\,x = f$, x is a regular epimorphism in the regular category $\mathbb{E}/Y$. Then so is the product

$x \times_Y x = R(x)$ in this same category. According to the 2-regularity of Σ, the right-hand side equivalence relation is a Σ-equivalence relation, and $x\colon f \to f' m$ is a regular epimorphism in $\Sigma l/Y$. Since $\mathbb{E}$ is regular, it is clear that these regular epimorphisms in $\Sigma l/Y$ are stable under pullbacks. Suppose now that $\mathbb{E}$ is Barr-exact and R is an equivalence relation on the object f in $\Sigma l/Y$, which means that the map $\phi = f\, d_0^R = f\, d_1^R$ is Σ-special. Since $\mathbb{E}$ is Barr-exact, there is some regular epimorphism h in $\mathbb{E}$ such that $R = R[h]$, whence the existence of a morphism f' as above, with f' Σ-special, according to the first part of this proof. Hence the equivalence relation R is effective in $\Sigma l/Y$. □

From now on, saying that an exact fork (3) is Σ-*exact* will mean that the morphism f is Σ-special.

Theorem 4.10. *Let $\mathbb{E}$ be a regular Σ-Mal'tsev category. Consider a weakly 3×3 diagram (4). Then:*

1. *if Σ is point-congruous, the map x is Σ-special, and the lower row is Σ-exact, then the upper row is Σ-exact;*
2. *if the class Σ is 2-regular, the maps f and x are in Σ, and the upper row is Σ-exact, then the lower row is Σ-exact.*

Proof.
1. We know, by Theorem 4.6, that the upper row is exact. Moreover, since Σ is point-congruous, the category $\Sigma l\mathbb{E}$ is stable under finite limits in $\mathbb{E}^2$. Accordingly, since both x and y are Σ-special, so is $R(x)$.
2. The map f being Σ-special and the upper row being exact, the lower row is exact by Theorem 4.6. Since x and $R(x)$ are Σ-special, $R[x]$ and $R[\phi] = R[R[x]]$ are Σ-equivalence relations. Since Σ is 2-regular, $W = R[y]$ is a Σ-equivalence relation, too. Consequently, the map y is Σ-special. □

There is a last situation dealing with the denormalized 3×3 lemma. We noticed that $\Sigma l/Y$ is a finitely complete category when Σ is point-congruous, and a Mal'tsev category when $\mathbb{E}$ is a Σ-Mal'tsev category. By Proposition 4.9, we know, moreover, that $\Sigma l/Y$ is regular when Σ is 2-regular.

Proposition 4.11. *Let $\mathbb{E}$ be a regular Σ-Mal'tsev category such that the class Σ is point-congruous and 2-regular. Any weakly 3×3 diagram (4) in $\mathbb{E}$ such that y, f', and $f'\,x = y\,f$ are Σ-special satisfies the denormalized 3×3 lemma.*

Proof. Let us think of diagram (4) as a diagram in the regular category $\mathbb{E}/Y'$. In this diagram, from any object there is a unique map to Y', so that any object in $\mathbb{E}/Y'$ can be identified with its domain. Here any object lies in $\Sigma l/Y'$ except $R[\phi]$ and W. When the lower row is exact, W is in $\Sigma l/Y'$, and so is $R[\phi]$. When the upper row is exact, $R[\phi]$ belongs to $\Sigma l/Y'$, and according to Proposition 4.9 so does W. So, under any of the conditions of the denormalized 3×3 lemma, the whole diagram lies in the regular Mal'tsev category $\Sigma l/Y'$, and the denormalized 3×3 lemma holds by Proposition 4.3. □

5. Σ-protomodular categories

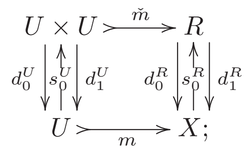
5.1. Preliminary observations

We recall that a category $\mathbb{E}$ is said to be *protomodular* [**1**] when any change-of-base functor with respect to the fibration of points is conservative. In the pointed context, this condition implies that the category $\mathbb{E}$ shares with the category Gp of groups the following well-known four properties:

(i) a morphism f is a monomorphism if and only if its kernel $K[f]$ is trivial; equivalently, pulling back reflects monomorphisms;

(ii) any regular epimorphism is the cokernel of its kernel; in other words, any regular epimorphism produces a short exact sequence;

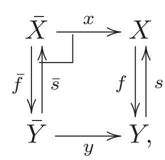
(iii) there is a very specific class of monomorphisms $m\colon U \rightarrowtail X$, the normal ones, namely those such that there exists a (necessarily unique) equivalence relation R on X such that $m^{-1}(R)$ is the indiscrete equivalence relation on U, and any commutative square in the following induced diagram is a pullback:



(iv) any reflexive relation is an equivalence relation, i.e. the category $\mathbb{E}$ is a Mal'tsev one.

We are now interested in seeing what remains of the properties above if we consider categories that are protomodular relatively to a pullback stable class Σ of points. To do that, let us first recall the following definition, see [**5**] and [**23**]:

Definition 5.2. *A point (f, s) in $\mathbb{E}$ is called* strong *whenever, given any pullback*



the pair (x, s) is jointly extremal epimorphic.

And also the following one, see [**6**], and [**13**] in the pointed case:

Definition 5.3. *Let $\mathbb{E}$ be a category endowed with a pullback stable class Σ of points. $\mathbb{E}$ is said to be* Σ-protomodular *when every point in Σ is strong.*

A category $\mathbb{E}$ is protomodular if and only if every point is strong. Let us now review the four previous properties with respect to this concept of partial protomodularity. As for (iv), we have the following:

Theorem 5.4 ([**6**])**.** *Let $\mathbb{E}$ be a category endowed with a pullback stable class Σ of points. Then:*

1. *when $\mathbb{E}$ is Σ-protomodular, it is a Σ-Mal'tsev category;*
2. *when, in addition, Σ is point-congruous, any change-of-base functor with respect to the subfibration $¶_{\mathbb{E}}^{\Sigma}$ of the fibration of points is conservative.*

As for (i) we get:

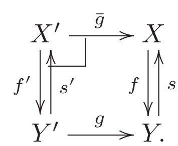
Proposition 5.5. *Let $\mathbb{E}$ be a Σ-protomodular category. Then:*

1. *the split epimorphic part of a point in Σ is an isomorphism if and only if any of its pullbacks is an isomorphism;*
2. *pulling back Σ-special morphisms reflects monomorphisms.*

In particular, when $\mathbb{E}$ is pointed, more classically we get:

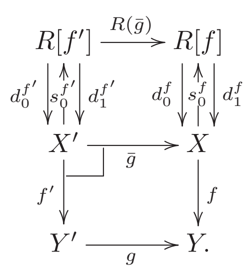
1. *the split epimorphic part of a point in Σ is an isomorphism if and only if its kernel is trivial;*
2. *a Σ-special morphism is a monomorphism if and only if its kernel is trivial.*

Proof. 1. Consider the following pullback, with (f, s) in Σ, and suppose f' is an isomorphism:



We get $s' f' = 1_{X'}$. We can check that $s\, f = 1_X$ by composing with the jointly extremal epimorphic pair $(s, \bar{g})$. The equality $s\, f\, s = 1_X\, s$ is straightforward. Moreover, $s\, f\, \bar{g} = \bar{g}\, s'\, f' = \bar{g} = 1_X\, \bar{g}$.

