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ERRATUM TO "JACOBI-ZARISKI EXACT SEQUENCE FOR HOCHSCHILD HOMOLOGY AND CYCLIC (CO)HOMOLOGY"

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(*communicated by Claude Cibils*)

Abstract

In our paper [**Kay12**] in Theorem 2.3 and all remaining results based on this theorem, it is stated that the Jacobi-Zariski long exact sequence works for the range $p \geqslant 0$. The range should be corrected to $p \geqslant 1$.

1. A counter-example

In [**Kay12**] in Theorem 2.3 we originally stated that if $\mathcal{B} \subseteq \mathcal{A}$ is an r-flat extension of unital algebras then we have a long exact sequence of the form

$$\cdots \to \mathrm{Tor}_{p+1}^{(\mathcal{A}|\mathcal{B})}(X,Y) \to \mathrm{Tor}_p^{\mathcal{B}}(X,Y) \to \mathrm{Tor}_p^{\mathcal{A}}(X,Y) \to \mathrm{Tor}_p^{(\mathcal{A}|\mathcal{B})}(X,Y) \to \cdots$$

for every $p \geqslant 0$. This statement is false as stated for $p = 0$. If the statement for Theorem 2.3 were true, then since we have

$$\mathrm{Tor}_0^{\mathcal{A}}(X,Y) = X \otimes_{\mathcal{A}} Y = \mathrm{Tor}_0^{(\mathcal{A}|\mathcal{B})}(X,Y)$$

the long exact sequence would have ended with

$$\cdots \to \mathrm{Tor}_1^{\mathcal{B}}(X,Y) \to \mathrm{Tor}_1^{\mathcal{A}}(X,Y) \to \mathrm{Tor}_1^{(\mathcal{A}|\mathcal{B})}(X,Y) \to X \otimes_{\mathcal{B}} Y \to 0.$$

If we take $\mathcal{B} = k$ and $X = \mathcal{A}$, the long exact sequence would dictate that the last term in the sequence $\mathcal{A} \otimes_k Y$ is trivial for every Y which is clearly not true.

We thank Claude Cibils for alerting us to the error.

2. The corrected statements and proofs

Theorem 2.3. *Assume $\mathcal{B} \subseteq \mathcal{A}$ is an r-flat extension of unital algebras. Then we have a long exact sequence of the form*

$$\cdots \to \mathrm{Tor}_{p+1}^{(\mathcal{A}|\mathcal{B})}(X,Y) \to \mathrm{Tor}_p^{\mathcal{B}}(X,Y) \to \mathrm{Tor}_p^{\mathcal{A}}(X,Y) \to \mathrm{Tor}_p^{(\mathcal{A}|\mathcal{B})}(X,Y) \to \cdots$$

for $p \geqslant 1$ where the last map in the sequence $\mathrm{Tor}_1^{\mathcal{A}}(X,Y) \to \mathrm{Tor}_1^{(\mathcal{A}|\mathcal{B})}(X,Y)$ is an epimorphism.

Proof. In the proof on page 72, we stated that

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For the subsequent pages in this spectral sequence, the only relevant differentials are

$$d^p_{p,0}\colon \operatorname{Tor}^{(\mathcal{A}|\mathcal{B})}_p(X,Y) \to \operatorname{Tor}^{\mathcal{B}}_{p-1}(X,Y)$$

which give us short exact sequences of the form

$$0 \to E^\infty_{p,0} \to \operatorname{Tor}^{(\mathcal{A}|\mathcal{B})}_p(X,Y) \to \operatorname{Tor}^{\mathcal{B}}_{p-1}(X,Y) \to E^\infty_{0,p-1} \to 0. \tag{1}$$

Unfortunately, these differentials work as prescribed only for $p \geqslant 2$ because we have a homological first quadrant spectral sequence, and we must deal with the cases $p = 0, 1$ separately. Then the rest of the proof works for the prescribed range without any modification and our long exact sequence ends with

$$\cdots \to \operatorname{Tor}^{(\mathcal{A}|\mathcal{B})}_2(X,Y) \to \operatorname{Tor}^{\mathcal{B}}_1(X,Y) \to E^\infty_{0,1} \to 0.$$

Now, we observe that the differentials originating at $E^1_{1,0}$ must all be zero, and therefore,

$$\operatorname{Tor}^{(\mathcal{A}|\mathcal{B})}_1(X,Y) = E^1_{1,0} = \cdots = E^\infty_{1,0}.$$

Since our spectral sequence converges to $\operatorname{Tor}^{\mathcal{A}}_*(X,Y)$ we have a short exact sequence

$$0 \to E^\infty_{0,1} \to \operatorname{Tor}^{\mathcal{A}}_1(X,Y) \to E^\infty_{1,0} = \operatorname{Tor}^{(\mathcal{A}|\mathcal{B})}_1(X,Y) \to 0 \tag{2}$$

coming from the filtration we defined on the two sided bar complex. Combining (1) and (2) we get the desired result. □

Here are the remaining statements on the Jacobi-Zariski long exact sequence with the corrected range.

