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GENERALIZED GOTTLIEB AND WHITEHEAD CENTER GROUPS OF SPACE FORMS

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Abstract

We extend Oprea's result that the Gottlieb group $G_1(\mathbb{S}^{2n+1}/H)$ is $\mathcal{Z}H$ (the center of H) and show that for a map $f\colon A \to \mathbb{S}^{2n+1}/H$, under some conditions on A, we have $G_1^f(\mathbb{S}^{2n+1}/H) = \mathcal{Z}_H f_*(\pi_1(A))$, the centralizer of the image $f_*(\pi_1(A))$ in H. Then, we compute or estimate the groups $G_m^f(\mathbb{S}^{2n+1}/H)$ and $P_m^f(\mathbb{S}^{2n+1}/H)$ for certain $m > 1$.

1. Introduction

Throughout this paper, all spaces are path-connected with the homotopy types of CW-complexes and maps and homotopies are based. We use the standard terminology and notations from homotopy theory, mainly from [**10**] and [**23**]. We do not distinguish between a map and its homotopy class.

Let X be a connected space and $\mathbb{S}^m$ the m-sphere. The m^{th} *Gottlieb group* $G_m(X)$ of X, defined first for $m = 1$ in [**11**] and then for $m \geqslant 1$ in [**12**], is the subgroup of the $m^{\rm th}$ homotopy group $\pi_m(X)$ consisting of all elements which can be represented by a map $\alpha\colon \mathbb{S}^m \to X$ such that $\mathrm{id}_X \vee \alpha\colon X \vee \mathbb{S}^m \to X$ extends (up to homotopy) to a map $F\colon X \times \mathbb{S}^m \to X$. Following [**12**], we recall that $P_m(X)$ is the set of elements of $\pi_m(X)$ whose Whitehead products with all elements of all homotopy groups of X are zero. It turns out that $P_m(X)$ forms a subgroup of $\pi_m(X)$ called the *Whitehead center group* and, by [**12**, Proposition 2.3], we also have $G_m(X) \subseteq P_m(X)$. Some advanced attempts to compute the groups $G_m(X)$ and $P_m(X)$ for spheres and projective spaces have been made in [**10**].

Now, given a map $f\colon A \to X$, in view of [**12**] (see also [**19**]), the m^{th} *generalized Gottlieb group* $G_m^f(X)$ is defined as the subgroup of the $m^{\rm th}$ homotopy group $\pi_m(X)$ consisting of all elements which can be represented by a map $\alpha\colon \mathbb{S}^m \to X$ such that $f \vee \alpha\colon A \vee \mathbb{S}^m \to X$ extends (up to homotopy) to a map $F\colon A \times \mathbb{S}^m \to X$. According to the literature, we say that the map F is *affiliated* to α. If $A = X$ then the group $G_1^f(X)$, also denoted by $J(f)$, is called the *Jiang group* of the map $f\colon X \to X$ in honor of Bo-Ju Jiang who recognized in [**17**] their importance to the Nielsen–Wecken

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theory of fixed point classes. The role the group $J(f)$ played in that theory has been intensively studied in the book [**7**] as well.

The m^{th} *generalized Whitehead center group* $P_m^f(X)$, as defined in [**19**], is the set of all elements $\alpha \in \pi_m(X)$ whose Whitehead products $[\alpha, f\beta]$ are zero for all $\beta \in \pi_l(A)$ with $l \geqslant 1$. It turns out that $P_m^f(X)$ forms a subgroup of $\pi_m(X)$ and $G_m^f(X) \subseteq P_m^f(X)$.

Given a free action $H \times \mathbb{S}^{2n+1} \to \mathbb{S}^{2n+1}$ of a finite group H on $\mathbb{S}^{2n+1}$, Oprea [**20**] has shown that $G_1(\mathbb{S}^{2n+1}/H) = \mathcal{Z}H$, the center of H. Further, in the special case of a linear action of H on $\mathbb{S}^{2n+1}$, a very nice representation-theoretic proof of that fact has been given in [**5**]. In this paper, we generalize Oprea's results to the case of a free action of a finite group H on a homotopy sphere $\Sigma(2n+1)$. We show that for a map $f\colon A \to \Sigma(2n+1)$ with $1 < \dim A = d \leqslant 2n+1$, $\pi_k(A) = 0$ for $1 < k < 2n+1$ and $H^{2n+1}(\pi_1(A); \mathbb{Z}) = 0$, we have $G_1^f(\Sigma(2n+1)/H) = \mathcal{Z}_H f_*(\pi_1(A))$, the centralizer of $f_*(\pi_1(A))$ in H. In particular, we obtain that the Jiang group obeys $J(f) = \mathcal{Z}_H f_*(H)$ for any $f\colon \Sigma(2n+1)/H \to \Sigma(2n+1)/H$.

Further, we compute or estimate the Gottlieb groups $G_m^f(\Sigma(2n+1)/H)$ and $P_m^f(\Sigma(2n+1)/H)$ for certain $m > 1$, finite groups H and $f\colon A \to \Sigma(2n+1)/H$.

In Section 2, Proposition 2.4 generalizes Gottlieb's result [**11**] and states that $G_1^f(K(\pi,1)) = \mathcal{Z}_\pi f_*(\pi_1(A))$ for $f\colon A \to K(\pi,1)$ and Theorem 2.10 states:

If $p\colon \tilde{X} \to X$ *is a covering map of a space* X *and* $f\colon A \to \tilde{X}$ *then the isomorphism* $p_*\colon \pi_m(\tilde{X}) \to \pi_m(X)$ *for* $m > 1$ *restricts to isomorphisms*

$$p_{*|}\colon G_m^f(\tilde{X}) \to G_m^{pf}(X) \text{ and } p_{*|}\colon P_m^f(\tilde{X}) \to P_m^{pf}(X)$$

for $m > 1$.

The main result of this section, generalizing [**20**, Theorem A], is Theorem 2.17 which implies that

$$G_1^f(\Sigma(2n+1)/H) = \mathcal{Z}_H f_*(K)$$

for $f\colon \Sigma(2d+1)/K \to \Sigma(2n+1)/H$ with $d \leqslant n$. Then, in Corollary 2.19, for the quotient map $\gamma_{2n+1}\colon \Sigma(2n+1) \to \Sigma(2n+1)/H$, we have:

(1) $P_1^{\gamma_{2n+1}}(\Sigma(2n+1)/H) = G_1^{\gamma_{2n+1}}(\Sigma(2n+1)/H) = H$ provided H is abelian;

(2) $P_m^{\gamma_{2n+1}}(\Sigma(2n+1)/H) = \gamma_{2n+1*}(P_m(\Sigma(2n+1))) = \gamma_{2n+1*}(G_m(\mathbb{S}^{2n+1}))$ for $m > 1$.

Section 3 is devoted to Whitehead center groups of projective spaces and some space forms. Subsection 3.1 makes use of some results from [**10**] to take up the systematic study of the groups $G_m(\Sigma(2n+1)/\mathbb{Z}_l)$ for certain $m > 1$.

Finally, Subsection 3.2 applies [**10**] to present computations of $G_m(\mathbb{S}^{2n+1}/H)$ for $m > 1$ and $H < \mathbb{S}^3$.

2. Generalized Gottlieb and Whitehead center groups

Given spaces A and X, write X^A for the space of continuous unbased maps from A into X with the compact-open topology. Write $\mathrm{ev}(g) = g(a_0)$ for the basepoint $a_0 \in A$. Let $f\colon A \to X$ with $f(a_0) = x_0 \in X$ and consider the component of X^A containing f,

(X^A, f). Then

$$G^f_m(X) = \mathrm{Im}(\mathrm{ev}_*\colon \pi_m(X^A, f) \to \pi_m(X, x_0))$$

for $m \geqslant 1$.

If $f \in P_n(X)$ for some $n \geqslant 1$ then $G^f_m(X) = P^f_m(X) = \pi_m(X)$ for any $m \geqslant 1$ because the Whitehead product $[\alpha, f]$ is the obstruction to extending the map $\mathbb{S}^m \vee \mathbb{S}^n \xrightarrow{\alpha\vee f} X$ to $\mathbb{S}^m \times \mathbb{S}^n \to X$ for $\alpha \in \pi_m(X)$.

Further, we have the following:

Proposition 2.1. *Let $f\colon A \to X$. Then:*

(1) *any map $g\colon A' \to A$ leads to the inclusion relations $G^f_m(X) \subseteq G^{fg}_m(X)$ and $P^f_m(X) \subseteq P^{fg}_m(X)$ for $m \geqslant 1$;*

(2) *if $g\colon A' \to A$ is a homotopy equivalence then $G^f_m(X) = G^{fg}_m(X)$ and $P^f_m(X) = P^{fg}_m(X)$ for $m \geqslant 1$;*

(3) *if $f_*\colon \pi_1(A) \to \pi_1(X)$ is the induced homomorphism by a map $f\colon A \to X$ and $\pi_1(X)$ acts trivially on $\pi_m(X)$ for all $m > 1$ then $P^f_1(X) = \mathcal{Z}_{\pi_1(X)} f_*(\pi_1(A))$, the centralizer of the image $f_*(\pi_1(A))$ in $\pi_1(X)$. If $f_*\colon \pi_*(A) \to \pi_*(X)$ is an epimorphism then $P^f_m(X) = P_m(X)$ for all $m \geqslant 1$;*

(4) *given a map $h\colon X \to Y$ the induced homomorphism $h_*\colon \pi_m(X) \to \pi_m(Y)$ restricts to homomorphisms $h_{*|}\colon G^f_m(X) \to G^{hf}_m(Y)$ and $h_{*|}\colon P^f_m(X) \to P^{hf}_m(Y)$. In particular, for $f = \mathrm{id}_X$, we obtain homomorphisms $h_{*|}\colon G_m(X) \to G^h_m(Y)$ and $h_{*|}\colon P_m(X) \to P^h_m(Y)$. Because $G_m(Y) \subseteq G^h_m(Y)$ and $P_m(Y) \subseteq P^h_m(Y)$, we conclude that $h^{-1}_{*|}(G_m(Y)) \subseteq G_m(X)$ and $h^{-1}_{*|}(P_m(Y)) \subseteq P_m(X)$;*