2. Consider the following diagram, where f is Σ-special and the lower square is a pullback:



Then the two upper commutative squares are pullbacks, and the pair (d_0^f, s_0^f) is in Σ. Suppose, moreover, that f' is a monomorphism; then $d_0^{f'}$ is an isomorphism. According to (1), the map d_0^f is itself an isomorphism, and f is a monomorphism. □

As for (ii), we get:

Proposition 5.6. *Let $\mathbb{E}$ be a Σ-protomodular category. Then:*

1. *given any pullback of the point (f, s) in Σ:*

$$\begin{array}{ccc} X' & \xrightarrow{\bar{g}} & X \\ {\scriptstyle f'}\downarrow\uparrow{\scriptstyle s'} & & {\scriptstyle f}\downarrow\uparrow{\scriptstyle s} \\ Y' & \xrightarrow[g]{} & Y, \end{array}$$

   the downward square is a pushout;

2. *consider any pullback of the form*

$$\begin{array}{ccc} X' & \xrightarrow{\bar{g}} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow[g]{} & Y. \end{array}$$

   If there exists a factorization $f\,h = g$ and f is a Σ-special regular epimorphism, the square is a pushout as well.

In particular, when $\mathbb{E}$ is pointed, more classically we get:

1. *any split epimorphism in Σ is the cokernel of is kernel;*
2. *any Σ-special regular epimorphism is the cokernel of its kernel.*

Proof. 1. Suppose we have a pair of morphisms $\alpha\colon X \to Z$, $\gamma\colon Y' \to Z$ such that $\alpha\,\bar{g} = \gamma\,f'$. We have that $\alpha\,s\,f = \alpha$: this can be checked by composing with the jointly extremal epimorphic pair $(s, \bar{g})$. Since f is an epimorphism, this factorization is unique. It remains to check that $\alpha\,s\,g = \beta$; this can be done by composing with the epimorphism f'.

2. Consider the same pullback without splittings, with f a Σ-special morphism, and $f\,h = g$. Then the map h produces a splitting $(1, h)\colon Y' \to X'$ of f', which is then an epimorphism. Now complete the diagram with the kernel equivalence relations:

$$\begin{array}{ccc} R[f'] & \xrightarrow{R(\bar{g})} & R[f] \\ {\scriptstyle d_0^{f'}}\downarrow\uparrow{\scriptstyle s_0^{f'}}\downarrow{\scriptstyle d_1^{f'}} & & {\scriptstyle d_0^{f}}\downarrow\uparrow{\scriptstyle s_0^{f}}\downarrow{\scriptstyle d_1^{f}} \\ X' & \xrightarrow{\bar{g}} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow[g]{} & Y. \end{array}$$

Any commutative square in the upper part is a pullback; moreover, the pair (d_0^f, s_0^f) is in Σ, because f is Σ-special. Suppose we have a pair of morphisms $\alpha\colon X \to Z$, $\gamma\colon Y' \to Z$ such that $\alpha\,\bar{g} = \gamma\,f'$. Let us show that α coequalizes the pair (d_0^f, d_1^f). This can be checked by composing with the jointly extremal epimorphic pair $(s_0^f, R(\bar{g}))$. We have

$$\alpha\, d_0^f\, s_0^f = \alpha = \alpha\, d_1^f\, s_0^f,$$

and

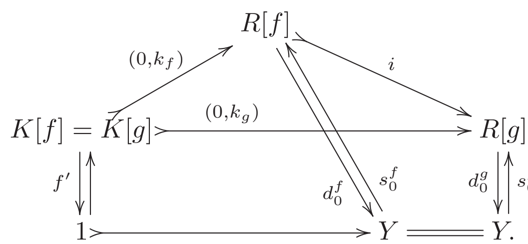
$$\alpha\, d_0^f\, R(\bar{g}) = \alpha\, \bar{g}\, d_0^{f'} = \gamma\, f'\, d_0^{f'} = \gamma\, f'\, d_1^{f'} = \alpha\, \bar{g}\, d_1^{f'} = \alpha\, d_1^f\, R(\bar{g}).$$

Since f is a regular epimorphism, there is a unique morphism $\bar{\alpha}\colon Y \to Z$ such that $f\,\bar{\alpha} = \alpha$. The equality $\bar{\alpha}\, g = \gamma$ can be checked by composing with the epimorphism f'. □

In the regular context we can add the following precision:

Proposition 5.7. *Let $\mathbb{E}$ be a pointed regular Σ-protomodular category and f a Σ-special regular epimorphism. Let $g\colon X \to Y'$ be a Σ-special morphism such that $k_g = k_f$, where k_g and k_f are the kernels of g and f, respectively. Then the unique arrow t such that $g = t\, f$ is a monomorphism.*

Proof. We have $0 = g\, k_g = g\, k_f$. Moreover, f, being Σ-special, is the cokernel of k_f (by Proposition 5.6). Hence we get the induced morphism t, and an inclusion $i\colon R[f] \rightarrowtail R[g]$ between two Σ-equivalence relations. Now consider the diagram



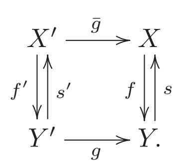
The rectangle is a pullback and (d_0^g, s_0^g) is in Σ, so the pair $(s_0^g, (0, k_g))$ is jointly extremal epimorphic. Since the pair $(s_0^f, (0, k_f))$ factors through the monomorphism i, i is an isomorphism. Hence we get $R[f] \simeq R[g]$, which implies, in the regular category $\mathbb{E}$, that t is a monomorphism. □

As for (iii), we recall from [**6**] the following:

Proposition 5.8. *Let $\mathbb{E}$ be a Σ-Mal'tsev category. When $m\colon U \rightarrowtail X$ is a monomorphism which is normal to a Σ-equivalence relation S on X, the object U is Σ-special. When $\mathbb{E}$ is Σ-protomodular, a monomorphism m is normal to at most one Σ-equivalence relation.*

As a first step toward the 3×3 lemma, we get:

Proposition 5.9. *Let $\mathbb{E}$ be a pointed Σ-protomodular category. Consider any morphism between points, with codomain (f, s) in Σ:*



(6)

