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A MOMENT-ANGLE MANIFOLD WHOSE COHOMOLOGY HAS TORSION

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Abstract

In this paper we give a method to construct moment-angle manifolds whose cohomology has torsion. We also give method to describe the corresponding simplicial sphere by its non-faces.

1. Introduction

Corresponding to every abstract simplicial complex L on the vertex set $[m] = \{1, 2, \dots, m\}$, there are the real and complex moment-angle complexes $\mathbb{R}\mathcal{Z}_L$ and $\mathcal{Z}_L$ (*cf.* [**3, 4**]). They are defined as

$$\mathbb{R}\mathcal{Z}_L = \bigcup_{\sigma\in L} \prod_{i\in\sigma} D_i^1 \times \prod_{i\notin\sigma} S_i^0 \subseteq D_1^1 \times D_2^1 \times \cdots \times D_m^1,$$
$$\mathcal{Z}_L = \bigcup_{\sigma\in L} \prod_{i\in\sigma} D_i^2 \times \prod_{i\notin\sigma} S_i^1 \subseteq D_1^2 \times D_2^2 \times \cdots \times D_m^2.$$

The cohomology groups of $\mathbb{R}\mathcal{Z}_L$ and $\mathcal{Z}_L$ are given by Hochster's theorem: Corresponding to every abstract simplicial complex L on the vertex set $[m] = \{1, 2, \dots, m\}$, there are the real and complex moment-angle complexes $\mathbb{R}\mathcal{Z}_L$ and $\mathcal{Z}_L$ (*cf.* [**3, 4**]). They are defined as

$$\mathbb{R}\mathcal{Z}_L = \bigcup_{\sigma\in L} \prod_{i\in\sigma} D_i^1 \times \prod_{i\notin\sigma} S_i^0 \subseteq D_1^1 \times D_2^1 \times \cdots \times D_m^1,$$
$$\mathcal{Z}_L = \bigcup_{\sigma\in L} \prod_{i\in\sigma} D_i^2 \times \prod_{i\notin\sigma} S_i^1 \subseteq D_1^2 \times D_2^2 \times \cdots \times D_m^2.$$

The cohomology groups of $\mathbb{R}\mathcal{Z}_L$ and $\mathcal{Z}_L$ are given by Hochster's theorem:

Theorem 1.1 ([**1, 3, 4**]). *Let L be a simplicial complex on the vertex set $[m]$, then*

$$H^*(\mathbb{R}\mathcal{Z}_L) \cong \bigoplus_{I\subset[m]} \widetilde{H}^{*-1}(L|_I),$$
$$H^*(\mathcal{Z}_L) \cong \bigoplus_{I\subset[m]} \widetilde{H}^{*-|I|-1}(L|_I),$$

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where $L|_I$ is the full subcomplex of L on subset I and I runs over all the subsets of $[m]$.

From [**5, 6, 7**] we know that both $\mathbb{R}\mathcal{Z}_L$ and $\mathcal{Z}_L$ are topological manifolds if L is a simplicial sphere, referred to as moment-angle manifolds. Furthermore if L is a polytopal sphere (the boundary complex of a simplicial polytope), then $\mathcal{Z}_L$ is a transverse intersection of real quadratic hypersurfaces (*cf.* [**2**]), while both $\mathbb{R}\mathcal{Z}_L$ and $\mathcal{Z}_L$ are framed differentiable manifolds.

Bosio and Meersseman in [**2**] announced that the cohomology groups of differentiable complex moment-angle manifolds may have any torsion $\mathbb{Z}/m$. Furthermore if L is $\mathbb{Z}/2$ colourable, Cai, Choi and Park in [**8, 9**] proved that the small cover under $\mathbb{R}\mathcal{Z}_L$ may have any torsion $\mathbb{Z}/m$.

From Hochster's theorem, it is easy to construct a moment-angle complex whose cohomology has torsion. But it is harder to construct such moment-angle manifolds, at least, the cohomology of all the moment-angle manifolds corresponding to dimensional 1, 2 and 3 simplicial spheres are torsion free (*cf.* [**2**, Corollary 11.1]).

Based on Hochster's theorem, our goal is to find a simplicial complex K whose cohomology has torsion and K is embedded in a polytopal sphere L as a full subcomplex. Then both the real and complex moment-angle complexes corresponding to L are differentiable manifolds and the cohomology of $\mathbb{R}\mathcal{Z}_L$ and $\mathcal{Z}_L$ have $\widetilde{H}^*(K)$ as a summand and then have torsion.

Theorem 3.2 (Construction)**.** *Let K be a subcomplex (not a full subcomplex) of a simplicial sphere L_0 on the vertex set $[m]$, $\mathbb{M} = \{\sigma_1, \sigma_2, \dots, \sigma_s\}$ be the set of missing faces of K, which are also simplices of L_0. On L_0, make stellar subdivisions at $\sigma_1, \sigma_2, \dots, \sigma_s$ one by one as follows*

$$L_1 = ss_{\sigma_1} L_0, \quad L_2 = ss_{\sigma_2} L_1, \quad \dots, \quad L_s = ss_{\sigma_s} L_{s-1}.$$

Then K becomes a full subcomplex of L_s, $K = L_s|_{[m]}$.

In fact, after making stellar subdivision on a polytopal (simplicial) sphere, it is still polytopal (simplicial) (see [**10**]). If L_0 is also a polytopal sphere, we thus obtain a polytopal sphere L_s by Theorem 3.2 such that K is a full subcomplex of L_s. The real and complex moment-angle complexes corresponding to L_s are differentiable manifolds. By Hochster's theorem both $H^*(\mathbb{R}\mathcal{Z}_{L_s})$ and $H^*(\mathcal{Z}_{L_s})$ have torsion if $\widetilde{H}^*(K)$ has torsion.

