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SPACES WITH COMPLEXITY ONE

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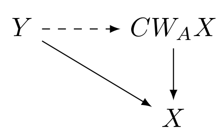
Abstract

An A-cellular space is a space built from a space A and its suspensions, analogous to the way that CW-complexes are built from S^0 and its suspensions. The A-cellular approximation of a space X is an A-cellular space $CW_A X$, which is closest to X among all A-cellular spaces. The A-complexity of a space X is an ordinal number that quantifies how difficult it is to build an A-cellular approximation of X. In this paper, we study spaces with low complexity. In particular, we show that if A is a sphere localized at a set of primes then the A-complexity of each space X is at most 1.

1. Introduction

Let A be a pointed CW-complex. An A-cellular space is a space built out of copies of A via iterated pointed homotopy colimit constructions. Given a pointed space X, the A-cellular approximation of X is an A-cellular space $CW_A X$ equipped with a map $CW_A X \to X$, such that the induced map of pointed mapping complexes $\mathrm{Map}_*(A, CW_A X) \to \mathrm{Map}_*(A, X)$ is a weak equivalence. For example, S^0-cellular spaces are CW-complexes and an S^0-cellular approximation of a space X is the CW-approximation of X. Farjoun [2] showed that the A-cellular approximation of X exists for any A and X and that it is unique up to homotopy equivalence. The space $CW_A X$ is the best possible approximation of X in the class of A-cellular spaces: for any A-cellular space Y and a map $Y \to X$ there exists a map $Y \to CW_A X$, unique up to homotopy, such that the following diagram is homotopy commutative.



In [1] Chachólski, Dwyer, and Intermont introduced the concept of the A-complexity of a space X, which is the minimum ordinal number of homotopy colimits necessary to produce $CW_A(X)$ from copies of A. More precisely, starting with $\mathbf{C}_0$ the full subcategory of pointed spaces such that objects of $\mathbf{C}_0$ have the homotopy type of a retract of wedges of A, one can construct an increasing chain of categories indexed

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by ordinal numbers α, where $\mathbf{C}_\alpha$ is the category of all spaces of the homotopy type of $\mathrm{hocolim}_{\mathbf{D}} F$ for some small category $\mathbf{D}$ and a functor $F\colon \mathbf{D} \to \bigcup_{\beta<\alpha} \mathbf{C}_\beta$ and their retracts. The A-complexity of X is the minimal ordinal $\kappa_A(X)$ such that $X \in \mathbf{C}_{\kappa_A(X)}$.

A CW-approximation of a space X can be constructed as $|\mathrm{sing} X|$, which is a homotopy colimit of discrete spaces. Thus, in the case of $A = S^0$ the category $\mathbf{C}_1$ consists of all spaces having the homotopy type of a CW-complex, which implies that $\kappa_{S^0}(X) \leqslant 1$ for any space X. This is, however, not the case for other choices of A, and in general the complexity of a space need not even be finite. For example, if $A = M(\mathbb{Z}/p, n)$ is a Moore space for some $n \geqslant 1$ with p a prime and $X = M(\mathbb{Z}/p^\infty, n+1)$ then $\kappa_A(X) = \omega$ (see [**1**, Proposition 9.3]).

Results of Stover [**7**] imply that if $A = S^n$ then $\kappa_A(X) \leqslant 2$ for all X. Chachólski, Dwyer and Intermont observed for $A = S^1$ we have $\kappa_A(X) \leqslant 1$ for all X and suggested that it should be possible to similarly lower the bound for all spheres $A = S^n$ [**1**, §9.3]. The goal of this note is to show that this is indeed the case, and that this result holds in even greater generality:

Theorem 1.1. *If A is a sphere or a sphere localized at a set of primes then $\kappa_A(X) \leqslant 1$ for all spaces X.*

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2. Algebras of mapping spaces

Our proof of Theorem 1.1 will be based on the following algebraic description of mapping spaces obtained by Sartwell [**5**]. For pointed spaces X, Y let $\mathrm{Map}_*^\Delta(X, Y)$ denote the simplicial mapping complex of pointed maps $X \to Y$. Given a pointed CW-complex A let $\mathbf{T}^A$ denote a simplicial category on objects T_n for $n = 0, 1, \ldots,$ and such that

$$\mathrm{Hom}_{\mathbf{T}^A}(T_n, T_m) = \mathrm{Map}_*^\Delta\left(\bigvee^m A, \bigvee^n A\right).$$

Notice that in this category T_n is the n-fold product of T_1. A $\mathbf{T}^A$-algebra is a product preserving simplicial functor $\mathbf{T}^A \to \mathbf{sSet}_*$ where $\mathbf{sSet}_*$ is the category of pointed simplicial sets. Let $\mathbf{Alg}^{\mathbf{T}^A}$ denote the category of all $\mathbf{T}^A$-algebras with natural transformations as morphisms.