Suppose, moreover, that the induced arrow $\phi\colon X' \to \check{X}$, where $\check{X}$ is the pullback of

(f,s) along g, is a monomorphism. Then the square (6) *is a pullback if and only if the map $K(\bar{g})\colon K[f'] \to K[f]$ is an isomorphism. It is clear that the condition on ϕ holds as soon as $\bar{g}$ is a monomorphism (the splittings making then g itself a monomorphism).*

Proof. Clearly if (6) is a pullback, then $K(\bar{g})$ is an isomorphism. Conversely, consider the following diagram with exact columns:

$$\begin{array}{ccccc}
K[f'] & \xrightarrow{K(\bar{g})} & K[\check{f}] & = & K[f] \\
{\scriptstyle k_{f'}}\downarrow & & {\scriptstyle (0,k_f)}\downarrow & & {\scriptstyle k_f}\downarrow \\
X' & \xrightarrow{\phi} & \check{X} & \xrightarrow{\check{g}} & X \\
{\scriptstyle f'}\downarrow\uparrow{\scriptstyle s'} & & {\scriptstyle \check{f}}\downarrow\uparrow{\scriptstyle \check{s}} & & {\scriptstyle f}\downarrow\uparrow{\scriptstyle s} \\
Y' & = & Y' & \xrightarrow{g} & Y.
\end{array}$$

Since $\mathbb{E}$ is Σ-protomodular and (f,s) in Σ, $(\check{f},\check{s})$ is in Σ, too, and the pair $(\check{s},(0,k_f))$ is jointly extremal epimorphic. Since s' and $k_{f'}\,K(\bar{g})^{-1}$ produce factorizations through the monomorphism ϕ, ϕ is an isomorphism, and the square (6) is a pullback. □

Corollary 5.10. *Let $\mathbb{E}$ be a pointed regular Σ-protomodular category.*

1. *Let f' be a regular epimorphism such that $f' = f\,m$ with m a monomorphism and f a Σ-special morphism. If $k(m)\colon K[f'] \to K[f]$ is an isomorphism, then m is an isomorphism, too.*
2. *Consider a commutative diagram*

$$\begin{array}{ccccc}
K[f'] & \xrightarrow{k_{f'}} & X' & \xrightarrow{f'} & Y' \\
{\scriptstyle K(\bar{g})}\downarrow & & {\scriptstyle \bar{g}}\downarrow & & \downarrow{\scriptstyle g} \\
K[f] & \xrightarrow[k_f]{} & X & \xrightarrow[f]{} & Y,
\end{array} \tag{7}$$

where f is a Σ-special regular epimorphism and f' is a regular epimorphism. Then $\bar{g}$ is a regular epimorphism and the right-hand side square is a regular pushout as soon as both g and $K(\bar{g})$ are regular epimorphisms. Accordingly the map $K(f')\colon K[\bar{g}] \to K[g]$ is a regular epimorphism as well.

Proof. 1. Consider the diagram

$$\begin{array}{ccccccc}
K[f'] & \xrightarrow{(0,k_{f'})} & R[f'] & \overset{d_0^{f'}}{\underset{d_1^{f'}}{\rightrightarrows}}\ \overleftarrow{s_0^{f'}} & X' & \xrightarrow{f'} & Y \\
{\scriptstyle K(m)}\downarrow{\scriptstyle \simeq} & & {\scriptstyle R(m)}\downarrow & & {\scriptstyle m}\downarrow & & \parallel \\
K[f] & \xrightarrow[(0,k_f)]{} & R[f] & \overset{d_0^{f}}{\underset{d_1^{f}}{\rightrightarrows}}\ \overleftarrow{s_0^{f}} & X & \xrightarrow[f]{} & Y.
\end{array}$$

We can apply the previous proposition to the central square, because $K(m)$ is an isomorphism. The point (d_0^f, s_0^f) is in Σ, since f is Σ-special. Consequently, the central square indexed by 0 is a pullback, and this central part of the diagram becomes a discrete fibration between equivalence relations. According to the Barr-Kock Theorem valid in any regular category (see, for example, [**10**]), the right-hand side square is a pullback as well, since f and f' are regular epimorphisms. Accordingly m is an isomorphism.

2. Consider the diagram (7). Let $\bar{g} = m\,\breve{g}$ be the decomposition of $\bar{g}$ with $m\colon U \rightarrowtail X$ a monomorphism and $\breve{g}$ a regular epimorphism. Then $f\,m\colon U \to Y$ is a regular epimorphism, because g and f' are. Since $K(\bar{g})$ is a regular epimorphism, and hence a strong epimorphism, there is a unique arrow $k\colon K[f] \to U$ such that $m\,k = k_f$, and this k is necessarily the kernel of the regular epimorphism $f\,m$. Whence $K(m) = 1_{K[f]}$, so that, according to the first point, m is an isomorphism and $\bar{g}$ is a regular epimorphism.
Now take the pullback $\bar{f}$ of f and g:

$$\begin{array}{ccc} \bar{X} & \xrightarrow{\bar{f}} & Y' \\ {\scriptstyle \breve{g}}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow[f]{} & Y. \end{array}$$

$\bar{f}$ is a Σ-special regular epimorphism, since f is. Let $\bar{g} = \breve{g}\,\psi$ be the induced decomposition through this pullback. Consider now the following commutative diagram, where the lower row is a Σ-special exact sequence and f' is a regular epimorphism:

$$\begin{array}{ccccc} K[f'] & \overset{k_{f'}}{\rightarrowtail} & X' & \overset{f'}{\twoheadrightarrow} & Y \\ {\scriptstyle K(\bar{g})=K(\psi)}\downarrow & & \downarrow{\scriptstyle \psi} & & \| \\ K[f] & \underset{(k_f,0)}{\rightarrowtail} & \bar{X} & \underset{\bar{f}}{\twoheadrightarrow} & Y. \end{array}$$

The map ψ is a regular epimorphism, since $K(\psi) = K(\bar{g})$ is a regular epimorphism. Accordingly, the square $g\,f' = f\,\bar{g}$ is a regular pushout. □

6. Aspects of the 3×3 lemma

In a pointed category, a short exact sequence is a sequence $K[f] \overset{k_f}{\rightarrowtail} X \overset{f}{\twoheadrightarrow} Y$, where k_f is the kernel of f and f the cokernel of k_f. In a pointed protomodular category, a regular epimorphism f is the cokernel of its kernel, and so it gives rise to an exact sequence. In a Σ-protomodular category, Proposition 5.6 shows that any Σ-special regular epimorphism f determines an exact sequence, as well.