At last in Section 4, we give a differentiable moment-angle manifold whose cohomology has $\mathbb{Z}/3$ as a summand. This is done as follows:

Triangulate the mod 3 Moore space K which has 8 vertices, 17 2-dimensional facets and 22 missing faces (see Figure 3). It can be embedded in $\partial\Delta^7 = L_0$. After making 22 stellar subdivisions on it, K becomes a full subcomplex of the polytopal sphere L_{22}. Then L_{22} is a 6-dimensional polytopal sphere with 30 vertices. $\mathcal{Z}_{L_{22}}$ is a 37-dimensional differentiable manifold and $H^{11}(\mathcal{Z}_{L_{22}})$ has $\widetilde{H}^2(K) = \mathbb{Z}/3$ as a summand.

It is notable that Bosio and Meersseman's construction in [**2**, Theorem 11.12] applied to the same example does not give a moment-angle manifold whose cohomology has $\mathbb{Z}/3$ torsion.

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2. Simplicial complement

An abstract simplicial complex K on the vertex set I is a collection of simplices that satisfies: for any simplex (face) $\sigma \in K$, all of its proper subsets (proper faces) are simplices of K.

An abstract simplicial complex K could also be given by all of its *non-faces*

$$\mathbb{A} = 2^I \setminus K$$

and

$$K = 2^I \setminus \mathbb{A}$$

that satisfies: if $\sigma \in \mathbb{A}$ is not a simplex of K and $\sigma' \supset \sigma$ then $\sigma' \in \mathbb{A}$ is not a simplex of K.

A simplex $\sigma = (i_1, i_2, \ldots, i_k) \in 2^I$ is called a *missing face* (or minimal non-face) of K if it is not a face of K, but all of its proper subsets are faces of K, i.e. $\sigma \notin K$ but every $\sigma_j = (i_1, \ldots, \widehat{i_j}, \ldots, i_k) \in K,\ j = 1, 2, \ldots, k$. An abstract simplicial complex could also be given by its set of missing faces

$$\mathbb{M} = \{\sigma \in 2^I \mid \sigma \text{ is a missing face of } K\}$$

and

$$K = \{\tau \in 2^I \mid \tau \text{ does not contain any } \sigma \in \mathbb{M}\}.$$

A subset σ' of I is not a simplex of K if and only if it contains a missing face $\sigma \in \mathbb{M}$ as a subset.

Definition 2.1. Let K be a simplicial complex on the vertex set I and $\mathbb{M}$, $\mathbb{A}$ be the sets of missing faces and non-faces of K respectively. We define a simplicial complement of K, denoted by

$$\mathbb{P} = \{\sigma_1, \sigma_2, \ldots, \sigma_s\},$$

to be a collection of non-faces that includes all the missing faces $\mathbb{M}$ i.e.

$$\mathbb{M} \subseteq \mathbb{P} \subseteq \mathbb{A}.$$

Similar to the set of missing faces $\mathbb{M}$, given a simplicial complement $\mathbb{P}$ (collection of non-faces) on the vertex set I, one can obtain a simplicial complex $K_{\mathbb{P}}$ on I by:

$$K_{\mathbb{P}}(I) = \{\tau \subset I \mid \tau \text{ does not contain any } \sigma_i \in \mathbb{P}\} \tag{1}$$

or by all of its non-faces

$$2^I \setminus K_{\mathbb{P}}(I) = \{\tau \subset I \mid \tau \text{ contains a } \sigma_i \in \mathbb{P}\}.$$

A subset σ of I is not a simplex of $K_{\mathbb{P}}(I)$ if and only if it contains a non-face σ_i in the simplicial complement $\mathbb{P}$.

Definition 2.2. Let $\mathbb{P}, \mathbb{P}'$ be two simplicial complements on the vertex set I, if they can obtain the same simplicial complex i.e. $K_{\mathbb{P}}(I) = K_{\mathbb{P}'}(I)$, we say that $\mathbb{P}$ and $\mathbb{P}'$ are equivalent, denoted by $\mathbb{P} \simeq \mathbb{P}'$.

It is easy to see that: Two simplicial complements $\mathbb{P}$, $\mathbb{P}^{'}$ on I are equivalent if and only if for every non-face $\sigma \in \mathbb{P}$ there exists a $\sigma' \in \mathbb{P}'$ such that $\sigma' \subseteq \sigma$ and for every non-face $\sigma' \in \mathbb{P}'$ there exists a $\sigma \in \mathbb{P}$ such that $\sigma \subseteq \sigma'$.

Proposition 2.3. *Let $\mathbb{P} = \{\sigma_1, \sigma_2, \ldots, \sigma_s\}$ be a simplicial complement of K on I. For a non-face $\sigma_j \in \mathbb{P}$ if there exists a $\sigma_i \in \mathbb{P}, i \neq j$ such that $\sigma_i \subseteq \sigma_j$, then we can remove σ_j from $\mathbb{P}$ and the resulting simplicial complement*

$$\mathbb{P}' = \{\sigma_1, \sigma_2, \ldots, \widehat{\sigma}_j, \ldots, \sigma_s\}$$

is equivalent to $\mathbb{P}$. In this case we call that $\mathbb{P}$ is reduced to $\mathbb{P}'$.

Every simplicial complement of K could be reduced to the set of missing faces by removing all the larger non-faces.

Example 2.4. The simplicial complex K is determined by the maximal simplices $(1,3)$, $(2,3)$, $(1,2,4)$, $(1,2,5)$, $(1,4,5)$, $(2,4,5)$ and their proper subsets on the vertex set $[5] = \{1,2,3,4,5\}$ (see Figure 1)

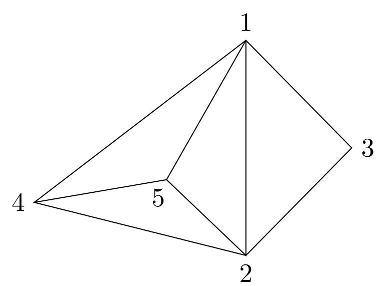

Figure 1: The simplicial complex K

$$\mathbb{P} = \left\{ \begin{array}{l} \sigma_1 = (1,2,4,5), \sigma_2 = (1,2,3), \sigma_3 = (3,4), \\ \sigma_4 = (3,5), \sigma_5 = (1,3,4), \sigma_6 = (3,4) \end{array} \right\}$$

is a simplicial complement of K on the vertex set $[5]$ where $\sigma_3 = (3,4) = \sigma_6$ appeared twice and $\sigma_3 = (3,4) \subset \sigma_5 = (1,3,4)$. So $\sigma_5 = (1,3,4)$ and $\sigma_6 = (3,4)$ could be removed from $\mathbb{P}$ to reduce to the set of missing faces $\mathbb{M} = \{(1,2,4,5), (1,2,3), (3,4), (3,5)\}$.