(5) *if $A = \mathbb{S}^k$ then $G^f_m(X) = P^f_m(X) = \mathrm{Ker}[f, -]$ for $m \geqslant 1$;*

(6) *if $\alpha\colon \mathbb{S}^l \to \mathbb{S}^n$ then the induced map $\alpha^*\colon \pi_n(X) \to \pi_l(X)$ restricts to maps*

$$\alpha^*_|\colon G^f_n(X) \to G^f_l(X) \text{ and } \alpha^*_|\colon P^f_n(X) \to P^f_l(X).$$

If a map $\alpha = E\beta$ is a suspension, then:

(i) *the restricted maps $\alpha^*_|$ are homomorphisms;*

(ii) *$(E^{m-n}\alpha)^*_|\colon G^\alpha_m(\mathbb{S}^n) \to G_{m-n+l}(\mathbb{S}^n)$ for $m \geqslant n$.*

Proof. Because (1)–(6)(i) are obvious, we show only (6)(ii). We write E for the suspension functor on the category of pointed spaces and apply the following property [**25**, Chapter X, (8.18) Theorem] of the Whitehead product:

$$[\gamma E\delta, \gamma' E\delta'] = [\gamma, \gamma']E(\delta \wedge \delta')$$

for $\gamma \in \pi_s(X)$, $\gamma' \in \pi_t(X)$ and maps $\delta\colon \mathbb{S}^{r-1} \to \mathbb{S}^{s-1}$ and $\delta'\colon \mathbb{S}^{r'-1} \to \mathbb{S}^{t-1}$.

Now, notice that we may assume $l, m \geqslant n$ and write ι_k for the identity map of the sphere $\mathbb{S}^k$. Then, given $\nu \in G^\alpha_m(\mathbb{S}^n)$, we obtain:

$$\begin{aligned} 0 &= [\nu, \alpha] = [\nu E\iota_{m-1}, \iota_n E\beta] = [\nu, \iota_n]E(\iota_{m-1} \wedge \beta) \\ &= [\nu, \iota_n]E^m\beta = [\nu, \iota_n]E(E^{m-n}\beta \wedge \iota_{n-1}) = [\nu E^{m-n+1}\beta, \iota_n] = [\nu E^{m-n}\alpha, \iota_n]. \end{aligned}$$

Consequently,

$$(E^{m-n}\alpha)^*_|\colon G^\alpha_m(\mathbb{S}^n) \to G_{m-n+l}(\mathbb{S}^n)$$

and the proof is complete. □

Example 2.2.

(1) Recall (see e.g., [**10**, Chapter 1, (1.2)]) that the order of the Whitehead product

$$\sharp[\iota_n,\iota_n]=\begin{cases}1, & \text{for } n=1,3,7;\\ 2, & \text{for } n\neq 1,3,7 \text{ odd};\\ \infty, & \text{for } n \text{ even}.\end{cases}$$

This implies that

$$G_n(\mathbb{S}^n)=P_n(\mathbb{S}^n)=\begin{cases}\pi_n(\mathbb{S}^n), & \text{for } n=1,3,7;\\ 2\pi_n(\mathbb{S}^n), & \text{for } n\neq 1,3,7 \text{ odd}\end{cases}$$

and $G_n(\mathbb{S}^n)=0$ provided n is even.
Given a map $f\colon \mathbb{S}^n\to\mathbb{S}^n$, write $\deg f$ for its degree and notice that $\alpha\in G^f_n(\mathbb{S}^n)=P^f_n(\mathbb{S}^n)$ if and only if $(\deg f)\alpha\in G_n(\mathbb{S}^n)$. Applying the order of $[\iota_n,\iota_n]$, we deduce that

$$G^f_n(\mathbb{S}^n)=P^f_n(\mathbb{S}^n)=\begin{cases}\pi_n(\mathbb{S}^n), & \text{for } \deg f \text{ even};\\ G_n(\mathbb{S}^n), & \text{for } \deg f \text{ odd},\end{cases}$$

if n is odd. Further, $G^f_n(\mathbb{S}^n)=0$ provided n is even.

(2) Consider a finite group H with a free action $H\times\mathbb{S}^n\to\mathbb{S}^n$ for an odd $n\geqslant 1$, write $\mathbb{S}^n/H$ for the associated orbit space and $\gamma\colon\mathbb{S}^n\to\mathbb{S}^n/H$ for the quotient map.
Let $f\colon\mathbb{S}^k\to\mathbb{S}^n/H$ be a pointed map with $k\geqslant 1$. Notice that for $k>1$ there is a unique map $f'\colon\mathbb{S}^k\to\mathbb{S}^n$ with $f=\gamma f'$. It is well-known (see e.g., [**6**, Chapter VII, Proposition 10.2]) that $\pi_1(\mathbb{S}^n/H)\cong H$ acts trivially on $\pi_m(\mathbb{S}^n/H)\cong\pi_m(\mathbb{S}^n)$ for $m>1$. Then, by Proposition 2.1(5), we get:

$$G^f_m(\mathbb{S}^n/H)=P^f_m(\mathbb{S}^n/H)=\operatorname{Ker}[f,-]=\begin{cases}\gamma_*G^{f'}_m(\mathbb{S}^n), & \text{if } k>1;\\ \gamma_*\pi_m(\mathbb{S}^n)=\pi_m(\mathbb{S}^n/H), & \text{if } k=1,\end{cases}$$

for $m>1$ and

$$G^f_1(\mathbb{S}^n/H)=P^f_1(\mathbb{S}^n/H)=\begin{cases}H, & \text{if } k>1;\\ \mathcal{Z}_H\langle f\rangle, & \text{if } k=1.\end{cases}$$

Now, we show:

Lemma 2.3. *If $f\colon A\to X$ then $P^f_1(X)\subseteq\mathcal{Z}_{\pi_1(X)}f_*(\pi_1(A))$.*

Proof. Take $\alpha\in P^f_1(X)$. Then, the Whitehead product $[\alpha,f_*(\beta)]=0$ for all $\beta\in\pi_1(A)$. This implies $\alpha f_*(\beta)=f_*(\beta)\alpha$ and the proof follows. □

Because $G^f_1(X)\subseteq P^f_1(X)$, we see that

$$G^f_1(X)\subseteq\mathcal{Z}_{\pi_1(X)}f_*(\pi_1(A)),$$

which for $f=\mathrm{id}_X$ implies the result of Gottlieb [**11**].

Gottlieb [**11**, Corollary I.13] has shown that $G_1(K(\pi,1))=\mathcal{Z}\pi$, the center of the group π. We generalize that result as follows:

Proposition 2.4. *If* $f\colon A \to K(\pi,1)$ *then* $G_1^f(K(\pi,1)) = \mathcal{Z}_\pi f_*(\pi_1(A))$.

Proof. By the above, we have $G_1^f(K(\pi,1)) \subseteq \mathcal{Z}_\pi f_*(\pi_1(A))$.

To show the opposite inclusion, take $\alpha \in \mathcal{Z}_\pi f_*(\pi_1(A))$ and consider the homomorphism $\varphi\colon \pi_1(A\times \mathbb{S}^1) = \pi_1(A)\times \mathbb{Z} \to \pi$ given by $\varphi(g,n) = \alpha^n f_*(g)$ for $(g,n)\in \pi_1(A)\times\mathbb{Z}$, where $\mathbb{Z}$ is the group of integers. Since homotopy classes of maps into a $K(\pi,1)$ are determined at the fundamental group level (see, e.g., [**25**, Chapter V, (4.3) Theorem]), we obtain the required map $A\times\mathbb{S}^1 \to K(\pi,1)$. □

We point out that a similar result as stated above has been already obtained in [**13**, Lemma 2] and the inclusion $\mathcal{Z}_\pi f_*(\pi) \subseteq G_1^f(K(\pi,1))$ for any self-map $f\colon K(\pi,1)\to K(\pi,1)$ shown in [**7**, Chapter VII, Theorem 10].

Example 2.5. (1) If π is an abelian group then $G_1(K(\pi,1)) = G_1^f(K(\pi,1)) = \pi$ for any $f\colon A\to K(\pi,1)$.

(2) Let $Q_8 = \langle i,j\rangle$ be the quaternionic group. Because the center $\mathcal{Z}(Q_8) = \mathbb{Z}_2$, we see that $G_1(K(Q_8,1)) = \mathbb{Z}_2$. Let $f\colon \mathbb{S}^1 \to K(Q_8,1)$ be the map determined by the homomorphism $\mathbb{Z}\to Q_8$ such that $1\mapsto i$. Then, $G_1^f(K(Q_8,1)) = \mathcal{Z}_{Q_8}\langle i\rangle = \langle i\rangle$ and so we have the proper inclusion $G_1(K(Q_8,1)) \subsetneqq G_1^f(K(Q_8,1))$.

Recall that a space X is said to be *aspherical* if $\pi_m(X,x) = 0$ for $m>1$ and all $x\in X$. Following the ideas stated in [**11**, Section III, X^X], we can easily generalize [**11**, Theorem III.2], [**13**, Lemma 2] and [**15**, Theorems 6.1 and 6.2] as follows:

Proposition 2.6. *If X is a locally finite, aspherical, path-connected space and* $f\colon A\to X$ *then:*

(1) $\mathrm{ev}_*\colon \pi_1(X^A,f) \xrightarrow{\cong} G_1^f(X) = \mathcal{Z}_{\pi_1(X)} f_*(\pi_1(A))$;

(2) $\pi_m(X^A,f) = 0$ *for* $m>1$.

In particular, if A is a 1-connected space then the space X^A is path-connected and $\pi_1(X^A,f)\cong G_1^f(X) = \pi_1(X)$ for any $f\colon A\to X$ provided X is a locally finite, aspherical and path-connected space.

