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ON THE HOMOLOGY OF LIE ALGEBRAS LIKE $\mathfrak{gl}(\infty, R)$

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Abstract

We revisit a recent paper of Fialowski and Iohara. They compute the homology of the Lie algebra $\mathfrak{gl}(\infty, R)$ for R an associative unital algebra over a field of characteristic zero. We explain how to obtain essentially the same results by a completely different method.

1. Introduction

This note is inspired by the recent paper [**FI17**] of Fialowski and Iohara. They compute the Lie algebra homology of the infinite matrix algebra $\mathfrak{gl}(\infty, R)$. We shall recall below how this Lie algebra is defined. Their paper takes up a thread of research initiated by Feigin and Tsygan, and generalizes one of the main results of [**FT83**].

While $\mathfrak{gl}(\infty, R)$ naturally acts on Laurent polynomials $R[t, t^{-1}]$, in this note we consider a very closely related variant, which acts in an analogous fashion on formal Laurent series $R((t))$. This leads to a *topologically completed* variant of the same Lie algebra, call it $\mathfrak{gl}_{\mathrm{top}}(\infty, R)$. However, this little change of perspective is only made for convenience, it is really not the main point of this text. Instead, our focus is on computing the homology of $\mathfrak{gl}_{\mathrm{top}}(\infty, R)$ in a completely different fashion than the methods used by Fialowski and Iohara.

Nonetheless, a posteriori it will turn out that $\mathfrak{gl}(\infty, R)$ and $\mathfrak{gl}_{\mathrm{top}}(\infty, R)$ have the same homology.

2. Statement of the results

Suppose k is a field of characteristic zero and R a unital associative k-algebra. Define the *algebra of generalized Jacobi matrices* (following [**FT83**])

$$J(R) := \{(a_{i,j})_{i,j\in\mathbb{Z}} \text{ with } a_{i,j} \in R \mid \exists N : a_{i,j} = 0 \text{ for all } i, j \text{ with } |i - j| > N\}.$$

This is again a unital associative k-algebra if we equip it with the usual matrix multiplication. The finiteness condition on the width of the diagonal support ensures that this multiplication is well-defined.

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Write $\mathfrak{gl}(\infty, R)$ to denote the Lie algebra of $J(R)$. This Lie algebra arose in the context of soliton theory, e.g., see [**MJD00**, Section 6.2], or [**KR87**, Lecture 4], for a textbook account. We refer the reader to these accounts for background on the use of this Lie algebra. Moreover, let $\mathfrak{gl}_n(R)$ denote the usual Lie algebra of $(n \times n)$-matrices with entries in R, and $\mathfrak{gl}_\infty(R) := \operatorname{colim}_{n \longrightarrow \infty} \mathfrak{gl}_n(R)$, where $\mathfrak{gl}_n(R)$ is embedded into $\mathfrak{gl}_{n+1}(R)$ as the upper-left $(n \times n)$-minor, and the remaining $(n+1)$-st row and column being set to zero. Again, $\mathfrak{gl}_\infty(R)$ is a Lie algebra. Despite the similar notation, it is very different from $\mathfrak{gl}(\infty, R)$.

The main results of [**FI17**] are:

Theorem 2.1 (Fialowski–Iohara [**FI17**])**.** *Let R be an associative unital k-algebra, where k is a field of characteristic zero. There are canonical isomorphisms:*

1. *of graded k-vector spaces $HC_\bullet(J(R)) \cong HC_{\bullet-1}(R)$,*
2. *of graded k-vector spaces $HH_\bullet(J(R)) \cong HH_{\bullet-1}(R)$,*
3. *of the primitive parts* $\operatorname{Prim} H_\bullet(\mathfrak{gl}(\infty, R), k) \cong \operatorname{Prim} H_\bullet(\mathfrak{gl}_\infty(R), k)[-1]$.

By speaking of the primitive part, we refer to the fact that the Lie homology algebra $H_\bullet(\mathfrak{g}, k)$, for an arbitrary Lie algebra $\mathfrak{g}$ over a field k, has a graded cocommutative coalgebra structure with counit, induced from the comultiplication

$$\triangle \colon H_\bullet(\mathfrak{g}, k) \xrightarrow{\delta} H_\bullet(\mathfrak{g} \oplus \mathfrak{g}, k) \xrightarrow{\sigma} H_\bullet(\mathfrak{g}, k) \otimes_k H_\bullet(\mathfrak{g}, k),$$

where $\delta \colon \mathfrak{g} \to \mathfrak{g} \oplus \mathfrak{g}$ is the diagonal $g \mapsto (g, g)$, and σ denotes the Künneth isomorphism for Lie homology. Thus, it has a notion of primitive elements, namely those x satisfying $\triangle x = 1 \otimes x + x \otimes 1$.

We define a variation of $J(R)$: We may take

$$E(R) := \left\{ \varphi \in \operatorname{End}_R(R((t))) \;\middle|\; \begin{array}{c} \text{(1) for every } n \in \mathbb{Z} \text{ there exists some } n' \in \mathbb{Z} \\ \text{such that } \varphi(t^n R[[t]]) \subseteq t^{n'} R[[t]], \textit{ and} \\ \text{(2) for every } m \in \mathbb{Z} \text{ there exists some } m' \in \mathbb{Z} \\ \text{such that } \varphi(t^{m'} R[[t]]) \subseteq t^m R[[t]] \end{array} \right\}$$

and let $\mathfrak{gl}_{\mathrm{top}}(\infty, R)$ be its Lie algebra.