The readers should be aware that the empty simplex $\{\emptyset\}$ (only the empty set is a simplex) is different from the empty complex $\emptyset$ (the empty set is not a simplex of $\emptyset$). $\mathbb{M} = \{(1), (2), \ldots, (m)\}$ is the set of missing faces of the empty simplex $\{\emptyset\}$ while $\mathbb{M}_1 = \{\emptyset\}$ is the set of missing faces of the empty complex $\emptyset$.

Let K be a simplicial complex on the vertex set I and σ be a simplex of K. The *link* and *star* of σ are defined to be the simplicial complexes

$$link_K\sigma = \{\tau \in K \mid \sigma \cup \tau \in K, \sigma \cap \tau = \emptyset\}, \qquad star_K\sigma = \{\tau \in K \mid \sigma \cup \tau \in K\}.$$

The interior (open) *star* is defined to be the set of simplices (do not contain their proper faces)

$$Intstar_K\sigma = \{\tau \in K \mid \sigma \subset \tau\}$$

and the boundary of *star* is the simplicial complex (*cf.* [**12**, Lemma 62.6])

$$\partial star_K\sigma = star_K\sigma \setminus Intstar_K\sigma = \{\tau \in K \mid \sigma \cup \tau \in K, \sigma \not\subset \tau\}.$$

Let K_1 and K_2 be two simplicial complexes on the vertex set I and J, where $I \cap J = \emptyset$. The join of K_1 and K_2 is defined to be the simplicial complex on the vertex set $I \cup J$

$$K_1 * K_2 = \{\sigma \cup \tau \in 2^{I\cup J} \mid \sigma \in K_1, \tau \in K_2\}.$$

Let $\mathbb{P} = \{\sigma_1, \sigma_2, \ldots, \sigma_s\}$ be a simplicial complement of K on the vertex set I and $\sigma \in K$ be a simplex. We define

$$\mathbb{P} - \sigma = \{\sigma_1 \setminus \sigma, \sigma_2 \setminus \sigma, \ldots, \sigma_s \setminus \sigma\}$$

which is a sequence of subsets on $I \setminus \sigma$.

Lemma 2.5. *Let $\mathbb{P} = \{\sigma_1, \sigma_2, \ldots, \sigma_s\}$ be a simplicial complement of K on the vertex set I. Then*

1. *$\mathbb{P} - \sigma = \{\sigma_1 \setminus \sigma, \sigma_2 \setminus \sigma, \ldots, \sigma_s \setminus \sigma\}$ is a simplicial complement of $link_K\sigma$ on the vertex set $I \setminus \sigma$, i.e. by (1)*

   $$link_K\sigma = K_{\mathbb{P}-\sigma}(I \setminus \sigma) = \{\tau \subset (I \setminus \sigma) \mid \sigma \text{ does not contain any } \sigma_i \setminus \sigma \in \mathbb{P} - \sigma\}.$$

2. *If we consider $\mathbb{P} - \sigma$ as a sequence of non-faces on the vertex set I, then it is a simplicial complement of $star_K\sigma$ on I, i.e. by (1)*

   $$star_K\sigma = K_{\mathbb{P}-\sigma}(I) = \{\tau \subset I \mid \sigma \text{ does not contain any } \sigma_i \setminus \sigma \in \mathbb{P} - \sigma\}.$$

Proof. We prove this lemma by showing that they have the same non-faces

$$2^{I\setminus\sigma} \setminus link_K\sigma = 2^{I\setminus\sigma} \setminus K_{\mathbb{P}-\sigma}(I \setminus \sigma) = \{\tau \subset (I \setminus \sigma) \mid \tau \text{ contains a } \sigma_i \setminus \sigma \in \mathbb{P} - \sigma\}$$

and

$$2^I \setminus star_K\sigma = 2^I \setminus K_{\mathbb{P}-\sigma}(I) = \{\tau \subset I \mid \tau \text{ contains a } \sigma_i \setminus \sigma \in \mathbb{P} - \sigma\}.$$

1. From its definition, we know that a simplex τ on the vertex set $I \setminus \sigma$ is not a simplex of $link_K\sigma$ if and only if $\sigma \cup \tau$ is not a simplex of K. In other words, there exists a $\sigma_i \in \mathbb{P}$ such that $\sigma_i \subseteq \tau \cup \sigma$. This is equivalent to say that $\sigma_i \setminus \sigma \subseteq \tau \cup \sigma \setminus \sigma = \tau$, every non-face $\tau \in 2^{I\setminus\sigma} \setminus link_K\sigma$ is a non-face of $K_{\mathbb{P}-\sigma}(I \setminus \sigma)$, i.e. $\tau \in 2^{I\setminus\sigma} \setminus K_{\mathbb{P}-\sigma}(I \setminus \sigma)$, so

   $$2^{I\setminus\sigma} \setminus link_K\sigma \subseteq 2^{I\setminus\sigma} \setminus K_{\mathbb{P}-\sigma}(I \setminus \sigma).$$

2. If a simplex τ on the vertex set $I \setminus \sigma$ contains a $\sigma_i \setminus \sigma$, then $\tau \cup \sigma \supseteq (\sigma_i \setminus \sigma) \cup \sigma \supseteq \sigma_i$, so such τ is not a simplex of $link_K\sigma$. This is equivalent to say that every

non-face $\tau \in 2^{I\setminus\sigma} \setminus K_{\mathbb{P}-\sigma}(I \setminus \sigma)$ is a non-face of $link_K\sigma$, i.e. $\tau \in 2^{I\setminus\sigma} \setminus link_K\sigma$, so

$$2^{I\setminus\sigma} \setminus K_{\mathbb{P}-\sigma}(I \setminus \sigma) \subseteq 2^{I\setminus\sigma} \setminus link_K\sigma.$$

Thus $\mathbb{P} - \sigma = \{\sigma_1 \setminus \sigma, \sigma_2 \setminus \sigma, \ldots, \sigma_s \setminus \sigma\}$ is a simplicial complement of $link_K\sigma$ on the vertex set $I \setminus \sigma$.