Further, Gottlieb [**12**, Theorems 6-1 and 6-2] has shown:

Proposition 2.7. *If* $p\colon \tilde{X}\to X$ *is a covering map then*

$$p_*^{-1}(G_m(X)) \subseteq G_m(\tilde{X}) \text{ for } m\geqslant 1.$$

For Whitehead center groups, we have:

Proposition 2.8. *If* $p\colon \tilde{X}\to X$ *is a covering map then*

$$p_*^{-1}(P_m(X)) \subseteq P_m(\tilde{X}) \text{ for } m\geqslant 1.$$

If X is a simple space then

$$p_*^{-1}(P_m(X)) = P_m(\tilde{X}) \text{ for } m\geqslant 1.$$

Proof. Since $p_*\colon \pi_*(\tilde{X})\to\pi_*(X)$ is a monomorphism, the inclusion $p_*^{-1}(P_m(X)) \subseteq P_m(\tilde{X})$ for $m\geqslant 1$ is straightforward. Let now X be a simple space and take $\alpha\in P_m(\tilde{X})$, and $\beta\in\pi_k(X)$. If $k=1$ then $[p\alpha,\beta]=0$ since X is a simple space. If $k>1$ then there is $\gamma\in\pi_k(\tilde{X})$ such that $p\gamma=\beta$. Hence, $[p\alpha,\beta] = [p\alpha,p\gamma] = p_*[\alpha,\gamma] = 0$ and the proof follows. □

To state the next result, we prove:

Lemma 2.9. *If $p\colon \tilde{X} \to X$ is a covering map and $f\colon A \to \tilde{X}$ then:*

(1) $G_1^f(\tilde{X}) = p_*^{-1}(G_1^{pf}(X))$;

(2) $P_1^f(\tilde{X}) = p_*^{-1}(P_1^{pf}(X))$.

Proof. Because $p_*\colon \pi_1(\tilde{X}) \to \pi_1(X)$ restricts to $p_{*|}\colon G_1^f(\tilde{X}) \to G_1^{pf}(X)$ and $p_{*|}\colon P_1^f(\tilde{X}) \to P_1^{pf}(X)$, we deduce that $G_1^f(\tilde{X}) \subseteq p_*^{-1}(G_1^{pf}(X))$ and $P_1^f(\tilde{X}) \subseteq p_*^{-1}(P_1^{pf}(X))$.

(1): We prove $p_*^{-1}(G_1^{pf}(X)) \subseteq G_1^f(\tilde{X})$. Given $\alpha \in p_*^{-1}(G_1^{pf}(X))$, we have a map $F\colon A \times \mathbb{S}^1 \to X$ such that the diagram

$$\begin{array}{ccc} A \vee \mathbb{S}^1 & \xrightarrow{pf \vee p\alpha} & X \\ \big\downarrow & \nearrow_{F} & \\ A \times \mathbb{S}^1 & & \end{array}$$

commutes up to homotopy. But, for any $\beta\colon \mathbb{S}^1 \to A \times \mathbb{S}^1$ there is a map $\beta'\colon \mathbb{S}^1 \to A \vee \mathbb{S}^1$ such that the composition $\mathbb{S}^1 \xrightarrow{\beta'} A \vee \mathbb{S}^1 \overset{i}{\hookrightarrow} A \times \mathbb{S}^1$ is homotopic to β. Then, $F\beta = p(f \vee \alpha)\beta'$ and, consequently, $F_*(\pi_1(A \times \mathbb{S}^1)) \subseteq p_*(\pi_1(\tilde{X}))$. Hence, there is a lifting $\tilde{F}\colon A \times \mathbb{S}^1 \to \tilde{X}$ of the map F. Because $p\tilde{F}_{|A\vee\mathbb{S}^1} = F_{|A\vee\mathbb{S}^1} = p(f \vee \alpha)$, we see that $\tilde{F}_{|A\vee\mathbb{S}^1} = f \vee \alpha$ and so $\alpha \in G_1^f(\tilde{X})$.

(2): We show $p_*^{-1}(P_1^{pf}(X)) \subseteq P_1^f(\tilde{X})$. Given $\alpha \in p_*^{-1}(P_1^{pf}(X))$, we have $p\alpha \in P_1^{pf}(X)$. Then, for any $\gamma \in \pi_m(A)$ with $m \geqslant 1$, we obtain $p_*[\alpha, f\gamma] = [p\alpha, pf\gamma] = 0$. Because p_* is a monomorphism, $[\alpha, f\gamma] = 0$ and so $\alpha \in P_1^f(\tilde{X})$ and the proof follows. □

Theorem 2.10. *If $p\colon \tilde{X} \to X$ is a covering map and $f\colon A \to \tilde{X}$ then the monomorphism $p_*\colon \pi_m(\tilde{X}) \to \pi_m(X)$ for $m \geqslant 1$ yields:*

(1) $G_m^f(\tilde{X}) = p_*^{-1}(G_m^{pf}(X))$;

(2) $P_m^f(\tilde{X}) = p_*^{-1}(P_m^{pf}(X))$.

Proof. In view of Lemma 2.9, we may assume that $m > 1$. Because $p_*\colon \pi_m(\tilde{X}) \to \pi_m(X)$ is an isomorphism, we see that its restrictions

$$p_{*|}\colon G_m^f(\tilde{X}) \to G_m^{pf}(X) \text{ and } p_{*|}\colon P_m^f(\tilde{X}) \to P_m^{pf}(X)$$

are monomorphisms.

First, we prove that $p_{*|}\colon G_m^f(\tilde{X}) \to G_m^{pf}(X)$ is surjective for $m > 1$. Given $\alpha \in G_m^{pf}(X)$, there are $\beta \in \pi_m(\tilde{X})$ such that $p\beta = \alpha$ and $F\colon A \times \mathbb{S}^m \to X$ extending $pf \vee \alpha\colon A \vee \mathbb{S}^m \to X$. But the 2-skeleton $(A \times \mathbb{S}^m)^{(2)} = A^{(2)} \vee (\mathbb{S}^m)^{(2)}$, so we obtain that $\pi_1(A \times \mathbb{S}^m) = \pi_1(A^{(2)} \vee (\mathbb{S}^m)^{(2)}) = \pi_1(A^{(2)}) = \pi_1(A)$. Because $F_{|A\vee\mathbb{S}^m} = pf \vee \alpha$, this implies that $F_*(\pi_1(A \times \mathbb{S}^m)) = (pf)_*(\pi_1(A)) \subseteq p_*(\pi_1(\tilde{X}))$. Hence, the map $F\colon A \times \mathbb{S}^m \to X$ lifts to $\tilde{F}\colon A \times \mathbb{S}^m \to \tilde{X}$. Because $p\tilde{F}_{|A\vee\mathbb{S}^m} = F_{|A\vee\mathbb{S}^m} = p(f \vee \beta)$, the map $\tilde{F}$ extends $f \vee \beta\colon A \vee \mathbb{S}^m \to \tilde{X}$ and so $\beta \in G_m^f(\tilde{X})$.

Now, we show that $p_{*|}\colon P_m^f(\tilde{X}) \to P_m^{pf}(X)$ is surjective for $m > 1$. Given $\alpha \in P_m^{pf}(X)$, there is $\beta \in \pi_m(\tilde{X})$ such that $p\beta = \alpha$. Then, $p_*[\beta, f\gamma] = [p\beta, pf\gamma] = [\alpha, pf\gamma] = 0$ for any $\gamma \in \pi_l(\tilde{X})$ which implies $[\beta, f\gamma] = 0$ and so $\beta \in P_m^f(\tilde{X})$, and the proof is complete. □

Then, Proposition 2.8 and Theorem 2.10 yield:

Corollary 2.11. *If* $p\colon \tilde{X} \to X$ *is a covering map then* $G_m(\tilde{X}) = p_*^{-1}(G_m^p(X))$ *and* $P_m(\tilde{X}) = p_*^{-1}(P_m^p(X))$ *for* $m \geqslant 1$. *Further, if* X *is a simple space then* $P_m(\tilde{X}) = p_*^{-1}(P_m^p(X)) = p_*^{-1}(P_m(X))$ *for* $m \geqslant 1$.

To extend [**20**, Theorem A] and state our main result, we need some prerequisites. Let π be an abelian group. Given $\Phi\colon A \times \mathbb{S}^1 \to X$, write $f = \Phi_{|A}$ for its restriction to the space A. The induced map $\Phi^*\colon H^n(X;\pi) \to H^n(A \times \mathbb{S}^1;\pi)$ gives

$$\Phi^*(x) = f^*(x) \otimes 1 + x_\Phi \otimes \lambda,$$

where $x \in H^n(X;\pi)$, the element λ is a chosen generator of $H^1(\mathbb{S}^1;\mathbb{Z})$ and $x_\Phi \in H^{n-1}(A;\pi)$.

Now, take an integer $m > 0$. Recall that a map $p\colon E \to B$ is called a *principal* $K(\pi,m)$*-fibration* if it is a pullback of the path fibration $K(\pi,m) \to PK(\pi,m+1) \to K(\pi,m+1)$ via the classifying map $k\colon B \to K(\pi,m+1)$. If $\iota \in H^{m+1}(K(\pi,m+1);\pi)$ is the fundamental class of $K(\pi,m+1)$, let $k^*(\iota) = \mu \in H^{m+1}(B;\pi)$ and recall that a map $\varphi\colon Y \to B$ has a lifting $\bar{\varphi}\colon Y \to E$ if and only if $\varphi^*(\mu) = 0$.