Theorem 2.2 (Theorem 4.5)**.** *Let R be an associative unital k-algebra, where k is a field of characteristic zero. There are canonical isomorphisms:*

1. *of graded k-vector spaces $HC_\bullet(E(R)) \cong HC_{\bullet-1}(R)$,*
2. *of graded k-vector spaces $HH_\bullet(E(R)) \cong HH_{\bullet-1}(R)$,*
3. *of the primitive parts* $\operatorname{Prim} H_\bullet(\mathfrak{gl}_{\mathrm{top}}(\infty, R), k) \cong \operatorname{Prim} H_\bullet(\mathfrak{gl}_\infty(R), k)[-1]$.

Of course this statement has exactly the same shape as the theorem of Fialowski–Iohara. But even though this is a statement entirely about associative algebras and Lie algebras, we shall prove it using a detour through some *category-theoretic concepts*. The key input for the proof will be a localization theorem for categories. No spectral sequences or auxiliary homology computations as in [**FI17**] will appear. So, the method of proof is quite different. We will explain below how E and $\mathfrak{gl}_{\mathrm{top}}(\infty, R)$ are just topologically completed variants of J and $\mathfrak{gl}(\infty, R)$; and note that since the right-hand side terms in our theorem are precisely those as in the one of Fialowski–Iohara, this topological completion curiously does not affect the homology at all.

Mild variations of the same proof also give a variant for group homology. Write $\mathrm{GL}_\infty(R)$ for $\operatorname*{colim}_{n\longrightarrow\infty} \mathrm{GL}_n(R)$, where $\mathrm{GL}_n(R)$ is embedded into $\mathrm{GL}_{n+1}(R)$ as the upper-left $(n \times n)$-minor, and the remaining $(n+1)$-st row and column being set to zero, except for the $(n+1, n+1)$ entry, which is set to the identity.

Although there does not exist an equally close connection between $\mathrm{GL}_\infty(R)$ and $\mathfrak{gl}_\infty(R)$ as in the world of real Lie groups and their Lie algebras, the analogous group homology fact still holds true.

Theorem 2.3 (Theorem 4.7)**.** *Let R be an associative unital k-algebra, where k is a field of any characteristic. Then there is a canonical isomorphism of primitive parts*

$$\operatorname{Prim} H_n(\mathrm{GL}_\infty(E(R)), \mathbb{Q}) \cong \operatorname{Prim} H_{n-1}(\mathrm{GL}_\infty(R), \mathbb{Q})$$

for all $n \geqslant 2$.

We believe that the same statement is probably also true for $J(R)$ instead of $E(R)$, but we have no proof.

3. The Lie algebras $\mathfrak{gl}(\infty, R)$ and $\mathfrak{gl}_{\mathrm{top}}(\infty, R)$

We have recalled Feigin and Tsygan's Lie algebra $\mathfrak{gl}(\infty, R)$ in the previous section. Now let us define $\mathfrak{gl}_{\mathrm{top}}(\infty, R)$. We shall use a slightly more complicated approach than we have used in Section 2 (we shall prove in Proposition 3.4 that the definition in Section 2 is equivalent). Even though we only need a Lie algebra, let us take a quick detour through a category-theoretic approach: For every exact category $\mathcal{C}$, there is a commutative diagram of exact functors

$$\begin{tikzcd} \mathcal{C} \arrow[r] \arrow[d] & \mathsf{Ind}^a_{\aleph_0}\mathcal{C} \arrow[d] \\ \mathsf{Pro}^a_{\aleph_0}\mathcal{C} \arrow[r] & \mathsf{Tate}^{el}_{\aleph_0}\mathcal{C}, \end{tikzcd}$$

where $\mathsf{Ind}^a_{\aleph_0}(\mathcal{C})$, $\mathsf{Pro}^a_{\aleph_0}(\mathcal{C})$ refer to certain variations of the classical Ind- and Pro-categories (the main difference is that we only allow defining Ind-diagrams whose transition morphisms are admissible monics, resp. Pro-diagrams whose transition morphisms are admissible epics with respect to the exact structure of $\mathcal{C}$). The category $\mathsf{Tate}^{el}_{\aleph_0}(\mathcal{C})$ can be defined as the full subcategory of $\mathsf{Ind}^a_{\aleph_0}(\mathsf{Pro}^a_{\aleph_0}(\mathcal{C}))$ consisting of those objects which admit a Pro-subobject with an Ind-object quotient, i.e. which can be presented as

$$L \hookrightarrow X \twoheadrightarrow X/L,$$

where $L \in \mathsf{Pro}^a_{\aleph_0}(\mathcal{C})$ and $X/L \in \mathsf{Ind}^a_{\aleph_0}(\mathcal{C})$; such a subobject is known as a *lattice*. Here we understand this Ind- and Pro-category as canonically embedded into the category $\mathsf{Ind}^a_{\aleph_0}(\mathsf{Pro}^a_{\aleph_0}(\mathcal{C}))$ by writing the Ind-diagram as an Ind-Pro diagram which is constant in the Pro-direction, resp. a Pro-diagram as an Ind-Pro diagram which is constant in the Ind-direction. We refer to [**Pre11**] or [**BGW16b**] for precise definitions. If $\mathcal{C}$ is a k-linear exact category, then so is $\mathsf{Tate}^{el}_{\aleph_0}(\mathcal{C})$, in a canonical fashion.

If $\mathcal{C} = \mathsf{P}_f(R)$ is the k-linear exact category of finitely generated projective R-modules, there is a special object in $\mathsf{Tate}^{el}_{\aleph_0}(\mathcal{C})$:

$$\text{“}R((t))\text{”} := \coprod_{n\in\mathbb{N}} R \oplus \prod_{n\in\mathbb{N}} R \tag{3.1}$$

or equivalently (isomorphically)

$$\text{“}R((t))\text{”} \cong \varinjlim_{n} \varprojlim_{m} \frac{1}{t^n} R[t] \Big/ t^m R[t], \tag{3.2}$$

which might be a more suggestive way of defining an object which we can think of as a formal incarnation of Laurent series. The former presentation makes it clear that “$R((t))$” lies in $\mathsf{Ind}^a_{\aleph_0}(\mathsf{Pro}^a_{\aleph_0}(\mathcal{C}))$, and since $\prod_{n\in\mathbb{N}} R$ is visibly a lattice, “$R((t))$” lies in $\mathsf{Tate}^{el}_{\aleph_0}(\mathcal{C})$.