A commutative diagram

$$\begin{array}{ccccc} K[\phi] & \overset{K(k_x)}{\rightarrowtail} & K[f] & \xrightarrow{K(x)} & K[f'] \\ {\scriptstyle k_\phi}\downarrow & & \downarrow{\scriptstyle k_f} & & \downarrow{\scriptstyle k_{f'}} \\ K[x] & \underset{k_x}{\rightarrowtail} & X & \underset{x}{\twoheadrightarrow} & X' \\ {\scriptstyle \phi}\downarrow & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f'} \\ U & \xrightarrow[u]{} & Y & \underset{y}{\twoheadrightarrow} & Y' \end{array} \qquad (8)$$

in a pointed category $\mathbb{E}$ is said to be a *weakly* 3×3 *diagram* when the three columns and the middle row are exact, and a 3×3 *diagram* when all the columns and rows are exact. In a pointed regular protomodular category the 3×3 *lemma* holds:

Proposition 6.1 ([**3**]). *Given a weakly* 3×3 *diagram* (8) *in a pointed regular protomodular category* $\mathbb{E}$*, the following conditions are equivalent:*

1. *the upper row is exact;*
2. *the lower row is exact;*
3. (8) *is* 3×3 *diagram.*

Given a weakly 3×3 diagram (8) in a pointed category, we say that it satisfies the 3×3 lemma when the three previous conditions are equivalent for the diagram.

Suppose now that $\mathbb{E}$ is a pointed regular Σ-protomodular category. We shall be interested in the *weakly* Σ*-special* 3×3 *diagrams*, namely those diagrams whose three columns and middle row are Σ-exact, i.e. where f, f', ϕ, and x are Σ-special regular epimorphisms, and consequently, produce exact sequences.

The next proposition is a version for Σ-protomodular categories of the so-called *upper* 3×3 *lemma* (the terminology is borrowed from [**18**]):

Proposition 6.2. *Let* $\mathbb{E}$ *be a pointed regular* Σ*-protomodular category and let* (8) *be a weakly* Σ*-special* 3×3 *diagram. Suppose, moreover, that* Σ *is point-congruous. When the lower row is* Σ*-exact, so is the upper row.*

Proof. By assumption, y is a Σ-special regular epimorphism, u is the kernel of y, and $\phi = K(f)$. This last equality implies that the upper row is left exact by commutation of limits, namely that $K(k_x)$ is the kernel of $K(x)$.

According to Corollary 5.10, this implies that the lower right-hand side square in (8) is a regular pushout, and hence $K(x)$ is a regular epimorphism. When Σ is point-congruous, the map $K(x)$ is in Σ, because x and y are. □

The converse implication, the so-called *lower* 3×3 *lemma*, requires further information, pointing out an asymmetric situation when compared to the *upper* 3×3 *lemma*.

Definition 6.3. *Let* Σ *be a pullback stable class of points in a pointed category* $\mathbb{E}$*. We say that this class is* equi-consistent *when, given any split epimorphism of equivalence relations*

$$\begin{array}{ccc} R & \overset{\bar{t}}{\underset{\bar{g}}{\leftrightarrows}} & S \\ d_0^R \downarrow \uparrow \downarrow d_1^R & & d_0^S \downarrow \uparrow \downarrow d_1^S \\ X & \overset{t}{\underset{g}{\leftrightarrows}} & Y, \end{array} \qquad (9)$$

where R *a* Σ*-equivalence relation and the split epimorphism* (g, t) *is in* Σ*,* $(\bar{g}, \bar{t})$ *is in* Σ *as soon as* $(K_0(\bar{g}), K_0(\bar{t}))$ *is in* Σ*, where* $K_0(\bar{g})$ *and* $K_0(\bar{t})$ *are the restrictions of* $\bar{g}$ *and* $\bar{t}$ *to the kernels of* d_0^R *and* d_0^S*.*

When Σ is point-congruous, the equivalence relation S is a Σ-one as well, since the morphism $(t, \bar{t})$ is an equalizer in $Pt\mathbb{E}$. Hence the previous condition becomes a characteristic one: namely, under the same assumptions of the previous proposition, the point $(\bar{g}, \bar{t})$ is in Σ if and only if the point $(K_0(\bar{g}), K_0(\bar{t}))$ is in Σ. Indeed, being Σ a point-congruous class, the kernel of a map in $\Sigma Pt\mathbb{E}$ is still in $\Sigma Pt\mathbb{E}$.

Proposition 6.4. *The class of Schreier points in Mon is equi-consistent. The same holds for the class of internal Schreier points in $Mon\mathbb{E}$.*

Proof. Consider a commutative diagram like (9) in Mon. Denote by q the Schreier retraction of the Schreier point (g, t); thanks to Proposition 2.4 in [**12**], we know that the map q satisfies the equalities $x = q(x) \cdot tg(x)$ for all $x \in X$, and $q(k \cdot t(y)) = k$ for all $(k, y) \in K[g] \times Y$. Proposition 2.1.5 in [**11**] tells us that the following two equalities also hold:

1. $q(x \cdot x') = q(x) \cdot q(tg(x) \cdot q(x')) \quad \text{for all}\, (x, x') \in X \times X;$
2. $q(t(y) \cdot k) \cdot t(y) = t(y) \cdot k \quad \text{for all}\, (k, y) \in K[g] \times Y.$

Because of the uniqueness of q, it suffices to show that $q(x)Rq(x')$ whenever xRx'. The equi-consistent assumption means that, when $x = 1$, we have $1Rq(x')$. Since R is a Schreier equivalence relation, there is a Schreier retraction $\chi \colon R \to K[d_0^R]$ satisfying the conditions $1R\chi(xRx')$, $x' = \chi(xRx') \cdot x$, and $\chi(zR(u \cdot z)) = u$ for all $z \in X$, whenever we have $1Ru$. Starting with xRx', we get $1R\chi(xRx')$ and $x' = \chi(xRx') \cdot x'$. Hence we get from (1) that:

$$q(x') = q(\chi(x, x')) \cdot q(tg\chi(x, x') \cdot q(x)).$$

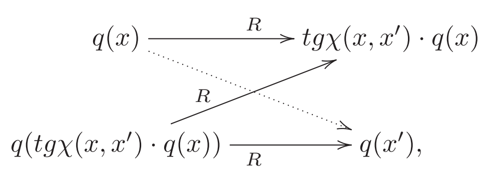
So, from $1Rq\chi(xRx')$, we also get $(q(tg\chi(x, x') \cdot q(x)))Rq(x')$. Now, from (2) we have:

$$q(tg\chi(x, x') \cdot q(x)) \cdot tg\chi(x, x') = tg\chi(x, x') \cdot q(x).$$

So, from $1Rtg\chi(x, x')$, we get

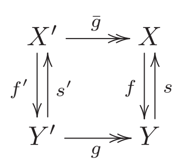
$$q(x)Rtg\chi(x, x') \cdot q(x) \,\text{and}\, q(tg\chi(x, x') \cdot q(x))Rtg\chi(x, x') \cdot q(x).$$

Whence the following situation:



and $q(x)Rq(x')$ as desired. The second assertion is a straightforward consequence of the Yoneda embedding. □

Proposition 6.5. *Let Σ be a pullback stable, 2-regular, equi-consistent class of points in a pointed regular category $\mathbb{E}$. Consider any regular epimorphism in $Pt\mathbb{E}$*



with domain (f', s') in Σ and the morphisms g and $\bar{g}$ Σ-special. Then (f, s) is in Σ as soon as the restriction $(K(f'), K(s'))$ to the kernels of g and $\bar{g}$ is in Σ.