Similarly, if we consider $\mathbb{P} - \sigma$ as a simplicial complement on the vertex set I, then

$$2^I \setminus star_K\sigma = 2^I \setminus K_{\mathbb{P}-\sigma}(I) = \{\tau \subset I \mid \tau \text{ contains a } \sigma_i \setminus \sigma \in \mathbb{P} - \sigma\}.$$

The lemma follows. □

Example 2.6. In Example 2.4, the *link* of the simplex $(1,2)$ is the pair of vertices $link_K(1,2) = \{(4),(5)\}$ and $star_K(1,2)$ is composed of two 2-simplices $(1,2,4)$, $(1,2,5)$ and its proper subsets.

$$\begin{aligned} \mathbb{M} - (1,2) &= \{(1,2,4,5) \setminus (1,2),\ \ (1,2,3) \setminus (1,2),\ \ (3,4) \setminus (1,2),\ \ (3,5) \setminus (2)\} \\ &= \{(4,5),\ (3),\ (3,4),\ (3,5)\} \\ &\simeq \{(4,5),(3)\} \end{aligned}$$

is a simplicial complement of $link_K(1,2)$ on the vertex set $\{3,4,5\}$. Consider $\mathbb{M} - \sigma$ as a sequence of non-faces on the vertex set $[5] = \{1,2,3,4,5\}$, it becomes the simplicial complement of $star_K(1,2)$.

Let $\mathbb{P}_1 = \{\sigma_1, \sigma_2, \ldots, \sigma_s\}$ and $\mathbb{P}_2 = \{\tau_1, \tau_2, \ldots, \tau_t\}$ be the simplicial complements of K_1 and K_2 on the vertex set I. We define their join $\mathbb{P}_1 * \mathbb{P}_2$ to be

$$\mathbb{P}_1 * \mathbb{P}_2 = \{\sigma_i \cup \tau_j \mid \sigma_i \in \mathbb{P}_1, \tau_j \in \mathbb{P}_2\},$$

which is a sequence of subsets on I.

Lemma 2.7. *Let K_1 and K_2 be two simplicial complexes on the vertex set I, $\mathbb{P}_1 = \{\sigma_1, \sigma_2, \ldots, \sigma_s\}$ and $\mathbb{P}_2 = \{\tau_1, \tau_2, \ldots, \tau_t\}$ be simplicial complements of K_1 and K_2 respectively. Then $\mathbb{P}_1 * \mathbb{P}_2 = \{\sigma_i \cup \tau_j \mid \sigma_i \in \mathbb{P}_1, \tau_j \in \mathbb{P}_2\}$ is a simplicial complement of $K_1 \cup K_2$ on the vertex set I,*

$$K_1 \cup K_2 = K_{\mathbb{P}_1 * \mathbb{P}_2}(I) = \{\tau \subset I \mid \tau \textit{ does not contain any } \sigma_i \cup \tau_j \in \mathbb{P}_1 * \mathbb{P}_2\}.$$

Proof. We prove this lemma in the same way as the proof of Lemma 2.5.

1. It is easy to see that a simplex τ on the vertex set I is not a simplex of $K_1 \cup K_2$ if and only if it is not a simplex of either K_1 or K_2. This implies that there exists a $\sigma_i \in \mathbb{P}_1$ such that $\sigma_i \subseteq \tau$ and also exists a $\tau_j \in \mathbb{P}_2$ such that $\tau_j \subseteq \tau$. This is equivalent to say that $\sigma_i \cup \tau_j \subseteq \tau$, every non-face τ of $K_1 \cup K_2$ contains a $\sigma_i \cup \tau_j \in \mathbb{P}_1 * \mathbb{P}_2$, so
$$2^I \setminus K_1 \cup K_2 \subseteq 2^I \setminus K_{\mathbb{P}_1 * \mathbb{P}_2}(I).$$
2. If a simplex τ on I contains a non-face $\sigma_i \cup \tau_j \in \mathbb{P}_1 * \mathbb{P}_2$, then $\sigma_i \subseteq \tau$ and $\tau_j \subseteq \tau$. This τ is neither a simplex of K_1 nor a simplex of K_2, so
$$2^I \setminus K_{\mathbb{P}_1 * \mathbb{P}_2}(I) \subseteq 2^I \setminus K_1 \cup K_2.$$

The lemma follows. □

Corollary 2.8. *If the simplicial complement $\mathbb{P}$ is equivalent to $\mathbb{P}'$, then for any simplex σ and simplicial complement $\mathbb{P}_2$*

$$\mathbb{P} - \sigma \simeq \mathbb{P}' - \sigma, \qquad \mathbb{P} * \mathbb{P}_2 \simeq \mathbb{P}' * \mathbb{P}_2.$$

Let σ be a simplex of a simplicial complex K on $[m]$. The stellar subdivision at σ on K is defined to be the union of the simplicial complexes $K \setminus \mathit{Intstar}_K\sigma$ and the cone $\mathit{cone}\partial\mathit{star}_K\sigma$ along their boundary $\partial\mathit{star}_K\sigma$, denoted by

$$ss_\sigma K = (K \setminus \mathit{Intstar}_K\sigma) \cup (\mathit{cone}\partial\mathit{star}_K\sigma),$$

where

$$K \setminus \mathit{Intstar}_K\sigma = \{\tau \in K \mid \sigma \not\subset \tau\}$$

and

$$\mathit{cone}\partial\mathit{star}_K\sigma = (m+1) * \partial\mathit{star}_K\sigma.$$

After stellar subdivision, one more vertex is added which is the vertex of the cone (*cf.* [**2**]).