Despite the fact that these category-theoretic constructions might appear quite fancy, perhaps even too fancy, we only need them in a very special situation where all the categories boil down to be equivalent to categories of projective modules:

Let $\mathsf{Tate}_{\aleph_0}(\mathcal{C})$ denote the idempotent completion of the exact category $\mathsf{Tate}^{el}_{\aleph_0}(\mathcal{C})$.

Lemma 3.1 (Key Lemma)**.** *If $\mathcal{C} = \mathsf{P}_f(R)$ is the k-linear exact category of finitely generated projective R-modules, then there is a k-linear exact equivalence of k-linear exact categories $\mathsf{Tate}_{\aleph_0}(\mathsf{P}_f(R)) \cong \mathsf{P}_f(E(R))$, where*

$$E(R) := \mathrm{End}_{\mathsf{Tate}_{\aleph_0}(\mathsf{P}_f(R))}(\text{“}R((t))\text{”}). \tag{3.3}$$

Indeed, “$R((t))$” is a projective generator of $\mathsf{Tate}_{\aleph_0}(\mathsf{P}_f(R))$.

Proof. This is [**BGW16a**, Theorem 1], applied for $\mathcal{C} = \mathsf{P}_f(R)$ and $n = 1$. Note that the countable collection $\{S_i\}$ of the theorem loc. cit. can be taken to be the single object R (as a module). The reference does not discuss the additional k-linear structure, but an inspection of the proof loc. cit. easily shows that this structure is also compatible under the given equivalence. □

We take the Lie algebra of the associative algebra underlying this projective module category as our definition for a topologically completed analogue of $\mathfrak{gl}(\infty, R)$:

Definition 3.2. Let k be a field and R a unital associative k-algebra. Then we define $\mathfrak{gl}_{\mathrm{top}}(\infty, R)$ as the Lie algebra of the unital associative k-algebra $E(R)$.

We need to justify how $\mathfrak{gl}_{\mathrm{top}}(\infty, R)$ resembles the infinite matrix algebra $\mathfrak{gl}(\infty, R)$ of Fialowski and Iohara.

Remark 3.3. Let us dwell on a bit more about the differences of $R[t, t^{-1}]$ (the situation of Feigin–Tsygan [**FT83**] and Fialowski–Iohara [**FI17**]), in comparison to $R((t))$. For simplicitly, let us focus on $R := k$. Then the big advantage of $k[t, t^{-1}]$ is that we can specify an explicit k-vector space basis. This is not possible for $k((t))$. However, $k((t))$ has the advantage to be closed under dualization; noting that $k((t)) \simeq tk[[t]] \oplus k[t^{-1}]$ as a k-vector space, and the two direct summands are dual to each other under taking the continuous k-dual, and equipping $k[[t]]$ with the natural t-adic linear topology, and $k[t^{-1}]$ with the discrete topology.

Let us give a more explicit description of the − quite abstract − definition in Equation 3.3. After all, we prefer to be able to specify $E(R)$ without having to rely on a category-theoretic picture:

Proposition 3.4. *There is a canonical isomorphism of unital associative k-algebras,*

$$E(R) \xrightarrow{\sim} \left\{ \varphi \in \mathrm{End}_R(R((t))) \;\middle|\; \begin{array}{c} \textit{(1) for every } n \in \mathbb{Z} \textit{ there exists some } n' \in \mathbb{Z} \\ \textit{such that } \varphi(t^n R[[t]]) \subseteq t^{n'} R[[t]], \textit{ and} \\ \textit{(2) for every } m \in \mathbb{Z} \textit{ there exists some } m' \in \mathbb{Z} \\ \textit{such that } \varphi(t^{m'} R[[t]]) \subseteq t^m R[[t]] \end{array} \right\}. \tag{3.4}$$

In particular, $\mathfrak{gl}_{\mathrm{top}}(\infty, R)$ is the Lie algebra of the right-hand side subalgebra of $\mathrm{End}_R(R((t)))$ under the commutator bracket.

Proof. (Step 1) First of all, define an exact functor

$$G\colon \mathsf{Tate}_{\aleph_0}(\mathsf{P}_f(R)) \longrightarrow \mathsf{Mod}(R), \qquad \text{“}R((t))\text{”} \longmapsto R((t)).$$

As “$R((t))$” is a projective generator, for defining an exact functor it suffices to prescribe the object this generator is sent to, and, moreover, make sure that every endomorphism of the projective generator also defines an endomorphism in $\mathsf{Mod}(R)$. To this end, we unravel the definition of morphisms in the category on the left: Recall that $\mathsf{Tate}_{\aleph_0}(\mathsf{P}_f(R))$ is the idempotent completion of $\mathsf{Tate}^{el}_{\aleph_0}(\mathsf{P}_f(R))$, and “$R((t))$” lies in the latter, so it suffices to understand the Hom-sets in $\mathsf{Tate}^{el}_{\aleph_0}(\mathsf{P}_f(R))$ itself. By definition,

$$\mathsf{Tate}^{el}_{\aleph_0}(\mathsf{P}_f(R)) \subset \mathsf{Ind}^a_{\aleph_0}\mathsf{Pro}^a_{\aleph_0}(\mathsf{P}_f(R))$$

is a full subcategory, so this reduces to evaluating the Hom-sets of Ind- resp. Pro-objects individually. For admissible Ind-objects of an arbitrary exact category $\mathcal{C}$ we have