Proof. Consider the following diagram:

$$\begin{array}{ccccccc} K[\bar g] & \xrightarrow{(0,k_{\bar g})} & R[\bar g] & \underset{d_1^{\bar g}}{\overset{d_0^{\bar g}}{\rightrightarrows}} & X' & \xrightarrow{\bar g} & X \\ {\scriptstyle K(f')}\downarrow\uparrow{\scriptstyle K(s')} & & {\scriptstyle R(f')}\downarrow\uparrow{\scriptstyle R(s')} & & {\scriptstyle f'}\downarrow\uparrow{\scriptstyle s'} & & {\scriptstyle f}\downarrow\uparrow{\scriptstyle s} \\ K[g] & \xrightarrow[(0,k_g)]{} & R[g] & \underset{d_1^{g}}{\overset{d_0^{g}}{\rightrightarrows}} & Y' & \xrightarrow[g]{} & Y. \end{array}$$

Since $R[g]$ and $R[\bar g]$ are Σ-equivalence relations and the class Σ is equi-consistent, the pair $(R(f'), R(s'))$ is in Σ. Moreover, since Σ is 2-regular, the pair (f, s) is in Σ as well. □

Corollary 6.6. *Let Σ be a pullback stable, point-congruous, 2-regular, equi-consistent class of points in a pointed regular category $\mathbb{E}$. Consider any regular pushout*

$$\begin{array}{ccc} X' & \xrightarrow{x} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow[y]{} & Y \end{array}$$

such that the morphisms f', y, and x are Σ-special. Then the map f is Σ-special as soon as the restriction $K(f') \colon K[x] \to K[y]$ is.

Proof. Consider the following diagram:

$$\begin{array}{ccccc} R[K(f')] & \xrightarrow{R(k_x)} & R[f'] & \xrightarrow{R(x)} & R[f] \\ {\scriptstyle d_0^{K(f')}}\downarrow\uparrow\downarrow{\scriptstyle d_1^{K(f')}} & & {\scriptstyle d_0^{f'}}\downarrow\uparrow\downarrow{\scriptstyle d_1^{f'}} & & {\scriptstyle d_0^{f}}\downarrow\uparrow\downarrow{\scriptstyle d_1^{f}} \\ K[x] & \xrightarrow{k_x} & X' & \xrightarrow{x} & X \\ {\scriptstyle K(f')}\downarrow & & {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ K[y] & \xrightarrow[k_y]{} & Y' & \xrightarrow[y]{} & Y. \end{array}$$

The maps $R(x)$ and $K(f')$ are regular epimorphisms, since the lower right-hand side square is a regular pushout. The upper row is left exact by commutation of limits. The map $R(x)$ is Σ-special, because y and x are Σ-special and Σ is point-congruous.

Since f' and $K(f')$ are Σ-special, the pairs $(d_0^{f'}, s_0^{f'})$ and $(d_0^{K(f')}, s_0^{K(f')})$ are in Σ. According to the previous proposition (d_0^f, s_0^f) is in Σ, and hence f is Σ-special. □

Now we can state the version for Σ-protomodular categories of the so-called *lower* 3 × 3 *lemma*:

Theorem 6.7. *Let $\mathbb{E}$ be a pointed regular Σ-protomodular category for a point-congruous, 2-regular, equi-consistent class Σ of points. Consider a weakly Σ-special* 3 × 3 *diagram* (8). *When the upper row is Σ-exact, so is the lower one.*

Proof. When Σ is point-congruous, the map $K(f) \colon K[x] \to K[y]$ is Σ-special, since f and f' are. When the upper row is left exact, we get $k_\phi = k_{K(f)}$. By Proposition 5.7, the morphism t such that $K(f) = t\,\phi$ is a monomorphism, and we get $u = k_y\, t$.

Since f' is Σ-special, and y and $K(x)$ are regular epimorphisms, Corollary 5.10 implies that the lower right-hand side square is a regular pushout, so that $K(f)$ is a regular epimorphism. Accordingly t is a regular epimorphism and consequently, an isomorphism, so that u is the kernel of y.

It remains to show that the regular epimorphism y is Σ-special. This is a consequence of Corollary 6.6, since the lower right-hand side square is a regular pushout, f and f' are Σ-special, and x and $K(x)$ are Σ-special, this last point thanks to the assumption that the upper row is Σ-exact. □

There is a last situation dealing with the 3×3 lemma. We observed that $\Sigma l/Y$ is finitely complete when Σ is point-congruous, and regular when Σ is 2-regular. From [**8**], we know, moreover, that $\Sigma l/Y$ is protomodular when $\mathbb{E}$ is Σ-protomodular. Recall from [**3**] the following:

Definition 6.8. *A category* $\mathbb{E}$ *is said to be* quasi-pointed *when it has an initial object* 0 *and the terminal map* $\tau_0\colon 0 \rightarrowtail 1$ *is a monomorphism. A category* $\mathbb{E}$ *is said to be* sequentiable *when it is quasi-pointed, protomodular and regular.*

Any fiber Cat_B (see Example 2.5 (5)) is quasi-pointed. The category Grd_B of groupoids with set of objects B is sequentiable. If a category $\mathbb{E}$ is pointed, regular and Σ-protomodular with respect to a point-congruous and 2-regular class Σ, then the category $\Sigma l/Y$ is sequentiable.