Lemma 3.2. *If $\mathcal{A}$ is a finite dimensional $\Bbbk$-algebra and $\mathcal{B} \subseteq \mathcal{A}$ is an r-flat extension then for any finitely generated $\mathcal{A}$-modules X and Y we have a long exact sequence in cohomology of the form*

$$\cdots \to \operatorname{Ext}^p_{(\mathcal{A}|\mathcal{B})}(X,Y) \to \operatorname{Ext}^p_{\mathcal{A}}(X,Y) \to \operatorname{Ext}^p_{\mathcal{B}}(X,Y) \to \operatorname{Ext}^{p+1}_{(\mathcal{A}|\mathcal{B})}(X,Y) \to \cdots \tag{3}$$

for every $p \geqslant 1$, and the first map in the sequence $\operatorname{Ext}^1_{(\mathcal{A}|\mathcal{B})}(X,Y) \to \operatorname{Ext}^1_{\mathcal{A}}(X,Y)$ is a monomorphism.

Theorem 3.5. *Let $\mathcal{A}$ be an arbitrary unital $\Bbbk$-algebra where we make no assumption on the $\Bbbk$-dimension of $\mathcal{A}$. Let $\mathcal{B} \subseteq \mathcal{A}$ be an r-flat extension and assume Y is an approximately finite dimensional $\mathcal{A}$-module. Then for every $\mathcal{A}$-module X we have a long exact sequence of the form (3) for $p \geqslant 1$ where the first map in the sequence for $p = 1$ is a monomorphism.*

Theorem 4.1. *Assume $\mathcal{B} \subseteq \mathcal{A}$ is an r-flat extension of unital, associative but not necessarily commutative $\Bbbk$-algebras. Then for any $\mathcal{A}$-bimodule M we have a long exact sequence of the form*

$$\cdots \to HH_{p+1}(\mathcal{B},M) \to HH_p(\mathcal{B},M) \to HH_p(\mathcal{A},M) \to HH_p(\mathcal{A}|\mathcal{B},M) \to \cdots$$

for $p \geqslant 1$ where the last map in the sequence for $p = 1$ is an epimorphism. If we assume M is approximately finite dimensional then we obtain have a long exact *sequence in cohomology of the form*

$$\cdots \to HH^p(\mathcal{A}|\mathcal{B}, M) \to HH^p(\mathcal{A}, M) \to HH^p(\mathcal{B}, M) \to HH^{p+1}(\mathcal{A}|\mathcal{B}, M) \to \cdots$$

for every $p \geqslant 1$ *where the first map in the sequence for* $p = 1$ *is a monomorphism.*

Theorem 4.2. *Assume* $\mathcal{A}$, $\mathcal{B}$ *and* $\Bbbk$ *are as before. Then we have the following long exact sequences:*

$$\cdots \to HH_{p+1}(\mathcal{A}|\mathcal{B}) \to HH_p(\mathcal{B}) \to HH_p(\mathcal{A}) \to HH_p(\mathcal{A}|\mathcal{B}) \to \cdots,$$
$$\cdots \to HC_{p+1}(\mathcal{A}|\mathcal{B}) \to HC_p(\mathcal{B}) \to HC_p(\mathcal{A}) \to HC_p(\mathcal{A}|\mathcal{B}) \to \cdots,$$
$$\cdots \to HC^p(\mathcal{A}|\mathcal{B}) \to HC^p(\mathcal{A}) \to HC^p(\mathcal{B}) \to HC^{p+1}(\mathcal{A}|\mathcal{B}) \to \cdots$$

for $p \geqslant 1$*. For Hochschild and cyclic homologies the last map in the sequence for* $p = 1$ *is an epimorphism whereas for cyclic cohomology the first map in the sequence for* $p = 1$ *is a monomorphism.*

Theorem 4.3. *Let* $\varphi\colon \mathcal{B} \to \mathcal{A}$ *be an r-flat morphism of unital* $\Bbbk$*-algebras such that* $\mathcal{I} := ker(\varphi)$ *is H-unital. Let* $\mathrm{CH}_*(\mathcal{A}, \mathcal{B})$ *be the homotopy cofiber of the morphism* $\varphi_*\colon \mathrm{CH}_*(\mathcal{B}) \to \mathrm{CH}_*(\mathcal{A})$*. Then there is a homotopy cofibration sequence of the form*

$$\Sigma\mathrm{CH}_{*\geqslant 1}(\mathcal{I}) \to \mathrm{CH}_{*\geqslant 1}(\mathcal{A}, \mathcal{B}) \to \mathrm{CH}_{*\geqslant 1}(\mathcal{A}|\mathcal{B})$$

which induces an isomorphism $HH_1(\mathcal{A}, \mathcal{B}) \cong HH_1(\mathcal{A}|\mathcal{B})$*, and a long exact sequence of the form*

$$\cdots \to HH_{p+2}(\mathcal{A}|\mathcal{B}) \to HH_p(\mathcal{I}) \to HH_{p+1}(\mathcal{A}, \mathcal{B}) \to HH_{p+1}(\mathcal{A}|\mathcal{B}) \to \cdots$$

for $p \geqslant 1$*. Analogous sequences exist for the cyclic homology and cohomology with no additional hypothesis.*

Proof. The Wodzicki excision sequence works for the complete range $p \geqslant 0$, but the Jacobi-Zariski sequence works only for $p \geqslant 1$. This means our original proof works provided we replace each complex C_* in the proof with its good truncation $C_{*\geqslant 1}$. □

References

[Kay12] A. Kaygun. Jacobi-Zariski exact sequence for Hochschild homology and cyclic (co)homology. *Homology Homotopy Appl.*, 14(1):65–78, 2012.

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