$$\mathrm{Hom}_{\mathsf{Ind}^a_{\aleph_0}(\mathcal{C})}(X_\bullet, Y_\bullet) = \varprojlim_i \varinjlim_j \mathrm{Hom}_{\mathcal{C}}(X_i, X_j),$$

where $X_\bullet\colon I \to \mathcal{C}$ is an admissible Ind-diagram (analogously for $Y_\bullet$) representing the object, with I the countable index category. Since $\mathsf{Pro}^a(\mathcal{C}) = \mathsf{Ind}^a(\mathcal{C}^{op})^{op}$, this also describes the Hom-sets of Pro-objects. In total, we find

$$\mathrm{Hom}_{\mathsf{Tate}^{el}_{\aleph_0}(\mathsf{P}_f(R))}(X_{\bullet,\bullet}, Y_{\bullet,\bullet}) = \varprojlim_I \varinjlim_J \varprojlim_j \varinjlim_i \mathrm{Hom}_R(X_{I,i}, X_{J,j}). \tag{3.5}$$

If we write

$$R((t)) = \varinjlim_n \varprojlim_m \frac{1}{t^n} R[t] \Big/ t^m R[t]$$

in $\mathsf{Mod}(R)$, but not as an Ind-Pro object, but as a concrete colimit and limit in the bicomplete category $\mathsf{Mod}(R)$, we see that an element of Equation 3.5 for the Tate object $X = Y =$ “$R((t))$” indeed defines an R-module endomorphism of $R((t))$. Moreover, the colimits in Equation 3.5 are along (split) injections, and the limits along split

surjections, so the map

$$\mathrm{Hom}_{\mathsf{Tate}^{el}_{\aleph_0}(\mathsf{P}_f(R))}(X,Y) \longrightarrow \mathrm{Hom}_R(GX,GY)$$

is not just well-defined, it is also injective. Thus, G is an exact and faithful functor. *(Step 2)* The faithfulness implies that we have an inclusion of unital associative algebras $E(R) \subseteq \mathrm{End}_R(\,R((t))\,)$. It remains to show that the image agrees with the description which we give in our claim. We note that for all $n \in \mathbb{Z}$ the subobject "$t^n \cdot R[[t]]$", which is a Pro-object, is indeed a lattice in "$R((t))$", because

$$\text{“}t^n \cdot R[[t]]\text{”} \hookrightarrow \text{“}R((t))\text{”} \twoheadrightarrow \text{“}R((t))\,/t^n \cdot R[[t]]\text{ ”}$$

and the latter is isomorphic to "$t^{n-1} \cdot R[t^{-1}]$", which is an Ind-object in turn. So all "$t^n \cdot R[[t]]$" are lattices, and they are bi-final in the poset of lattices, meaning that every lattice L admits some $n, n' \in \mathbb{Z}$ with

$$\text{“}t^n \cdot R[[t]]\text{”} \hookrightarrow L \hookrightarrow \text{“}t^{n'} \cdot R[[t]]\text{”}. \tag{3.6}$$

Now suppose $\varphi \in E(R) = \mathrm{End}_{\mathsf{Tate}_{\aleph_0}(\mathsf{P}_f(R))}(\text{“}R((t))\text{”})$. Since "$t^n \cdot R[[t]]$" is a lattice in "$R((t))$", [**BGW16a**, Lemma 1, (1)] implies that there exists a lattice L in "$R((t))$" such that $\varphi(\text{“}t^n \cdot R[[t]]\text{”}) \subseteq L$ and since the lattices of the shape "$t^{\mathbb{Z}} \cdot R[[t]]$" are cofinal among all lattices, this L is contained in some "$t^{n'} \cdot R[[t]]$" for a suitable n', as in Equation 3.6. It follows that Condition (1) in Equation 3.4 is met. For Condition (2) argue analogously, this time using [**BGW16a**, Lemma 1, (2)] instead. This proves that under the inclusion $E(R) \subseteq \mathrm{End}_R(\,R((t))\,)$ the elements $\varphi \in E(R)$ indeed meet the Conditions (1), (2) of our claim. It remains to show that whenever we are given a $\varphi \in \mathrm{End}_R(\,R((t))\,)$ meeting Conditions (1), (2), that this defines an endomorphism in $\mathrm{End}_{\mathsf{Tate}_{\aleph_0}(\mathsf{P}_f(R))}(\text{“}R((t))\text{”})$. We verify this: Observing Equation 3.2, Conditions (1), (2) just mean that, up to re-indexing n, m, φ induces a (possibly non-straight) morphism in the underlying Ind- and Pro-category. This proves that Conditions (1), (2) precisely determine the image of $E(R)$ inside $\mathrm{End}_R(\,R((t))\,)$. Thus, for the isomorphism of unital associative algebras which we claim to exist in Equation 3.4, we may just take the inclusion map. We already know that this is an algebra homomorphism, Step 1 shows that it is injective, and Step 2 that it is surjective. The claim follows. □

We may also compare $J(R)$ and $E(R)$ on a more structural level:
Firstly, the $J(R)$ of [**FI17**] comes equipped with two-sided ideals;

$$\begin{aligned} I^+(R) &:= \{(a_{i,j}) \in J(R) \mid \exists B_a : i < B_a \Rightarrow a_{i,j} = 0\}, \\ I^-(R) &:= \{(a_{i,j}) \in J(R) \mid \exists B_a : j > B_a \Rightarrow a_{i,j} = 0\}, \\ I^0(R) &:= I^+(R) \cap I^-(R). \end{aligned} \tag{3.7}$$

These have the following property:

$$I^+(R) + I^-(R) = J(R).$$

See [**Bra14**, §1.1] for more on this. Loc. cit. we write $E(R)$ instead of $J(R)$, but for the present text we prefer to follow Feigin–Tsygan's notation. The ideal $I^0(R)$ consists precisely of those matrices with only finitely many non-zero entries, so the usual definition of a matrix trace makes sense on it. We could call $I^0(R)$ the (non-unital) subalgebra of trace-class elements. In fact, all this means that the generalized

Jacobi matrices are an example of a 1-fold Beilinson cubical algebra in the sense of [**BGW16a**, Definition 1].