In [**4**, Definition 2.7.1], the stellar subdivision is defined to be

$$ss_\sigma K = (K \setminus \mathit{star}_K\sigma) \cup (\mathit{cone}\partial\mathit{star}_K\sigma),$$

where $K \setminus \mathit{star}_K\sigma$ is not a simplicial complex. Note that

$$K \setminus \mathit{star}_K\sigma = (K \setminus \mathit{Intstar}_K\sigma) \setminus \partial\mathit{star}_K\sigma$$

and

$$(K \setminus \mathit{Intstar}_K\sigma) \cap (\mathit{cone}\partial\mathit{star}_K\sigma) = \partial\mathit{star}_K\sigma,$$

so our definition coincides with that in [**4**].

Theorem 2.9. *Let $\mathbb{P} = \{\sigma_1, \sigma_2, \ldots, \sigma_s\}$ be a simplicial complement of K. Then $\{\mathbb{P}, \sigma, (\mathbb{P} - \sigma) * (m+1)\}$ is a simplicial complement of $ss_\sigma K$ on the vertex set $[m+1]$, where*

$$(\mathbb{P} - \sigma) * (m+1) = \{(\sigma_1 \setminus \sigma, m+1), (\sigma_2 \setminus \sigma, m+1), \ldots, (\sigma_s \setminus \sigma, m+1)\}.$$

Proof. First we prove that $\{\sigma, \mathbb{P} - \sigma\}$ is a simplicial complement of $\mathit{cone}\partial\mathit{star}_K\sigma$ on the vertex set $[m+1] = \{1, 2, \ldots, m, m+1\}$.

From Lemma 2.5 we know that $\mathbb{P} - \sigma$ is a simplicial complement of $\mathit{star}_K\sigma$ on the vertex set $[m]$.

A simplex τ on the vertex set $[m]$ is not a simplex of $\partial\mathit{star}_K\sigma = \mathit{star}_K\sigma \setminus \mathit{Intstar}_K\sigma$ if and only if $\tau \in \mathit{Intstar}_K\sigma$ or $\tau \notin \mathit{star}_K\sigma$, i.e. $\sigma \subset \tau$ or there exists a $\sigma_i \setminus \sigma$ such that $\sigma_i \setminus \sigma \subset \tau$, so

$$2^{[m]} \setminus \partial\mathit{star}_K\sigma = 2^{[m]} \setminus K_{\{\sigma, \mathbb{P}-\sigma\}}([m]),$$

$\{\sigma, \mathbb{P} - \sigma\}$ is a simplicial complement of $\partial\mathit{star}_K\sigma$ on the vertex set $[m]$.

Take the cone of $\partial\mathit{star}_K\sigma$ on the vertex set $[m+1]$, a simplex $\tau \subset [m]$ or $(\tau, m+1) \subset [m+1]$ is not a simplex of $\mathit{cone}\partial\mathit{star}_K\sigma = (m+1) * \partial\mathit{star}_K\sigma$ if and only if τ is not a simplex of $\partial\mathit{star}_K\sigma$, i.e.

$$2^{[m+1]} \setminus \mathit{cone}\partial\mathit{star}_K\sigma = 2^{[m+1]} \setminus K_{\{\sigma, \mathbb{P}-\sigma\}}([m+1]),$$

$\{\sigma, \mathbb{P} - \sigma\}$ is a simplicial complement of $\mathit{cone}\partial\mathit{star}_K\sigma$ on the vertex set $[m+1]$.

Second, we prove that $\{\mathbb{P}, \sigma, (m+1)\}$ is a simplicial complement of $K \setminus Intstar_K\sigma$ on the vertex set $[m+1]$.

A simplex τ on the vertex set $[m]$ is not a simplex of $K \setminus Intstar_K\sigma$ if and only if $\tau \notin K$ or $\tau \in Intstar_K\sigma$, i.e. there exists a $\sigma_i \in \mathbb{P}$ such that $\sigma_i \subset \tau$ or $\sigma \subset \tau$. $\{\mathbb{P}, \sigma\}$ is a simplicial complement of $K \setminus Intstar_K\sigma$ on the vertex set $[m]$.

Consider $K \setminus Intstar_K\sigma$ as a simplicial complex on the vertex set $[m+1]$, $(m+1)$ does not appear in $K \setminus Intstar_K\sigma$. It is a ghost vertex and $(m+1)$ is a missing face. So

$$\{\mathbb{P}, \sigma, (m+1)\}$$

is a simplicial complement of $K \setminus Intstar_K\sigma$ on the vertex set $[m+1]$.

From Lemma 2.7, we know that $\{\mathbb{P}, \sigma, (m+1)\} * \{\sigma, \mathbb{P} - \sigma\}$ is a simplicial complement of $ss_\sigma K = (K \setminus Intstar_K\sigma) \cup (cone\partial star_K\sigma)$, where

$$\{\mathbb{P}, \sigma, (m+1)\} * \{\sigma, \mathbb{P} - \sigma\} = \left\{ \begin{array}{ll} \mathbb{P} * \sigma, & \mathbb{P} * \{\mathbb{P} - \sigma\}, \\ \sigma * \sigma, & \sigma * \{\mathbb{P} - \sigma\}, \\ (m+1) * \sigma, & (m+1) * \{\mathbb{P} - \sigma\} \end{array} \right\}.$$

At last, we complete the proof by showing that the simplicial complement $\{\mathbb{P}, \sigma, (m+1)\} * \{\sigma, \mathbb{P} - \sigma\}$ is equivalent to $\{\mathbb{P}, \sigma, \{\mathbb{P} - \sigma\} * (m+1)\}$, i.e.

$$ss_\sigma K = K_{\{\mathbb{P},\sigma,(m+1)\}*\{\sigma,\mathbb{P}-\sigma\}}([m+1]) = K_{\{\mathbb{P},\sigma,\{\mathbb{P}-\sigma\}*(m+1)\}}([m+1]),$$

First,

$$\sigma * \sigma = \sigma \in \{\mathbb{P}, \sigma, (m+1)\} * \{\sigma, \mathbb{P} - \sigma\}.$$