Then, following *mutatis mutandis* the proof of [**20**, Theorem 1], we can show a fundamental lifting result due to Gottlieb [**12**]:

Lemma 2.12. *Let* $p\colon E \to B$ *and* $p'\colon E' \to B'$ *be principal* $K(\pi,m)$- *and* $K(\pi',m)$-*fibrations, respectively with a commutative diagram*

$$\begin{array}{ccc} E & \xrightarrow{\ \bar{f}\ } & E' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ B & \xrightarrow[\ f\ ]{} & B' \end{array}$$

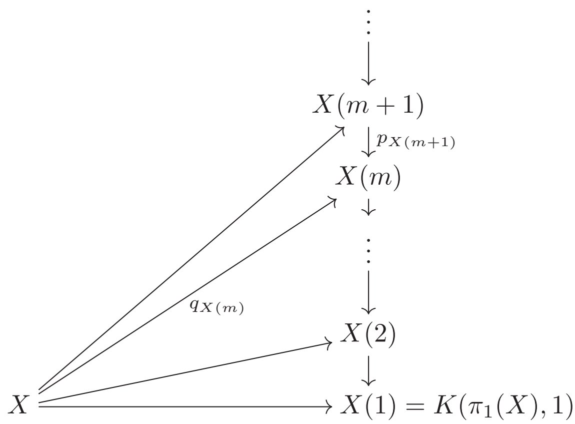
and $\Phi\colon B \times \mathbb{S}^1 \to B'$ *such that* $\Phi_{|B} = f$. *If* $\mu' = k'^*(\iota')$, *where* k' *classifies* p' *and* ι' *is the fundamental class of* $K(\pi',m+1)$ *then there exists a map* $\bar{\Phi}\colon E \times \mathbb{S}^1 \to E'$ *such that* $\bar{\Phi}_{|E} = \bar{f}$ *and the diagram*

$$\begin{array}{ccc} E \times \mathbb{S}^1 & \overset{\bar{\Phi}}{- - - \to} & E' \\ {\scriptstyle p \times \mathrm{id}_{\mathbb{S}^1}}\downarrow & & \downarrow{\scriptstyle p'} \\ B \times \mathbb{S}^1 & \xrightarrow[\ \Phi\ ]{} & B' \end{array}$$

commutes if and only if $\mu'_\Phi = 0 \in H^m(B;\pi')$ *with* $\Phi^*(\mu') = f^*(\mu') \otimes 1 + \mu'_\Phi \otimes \lambda$ *for a chosen generator* λ *of* $H^1(\mathbb{S}^1;\mathbb{Z})$.

Remark 2.13. As in [**20**, Remark, p. 68], we notice that without loss of generality we can take the diagrams to be homotopy commutative.

Next, given a space X, consider its universal covering $p\colon \tilde{X} \to X$. As usual, we can take its classifying map $\kappa\colon X \to K(\pi_1(X),1)$ to be an inclusion. If $\pi_1(X)$ acts trivially on $\pi_m(X)$ for $m > 1$ then the pair $(K(\pi_1(X),1),X)$ is simple. Hence, according to [**3**, Chapter 7, Section 7.4], the Moore–Postnikov tower



for the classifying map $\kappa\colon X \to K(\pi_1(X),1)$ of the covering p exists, where $X(m)$ is called the m^{th} *stage* of this tower for $m \geqslant 1$. Recall that the map $q_{X(m)}\colon X \to X(m)$ is an $(m+1)$-equivalence for $m \geqslant 1$.

From now on, we assume that A and X are spaces such that $\pi_1(A)$ and $\pi_1(X)$ act trivially on $\pi_m(A)$ and $\pi_m(X)$ for $m > 1$, respectively. Given a map $f\colon A \to X$, write $f(m)\colon A(m) \to X(m)$ for the induced map of the m^{th} stages for $m \geqslant 0$.

If $\alpha \in G_1^{f(m+1)}(X(m+1))$ and $\bar{\Phi}\colon A(m+1) \times \mathbb{S}^1 \to X(m+1)$ is the associated map then the naturality (see, e.g., [**3**, Proposition 7.2.11] or [**18**, Theorem 2.1]) of the Moore–Postnikov tower provides a homotopy commutative diagram

$$\begin{array}{ccc} A(m+1) \times \mathbb{S}^1 & \xrightarrow{\ \bar{\Phi}\ } & X(m+1) \\ {\scriptstyle p_{A(m+1)} \times \mathrm{id}_{\mathbb{S}^1}} \big\downarrow & & \big\downarrow {\scriptstyle p_{X(m+1)}} \\ A(m) \times \mathbb{S}^1 & \xrightarrow[\ \Phi\ ]{} & X(m). \end{array}$$

Taking $p_{A(m+1)}\colon A(m+1) \to A(m)$ to be an inclusion, the obstructions to the existence of a relative homotopy from $\Phi_{|A(m)}$ to $f(m)$ lie in $H^k(A(m), A(m+1); \pi_k(X(m))) = 0$ for $k \geqslant 1$. Hence, $\Phi_{|A(m)}$ is homotopic to $f(m)$ and Φ is the affiliated map to $p_{X(m+1)*}(\alpha) \in G_1^{f(m)}(X(m))$.

Then, Lemma 2.12 leads to the following generalization of [**14**, Lemma 7] and [**20**, Theorem 2]:

Proposition 2.14. *The map $p_{X(m+1)}\colon X(m+1) \to X(m)$ induces a homomorphism $p_{X(m+1)*}\colon G_1^{f(m+1)}(X(m+1)) \to G_1^{f(m)}(X(m))$.*

Now, for $\alpha \in \mathcal{Z}_{\pi_1(X)} f_*(\pi_1(A))$, consider the map $\Phi_\alpha\colon K(\pi_1(A),1) \times \mathbb{S}^1 \to K(\pi_1(X),1)$ corresponding to the homomorphism

$$\pi_1(A) \times \mathbb{Z} \to \pi_1(X)$$

given by $(g,n) \mapsto \alpha^n f_*(g)$ for $(g,n) \in \pi_1(A) \times \mathbb{Z}$.

If $H^m(A(m-1);\pi_m(X)) = 0$ for $m > 1$ then Lemma 2.12 leads to commutative diagrams

$$\begin{array}{ccc} A(m+1)\times\mathbb{S}^1 & \xrightarrow{\Phi_\alpha(m+1)} & X(m+1) \\ {\scriptstyle p_{A(m+1)}\times\mathrm{id}_{\mathbb{S}^1}}\downarrow & & \downarrow{\scriptstyle p_{X(m+1)}} \\ A(m)\times\mathbb{S}^1 & \xrightarrow[\Phi_\alpha(m)]{} & X(m) \end{array}$$

with $\Phi_\alpha(m)_{|A(m)} = f(m)$. Hence, we obtain a map

$$\lim_{\leftarrow}\Phi_\alpha(m)\colon \lim_{\leftarrow} A(m)\times\mathbb{S}^1 \to \lim_{\leftarrow} X(m).$$

Let $\phi(A)\colon A\to \lim\limits_{\leftarrow} A(m)$ and $\phi(X)\colon X\to\lim\limits_{\leftarrow} X(m)$ denote the standard weak homotopy equivalences. Then, there is a unique (up to homotopy) map $\hat{\Phi}_\alpha\colon A\times\mathbb{S}^1\to X$ which makes the diagram

$$\begin{array}{ccc} A\times\mathbb{S}^1 & \overset{\hat{\Phi}_\alpha}{\dashrightarrow} & X \\ {\scriptstyle \phi(A)\times\mathrm{id}_{\mathbb{S}^1}}\downarrow & & \downarrow{\scriptstyle \phi(X)} \\ \lim\limits_{\leftarrow} A(m)\times\mathbb{S}^1 & \xrightarrow[\lim\limits_{\leftarrow}\Phi_\alpha(m)]{} & \lim\limits_{\leftarrow} X(m) \end{array}$$

commutative. Certainly, $\hat{\Phi}_{\alpha|\mathbb{S}^1} = \alpha$. To see that $\hat{\Phi}_{\alpha|A} = f$, observe that if $\dim A = d < \infty$, then there is a bijection of homotopy classes $[A,X]\cong[A,X(d)]\cong[A(d),X(d)]$ and $\hat{\Phi}_{\alpha|A}$ corresponds to $\Phi_\alpha(d)_{|A(d)} = f(d)\colon A(d)\to X(d)$. Thus, this bijection implies $\hat{\Phi}_{\alpha|A} = f$ and we may state:

Proposition 2.15. *If $f\colon A\to X$ with $\dim A = d<\infty$, $\alpha\in\mathcal{Z}_{\pi_1(X)}f_*(\pi_1(A))$ and $H^m(A(m-1);\pi_m(X)) = 0$ for $m>1$ then $\alpha\in G_1^f(X)$. Hence, we have $G_1^f(X) = \mathcal{Z}_{\pi_1(X)}f_*(\pi_1(A))$.*

Since $G_1^f(X)\subseteq P_1^f(X)\subseteq\mathcal{Z}_{\pi_1(X)}f_*(\pi_1(A))$, we find that

$$P_1^f(X) = \mathcal{Z}_{\pi_1(X)}f_*(\pi_1(A))$$

under the hypothesis of Proposition 2.15.

Example 2.16. (1) If $X = K(\pi,1)$ for some group π then $H^m(A(m-1);\pi_m(X)) = 0$ for $m>1$.

(2) If $\dim A = 1$ then there exists a homotopy equivalence $A\simeq K(\pi,1)$, where π is a free group. Hence $A(m-1) = K(\pi,1)$ and, consequently, $H^m(A(m-1);\pi_m(X)) = H^m(\pi;\pi_m(X)) = 0$ for $m>1$.

Now, we are in a position to state the main result of this section:

Theorem 2.17. *If $f\colon A\to X$ with $1<\dim A = d<\infty$ and $H^m(A(m-1);\pi_m(X)) = 0$ for $1<m\leqslant d$ then $G_1^f(X) = \mathcal{Z}_{\pi_1(X)}f_*(\pi_1(A))$.*

Proof. Since $H^m(A(m-1);\pi_m(X)) = 0$ for $1<m\leqslant d$, in view of Proposition 2.15, we have to show that $H^m(A(m-1);\pi_m(X)) = 0$ for $m>d$.