The algebra $J(R)$ naturally acts on the Laurent polynomial ring $R[t, t^{-1}]$. Write polynomials as $f = \sum_{i\in\mathbb{Z}} f_i t^i$, also denoted $f = (f_i)_i$ with $f_i \in R$, and let $a = (a_{i,j})$ act by $(a \cdot f)_i := \sum_{k\in\mathbb{Z}} a_{i,k} f_k$. This makes $J(R)$ a submodule of $\mathrm{End}_R(R[t, t^{-1}])$.

Now, let us explain how this differs from $E(R)$: Instead of Laurent polynomials, we could also consider formal Laurent series, that is $R((t)) := R[[t]][t^{-1}]$. Clearly, $R[t, t^{-1}] \subseteq R((t))$. As in Equation 3.3, we define $E(R)$ to be the endomorphism algebra of "$R((t))$" in the category $\mathsf{Tate}_{\aleph_0}(\mathsf{P}_f(R))$. Then,

$$\begin{aligned} I^+(R) &:= \{a \in \mathrm{End}(\text{“}R((t))\text{”}) \mid \text{the image of } a \text{ is contained in a lattice}\}, \\ I^-(R) &:= \{a \in \mathrm{End}(\text{“}R((t))\text{”}) \mid \text{a lattice is mapped to zero under } a\}, \\ I^0(R) &:= I^+(R) \cap I^-(R) \end{aligned}$$

are two-sided ideals, now inside of $E(R)$ instead of $J(R)$, and again

$$I^+(R) + I^-(R) = E(R)$$

holds. We chose to pick the same notation for the ideals to stress the analogy with Equations 3.7. In particular, we again obtain an example of a 1-fold Beilinson cubical algebra. All these properties are a special case of [**BGW16a**, Theorem 1], using that the projective modules $\mathsf{P}_f(R)$ form an idempotent complete split exact category.

4. Proof

Fix a field k of arbitrary characteristic. Below, whenever we speak of cyclic or Hochschild homology, we tacitly regard k as the base field, even if we do not repeat mentioning k in the notation.

Cyclic homology is classically only defined for k-algebras, but thanks to the work of Keller [**Kel99**], we also have a concept of cyclic homology for k-linear exact categories. In the case of k-algebras, we have agreement in the sense that

$$HC(R) \cong HC(\mathsf{P}_f(R)),$$

where on the left-hand side $HC(R)$ denotes the ordinary definition of cyclic homology, for example as in [**Lod92**], whereas on the right-hand side we use Keller's definition. For the proof of agreement see [**Kel99**, §1.5, Theorem, (a)].

Let us briefly recall the Eilenberg swindle, in a quite general format, using the language of [**BGT13**]:

Lemma 4.1 (Eilenberg swindle)**.** *Suppose an exact category $\mathcal{C}$ is closed under countable products or coproducts and $I\colon \mathrm{Cat}_\infty^{\mathrm{ex}} \to \mathsf{Sp}$ is any additive invariant with values in spectra Sp (e.g., Hochschild homology or cyclic homology), then $I(\mathcal{C}) = 0$.*

We formulate this in terms of spectra here, but since we only care about Hochschild or cyclic homology, simplicial abelian groups would suffice.

Proof. Write $\mathcal{EC}$ to denote the exact category of exact sequences in $\mathcal{C}$. We assume

that $\mathcal{C}$ has countable coproducts. Define

$$E\colon \mathcal{C} \longrightarrow \mathcal{EC}, \quad X \mapsto (X \hookrightarrow \coprod\nolimits_{i\in\mathbb{Z}} X \twoheadrightarrow \coprod\nolimits_{i\in\mathbb{Z}} X).$$

The admissible epic on the right is the shift-one-to-the-left functor in the indexing of the coproducts. We can project to the left, middle or right term of $\mathcal{EC}$, write $p_i\colon \mathcal{C} \to \mathcal{C}$ with $i = 0, 1, 2$ for these functors. The additivity of the invariant I implies that $p_1 = p_0 + p_2$ holds for the morphisms functorially induced on $I(\mathcal{C}) \to I(\mathcal{C})$. Since $p_1 = p_2$, we must have $p_0 = 0$. Since p_0 is induced from the identity functor, p_0 must also be the identity, and thus $I(\mathcal{C}) = 0$. The argument in the case of $\mathcal{C}$ having countable products is analogous. □

That cyclic homology and Hochschild homology are additive invariants in the sense of the above lemma, was proven by Keller [**Kel99**, §1.1.2, Theorem].

If $\mathcal{C}$ is an idempotent complete exact category and $\mathcal{C} \hookrightarrow \mathcal{D}$ a right s-filtering inclusion as a full subcategory of an exact category $\mathcal{D}$, then

$$D^b(\mathcal{C}) \hookrightarrow D^b(\mathcal{D}) \twoheadrightarrow D^b(\mathcal{D}/\mathcal{C}) \tag{4.1}$$

is an exact sequence of triangulated categories by work of Schlichting, see [**Sch04**, Prop. 2.6]. From this one can deduce the following variant of a theorem of Keller:

Theorem 4.2 (Keller's Localization Theorem)**.** *Let $\mathcal{C}$ be an exact category and $\mathcal{C} \hookrightarrow \mathcal{D}$ a right s-filtering inclusion as a full subcategory of an exact category $\mathcal{D}$. Then*

$$HC(\mathcal{C}) \longrightarrow HC(\mathcal{D}) \longrightarrow HC(\mathcal{D}/\mathcal{C})$$

is a fiber sequence in cyclic homology. The analogous statement for Hochschild homology is also true.