Every subset $\sigma_i \cup \sigma \in \mathbb{P} * \sigma$, $(\sigma, m+1) \in (m+1) * \sigma$ and $\sigma \cup (\sigma_i \setminus \sigma) \in \sigma * \{\mathbb{P} - \sigma\}$ contain σ. They could be removed from $\{\mathbb{P}, \sigma, (m+1)\} * \{\sigma, \mathbb{P} - \sigma\}$, so

$$\{\mathbb{P}, \sigma, (m+1)\} * \{\sigma, \mathbb{P} - \sigma\} \simeq \left\{ \begin{array}{l} \mathbb{P} * \{\mathbb{P} - \sigma\}, \\ \sigma, \\ (m+1) * \{\mathbb{P} - \sigma\} \end{array} \right\}.$$

Then for any $\sigma_i \in \mathbb{P}$, one has $\sigma_i \setminus \sigma \in \mathbb{P} - \sigma$. So

$$\sigma_i = \sigma_i \cup (\sigma_i \setminus \sigma) \in \mathbb{P} * \{\mathbb{P} - \sigma\}.$$

Any other $\sigma_i \cup (\sigma_j \setminus \sigma) \in \mathbb{P} * \{\mathbb{P} - \sigma\}$ contains σ_i, they could be removed from $\mathbb{P} * \{\mathbb{P} - \sigma\}$. Thus $\mathbb{P} * \{\mathbb{P} - \sigma\}$ is equivalent to $\mathbb{P}$ and $\{\mathbb{P}, \sigma, (m+1)\} * \{\sigma, \mathbb{P} - \sigma\}$ could be reduced to

$$\{\mathbb{P}, \sigma, \{\mathbb{P} - \sigma\} * (m+1)\} = \{\mathbb{P}, \sigma, (\sigma_1 \setminus \sigma, m+1), (\sigma_2 \setminus \sigma, m+1), \ldots, (\sigma_s \setminus \sigma, m+1)\}.$$

The theorem follows. □

Remark 2.10. If σ is not a simplex of K, we still have $\{\mathbb{P}, \sigma, \{\mathbb{P} - \sigma\} * (m+1)\}$ as a simplicial complement of a simplicial complex $ss_\sigma K$. In that case, there exists a $\sigma_i \in \mathbb{P}$ such that $\sigma_i \subseteq \sigma$. So σ could be removed from $\{\mathbb{P}, \sigma, \{\mathbb{P} - \sigma\} * (m+1)\}$ and $\sigma_i \setminus \sigma = \emptyset \in \mathbb{P} - \sigma$. Thus $(\sigma_i \setminus \sigma, m+1) = (m+1) \in \{\mathbb{P} - \sigma\} * (m+1)$ and all the other $(\sigma_j \setminus \sigma, m+1)$ could be removed from $\{\mathbb{P}, \sigma, \{\mathbb{P} - \sigma\} * (m+1)\}$. That is to say that $(m+1)$ is a missing face and

$$\{\mathbb{P}, \sigma, \{\mathbb{P}-\sigma\} * (m+1)\} \simeq \{\mathbb{P}, (m+1)\}$$

is still a simplicial complement of $ss_\sigma K = K$ but on the vertex set $[m+1]$ and a ghost vertex $(m+1)$ is added.

We still call it the stellar subdivision at σ on K.

Example 2.11. In Example 2.4, we make stellar subdivision at $\sigma = (1, 2)$ on K (see Figure 2).

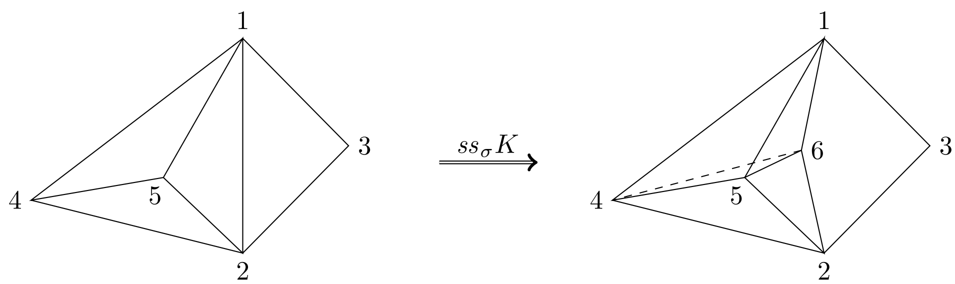

Figure 2: The stellar subdivision at σ on K

$\mathbb{P} = \{(1, 2, 4, 5), (1, 2, 3), (3, 4), (3, 5)\}$ is a simplicial complement of K, $\sigma = (1, 2)$, so

$$\{\mathbb{P}-\sigma\} * (6) = \{(4, 5), (3), (3, 4), (3, 5)\} * (6) = \{(4, 5, 6), (3, 6), (3, 4, 6), (3, 5, 6)\}.$$

$$\{\mathbb{P}, \sigma, \{\mathbb{P}-\sigma\} * (6)\} = \left\{ \begin{array}{l} (1, 2, 4, 5), (1, 2, 3), (3, 4), (3, 5), \\ (1, 2) = \sigma, \\ (4, 5, 6), (3, 6), (3, 4, 6), (3, 5, 6) \end{array} \right\}$$

is a simplicial complement of $ss_\sigma K$. The maximal simplices of $ss_\sigma K$ are

$$\left\{ \begin{array}{l} (1, 3), (2, 3), (1, 4, 5), (2, 4, 5), \\ (1, 4, 6), (1, 5, 6), (2, 4, 6), (2, 5, 6) \end{array} \right\}.$$

3. Construction

After given the simplicial complement of stellar subdivision, we construct our moment-angle manifolds whose cohomology has torsion.

Lemma 3.1. *Let K be a simplicial complex on the vertex set $[m]$ and*

$$\mathbb{P} = \{\sigma_1, \sigma_2, \ldots, \sigma_s\}$$

be a simplicial complement of it. Let I be a subset of the vertex set $[m]$. Then

$$\mathbb{P}|_I = \{\sigma_i \in \mathbb{P} \mid \sigma_i \subset I\}$$

is a simplicial complement of the full subcomplex $K|_I$ on the vertex set I.