In a sequentiable category, the kernel of a map f is defined as the pullback of $f\colon X \to Y$ along the monomorphic initial map $\alpha_Y\colon 0 \rightarrowtail Y$; a sequence is said to be exact when f is a regular epimorphism and the following pullback is a pushout as well:

$$\begin{array}{ccc} K[f] & \overset{k_f}{\rightarrowtail} & X \\ \downarrow & & \downarrow f \\ 0 & \underset{\alpha_Y}{\rightarrowtail} & Y. \end{array}$$

It was proved in [**3**] that, in a sequentiable category, any weakly 3×3 diagram satisfies the 3×3 lemma with respect to this extended notion of exact sequence. Whence the following:

Proposition 6.9. *Let* Σ *be a point-congruous and* 2*-regular class of points in a pointed regular* Σ*-protomodular category* $\mathbb{E}$*. Any weakly* 3×3 *diagram* (8) *in* $\mathbb{E}$ *such that the morphisms* y*,* f'*, and* $f'\,x = y\,f$ *are* Σ*-special satisfies the* 3×3 *lemma.*

Proof. The proof is exactly on the same model as the one of Proposition 4.11. Let us think of diagram (8) as a diagram in the regular category $\mathbb{E}/Y'$. In this diagram, from any object there is a unique morphism to Y', so that any object in $\mathbb{E}/Y'$ can be identified with its domain. In our case, any object lies in $\Sigma l/Y'$ except $K[\phi]$ and U. When the lower row is exact, U is in $\Sigma l/Y'$, and so is $K[\phi]$. When the upper row is exact, the upper left-hand side square is a pullback, since $k_{f'}$ is a monomorphism. Accordingly, k_ϕ is also the kernel of $f\,k_x (= u\,\phi)$, which is Σ-special, being in $\Sigma l/Y'$. Hence the morphism u is a monomorphism by Proposition 5.7, and $R[\phi] = R[\phi\,u] = R[f\,k_x]$ is in $\Sigma l/Y'$; consequently, U is also in $\Sigma l/Y'$ by Proposition 4.9. In this way, under any of the conditions of the 3×3 lemma, the whole diagram lies in the sequentiable category $\Sigma l/Y'$, and the 3×3 lemma holds. □

6.10. The fibers $Cat_Y\mathbb{E}$

We recalled in Section 2 what is the class Σ of points with fibrant splittings in the category Cat_Y. As in $Cat_1 = Mon$, there is a description of these points in terms of Schreier retractions: they are those points $(F, S)\colon \mathbb{A} \rightleftarrows \mathbb{B}$ which are equipped with a function $q\colon \mathbb{A} \to K[F]$ such that $\phi = q(\phi)\, SF(\phi)$ for all $\phi \in \mathbb{B}$, and $q(k\, S(\psi)) = k$ for all $(k, \psi) \in K[F] \times \mathbb{B}$. q is called the *internal Schreier retraction* of the point (F, S) (notice that, consequently, any split epimorphism above a groupoid has cofibrant splittings: in this case we have $q(\phi) = \phi\, SF(\phi^{-1})$).

Given any category $\mathbb{E}$, we denote by $(\ )_0\colon Cat\mathbb{E} \to \mathbb{E}$ the fibration associating its object of objects with any internal category, and by $Cat_Y\mathbb{E}$ the fiber above Y. When $\mathbb{E}$ is a regular category, the category $Cat\mathbb{E}$ of internal categories in $\mathbb{E}$ is no longer regular, in general; however, so is any fiber $Cat_Y\mathbb{E}$, which is also quasi-pointed.

We say that an internal split epimorphic functor $(\underline{F}_1, \underline{S}_1)\colon \underline{X}_1 \rightleftarrows \underline{Y}_1$ *has cofibrant splittings* when there is a morphism $q_1\colon X_1 \to K[F_1]$ in the underlying category $\mathbb{E}$ such that $m_{X_1}(k_{F_1}\, q_1, S_1F_1) = 1_{X_1}$ and $q_1\, m_{X_1}(k_{F_1}, S_1) = p_0^{K[F_1]}$, where m_{X_1} denotes the composition map of the internal category $\underline{X}_1$. Denote by Σ_Y the class of points with cofibrant splittings. By the Yoneda embedding, we get (see [**8**]): 1) any split epimorphism above an internal groupoid has cofibrant splittings; 2) the class Σ_Y is stable under pullbacks and point-congruous; and 3) any fiber $Cat_Y\mathbb{E}$ is Σ_Y-protomodular, and accordingly Σ_Y-Mal'tsev.

Proposition 6.11. *Given any regular category $\mathbb{E}$, any fiber $Cat_Y\mathbb{E}$ is such that the class of points having cofibrant splittings is* 2*-regular.*

Proof. Thanks to the internal Schreier retractions, the same proof as in Proposition 4.8 holds. □

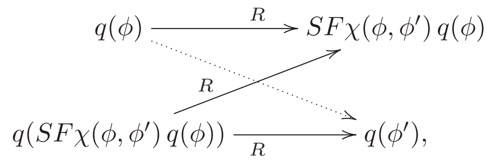
Accordingly, Theorems 4.6 and 4.10, as well as Proposition 4.11, hold for the fibers $Cat_Y\mathbb{E}$. As for the quasi-pointed version of the 3×3 lemma, Propositions 6.2 and 6.9 hold as well. Let us conclude this section with the following observation:

Proposition 6.12. *The class Σ_Y of points with fibrant splittings, in any fiber Cat_Y, is equi-consistent. Given any category $\mathbb{E}$, the same holds for any fiber $Cat_Y\mathbb{E}$.*

Proof. The proof of the first assertion mimics the proof of Proposition 6.4. Starting with a parallel pair $(\phi, \phi')\colon a \rightrightarrows b$ such that $\phi R \phi'$, we have to show that $q(\phi) R q(\phi')$. Since R is a Σ_Y-equivalence relation, there is an endo map $\chi(\phi, \phi')\colon b \to b$ such that $\phi' = \chi(\phi, \phi')\, \phi$ and $1_B R \chi(\phi, \phi')$; whence $1 R q\chi(\phi, \phi')$, since the point (K_0F, K_0S) has fibrant splittings. As in the proof of Proposition 6.4, the identities

1. $q(\phi') = q\chi(\phi, \phi')\, q(SF\chi(\phi, \phi')\, q(\phi))$;
2. $q(SF\chi(\phi, \phi')\, q(\phi))\, SF(\chi(\phi, \phi')) = SF\chi(\phi, \phi')$

produce the diagram



and so $q(\phi)Rq(\phi')$ as desired.

The proof of the second assertion is a consequence of the Yoneda embedding and of the equations satisfied by the internal Schreier retractions. □

Accordingly, Proposition 6.7 holds for any fiber $Cat_Y\mathbb{E}$, when $\mathbb{E}$ is regular.

7. A remark on Baer sums

In [**25**], given any Schreier exact sequence with abelian kernel in Mon

$$A \overset{k_f}{\rightarrowtail} X \overset{f}{\twoheadrightarrow} Y,$$

a monoid action $\phi\colon Y \to End(A)$ of Y on A is produced, giving rise to an abelian group $Y \ltimes A \rightleftarrows Y$ in the slice category Mon/Y. Then a Baer sum construction of exact sequences giving rise to the same monoid action as above is described, together with a so-called push forward construction of Baer sums.

In [**7**], a similar Baer sum construction is given for quandles, concerning Σ-special exact sequences with *abelian kernel equivalence relations*, where Σ is the class of acupuncturing points of quandles. Recall that a Σ-special map $f\colon X \to Y$ has an abelian kernel equivalence relation when there is a Mal'tsev operation $p\colon R[f] \times_X R[f] \to X$ in $\Sigma l/Y$ (see the proof of the lemma below for more details).