Proof. Let us provide some details to translate this formulation into the one of Keller's paper. If $\mathcal{C}$ is idempotent complete, the claim literally follows from Schlichting's result, Equation 4.1, and Keller's [**Kel99**, §1.5, Theorem]. If $\mathcal{C}$ is not idempotent complete, we need to work a little harder, but the necessary tools are all available thanks to [**Sch04**]: Instead of Equation 4.1, Schlichting also provides a slightly more general variant: If $\mathcal{C}$ is not necessarily idempotent complete, he constructs an exact category $\widetilde{\mathcal{D}}^{\mathcal{C}}$ (cf. [**Sch04**, Lemma 1.20]) such that

$$\mathcal{D} \hookrightarrow \widetilde{\mathcal{D}}^{\mathcal{C}} \hookrightarrow \mathcal{D}^{ic},$$

with (1) $\widetilde{\mathcal{D}}^{\mathcal{C}} \hookrightarrow \mathcal{D}^{ic}$ an extension-closed full subcategory, and (2) $\mathcal{D} \hookrightarrow \widetilde{\mathcal{D}}^{\mathcal{C}}$ co-final (a.k.a. 'factor-dense' in Keller's language [**Kel99**, §1.5]), (3) the right s-filtering embedding $\mathcal{C} \hookrightarrow \mathcal{D}$, inducing $\mathcal{C}^{ic} \hookrightarrow \mathcal{D}^{ic}$ by 2-functoriality, factors over a right s-filtering embedding $\mathcal{C}^{ic} \hookrightarrow \widetilde{\mathcal{D}}^{\mathcal{C}}$ and, finally, (4), there is an exact equivalence of exact categories $\mathcal{D}/\mathcal{C} \xrightarrow{\sim} \widetilde{\mathcal{D}}^{\mathcal{C}}/\mathcal{C}^{ic}$. See Schlichting's paper [**Sch04**, Lemma 1.20]. We can use this tool now and proceed as follows: From the assumptions of the theorem as our input, we begin with Schlichting's construct $\mathcal{C}^{ic} \hookrightarrow \widetilde{\mathcal{D}}^{\mathcal{C}} \twoheadrightarrow \widetilde{\mathcal{D}}^{\mathcal{C}}/\mathcal{C}^{ic}$, so that Equation 4.1 yields the exact sequence of triangulated categories $D^b(\mathcal{C}^{ic}) \hookrightarrow D^b(\widetilde{\mathcal{D}}^{\mathcal{C}}) \twoheadrightarrow D^b(\widetilde{\mathcal{D}}^{\mathcal{C}}/\mathcal{C}^{ic})$. Keller's original formulation in [**Kel99**] takes such a sequence as its

input, giving the fiber sequence

$$HC(\mathcal{C}^{ic}) \longrightarrow HC(\widetilde{\mathcal{D}}^{\mathcal{C}}) \longrightarrow HC(\widetilde{\mathcal{D}}^{\mathcal{C}}/\mathcal{C}^{ic}).$$

But Keller's cyclic homology is invariant up to exact equivalence up to factors/co-finally (see [**Kel99**, §1.5, Theorem, (b)]), so

$$HC(\widetilde{\mathcal{D}}^{\mathcal{C}}) \cong HC(\mathcal{D}) \qquad \text{and} \qquad HC(\widetilde{\mathcal{D}}^{\mathcal{C}}/\mathcal{C}^{ic}) \cong HC(\mathcal{D}/\mathcal{C}).$$

Finally, $\mathcal{C} \hookrightarrow \mathcal{C}^{ic}$ is co-final, and the idempotent completion of an exact category commutes with the idempotent completion of its bounded derived category, $D^b(\mathcal{C}^{ic}) = D^b(\mathcal{C})^{ic}$, by Balmer and Schlichting [**BS01**, Corollary 2.12], so that $D^b(\mathcal{C}) \to D^b(\mathcal{C}^{ic})$ is also co-final and thus an equivalence up to factors itself. □

Based on the Eilenberg swindle, we obtain the fundamental delooping property of Tate categories. This transports an idea of Sho Saito from algebraic K-theory into the setup of the present note:

Theorem 4.3. *Let k be any field. Suppose $\mathcal{C}$ is an idempotent complete k-linear exact category. Then there are canonical isomorphisms*

$$HC_\bullet(\mathsf{Tate}_{\aleph_0}(\mathcal{C})) \cong HC_{\bullet-1}(\mathcal{C}) \qquad \textit{and} \qquad HH_\bullet(\mathsf{Tate}_{\aleph_0}(\mathcal{C})) \cong HH_{\bullet-1}(\mathcal{C}).$$

Proof. The delooping property of (countably indexed) Tate categories *for algebraic K-theory* was first conjectured by Kapranov in an unpublished letter to Brylinski, later first steps to a proof were taken in Previdi's thesis, and eventually proven in full generality by Sho Saito [**Sai15**]. By inspecting Sho Saito's proof, one quickly realizes that it can be adapted to prove the same delooping property for cyclic homology and Hochschild homology. Let us explain this: Using that $\mathcal{C} \hookrightarrow \mathsf{Pro}^a_{\aleph_0}(\mathcal{C})$ is right s-filtering (see loc. cit., or [**BGW16b**, Theorem 4.2]), and that $\mathsf{Ind}^a_{\aleph_0}(\mathcal{C}) \hookrightarrow \mathsf{Tate}^{el}_{\aleph_0}(\mathcal{C})$ is right s-filtering (loc. cit., or [**BGW16b**, Remark 5.3.5]), we get the homotopy commutative diagram