Let $m > d$. Then, in view of Whitehead Theorem, it follows that the m-equivalence $q_{A(m-1)}\colon A \to A(m-1)$ induces isomorphisms $H_k(A;\mathbb{Z}) \xrightarrow{\simeq} H_k(A(m-1);\mathbb{Z})$ for $k < m$ and a surjection $H_m(A;\mathbb{Z}) \to H_m(A(m-1);\mathbb{Z})$. Hence the group $H_d(A(d);\mathbb{Z})$ is free, $H_{m-1}(A(m-1);\mathbb{Z}) = 0$ for $m > d+1$ and $H_m(A(m-1);\mathbb{Z}) = 0$ for $m > d$. Thus, by the universal coefficient theorem, we have

$$\begin{aligned} H^m(A(m-1);\pi_m(X)) \simeq \operatorname{Hom}(&H_m(A(m-1);\mathbb{Z}),\pi_m(X)) \\ &\oplus \operatorname{Ext}(H_{m-1}(A(m-1);\mathbb{Z}),\pi_m(X)) = 0 \end{aligned}$$

for $m > d$ and the proof is complete. □

Given a finite group H acting freely and cellularly on $\mathbb{S}^{2n+1}$ write $\mathbb{S}^{2n+1}/H$ for the orbit space. Oprea [**20**, Theorem A] has shown that $G_1(\mathbb{S}^{2n+1}/H) = \mathcal{Z}H$. Then, the relations

$$\mathcal{Z}H = G_1(\mathbb{S}^{2n+1}/H) \subseteq P_1(\mathbb{S}^{2n+1}/H) \subseteq \mathcal{Z}H$$

imply that $P_1(\mathbb{S}^{2n+1}/H) = \mathcal{Z}H$, what was already observed by Gottlieb [**11**, §3] and follows from Proposition 2.1(3) as well.

Recall that a finite dimensional CW-complex $\Sigma(n)$ with the homotopy type of the n-sphere $\mathbb{S}^n$ is called an *n-homotopy sphere* for $n \geqslant 1$. The finite periodic groups, which are the only finite groups that can act freely and cellularly on some homotopy sphere, have been fully classified by Suzuki–Zassenhaus, see e.g., [**2**, Chapter IV, Theorem 6.15]. It is well-known that the only finite groups acting freely and cellularly on $\Sigma(2n)$ are $\mathbb{Z}_2$ and the trivial group.

Given a free and cellular action $H \times \Sigma(2m+1) \to \Sigma(2m+1)$ of a finite group H on a homotopy sphere $\Sigma(2m+1)$, write $\gamma_{2n+1}\colon \Sigma(2n+1) \to \Sigma(2n+1)/H$ for the quotient map. It is well-known [**8**, Chapter XVI, §4, Application 4] that H must have periodic cohomology with period $2m+2$. Then, in view of [**6**, Chapter VII, Proposition 10.2], the action of H on $\pi_k(\Sigma(2m+1)/H)$ is trivial for $k > 1$. Because $H^{2k+1}(H;\mathbb{Z}) = 0$ for $k \geqslant 1$ (see e.g., [**20**, Lemma 6]), Theorem 2.17 yields the following generalization of [**20**, Theorem A]:

Corollary 2.18. *If $f\colon A \to \Sigma(2n+1)/H$ with $1 < \dim A = d \leqslant 2n+1$, $\pi_k(A) = 0$ for $1 < k < 2n+1$ and $H^{2n+1}(\pi_1(A);\mathbb{Z}) = 0$ then $G_1^f(\Sigma(2n+1)/H) = \mathcal{Z}_H f_*(\pi_1(A))$.*

In particular, if $f\colon \Sigma(2d+1)/K \to \Sigma(2n+1)/H$ and $d \leqslant n$ then $G_1^f(\Sigma(2n+1)/H) = \mathcal{Z}_H f_*(K)$.

Proof. The first part is a direct conclusion from Theorem 2.17.

Let now $f\colon \Sigma(2d+1)/K \to \Sigma(2n+1)/H$ with $d \leqslant n$. Then $\pi_k(\Sigma(2d+1)/K) \cong \pi_k(\Sigma(2d+1)) = 0$ for $1 < k < 2d+1 \leqslant 2n+1$. Next, notice that by [**1**] or [**22**] there is a homotopy sphere $\Sigma'(2d+1)$ admitting a free action of the group K such that $\dim \Sigma'(2d+1) = 2d+1 \leqslant 2n+1$ and the space forms $\Sigma(2d+1)/K$ and $\Sigma'(2d+1)/K$ are homotopy equivalent.

Because $H^{2n+1}(K;\mathbb{Z}) = 0$ (see e.g., [**20**, Lemma 6]), the space $\Sigma'(2d+1)/K$ satisfies all required hypotheses of the first part and the proof follows. □

Suppose $f\colon A\to\Sigma(2n+1)/H$ satisfies the conditions above. First, notice that

$$G_1^f(\Sigma(2n+1)/H)=P_1^f(\Sigma(2n+1)/H)=\mathcal{Z}_H f_*(\pi_1(A))$$

and, if A is 1-connected,

$$G_1^f(\Sigma(2n+1)/H)=P_1^f(\Sigma(2n+1)/H)=H.$$

This implies

$$G_1^f(\Sigma(2n+1)/H)=P_1^f(\Sigma(2n+1)/H)=H$$

for any map $f\colon\Sigma(2d+1)\to\Sigma(2n+1)/H$ and $1\leqslant d\leqslant n$. Next,

$$P_1^f(\Sigma(2n+1)/H)=G_1^f(\Sigma(2n+1)/H)=P_1(\Sigma(2n+1)/H)=G_1(\Sigma(2n+1)/H)=H$$

provided the group H is abelian.

Further, notice that given $\gamma_{2n+1}\alpha\in P_m(\Sigma(2n+1)/H)$, we have $[\gamma_{2n+1}\alpha,\gamma_{2n+1}]=\gamma_{2n+1}[\alpha,\iota_{2n+1}]=0$. Since the map $\gamma_{2n+1*}\colon\pi_m(\Sigma(2n+1))\to\pi_m(\Sigma(2n+1)/H)$ is a monomorphism, we see that $\alpha\in P_m(\Sigma(2n+1))$ which leads to

$$P_m(\Sigma(2n+1)/H)\subseteq\gamma_{2n+1*}P_m(\Sigma(2n+1))=\gamma_{2n+1*}G_m(\Sigma(2n+1)).$$

But, $\pi_1(\Sigma(2n+1)/H)\cong H$ acts trivially on $\pi_m(\Sigma(2n+1)/H)$ for $m>1$. Consequently, the above yields

$$P_m(\Sigma(2n+1)/H)=\gamma_{2n+1*}P_m(\Sigma(2n+1))=\gamma_{2n+1*}G_m(\Sigma(2n+1))$$

provided H is abelian. Then, in view of Theorem 2.10, we have:

Corollary 2.19. *Let $H\times\Sigma(2m+1)\to\Sigma(2m+1)$ be a free and cellular action of a finite group H on a homotopy sphere $\Sigma(2m+1)$ and let $\gamma_{2n+1}\colon\Sigma(2n+1)\to\Sigma(2n+1)/H$ be the quotient map. Then:*

1. $P_1^{\gamma_{2n+1}}(\Sigma(2n+1)/H)=G_1^{\gamma_{2n+1}}(\Sigma(2n+1)/H)=H$ *provided H is abelian;*
2. $G_m(\Sigma(2n+1)/H)\subseteq G_m^{\gamma_{2n+1}}(\Sigma(2n+1)/H)=\gamma_{2n+1*}G_m(\Sigma(2n+1))$ *for $m>1$;*
3. *we have* $P_m(\Sigma(2n+1)/H)\subseteq P_m^{\gamma_{2n+1}}(\Sigma(2n+1)/H)=\gamma_{2n+1*}P_m(\Sigma(2n+1))=\gamma_{2n+1*}G_m(\Sigma(2n+1))$ *for $m>1$. In particular,*

$$\begin{aligned}P_m(\Sigma(2n+1)/H)&=P_m^{\gamma_{2n+1}}(\Sigma(2n+1)/H)\\&=\gamma_{2n+1*}P_m(\Sigma(2n+1))=\gamma_{2n+1*}G_m(\Sigma(2n+1))\end{aligned}$$

for $m>1$ provided H is abelian.

Given a space X and a prime number p, write $\pi_m(X;p)$ for the p-primary component of $\pi_m(X)$. Then, the result [**9**, Theorem 2.3] yields:

Corollary 2.20. *If p is a prime not dividing the order of the finite group H then*

$$\gamma_{2n+1*}(G_m(\Sigma(2n+1);p))=G_m(\Sigma(2n+1)/H;p)=\pi_m(\Sigma(2n+1)/H;p)$$

for $m>1$.

3. Whitehead center and Gottlieb groups of some space forms

Throughout this section, we use the standard terminology and notations from homotopy theory, mainly from [**10**] and [**23**]. We aim to compute or approximate Whitehead center groups of some space forms.

3.1. Whitehead center groups of $\mathbb{F}P^n$ and Gottlieb groups of $\Sigma(2n+1)/\mathbb{Z}_l$

First, let $\mathbb{F}$ denote the field of reals $\mathbb{R}$, complex numbers $\mathbb{C}$ or the skew $\mathbb{R}$-algebra of quaternions $\mathbb{H}$ and $\mathbb{F}P^n$ the appropriate projective n-space for $n \geqslant 1$.

Write $\gamma_n(\mathbb{F})\colon \mathbb{S}^{d(n+1)-1} \to \mathbb{F}P^n$ for the quotient map, where $d = \dim_{\mathbb{R}} \mathbb{F}$ and $i_n(\mathbb{F})\colon \mathbb{S}^d = \mathbb{F}P^1 \hookrightarrow \mathbb{F}P^n$ for the canonical inclusion with $n \geqslant 1$, and recall from [**4**] that

$$\pi_m(\mathbb{F}P^n) = \gamma_n(\mathbb{F})_*\pi_m(\mathbb{S}^{d(n+1)-1}) \oplus i_n(\mathbb{F})_* E\pi_{m-1}(\mathbb{S}^{d-1}).$$

The results below are direct consequences of Proposition 2.1(5) and the result [**4**, (4.1-3)] (see also [**10**, Lemma 2.4]).

