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STABLE INDECOMPOSABILITY OF THREE-MANIFOLDS

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Abstract

We explain that a variant of a recent result of Kwasik-Schultz [**3**] about stable indecomposability of three-manifolds is an immediate consequence of results of Kotschick, Löh and Neofytidis [**2, 1**].

The following result was proved recently by Kwasik and Schultz; see [**3**, Theorem A]:

Theorem 1.1. *Let M be a closed oriented 3-manifold that is not a Cartesian product. Then there is no closed oriented manifold N of dimension $\leqslant 3$ such that $M \times N$ decomposes as a Cartesian product of surfaces and the circle.*

We show that the following variation on the theme of this theorem follows directly from the results of Kotschick, Löh and Neofytidis in [**2, 1**]:

Theorem 1.2. *Let M be a closed oriented 3-manifold that is not finitely covered by a Cartesian product. Then there is no closed oriented manifold N, of any dimension, such that $M \times N$ decomposes as a Cartesian product of surfaces and the circle.*

This is both weaker and stronger than Theorem 1.1. It is weaker in that we not only assume that M is not a product, but make the stronger assumption that M is not finitely covered by a product. It is stronger in that it does not require the assumption $\dim(N) \leqslant 3$.

Proof of Theorem 1.2. Recall that a closed oriented 3-manifold is rationally essential if and only if it has an aspherical summand in its Kneser–Milnor decomposition; cf. [**2**, Theorem 3]. With the additional assumption of rational essentialness one has the following much stronger conclusion than in Theorems 1.1 and 1.2:

Proposition 1.3. *Let M be a rationally essential closed oriented 3-manifold that is not finitely covered by a Cartesian product. Then there is no closed oriented manifold N such that $M \times N$ admits a non-zero degree map from a Cartesian product of surfaces and the circle.*

Proof. Since M is rationally essential and not finitely covered by a product, it is not dominated by a product by [**2**, Proposition 1]. Therefore, the conclusion follows from [**1**, Theorem 2.3]. □

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Most cases of Theorem 1.2 follow from Proposition 1.3. The missing cases concern the rationally inessential manifolds that are not finitely covered by products. By [**2**, Theorem 3], a rationally inessential M is finitely covered by some $\#_k(S^1 \times S^2)$ for $k \geqslant 0$. Moreover, the case $k = 1$ is excluded by the assumption that M is not finitely covered by a product.

If $k \geqslant 2$, then $\pi_2(M) = H_2(\widetilde{M};\mathbb{Z})$ is not finitely generated since $\widetilde{M}$ is the universal covering of $\#_k(S^1 \times S^2)$. For such M the conclusion of Theorem 1.2 follows from the fact that for a product of closed orientable surfaces and the circle, the second homotopy group is finitely generated, with generators the S^2-factors in the product.

Finally, if $k = 0$, then M is finitely covered by S^3 and the universal covering of any $M \times N$ splits as $S^3 \times \widetilde{N}$. In particular, $H_3(\widetilde{M \times N};\mathbb{Z}) \neq 0$. For such an M the conclusion of Theorem 1.2 follows from the fact that for a product of closed orientable surfaces and the circle, the universal covering is a product of two-spheres and Euclidean spaces; in particular its third homology vanishes.

This completes the proof of Theorem 1.2. □

References

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