But, as proved in [**21**], a Schreier exact sequence in Mon has an abelian kernel if and only if it has an abelian kernel equivalence relation. We can be even more precise about this point:

Lemma 7.1. *Given any Schreier equivalence relation R on X in Mon, the following conditions are equivalent:*

(i) $K[d_0^R]$ is an abelian group;

(ii) for all $1Rt$ and xRx', we get $q(xR(x \cdot t)) = q(x'R(x' \cdot t))$, where q is the Schreier retraction of the point (d_0^R, s_0^R);

(iii) the Schreier equivalence relation R is abelian.

Proof. We start by observing that the map q satisfies the following equalities (see Corollary 2.8 in [**24**]):

(a) $q(aRb) \cdot b = q(bRa) \cdot b = a$;

(b) $q((b \cdot b')R(a \cdot a')) = q(bRa) \cdot q(bR(b \cdot q(b'Ra')))$.

Suppose (i). Thanks to the equality (a) above we get:

$$\begin{aligned} q(xR(x \cdot t)) \cdot x' &= q(xR(x \cdot t)) \cdot q(xRx') \cdot x = q(xRx') \cdot q(xR(x \cdot t)) \cdot x \\ &= q(xRx') \cdot x \cdot t = x' \cdot t. \end{aligned}$$

Whence $q(xR(x \cdot t)) = q(x'R(x' \cdot t))$ and (ii) holds.

Now, assume that (ii) holds, and define the Mal'tsev operation by $p(aRbRc) = q(bRa) \cdot c$. First we have to show the Mal'tsev axioms, namely: $p(aRaRc) = c = p(cRaRa)$. We have:

$$q(aRa) \cdot c = 1 \cdot c = c = q(aRc) \cdot a = p(cRaRa).$$

It remains to show that p is a monoid homomorphism, namely:

$$q(bRa) \cdot c \cdot q(b'Ra') \cdot c' = q((b \cdot b')R(a \cdot a')) \cdot c \cdot c'.$$

Using the equality (b) above, it is enough to check the following equality:

$$q(bR(b \cdot q(b'Ra'))) \cdot c = c \cdot q(b'Ra').$$

We have bRc, so that, by (ii), it remains to check that:

$$q(cR(c \cdot q(b'Ra'))) \cdot c = c \cdot q(b'Ra').$$

This is true since the point (d_0^R, s_0^R) is a Schreier point, so that (iii) holds.

The fact that condition (iii) implies (i) is true in any category. □

As shown in [**21**], the fact that a Schreier exact sequence has an abelian kernel if and only if it has an abelian kernel equivalence relation is actually true in every category of monoids with operations in the sense of [**22**], in particular, in the category *SRng* of semirings. An analogue of Lemma 7.1 can be proved in a very similar way for all such algebraic structures.

According to Lemma 7.1, a Schreier exact sequence has an abelian kernel if and only if it has an abelian kernel equivalence relation. This shows that both Baer sum constructions in the categories Mon and Qnd appear to be of the same nature.

Actually they are both particular instances of a very general situation described in [**2**] concerning Barr-exact Mal'tsev categories, which we are now going to recall. First observe that the category $\Sigma l/Y$, in both cases of Mon and Qnd is a Barr-exact category by Proposition 4.9, since so are Mon and Qnd because they are both varieties of universal algebras. Secondly $\Sigma l/Y$, in both cases, is a Mal'tsev category, since both Mon and Qnd are Σ-Mal'tsev ones.

Now, given any Mal'tsev category, we say that an object X is *affine* when it is endowed with a (necessarily unique) Mal'tsev operation, namely a ternary operation $p\colon X \times X \times X \to X$ satisfying $p(x, y, y) = x = p(y, y, x)$. Recall from [**14**] that a Mal'tsev operation in any Mal'tsev category is necessarily associative and commutative. Denote by $Aff\mathbb{E}$ the full subcategory of affine objects. An object X is *abelian* when it is endowed with a (necessarily unique and abelian) internal group structure. An abelian object is nothing but an affine object X equipped with a point $0_X : 1 \rightarrowtail X$; in set-theoretical terms, the abelian group operation $a + b$ is just $p(a, 0, b)$. Denote by $Ab\mathbb{E}$ the full subcategory of abelian objects, and by $U\colon Ab\mathbb{E} \to Aff\mathbb{E}$ the forgetful functor. An object X has *global support* if the terminal map $\tau_X : X \to 1$ is a regular epimorphism.

Definition 7.2 ([**2**]). *Given a Barr-exact Mal'tsev category* $\mathbb{E}$ *and an affine object* X *with global support, the* direction $d(X)$ *of* X *is the abelian object defined by the* *diagram*

$$\begin{array}{ccccc} X\times X\times X & \underset{(p_0p_0,p)}{\overset{p_2}{\rightrightarrows}} & X\times X & \overset{q_X}{\twoheadrightarrow} & d(X) \\ {\scriptstyle p_0}\downarrow\uparrow\downarrow{\scriptstyle (p,p_1p_2)} & & {\scriptstyle p_0}\downarrow\uparrow\downarrow{\scriptstyle p_1} & & {\scriptstyle o_X}\uparrow\downarrow{\scriptstyle \tau_{d(X)}} \\ X\times X & \underset{p_0}{\overset{p_1}{\rightrightarrows}} & X & \underset{\tau_X}{\twoheadrightarrow} & 1, \end{array}$$

where p is the Mal'tsev operation and q_X is the quotient of the upper horizontal equivalence relation.

The two left-hand side vertical groupoid structures associated with the structure of equivalence relation give rise (by passage to the quotient) to a vertical right-hand side groupoid structure above the terminal object 1, namely to a group structure in $\mathbb{E}$. To make it short, we shall denote by the only symbol $d(X)$ this whole group structure. It is clear that, when X is an abelian object, we have $d(U(X)) \simeq X$ in a natural way.

Take $\mathbb{E} = \Sigma l/Y$ in Mon; the objects of $\mathbb{E}$ are the special Schreier homomorphisms $f\colon X \to Y$ with codomain the monoid Y. The objects with global support are the special Schreier surjective homomorphisms over Y. An object f is affine in $\mathbb{E}$ if and only if it has an abelian kernel equivalence relation, i.e. if and only if it has an abelian kernel. In this case, its direction is precisely the abelian group $Y \ltimes A \rightleftarrows Y$ described at the beginning of this section.