$$\begin{tikzcd}
HC(\mathcal{C}) \arrow[r] \arrow[d] & HC(\mathsf{Pro}^a_{\aleph_0}(\mathcal{C})) \arrow[r] \arrow[d] & HC(\mathsf{Pro}^a_{\aleph_0}(\mathcal{C})/\mathcal{C}) \arrow[d] \\
HC(\mathsf{Ind}^a_{\aleph_0}(\mathcal{C})) \arrow[r] & HC(\mathsf{Tate}^{el}_{\aleph_0}(\mathcal{C})) \arrow[r] & HC(\mathsf{Tate}^{el}_{\aleph_0}(\mathcal{C})/\mathsf{Ind}^a_{\aleph_0}(\mathcal{C})),
\end{tikzcd}$$

where both rows are fiber sequences. This follows from Keller's Localization Theorem, in the special formulation we have given above as Theorem 4.2. The downward arrows are induced from the natural exact functors between the respective categories. Adapting Saito's idea, the right-hand side downward map stems from an exact equivalence,

$$\mathsf{Tate}^{el}_{\aleph_0}(\mathcal{C})/\mathsf{Ind}^a_{\aleph_0}(\mathcal{C}) \xrightarrow{\sim} \mathsf{Pro}^a_{\aleph_0}(\mathcal{C})/\mathcal{C}\ .$$

(See Saito's paper for a proof of the dual statement with the roles of Ind and Pro in reverse, [**Sai15**, Lemma 3.3]. A detailed proof in precisely the format we use here is given in [**BGW16b**, Proposition 5.34].) Thus, it is an equivalence in cyclic homology and Hochschild homology. Thus, the square on the left is homotopy bi-cartesian. Next,

note that

$$HC(\mathsf{Ind}^a_{\aleph_0}(\mathcal{C})) = 0 \qquad \text{and} \qquad HC(\mathsf{Pro}^a_{\aleph_0}(\mathcal{C})) = 0$$

by the Eilenberg swindle, Lemma 4.1. It is a rather easy exercise to show that the Ind-category is closed under countable coproducts, and the Pro-category under countable products. Thus, the homotopy bi-cartesian square turns into the classical loop space square,

$$\begin{tikzcd} HC(\mathcal{C}) \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & HC(\mathsf{Tate}^{el}_{\aleph_0}\mathcal{C}). \end{tikzcd}$$

We obtain the equivalence $HC(\mathcal{C}) \xrightarrow{\sim} \Omega HC(\mathsf{Tate}^{el}_{\aleph_0}(\mathcal{C}))$, and analogously we obtain this for Hochschild homology.[1] It remains to go to the idempotent completion on the right; we copy the same argument as in the end of the proof of Theorem 4.2: For every exact category $\mathcal{D}$ the idempotent completion $\mathcal{D} \hookrightarrow \mathcal{D}^{ic}$ induces $D^b(\mathcal{D}^{ic}) \hookrightarrow D^b(\mathcal{D})^{ic}$, being co-final and an equivalence up to factors. Thus, applied to $\mathcal{D} := \mathsf{Tate}^{el}_{\aleph_0}(\mathcal{C})$, we get $HC(\mathsf{Tate}^{el}_{\aleph_0}(\mathcal{C})) \cong HC(\mathsf{Tate}_{\aleph_0}(\mathcal{C}))$. □

Let us make the connection between this and the work of Fialowski and Iohara.

Corollary 4.4. *Let k be a field and R a unital associative k-algebra. Then there are canonical isomorphisms:*

1. $HC_\bullet(E(R)) \cong HC_{\bullet-1}(R)$,
2. $HH_\bullet(E(R)) \cong HH_{\bullet-1}(R)$.

We observe that we have proven the analogue of the delooping result of Fialowski and Iohara for E instead of J.

Proof. To this end, we use the category-level delooping theorem in the special case of rings, i.e. $\mathcal{C} = \mathsf{P}_f(R)$. Then

$$HC_\bullet(E(R)) \cong HC_\bullet(\mathsf{Tate}_{\aleph_0}(\mathsf{P}_f(R))) \cong HC_{\bullet-1}(\mathsf{P}_f(R)) \cong HC_{\bullet-1}(R).$$

Here the first isomorphism stems from Lemma 3.1, the second from the delooping property of Tate categories, Theorem 4.3, and the last from the agreement of the cyclic homology of the exact category of finitely generated projective modules with the classical definition for k-algebras. The same computation holds for Hochschild homology. □

Let us stress how different this proof is from the one given in [**FI17**]. While loc. cit. uses the degeneration of a Hochschild–Serre type spectral sequence, based on some explicit computations of terms, the main technical point in our approach is Keller's Localization Theorem, along with a few category-theoretic properties. In an at best vaguely precise comparison, the rôle of the homological vanishing statements in [**FI17**] appears to be replaced by the Eilenberg swindle.

[1] The same argument works for non-connective K-theory, this is the original argument of Saito in [**Sai15**], and, finally, all of this generalizes to any localizing invariant in the sense of Blumberg, Gepner, Tabuada [**BGT13**].