Proof. From its definition, we know that the full subcomplex

$$K|_I = \{\sigma \in K \mid \sigma \subset I\}$$

is a simplicial complex on the vertex set I. A subset τ on the vertex set I is not a simplex of $K|_I$ if and only if τ is not a simplex of K, i.e. there exists a non-face $\sigma_i \in \mathbb{P}$ such that $\tau_i \subset \tau$. Note that $\tau \subset I$, $\tau_i \subset \tau \subset I$. The lemma follows. □

Theorem 3.2 (Construction). *Let K be a subcomplex (not a full subcomplex) of a simplicial sphere L_0 on the vertex set $[m]$, $\mathbb{M} = \{\sigma_1, \sigma_2, \ldots, \sigma_s\}$ be the set of missing faces of K, which are also simplices of L_0. On L_0, make stellar subdivisions at $\sigma_1, \sigma_2, \ldots, \sigma_s$ one by one as follows*

$$L_1 = ss_{\sigma_1} L_0, \quad L_2 = ss_{\sigma_2} L_1, \quad \ldots, \quad L_s = ss_{\sigma_s} L_{s-1}.$$

Then K becomes a full subcomplex of L_s, $K = L_s|_{[m]}$.

Proof. Let $\mathbb{P}_0 = \{\tau_1, \tau_2, \ldots, \tau_r\}$ be a simplicial complement of L_0 on $[m]$. From Theorem 2.9 we know that

$$\mathbb{P}_1 = \{\mathbb{P}_0, \sigma_1, \mathbb{P}_1'\}$$

is a simplicial complement of $L_1 = ss_{\sigma_1} L_0$ on $[m+1]$, where

$$\mathbb{P}_1^{'} = \{\mathbb{P}_0 - \sigma_1\} * (m+1).$$

By induction, we get a simplicial complement of $L_s = ss_{\sigma_s} L_{s-1}$ on $[m+s]$ as

$$\begin{aligned}\mathbb{P}_s &= \{\mathbb{P}_{s-1}, \sigma_s, \mathbb{P}_s'\} \\ &= \{\mathbb{P}_0, \sigma_1, \sigma_2, \ldots, \sigma_s, \mathbb{P}_1', \mathbb{P}_2', \ldots, \mathbb{P}_s'\},\end{aligned}$$

where

$$\mathbb{P}_i' = \{\mathbb{P}_{i-1} - \sigma_i\} * (m+i).$$

Note that every non-face in $\mathbb{P}_i'$ contains $(m+i)$ as a vertex. From Lemma 3.1 we know that

$$\mathbb{P}_s|_{[m]} = \{\mathbb{P}_0, \sigma_1, \sigma_2, \ldots, \sigma_s\}$$

is a simplicial complement of the full subcomplex $L_s|_{[m]}$.

Finally, we consider the simplicial complement $\mathbb{P}_s|_{[m]}$. Note that K is a subcomplex of L_0, every non-face $\tau_i \in \mathbb{P}_0$ is not a simplex of K, so there exists a $\sigma_j \in \mathbb{M}$ such that $\sigma_j \subseteq \tau_i$. Then τ_i could be removed from $\{\mathbb{P}_0, \sigma_1, \sigma_2, \ldots, \sigma_s\}$.

Thus

$$\mathbb{P}_s|_{[m]} = \{\mathbb{P}_0, \sigma_1, \sigma_2, \ldots, \sigma_s\} \simeq \{\sigma_1, \sigma_2, \ldots, \sigma_s\} = \mathbb{M}$$

which is the set of missing faces of K. The theorem follows. □

Remark 3.3. If L_0 is also a polytopal sphere, the stellar subdivision of L_0 is also polytopal. It has been proved in a geometric sense by Ewald and Shephard in [**10**].

Let $\widetilde{L}_0$ be the simplicial polytope and its boundary $\partial\widetilde{L}_0 = L_0$ be the polytopal sphere. If σ is a simplex of L_0 and σ is the intersection of the facets (maximal simplices of L_0) $F_{i_1}, F_{i_2}, \ldots, F_{i_r}$, one can take any point p beyond the facets $F_{i_1}, F_{i_2}, \ldots, F_{i_r}$ and beneath the other facets (See [**11**, p. 78] for the definitions of "beyond" and "beneath"). The stellar subdivision $ss_\sigma \partial \widetilde{L}_0$ is the boundary of the convex hall of $\widetilde{L}'_0 = conv(\widetilde{L}_0 \cup p)$.

It could also be proved from the duality of polytopes.

Let $\widetilde{L}_0$ be the simplicial polytope corresponding to L_0, and P_0 be the dual simple polytope, (the vertex of L_0 corresponding to the facet while the facet of L_0 corresponding to the vertex of P_0). Let $\sigma = (i_1, i_2, \ldots, i_k)$ be a simplex of L_0, make a stellar subdivision at σ on L_0 is equivalent, though the duality of polytopes, to cutting off the face $\sigma^* = F_{i_1} \cap F_{i_2} \cap \cdots \cap F_{i_k}$ in P_0 by a generic hyperplane. The cutting off operation on a simple polytope is still simple, so $ss_\sigma \partial \widetilde{L}_0$ is polytopal.

4. Application

Proposition 4.1. *The cohomology of differentiable moment-angle manifolds may have torsion of any order.*

Proof. Let L_0 be a polytopal sphere and K be a subcomplex of L_0, whose cohomology has torsion. Construct a new polytopal sphere L_s by Theorem 3.2, then K becomes a full subcomplex of L_s, while both $\mathbb{R}\mathcal{Z}_{L_s}$ and $\mathcal{Z}_{L_s}$ are framed differentiable manifolds. From Hochster's Theorem, the cohomology of $\mathbb{R}\mathcal{Z}_{L_s}$ and $\mathcal{Z}_{L_s}$ has $\widetilde{H}^*(K)$ as a summand and then have torsion.

At least, every simplicial complex K with m vertexes is a subcomplex of the polytopal sphere $\partial\Delta^{m-1}$. So the cohomology of differentiable moment-angle manifolds could have any torsion. □

Here is an example.