Let us denote by $\mathbb{E}_g$ the full subcategory of $\mathbb{E}$ whose objects are the ones with global support. We have the following:

Proposition 7.3 ([**2**], Proposition 6 and Theorem 7)**.** *Let $\mathbb{E}$ be a Barr-exact Mal'tsev category. The construction of the direction gives rise to a finite products preserving functor $d\colon Aff\mathbb{E}_g \to Ab\mathbb{E}$ which is a cofibration whose morphisms in the fibers are isomorphisms.*

Suppose again that $\mathbb{E}$ is $\Sigma l/Y$ in Mon. The fact that the morphisms in the fibers are isomorphisms is nothing but the short five lemma for special Schreier exact sequences (see Proposition 7.2.2 in [**11**] for a proof of this version of the short five lemma). Moreover, the construction of the cocartesian map above a morphism in $Ab\mathbb{E}$ is precisely what is called the push forward construction described in [**25**].

Proposition 7.4 ([**2**], Theorem 9)**.** *Let $\mathbb{E}$ be a Barr-exact Mal'tsev category. Given any abelian object A in $\mathbb{E}$, the groupoid $d^{-1}(A)$ is canonically endowed with a closed symmetric monoidal structure, and any change-of-base functor of the cofibration d is monoidal.*

Proof. Given any pair of affine objects C and C', both with the abelian group A as direction, we first observe that $d(C \times C') = A \times A$; then their Baer sum is the codomain of the cocartesian map above the abelian group operation $+\colon A \times A \to A$, namely the codomain of the quotient map of the equivalence relation on $C \times C'$

determined by the following pullback in $\mathbb{E}$:

$$\begin{array}{ccc} R & \twoheadrightarrow & A \\ \downarrow & & \downarrow {\scriptstyle (1_A, -1_A)} \\ (C \times C') \times (C \times C') & \underset{q_{C \times C'}}{\twoheadrightarrow} & A \times A. \end{array}$$

□

When $\mathbb{E}$ is the category $\Sigma l/Y$ in Mon, the construction of this tensor product on the fiber $d^{-1}(Y \ltimes A \rightleftarrows Y)$ coincides with the Baer sum, described in [**24, 25**], of exact sequences whose associated monoid action corresponds to the split sequence $Y \ltimes A \rightleftarrows Y$. This is also the case of the construction of the Baer sum of special exact sequences of quandles described in [**7**].

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References

[**1**] D. Bourn, *Normalization equivalence, kernel equivalence and affine categories*, in: Lecture Notes in Math. 1488 (1991), Springer, Berlin, 43–62.

[**2**] D. Bourn, *Baer sums and fibered aspects of Mal'cev operations*, Cah. Topol. Géom. Différ. Catég. 40 (1999), 297–316.

[**3**] D. Bourn, 3 × 3 *lemma and protomodularity*, J. Algebra 236 (2001), 778–795.

[**4**] D. Bourn, *The denormalized* 3 × 3 *lemma*, J. Pure Appl. Algebra 177 (2003), 113–129.

[**5**] D. Bourn, *On the monad of internal groupoids*, Theory Appl. Categ. 28 (2013), 150–165.

[**6**] D. Bourn, *Mal'tsev reflection, S-Mal'tsev and S-protomodular categories*, preprint Cahiers LMPA, n. 497 (2014).

[**7**] D. Bourn, *A structural aspect of the category of quandles*, J. Knot Theory Ramifications 24, n. 12 (2015), 1550060.

[**8**] D. Bourn, *Partial Mal'tsevness and partial protomodularity*, preprint arXiv: 1507.02886v1, 2015.

[**9**] D. Bourn, *Suprema of equivalence relations and non-regular Goursat categories*, Cah. Topol. Géom. Différ. Catég. 54 (2018), 142–194.

[**10**] D. Bourn and M. Gran, *Regular, protomodular, and abelian categories*, in: M.C. Pedicchio and W. Tholen (eds.), Categorical Foundations, Cambridge University Press (2004).

[**11**] D. Bourn, N. Martins-Ferreira, A. Montoli, and M. Sobral, *Schreier split epimorphisms in monoids and in semirings*, Textos Mat. (B) 45 (2013), Departamento de Matemática da Universidade de Coimbra.

[**12**] D. Bourn, N. Martins-Ferreira, A. Montoli, and M. Sobral, *Schreier split epimorphisms between monoids*, Semigroup Forum 88 (2014), 739–752.

[**13**] D. Bourn, N. Martins-Ferreira, A. Montoli, and M. Sobral, *Monoids and pointed S-protomodular categories*, Homology Homotopy Appl. 18, n. 1 (2016), 151–172.

[**14**] A. Carboni, M.C. Pedicchio, and N. Pirovano, *Internal graphs and internal groupoids in Mal'cev categories*, CMS Conf. Proc. 13 (1992), 97–109.

[**15**] M. Gran, Z. Janelidze, and D. Rodelo, 3×3 *lemma for star-exact sequences*, Homology Homotopy Appl. 14, n. 2 (2012), 1–22.

[**16**] M. Gran and D. Rodelo, *A new characterization of Goursat categories*, Appl. Categ. Structures 20 (2012), 229–238.

[**17**] G. Janelidze, *What is a double central extension? (the question was asked by Ronald Brown)*, Cah. Topol. Géom. Différ. Catég. 32 (1991), 191–201.

[**18**] Z. Janelidze, *The pointed subobject functor,* 3×3 *lemmas and subtractivity of spans*, Theory Appl. Categ. 23 (2010), 221–242.

[**19**] D. Joyce, *A classifying invariant of knots, the knot quandle*, J. Pure Appl. Algebra 23 (1982), 37–65.

[**20**] S. Lack, *The 3-by-3 lemma for regular Goursat categories*, Homology Homotopy Appl. 6 (2004), 1–3.

[**21**] N. Martins-Ferreira and A. Montoli, *On the "Smith is Huq" condition in S-protomodular categories*, Appl. Categ. Structures 25 (2017), 59–75.

[**22**] N. Martins-Ferreira, A. Montoli, and M. Sobral, *Semidirect products and crossed modules in monoids with operations*, J. Pure Appl. Algebra 217 (2013), 334–347.

[**23**] N. Martins-Ferreira, A. Montoli, and M. Sobral, *Semidirect products and Split Short Five Lemma in normal categories*, Appl. Categ. Structures 22 (2014), 687–697.

[**24**] N. Martins-Ferreira, A. Montoli, and M. Sobral, *Baer sums of special Schreier extensions of monoids*, Semigroup Forum 93 (2016), 403–415.

[**25**] N. Martins-Ferreira, A. Montoli, and M. Sobral, *The nine lemma and the push forward construction for special Schreier extensions of monoids with operations*, Semigroup Forum 97 (2018), 325–352.

[**26**] S.V. Matveev, *Distributive groupoids in knot theory*, Mat. Sb. (N.S.) 119 (161), n. 1 (9) (1982), 78–88.

[**27**] A. Patchkoria, *Crossed semimodules and Schreier internal categories in the category of monoids*, Georgian Math. J. 5, n. 6 (1998), 575–581.

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