Finally, and for the first time in this note, let us restrict the characteristic of the base field to zero. Then the Loday–Quillen–Tsygan (LQT) theorem applies, telling us that for any unital associative k-algebra R, we have

$$\operatorname{Prim} H_\bullet(\mathfrak{gl}_\infty(R)) \cong HC_{\bullet-1}(R).$$

Theorem 4.5. *Suppose k is a field of characteristic zero and R a unital associative k-algebra. Then there is a canonical isomorphism of the primitive parts*

$$\operatorname{Prim} H_\bullet(\mathfrak{gl}(\infty, R), k) \cong \operatorname{Prim} H_\bullet(\mathfrak{gl}_\infty(R), k)[-1].$$

Proof. Write $\mathbb{M}_n(R)$ to denote the $(n \times n)$-matrix algebra over an associative algebra R. Using Loday–Quillen–Tsygan twice, we obtain the isomorphisms

$$\operatorname{Prim} H_\bullet(\mathfrak{gl}_\infty(E(R))) \underset{\text{(LQT)}}{\cong} HC_{\bullet-1}(E(R)) \cong HC_{\bullet-2}(R) \underset{\text{(LQT)}}{\cong} \operatorname{Prim} H_\bullet(\mathfrak{gl}_\infty(R))[-1],$$

where the middle isomorphism is Corollary 4.4. Moreover, if *Lie* denotes the functor sending an associative k-algebra to its underlying Lie algebra, then

$$\begin{aligned} \mathfrak{gl}_n(E(R)) &= Lie\,(\mathbb{M}_n(E(R))) \\ &= Lie\left(\mathbb{M}_n(\operatorname{End}_{\mathsf{Tate}_{\aleph_0}(\mathsf{P}_f(R))}(\text{“}R((t))\text{”}))\right) \\ &= Lie(\operatorname{End}_{\mathsf{Tate}_{\aleph_0}(\mathsf{P}_f(R))}(\text{“}R((t))\text{”}^{\oplus n})). \end{aligned} \tag{4.2}$$

However, when writing “$R((t))$” $:= \coprod_{n\in\mathbb{N}} R \oplus \prod_{n\in\mathbb{N}} R$ as in Equation 3.1, we can write down a system of isomorphisms

$$\text{“}R((t))\text{”} \cong \text{“}R((t))\text{”}^{\oplus 2} \cong \text{“}R((t))\text{”}^{\oplus 3} \cong \cdots$$

induced from maps between the countable index sets of the product and coproduct, as in [**FI17**, §1.3]. (That is, essentially we use that the disjoint union of $n \geqslant 1$ copies of $\mathbb{N}$ is in bijection to $\mathbb{N}$ itself.) Being isomorphic as objects in $\mathsf{Tate}_{\aleph_0}(\mathsf{P}_f(R))$, their endomorphism algebras are also isomorphic, and thus their Lie algebras. Hence, Equation 4.2 yields that the transition maps in

$$\cdots \longrightarrow H_\bullet(\mathfrak{gl}_n(E(R))) \longrightarrow H_\bullet(\mathfrak{gl}_{n+1}(E(R))) \longrightarrow \cdots$$

are all isomorphisms, and thus $H_\bullet(\mathfrak{gl}_1(E(R))) \cong H_\bullet(\mathfrak{gl}_\infty(E(R)))$ is an isomorphism. But clearly $\mathfrak{gl}_1(E(R))$ is just a different way to denote the Lie algebra of $E(R)$, so this is nothing but $\mathfrak{gl}_{\text{top}}(\infty, R)$. This finishes the proof. □

Note that this statement is precisely the same as was proven by Fialowski and Iohara for $\mathfrak{gl}(\infty, R)$ in their paper. In particular, combining both results, we obtain that the homology of the plain uncompleted Lie algebra $\mathfrak{gl}(\infty, R)$ acting on $R[t, t^{-1}]$, and the topologically completed variant $\mathfrak{gl}_{\text{top}}(\infty, R)$, acting on $R((t))$, does not differ at all:

Corollary 4.6. $H_\bullet(\mathfrak{gl}(\infty, R), k) \cong H_\bullet(\mathfrak{gl}_{\text{top}}(\infty, R), k)$.

One may speculate that this statement might also hold in positive characteristic, but we have no proof.

The delooping property for Hochschild homology would allow us to prove statements analogous to the Corollary for Leibniz homology; we leave this to the reader. Finally, the same methods work for group homology:

Theorem 4.7. *Let R be an associative unital k-algebra, where k is a field of any characteristic. There is a canonical isomorphism of primitive parts*

$$\operatorname{Prim} H_n(\mathrm{GL}_\infty(E(R)), \mathbb{Q}) \cong \operatorname{Prim} H_{n-1}(\mathrm{GL}_\infty(R), \mathbb{Q})$$

for all $n \geqslant 2$.

Proof. The proof follows exactly the same pattern as the one of Theorem 4.5; we only need to employ the picture (due to Feigin and Tsygan [**FT87**]) to view cyclic homology as the additive analogue of K-theory. Then take the arguments for cyclic homology and adapt them. In detail: The Loday–Quillen–Tsygan theorem is modelled after the following fact, essentially belonging to the realm of rational homotopy theory:

$$K_n(R) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \operatorname{Prim} H_n(\mathrm{GL}_\infty(R), \mathbb{Q}),$$

where the latter denotes the primitive part of the group homology of $\mathrm{GL}(R)$, equipped with the discrete topology. Moreover, $K_\bullet(-)$ denotes the usual Quillen (that is: connective) K-theory. Now replace Theorem 4.3 by Saito's original theorem for (non-connective) K-theory [**Sai15**]. We get $K_{n+1}(\mathsf{Tate}_{\aleph_0}(\mathsf{P}_f(R))) \cong K_n(\mathsf{P}_f(R))$ for all $n \geqslant 1$; because for $n \geqslant 1$ the connective and non-connective K-theories agree. Thus,

$$\begin{aligned}\operatorname{Prim} H_{n+1}(\mathrm{GL}_\infty(E(R)), \mathbb{Q}) &\cong K_{n+1}(E(R)) \otimes \mathbb{Q} \cong K_{n+1}(\mathsf{Tate}_{\aleph_0}(\mathsf{P}_f(R))) \otimes \mathbb{Q} \\ &\cong K_n(\mathsf{P}_f(R)) \otimes \mathbb{Q} \cong \operatorname{Prim} H_n(\mathrm{GL}_\infty(R), \mathbb{Q}).\end{aligned}$$

This finishes the proof. □

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