Example 4.2. Let K be the triangulated mod 3 Moore space (see Figure 3) which can be embedded in 6-dimensional polytopal sphere

$$L_0 = \partial\Delta^7 = \partial(1,2,3,4,5,6,7,8).$$

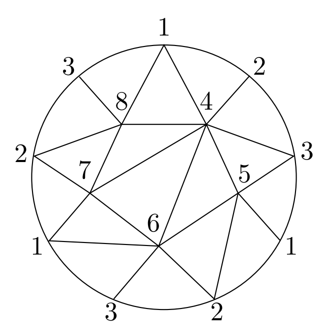

Figure 3: The triangulated mod 3 Moore space

The set of missing faces of L_0 is

$$\mathbb{P}_0 = \{(1,2,3,4,5,6,7,8)\}.$$

The set of missing faces of K is

$$\mathbb{M} = \left\{ \begin{array}{l} (1,2,3),(1,2,6),(1,2,8),(1,3,4), \\ (1,4,5),(1,4,6),(1,4,7),(1,5,6),(1,7,8), \\ (2,3,5),(2,4,5),(2,4,6),(2,4,7),(2,4,8),(2,6,7), \\ (3,4,6),(3,4,8),(3,5,6), \\ (3,7),(5,8),(5,7),(6,8) \end{array} \right\} \tag{2}$$

and the set of maximal simplices of K is

$$\left\{ \begin{array}{l} (1,2,4),(1,2,5),(1,2,7),(1,3,5), \\ (1,3,6),(1,3,8),(1,4,8),(1,6,7), \\ (2,3,4),(2,3,6),(2,3,8),(2,5,6),(2,7,8), \\ (3,4,5),(4,5,6),(4,6,7),(4,7,8) \end{array} \right\}.$$

Making 22 stellar subdivisions at missing faces of K on $\partial\Delta^7$, we thus obtain a 6-dimensional polytopal sphere L_{22} with 30 vertices which has K as a full subcomplex. The real moment-angle manifold corresponding to L_{22} is of 6-dimensional while the complex one is of 37-dimensional where $H^3(\mathbb{R}\mathcal{Z}_{L_{22}})$ and $H^{11}(\mathcal{Z}_{L_{22}})$ has $\widetilde{H}^2(K) = \mathbb{Z}/3$ as a summand.

Passing to the dual, Δ^7 is the dual simple polytope of $\partial\Delta^7$ with facets numbered as vertexes of $\partial\Delta^7$. Making stellar subdivision on $\partial\Delta^7$ at $\sigma = (i_1, i_2, \ldots, i_r)$ is dual to cutting off face $F_{i_1} \cap F_{i_2} \cap \cdots \cap F_{i_r}$ in Δ^7,

$$\begin{array}{ccc} K \hookrightarrow \partial\Delta^7 = \partial(\Delta^{7^*}) & \xrightarrow{\text{s.s.}} & L_{22} \\ \Updownarrow & & \Updownarrow \\ \Delta^7 & \xrightarrow{\text{cut off}} & P_{22}. \end{array}$$

After cutting off the faces $F_{i_1} \cap F_{i_2} \cap \cdots \cap F_{i_r}$ numbered at $\mathbb{M}$ in (2), one gets a simple polytope P_{22}. The cohomology of the moment-angle manifold corresponding to P_{22} has $H^2(K) = \mathbb{Z}/3$ as a summand and then has torsion. If we only cut off $\{1, 2, \ldots, 8\} \setminus \sigma$ for every maximal simplex σ of K in Δ^7 as Bosio and Meersseman did in [**2**, Theorem 11.12], we do not get torsion.

Compute the missing faces after making stellar subdivision at $(1,2,3)$ and $(3,7)$ on $\partial\Delta^7$ in different sequence, one has

1. We make stellar subdivision at $(1,2,3)$ on $L_0 = \partial\Delta^7$ at first, then make stellar subdivision at $(3,7)$.
   From Theorem 2.9 we know that,

$$\begin{aligned} \mathbb{P}_0 &= \{\underline{(1,2,3,4,5,6,7,8)}\}, \\ \sigma_1 &= (1,2,3), \\ (\mathbb{P}_0 - \sigma_1) * (9) &= \{(4,5,6,7,8,9)\} \end{aligned}$$

   is a simplicial complement of $L_1 = ss_{(1,2,3)}L_0$. After removing the larger non-faces $(1,2,3,4,5,6,7,8)$, we get the set of missing faces of L_1

$$\mathbb{M}_1 = \{(1,2,3),(4,5,6,7,8,9)\}.$$

   Then we make stellar subdivision at $(3,7)$ on L_1 and get the set of missing faces of $L_2 = ss_{(3,7)}L_1$

$$\mathbb{M}_2 = \{(1,2,3),(4,5,6,7,8,9),(3,7),(1,2,10),(4,5,6,8,9,10)\}.$$

2. Similarly, we make stellar subdivision at $(3,7)$ on L_0 at first, then make stellar subdivision at $(1,2,3)$, the resulting set of missing faces of L_2' is

$$\mathbb{M}_2' = \{(3,7),(1,2,4,5,6,8,9),(1,2,3),(7,10),(4,5,6,8,9,10)\}.$$

It is easy to see that two simplicial complexes K and K' on the vertex set I are combinatorially equivalent if and only if their sets of missing faces $\mathbb{M}$ and $\mathbb{M}'$ are equivalent, i.e. there exists a one to one correspondence $\phi : I \to I$ that gives a one to one correspondence between $\mathbb{M}$ and $\mathbb{M}'$.

Comparing with these two sequences, we can find that L_2 has one 2-vertex missing faces $(3,7)$ while L_2' has two $(3,7)$, $(7,10)$. This implies that L_2 is not combinatorially isomorphic to L_2' and this difference might persist during the later stellar subdivisions.

Remark 4.3. Though K will be a full subcomplex of L_s in every sequence of making stellar subdivisions at K's missing faces, the combinatorial structure of L_s may not be combinatorially isomorphic in different sequences.

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