Notice that any pointed space X defines a $\mathbf{T}^A$-algebra $\Omega^A(X)$ such that

$$\Omega^A(X)(T_n) = \mathrm{Map}_*^\Delta\left(\bigvee^n A, X\right).$$

The resulting functor $\Omega^A\colon \mathbf{Top}_* \to \mathbf{Alg}^{\mathbf{T}^A}$ has a left adjoint B^A.

The category $\mathbf{Alg}^{\mathbf{T}^A}$ can be equipped with a model category structure where weak equivalences and fibrations are defined as objectwise weak homotopy equivalence and Serre fibrations respectively. Denote also by $\mathbf{R}^A\mathbf{Top}_*$ the category $\mathbf{Top}_*$ taken with the model category structure where fibrations are Serre fibrations and weak equivalences are maps $f\colon X\to Y$ that induce weak equivalences $f_*\colon \mathrm{Map}_*^\Delta(A,X)\to \mathrm{Map}_*^\Delta(A,Y)$. In other words, $\mathbf{R}^A\mathbf{Top}_*$ is obtained by taking the right Bousfield localization of the usual model category structure on $\mathbf{Top}_*$ with respect to the space A [3, 5.1.1]. Sartwell showed that the following holds:

Theorem 2.1 ([5])**.** *Let A be a sphere or a sphere localized at a set of primes. The adjoint pair*

$$B^A\colon \mathbf{Alg}^{\mathbf{T}^A} \rightleftarrows \mathbf{R}^A\mathbf{Top}_*\colon \Omega^A$$

is a Quillen equivalence.

A simplicial version of this argument can be seen in [4].

3. Proof of Theorem 1.1

Directly from the definition of an A-cellular approximation it follows that if spaces X and Y are weakly equivalent in $R^A\mathbf{Top}$ then CW_AX and CW_AY are weakly homotopy equivalent, and so $\kappa_A(X)=\kappa_A(Y)$. Let A be a space as in Theorem 1.1. By Theorem 2.1 for any space X we get a weak equivalence $X\simeq B^AQ\Omega^AX$ in $R^A\mathbf{Top}$, where $Q\Omega^AX$ denotes a cofibrant replacement of Ω^AX in the category $\mathbf{Alg}^{\mathbf{T}^A}$. Therefore, it is only necessary to show that for any space X, $\kappa_A(B^AQ\Omega^AX)\leqslant 1$.

The algebra $Q\Omega^AX$ can be described more explicitly as follows. Let $U\colon \mathbf{Alg}^{\mathbf{T}^A}\to \mathbf{sSet}_*$ denote the forgetful functor, $U(\Phi)=\Phi(T_1)$. This functor has a left adjoint $F\colon \mathbf{sSet}_*\to \mathbf{Alg}^{\mathbf{T}^A}$ [6, 2.3]. For a $\mathbf{T}^A$-algebra Φ, let $FU_\bullet\Phi$ be the simplicial $\mathbf{T}^A$-algebra defined by the adjoint pair (F,U). Explicitly:

$$FU_n\Phi := (FU)^{n+1}\Phi.$$

By [5, 2.3.3] the natural map $|FU_\bullet\Phi|\to\Phi$ is a cofibrant replacement of Φ in the category $\mathbf{Alg}^{\mathbf{T}^A}$.

In view of this fact we need to show that for any space X the A-complexity of $B^A|FU_\bullet\Omega^AX|$ is at most one. The functor B^A is a left adjoint and so commutes with homotopy colimits, in particular, with the geometric realization functor. The functor $B^AF\colon \mathbf{sSet}_*\to\mathbf{Top}_*$ is left adjoint to $U\Omega^A$, and since $U\Omega^A(X)=\mathrm{Map}_*^\Delta(A,X)$, thus for Y a (pointed) simplicial set we get $B^AF(Y)=A\wedge|Y|$. Combining these observations we obtain

$$\begin{aligned}
B^A|FU_\bullet\Omega^AX|\simeq\ |B^AFU_\bullet\Omega^AX| &\simeq \mathrm{hocolim}_{n\in\Delta^{op}}|(B^AF)U((FU)^n\Omega^AX)|\\
&\simeq \mathrm{hocolim}_{n\in\Delta^{op}}|(B^AF)(FU)^n\Omega^AX(T_1)|\\
&\simeq \mathrm{hocolim}_{n\in\Delta^{op}}A\wedge|(FU)^n\Omega^AX(T_1)|\\
&\simeq \mathrm{hocolim}_{(n,m)\in\Delta^{op}\times\Delta^{op}}A\wedge((FU)^n\Omega^AX(T_1))_m\\
&\simeq \mathrm{hocolim}_{(n,m)\in\Delta^{op}\times\Delta^{op}}\bigvee_{\sigma\in S^n_m}A,
\end{aligned}$$

where S^n_m denotes the set $((FU)^n\Omega^AX(T_1))_m$.

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