Proposition 3.1.

(1) $G_m^{\gamma_n(\mathbb{F})}(\mathbb{F}P^n) = P_m^{\gamma_n(\mathbb{F})}(\mathbb{F}P^n) = \mathrm{Ker}[\gamma_n(\mathbb{F}), -]$ *for* $m, n \geqslant 1$.

(2) $P_m^{\gamma_n(\mathbb{R})}(\mathbb{R}P^n) = P_m(\mathbb{R}P^n) = \gamma_n(\mathbb{R})_* P_m(\mathbb{S}^n)$ *for* $m > 1$, $n \geqslant 1$ *and*

$$P_1^{\gamma_n(\mathbb{R})}(\mathbb{R}P^n) = \begin{cases} \pi_1(\mathbb{R}P^n), & \text{if } n \text{ is odd;} \\ 0, & \text{if } n \text{ is even.} \end{cases}$$

(3) (i) $P_m^{\gamma_n(\mathbb{C})}(\mathbb{C}P^n) = P_m(\mathbb{C}P^n) = \gamma_n(\mathbb{C})_* P_m(\mathbb{S}^{2n+1})$ *for* $m > 2$ *and* $n \geqslant 1$ *odd, and*

$$P_2^{\gamma_n(\mathbb{C})}(\mathbb{C}P^n) = \begin{cases} \pi_2(\mathbb{C}P^n), & \text{if } n \text{ is odd;} \\ 2\pi_2(\mathbb{C}P^n), & \text{if } n \text{ is even.} \end{cases}$$

In particular, $P_m^{\eta_2}(\mathbb{S}^2) = \pi_m(\mathbb{S}^2)$ *for* $m \geqslant 1$.

(ii) $P_{2n+1}^{\gamma_n(\mathbb{C})}(\mathbb{C}P^n) = P_{2n+1}(\mathbb{C}P^n) = \gamma_n(\mathbb{C})_* P_{2n+1}(\mathbb{S}^{2n+1})$ *and* $2P_m^{\gamma_n(\mathbb{C})}(\mathbb{C}P^n) \subseteq P_m(\mathbb{C}P^n) = \gamma_n(\mathbb{C})_* P_m(\mathbb{S}^{2n+1})$ *for* $m > 2n+1$ *and* n *even.*

(4) (i) $P_m^{\gamma_n(\mathbb{H})}(\mathbb{H}P^n; p) = \gamma_n(\mathbb{H})_*\pi_m(\mathbb{S}^{4n+3}; p) \oplus i_n(\mathbb{H})_* EL''_{m-1}(\mathbb{S}^3; p)$, *if* p *is an odd prime.*

(ii) $P_m^{\gamma_n(\mathbb{H})}(\mathbb{H}P^n; 2) = \gamma_n(\mathbb{H})_*\pi_m(\mathbb{S}^{4n+3}; 2) \oplus i_n(\mathbb{H})_* EL''(\mathbb{S}^3; 2)$, *provided*

$$[\iota_{4n+3}, \pi_m^{4n+3}] \cap (n+1)\nu_{4n+3} E^{4n+3}\pi_{m-1}^3 = 0,$$

where $L''_{m-1}(\mathbb{S}^3) = \{\beta \in \pi_{m-1}(\mathbb{S}^3);\ [i_n(\mathbb{H})E\beta, \gamma_n(\mathbb{H})] = 0\}$. *In particular,*

$$P_m^{\nu_4}(\mathbb{S}^4) = \begin{cases} \nu_{4*}\pi_m(\mathbb{S}^7; p) \oplus EL''_{m-1}(\mathbb{S}^3; p), & \text{if } p \text{ is an odd prime;} \\ \nu_{4*}\pi_m(\mathbb{S}^7; 2) \oplus EL''(\mathbb{S}^3; 2). \end{cases}$$

(5) $$P_m^{\sigma_8}(\mathbb{S}^8) = \begin{cases} \sigma_{8*}\pi_m(\mathbb{S}^{15}; p) \oplus EL''_{m-1}(\mathbb{S}^7; p), & \text{if } p \text{ is an odd prime;} \\ \sigma_{8*}\pi_m(\mathbb{S}^{15}; 2) \oplus EL''(\mathbb{S}^7; 2). \end{cases}$$

(6) $G_m^{i_n(\mathbb{F})}(\mathbb{F}P^n) = P_m^{i_n(\mathbb{F})}(\mathbb{F}P^n) = \mathrm{Ker}[i_n(\mathbb{F}), -]$ *for* $m, n \geqslant 1$.

(7) $P_m^{i_n(\mathbb{R})}(\mathbb{R}P^n) = \pi_m(\mathbb{R}P^n)$ *for* $m, n \geqslant 1$ *provided* n *is odd, and* $P_m^{i_n(\mathbb{R})}(\mathbb{R}P^n) = 0$ *for* $1 \leqslant m \leqslant n$ *provided* n *is even.*

(8) (i) $P_2^{i_n(\mathbb{C})}(\mathbb{C}P^1) = 0$ *and* $P_m^{i_n(\mathbb{C})}(\mathbb{C}P^n) = \pi_m(\mathbb{C}P^n)$ *for* $m > 2$ *provided* $n \geqslant 1$ *is odd.*

(ii) $P_m^{i_n(\mathbb{C})}(\mathbb{C}P^n) = \pi_m(\mathbb{C}P^n)$ *for* $1 \leqslant m < 2n+1$, $P_{2n+1}^{i_n(\mathbb{C})}(\mathbb{C}P^n) = 2\pi_{2n+1}(\mathbb{C}P^n)$ *and* $P_m^{i_n(\mathbb{C})}(\mathbb{C}P^n) \supseteq 2\pi_m(\mathbb{C}P^n)$ *for* $m > 2n+1$ *provided* n *is even.*

(9) $P_4^{i_n(\mathbb{H})}(\mathbb{H}P^1)=0$ *and* $P_m^{i_n(\mathbb{H})}(\mathbb{H}P^n)=\pi_m(\mathbb{H}P^n)$ *for* $4<m<4n+3$,

$$P_{4n+3}^{i_n(\mathbb{H})}(\mathbb{H}P^n)=\frac{24}{(n+1,24)}\gamma_n(\mathbb{H})_*\pi_{4n+3}(\mathbb{S}^{4n+3})\oplus i_n(\mathbb{H})_*E\pi_{4n+2}(\mathbb{S}^3)$$

and $P_m^{i_n(\mathbb{H})}(\mathbb{H}P^n)\supseteq\frac{24}{(n+1,24)}\gamma_n(\mathbb{H})_*\pi_m(\mathbb{S}^{4n+3})\oplus i_n(\mathbb{H})_*E\pi_{m-1}(\mathbb{S}^3)$ *for* $m>4n+3$ *and* $n\geqslant 1$.

Next, recall the well-known fact proved e.g., in [**24**] and needed in the sequel:

Theorem 3.2. *If a compact Lie group G acts on a compact smooth manifold M, then the manifold M admits an equivariant triangulation. In particular, it has the structure of a G-CW-complex.*

Consequently, any action of a finite group on the n-sphere $\mathbb{S}^n$ is cellular.

Next, given a free and cellular action $\mathbb{Z}_2\times\Sigma(n)\to\Sigma(n)$, by [**16**, Lemma 2.5], there is a homotopy equivalence $\Sigma(n)/\mathbb{Z}_2\simeq\mathbb{R}P^n$. Then, $G_m(\Sigma(n)/\mathbb{Z}_2)\cong G_m(\mathbb{R}P^n)$ and $G_m(\Sigma(n))\cong G_m(\mathbb{S}^n)$ for $m\geqslant 1$. Hence, [**21**] implies:

$$\begin{aligned}G_{2n+1}(\Sigma(2n+1)/\mathbb{Z}_2)&=\gamma_{2n+1*}G_{2n+1}(\Sigma(2n+1))\\&=\begin{cases}\pi_{2n+1}(\Sigma(2n+1)/\mathbb{Z}_2), & \text{for } n=0,1,3;\\ 2\pi_{2n+1}(\Sigma(2n+1)/\mathbb{Z}_2), & \text{for odd } n \text{ and } n\neq 0,1,3.\end{cases}\end{aligned}$$

In virtue of the inclusion $G_m(\Sigma(2n+1)/\mathbb{Z}_2)\subseteq\gamma_{2n+1*}G_m(\Sigma(2n+1))$ for $m>1$, the group $\gamma_{2n+1*}G_m(\Sigma(2n+1))$ is an upper bound of $G_m(\Sigma(2n+1)/\mathbb{Z}_2)$.

The results in the sequel mainly follows from [**10**]. More precisely, we make use of [**10**, Theorems 2.41 and 2.45, Propositions 2.42 and 2.46, and Corollary 2.47].

Theorem 3.3. *If $m\leqslant 7$ then we have the equality*

$$G_{m+2n+1}(\Sigma(2n+1)/\mathbb{Z}_2)=\gamma_{2n+1*}G_{m+2n+1}(\Sigma(2n+1))$$

except for the following pairs: $(m,2n+1)=(3,2^l-3)$ with $l\geqslant 4$, $(6,2^l-5)$ with $l\geqslant 5$ and $(7,11)$. Furthermore:

(1) $G_{2^l}(\Sigma(2^l-3)/\mathbb{Z}_2)\supseteq 2\pi_{2^l}(\Sigma(2^l-3)/\mathbb{Z}_2)$ *for* $l\geqslant 4$;

(2) $G_{18}(\Sigma(11)/\mathbb{Z}_2)\supseteq 2\pi_{18}(\Sigma(11)/\mathbb{Z}_2)$.

Remark 3.4. According to J. Mukai's conjecture,

$$G_{18}(\Sigma(11)/\mathbb{Z}_2)=2\pi_{18}(\Sigma(11)/\mathbb{Z}_2)=2\gamma_{18*}\pi_{18}(\Sigma(11))=2\gamma_{18*}G_{18}(\Sigma(11)).$$

This would imply that there is a proper inclusion $G_{18}(\Sigma(11)/\mathbb{Z}_2)\subsetneqq\gamma_{18*}G_{18}(\Sigma(11))$.

Since $\Sigma(2n+1)/\mathbb{Z}_2\simeq\mathbb{R}P^{2n+1}$, [**10**, Proposition 2.42] yields:

Proposition 3.5. *If one of the following conditions is satisfied:*

(1) $n\equiv 1 \pmod 2$ *and* $m=8,9$;

(2) $n\equiv 1 \pmod 2$ *with* $n\geqslant 3$ *and* $m=10$;

(3) $n\equiv 0 \pmod 2$;

(4) $n\equiv 3 \pmod 4$ *with* $n\geqslant 6$ *and* $m=11$;

(5) $n\geqslant 7$ *and* $m=13$

then $G_{m+2n+1}(\Sigma(2n+1)/\mathbb{Z}_2) = \pi_{m+2n+1}(\Sigma(2n+1)/\mathbb{Z}_2)$.

Now, we aim to study $G_m(\mathbb{S}^{2n+1}/\mathbb{Z}_l)$ for $l > 2$. First notice that the canonical fibration

$$\mathbb{S}^1 \to \mathbb{S}^{2n+1}/\mathbb{Z}_l \to \mathbb{C}P^n$$

yields an isomorphism $\pi_m(\mathbb{S}^{2n+1}/\mathbb{Z}_l) \cong \pi_m(\mathbb{C}P^n)$ for $m \geqslant 3$.

In the sequel we need the following. Let H be a closed subgroup of a Lie group K and write K/H for the associated orbit space. Then, $K/H \to BH \to BK$ is a fibre sequence and, by means of [**12**, Theorem 2-6], we obtain:

Lemma 3.6. *The quotient map* $p\colon K \to K/H$ *leads to* $p_*(\pi_m(K)) \subseteq G_m(K/H)$ *for* $m \geqslant 1$.

If H is a finite subgroup of K then Proposition 2.7 and Lemma 3.6 imply $p_*(\pi_m(K)) = G_m(K/H)$ for $m > 1$. In particular, if H is a finite subgroup of $\mathbb{S}^3$ then $p_*(\pi_m(\mathbb{S}^3)) = G_m(\mathbb{S}^3/H)$ for $m > 1$.

Let now $U(n)$ be the n^{th} unitary group and consider the canonical inclusion map $U(n) \times \mathbb{Z}_l \hookrightarrow U(n) \times U(1) \hookrightarrow U(n+1)$. Then, we may identify $\mathbb{S}^{2n+1} \simeq U(n+1)/U(n)$ and $\mathbb{S}^{2n+1}/\mathbb{Z}_l \simeq U(n+1)/U(n) \times \mathbb{Z}_l$. Set $p_n : U(n+1) \to \mathbb{S}^{2n+1}$ and $p'_n : U(n+1) \to \mathbb{S}^{2n+1}/\mathbb{Z}_l$ for the appropriate quotient maps being fibrations $U(n+1) \xrightarrow{U(n)} \mathbb{S}^{2n+1}$ and $U(n+1) \xrightarrow{U(n)\times\mathbb{Z}_l} \mathbb{S}^{2n+1}/\mathbb{Z}_l$, respectively. Next, consider the map of exact sequences induced by those fibrations:

$$\begin{array}{ccccc}
\pi_k(U(n+1)) & \xrightarrow{p_*} & \pi_k(\mathbb{S}^{2n+1}) & \xrightarrow{\Delta_{\mathbb{C}}} & \pi_{k-1}(U(n)) \\
\| & & \downarrow{\scriptstyle \gamma_{2n+1*}} & & \downarrow \\
\pi_k(U(n+1)) & \xrightarrow[p'_*]{} & \pi_k(\mathbb{S}^{2n+1}/\mathbb{Z}_l) & \xrightarrow[\Delta'_{\mathbb{C}}]{} & \pi_{k-1}(U(n) \times \mathbb{Z}_l),
\end{array}$$

where $\Delta_{\mathbb{C}}$ and $\Delta'_{\mathbb{C}}$ are the appropriate connecting maps. Then, by Lemma 3.6, we have:

$$\operatorname{Ker} \Delta'_{\mathbb{C}} = p'_* \pi_k(U(n+1)) \subseteq G_k(\mathbb{S}^{2n+1}/\mathbb{Z}_l).$$

Recall that $\pi_m(X;p)$ denotes the p-primary component of $\pi_m(X)$ for a prime p. Let $J\colon \pi_k(SO(n)) \to \pi_{k+n}(\mathbb{S}^n)$ be the J-homomorphism and $r\colon U(n) \hookrightarrow SO(2n)$ be the canonical inclusion. Taking $J_{\mathbb{C}} = J \circ r_*$, the result [**10**, Lemma 2.33] implies:

Proposition 3.7.

(1) $\operatorname{Ker}\big(\Delta_{\mathbb{C}}\colon \pi_m(\mathbb{S}^{2n+1}) \to \pi_{m-1}(U(n))\big) \subseteq \gamma_{2n+1*}^{-1} G_m(\mathbb{S}^{2n+1}/\mathbb{Z}_l)$. *In particular, we have* $\gamma_{2n+1*}\pi_m(\mathbb{S}^{2n+1};p) \subseteq G_m(\mathbb{S}^{2n+1}/\mathbb{Z}_l)$ *provided* $\pi_{m-1}(U(n);p) = 0$ *for a prime* p.

(2) *Let* $m \geqslant 3$. *If* $E \circ J_{\mathbb{C}|\Delta_{\mathbb{C}}(\pi_m(\mathbb{S}^{2n+1}))}\colon \Delta_{\mathbb{C}}(\pi_m(\mathbb{S}^{2n+1})) \to \pi_{m+2n}(\mathbb{S}^{2n+1})$ *is a monomorphism then* $\gamma_{2n+1*}G_m(\mathbb{S}^{2n+1}) \subseteq G_m(\mathbb{S}^{2n+1}/\mathbb{Z}_l)$, *where* E *is the suspension homomorphism. In particular, under this assumption,* Corollary 2.19(2) *yields that* $G_m(\mathbb{S}^{2n+1}/\mathbb{Z}_l) = \gamma_{2n+1*}G_m(\mathbb{S}^{2n+1})$.

Now, we make use of Proposition 3.7 to obtain, as [**10**, Theorem 2.45], the following:

Proposition 3.8.

(1) *Let* $m = 1, 2$. *Then,*

$$G_{m+2n+1}(\mathbb{S}^{2n+1}/\mathbb{Z}_l) = \begin{cases} 0, & \text{if } n \text{ is even,} \\ \pi_{m+2n+1}(\mathbb{S}^{2n+1}/\mathbb{Z}_l) \cong \mathbb{Z}_2, & \text{if } n \text{ is odd.} \end{cases}$$

(2) $G_{2n+4}(\mathbb{S}^{2n+1}/\mathbb{Z}_l) \supseteq$

$$\begin{cases} (24, n)\pi_{2n+4}(\mathbb{S}^{2n+1}/\mathbb{Z}_l) \cong \mathbb{Z}_{\frac{24}{(24,n)}}, & \text{if } n \text{ is even,} \\ \frac{(24,n+3)}{2}\pi_{2n+4}(\mathbb{S}^{2n+1}/\mathbb{Z}_l) \cong \mathbb{Z}_{\frac{48}{(24,n+3)}}, & \text{if } n \text{ is odd.} \end{cases}$$

In particular, $G_{2n+4}(\mathbb{S}^{2n+1}/\mathbb{Z}_l) = 2\pi_{2n+4}(\mathbb{S}^{2n+1}/\mathbb{Z}_l)$ *for* $n \equiv 2, 10 \pmod{12}$ *with* $n \geqslant 10$ *except for* $n = 2^{i-1} - 2$ *or* $n \equiv 1, 17 \pmod{24}$ *with* $n \geqslant 17$, *and* $G_{2n+4}(\mathbb{S}^{2n+1}/\mathbb{Z}_l) = \pi_{2n+4}(\mathbb{S}^{2n+1}/\mathbb{Z}_l)$ *for* $n \equiv 7, 11 \pmod{12}$.

(3) $G_{2n+6}(\mathbb{S}^{2n+1}/\mathbb{Z}_l) = \pi_{2n+6}(\mathbb{S}^{2n+1}/\mathbb{Z}_l) \cong \begin{cases} 0, & \text{for } n \geqslant 3, \\ \mathbb{Z}_2, & \text{for } n = 2. \end{cases}$

(4) $G_{2n+7}(\mathbb{S}^{2n+1}/\mathbb{Z}_l) = \pi_{2n+7}(\mathbb{S}^{2n+1}/\mathbb{Z}_l)$ *for* $n \equiv 2, 3 \pmod 4$.

Further, making use of Proposition 3.7, as in [**10**, Proposition 2.46], we can show:

Proposition 3.9. $G_m(\mathbb{S}^7/\mathbb{Z}_l; 2) = \pi_m(\mathbb{S}^7/\mathbb{Z}_l; 2)$ *for* $8 \leqslant m \leqslant 24$ *unless* $m = 15, 21$. *Further,* $\gamma_{7*}\{\sigma'\eta_{14}, \bar{\nu}_7 + \varepsilon_7\} \subseteq G_{15}(\mathbb{S}^7/\mathbb{Z}_l; 2)$ *and* $\gamma_{7*}\{\sigma'\sigma_{14}\} \subseteq G_{21}(\mathbb{S}^7/\mathbb{Z}_l; 2)$.

Then, as in [**10**, Corollary 2.47], we deduce:

Corollary 3.10.

(1) $G_m(\mathbb{S}^5/\mathbb{Z}_l) = \pi_m(\mathbb{S}^5/\mathbb{Z}_l)$ *for* $10 \leqslant m \leqslant 12$;
(2) $G_m(\mathbb{S}^5/\mathbb{Z}_l; p) = \pi_m(\mathbb{S}^5/\mathbb{Z}_l; p)$ *for an odd prime* p;
(3) $G_m(\mathbb{S}^{4n+3}/\mathbb{Z}_l) \supseteq (\gamma_{4n+3}\eta^m_{4n+3})_*\pi^{4n+k+3}_m$ *for* $m = 1, 2$;
(4) (i) $G_m(\mathbb{S}^{4n+1}/\mathbb{Z}_l) \supseteq 2(12, n)(\gamma_{2n}\nu^+_{4n+1})_*\pi_k(\mathbb{S}^{4n+4})$,
(ii) $G_m(\mathbb{S}^{4n+3}/\mathbb{Z}_l) \supseteq (12, n+2)(\gamma_{2n+1}\nu^+_{4n+3})_*\pi_m(\mathbb{S}^{4n+6})$;
(5) $G_m(\mathbb{S}^{8n+5}/\mathbb{Z}_l) \supseteq (\gamma_{4n+2}\nu^2_{8n+5})_*\pi^{8n+11}_m$;
(6) $G_{8n+11}(\mathbb{S}^{8n+3}/\mathbb{Z}_l) = \pi_{8n+11}(\mathbb{S}^{8n+3}/\mathbb{Z}_l)$ *for* $n \geqslant 2$;
(7) $G_{8n+k}(\mathbb{S}^{8n+7}/\mathbb{Z}_l) = \pi_{8n+k}(\mathbb{S}^{8n+7}/\mathbb{Z}_l)$ *for* $k = 28, 29$ *with* $n \geqslant 2$.

3.2. Gottlieb groups of $\mathbb{S}^{2n+1}/H$

Let $H \subseteq SO(2n+2)$ be a finite subgroup of the special orthogonal group $SO(2n+2)$ acting freely on the $(2n+1)$-sphere $\mathbb{S}^{2n+1}$. Certainly, a spherical $(2n+1)$-manifold $\mathbb{S}^{2n+1}/H$ has a finite fundamental group isomorphic to H itself. For $n = 1$, Thurston's *elliptization conjecture* (proved by Perelman) states that, conversely, all 3-manifolds with finite fundamental group are spherical manifolds.

Then, in view of Proposition 2.7, Corollary 2.19 and Lemma 3.6, we may state:

Remark 3.11. If $H \subseteq SO(2n+2)$ is a finite subgroup acting freely on $\mathbb{S}^{2n+1}$ then for $m \geqslant 1$ we have:

(1) $G^{\gamma_3}_m(\mathbb{S}^3/H) = P^{\gamma_3}_m(\mathbb{S}^3/H) = G_m(\mathbb{S}^3/H) = P_m(\mathbb{S}^3/H) = \gamma_{3*}\pi_m(\mathbb{S}^3)$ for $n = 1$;
(2) $G^{\gamma_{2n+1}}_m(\mathbb{S}^{2n+1}/H) = P^{\gamma_{2n+1}}_m(\mathbb{S}^{2n+1}/H) = \gamma_{2n+1*}G_m(\mathbb{S}^{2n+1})$ for $n > 1$.

Next, write $V_{n,k}$ for the Stiefel manifold given by orthonormal k-frames in an n-dimensional Euclidean space and recall that $V_{n,k} \simeq O(n)/O(n-k) \simeq SO(n)/SO(n-k)$. Then, a free action of a finite subgroup $H \subseteq SO(2n+2) \subseteq SO(2n+3)$ on $\mathbb{S}^{2n+1} \simeq SO(2n+2)/SO(2n+1)$ extends to a free action on $V_{2n+3,2} \simeq SO(2n+3)/SO(2n+1)$. By means of the fibration

$$\mathbb{S}^{2n+1}/H \to V_{2n+3,2}/H \to \mathbb{S}^{2n+2}$$

for $n \geqslant 1$, it was shown in [**9**] that

$$G_{2n+1}(\mathbb{S}^{2n+1}/H) = \begin{cases} \mathbb{Z}, & \text{for } n = 0, 1, 3; \\ 2\mathbb{Z}, & \text{for any other } n. \end{cases}$$

Now, let $Sp(n)$ be the n^{th} symplectic group and $H < Sp(1)$ be a finite subgroup. Write $i'_n \colon Sp(n) \times H \hookrightarrow Sp(n) \times Sp(1) \hookrightarrow Sp(n+1)$ for the canonical inclusion map. Then, we may identify $\mathbb{S}^{4n+3} \simeq Sp(n+1)/Sp(n)$ and $\mathbb{S}^{4n+3}/H \simeq Sp(n+1)/Sp(n) \times K$. Set $p_n \colon Sp(n+1) \to \mathbb{S}^{4n+3}$ and $p'_n \colon Sp(n+1) \to \mathbb{S}^{4n+3}/H$ for the quotient maps being fibrations $Sp(n+1) \xrightarrow{Sp(n)} \mathbb{S}^{4n+3}$ and $Sp(n+1) \xrightarrow{Sp(n)\times H} \mathbb{S}^{4n+3}/H$. We consider the map of exact sequences induced by those fibrations:

$$\begin{array}{ccccc} \pi_k(Sp(n+1)) & \xrightarrow{p_*} & \pi_k(\mathbb{S}^{4n+3}) & \xrightarrow{\Delta_{\mathbb{H}}} & \pi_{k-1}(Sp(n)) \\ \| & & \downarrow {\gamma_{4n+3}}_* & & \downarrow \\ \pi_k(Sp(n+1)) & \xrightarrow[p'_*]{} & \pi_k(\mathbb{S}^{4n+3}/H) & \xrightarrow[\Delta'_{\mathbb{H}}]{} & \pi_{k-1}(Sp(n) \times H), \end{array}$$

where $\Delta_{\mathbb{H}}$ and $\Delta'_{\mathbb{H}}$ are the appropriate connecting maps. Then, by Lemma 3.6, we have:

$$\operatorname{Ker} \Delta'_{\mathbb{H}} = p'_* \pi_k(Sp(n+1)) \subseteq G_k(\mathbb{S}^{4n+3}/H).$$

Write $c \colon Sp(n) \hookrightarrow SU(2n)$ for the canonical inclusion. Then, taking $J_{\mathbb{H}} = J \circ r_* \circ c_*$, by the result [**10**, Lemma 2.33], we obtain a key fact determining $G_k(\mathbb{S}^{4n+3}/H)$:

Proposition 3.12.

(1) $\operatorname{Ker}\big(\Delta_{\mathbb{H}} \colon \pi_k(\mathbb{S}^{4n+3}) \to \pi_{k-1}(Sp(n))\big) \subseteq \gamma^{-1}_{4n+3*} G_k(\mathbb{S}^{4n+3}/H)$. *In particular, we have* $\gamma_{4n+3*}\pi_k(\mathbb{S}^{4n+3}; p) \subseteq G_k(\mathbb{S}^{4n+3}/H)$ *provided* $\pi_{k-1}(Sp(n); p) = 0$ *for a prime* p.

(2) *Let* $k \geqslant 5$. *If* $E^3 \circ J_{\mathbb{H}|\Delta_{\mathbb{H}}(\pi_k(\mathbb{S}^{4n+3}))} \colon \Delta_{\mathbb{H}}(\pi_k(\mathbb{S}^{4n+3})) \to \pi_{k+4n+2}(\mathbb{S}^{4n+3})$ *is a monomorphism then* $\gamma_{4n+3*} G_k(\mathbb{S}^{4n+3}) \subseteq G_k(\mathbb{S}^{4n+3}/H)$, *where* E *is the suspension homomorphism. In particular, under this assumption,* Corollary 2.19(2) *yields* $G_k(\mathbb{S}^{4n+3}/H) = \gamma_{4n+3*} G_k(\mathbb{S}^{4n+3})$.

Now, we make use of Proposition 3.12 to obtain, as [**10**, Theorem 2.49], the following:

Theorem 3.13. $G_k(\mathbb{S}^{4n+3}/H) \supseteq (24, n+2)(\gamma_{4n+3}\nu^+_{4n+3})_* \pi_k(\mathbb{S}^{4n+6})$. *In particular, we have* $G_{4n+6}(\mathbb{S}^{4n+3}/H) \supseteq (24, n+2)\gamma_{4n+3*}\pi_{4n+6}(\mathbb{S}^{4n+3}) \cong \mathbb{Z}_{\frac{24}{(24,n+2)}}$ *for* $n \geqslant 2$.

Observe that Theorem 3.13 yields:

Corollary 3.14.

(1) $G_{8n+9}(\mathbb{S}^{8n+3}/H) = 0$ *and* $G_{8n+10}(\mathbb{S}^{84+7}/H) = \gamma_{8n+7*}\pi_{8n+10}(\mathbb{S}^{8n+7})$ *for* $n \not\equiv 0$ (mod 3).

(2) $G_{4n+14}(\mathbb{S}^{4n+3}/H) = \gamma_{4n+3*}\pi_{4n+14}(\mathbb{S}^{4n+3})$ *for* $n \equiv 5, 9 \pmod{12}$ *with* $n \geqslant 5$ *and* $n \equiv 15, 23 \pmod{24}$.

Next, applying Proposition 3.12, as [**10**, Corollary 2.53], we obtain:

Corollary 3.15. $G_{16n+k}(\mathbb{S}^{16n-1}/H) = \gamma_{16n-1*}\pi_{16n+k}(\mathbb{S}^{16n-1})$ *for* $k = 20$, $n \geqslant 1$ *and* $k = 21$, $n \geqslant 2$.

Finally, as [**10**, Proposition 2.54], Proposition 3.12 yields:

Proposition 3.16.
(1) $G_{21}(\mathbb{S}^{11}/H) \supseteq 2\gamma_{11*}\pi_{21}(\mathbb{S}^{11})$;
(2) $G_{22}(\mathbb{S}^{11}/H) \supseteq 8\gamma_{11*}\pi_{22}(\mathbb{S}^{11}) \cong \mathbb{Z}_{63}$;
(3) $G_{22}(\mathbb{S}^{15}/H) \supseteq 4\gamma_{15*}\pi_{22}(\mathbb{S}^{15}) \cong \mathbb{Z}_{60}$.

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