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INTEGRAL COHOMOLOGY OF CONFIGURATION SPACES OF THE SPHERE

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(*communicated by Nathalie Wahl*)

Abstract

We compute the cohomology of the unordered configuration spaces of the sphere S^2 with integral and with $\mathbb{Z}/p\mathbb{Z}$-coefficients using a cell complex due to Fox, Neuwirth, Fuks, Vainshtein and Napolitano.

Contents

1. Introduction

For any topological space X, let

$$F_n(X) = \{(x_1, \dots, x_n) \in X^n \,|\, x_i \neq x_j\}$$

be the ordered configuration space of n distinct points in X. The symmetric group S_n acts on $F_n(X)$ by permuting the points and the quotient

$$C_n(X) = F_n(X)/S_n$$

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is the unordered configuration space.

Despite their simple definition, getting a grasp of their topology is hard. The cohomology of configuration spaces has been widely studied (e.g., [**8, 27, 22, 1, 10, 24, 9**]) but only few cases have been computed explicitly. Usually this is only possible if the space X is very simple or one restricts to rational or mod p coefficients. Aside from the Euclidean case due to Arnold [**2**], vanishingly few complete integral homology calculations are available.

On the other hand, the cohomology of many configuration spaces satisfies interesting properties, for example with rational coefficients homological stability [**7**] or eventual periodicity with mod p coefficients [**17, 5, 16**].

In this paper, we will completely compute $H^*(C_n(S^2), \mathbb{Z}/p\mathbb{Z})$ and $H^*(C_n(S^2), \mathbb{Z})$.

Theorem 1.1. *Let*

$$B_p(n,r) = \left| \left\{ \begin{matrix} 1 \leqslant a_1 \leqslant a_2 \leqslant \cdots \leqslant a_g \\ 0 \leqslant b_1 < b_2 < \cdots < b_h \end{matrix} \middle| \begin{matrix} 2\sum_i p^{a_i} + 2\sum_j p^{b_j} - 2g - h = r \\ 2\sum_i p^{a_i} + 2\sum_j p^{b_j} \leqslant n \end{matrix} \right\} \right|$$

and

$$B'_p(n,r) = \left| \left\{ \begin{matrix} 1 \leqslant a_1 \leqslant a_2 \leqslant \cdots \leqslant a_g \\ 1 \leqslant b_1 < b_2 < \cdots < b_h \end{matrix} \middle| \begin{matrix} 2\sum_i p^{a_i} + 2\sum_j p^{b_j} + 1 - 2g - h = r \\ 2\sum_i p^{a_i} + 2\sum_j p^{b_j} + 2 \leqslant n \\ p \nmid 2(n - 2\sum_i p^{a_i} - 2\sum_j p^{b_j} - 1) \end{matrix} \right\} \right|.$$

Then

$$\dim H^r(C_n(S^2), \mathbb{Z}/p\mathbb{Z}) = B_p(n,r) + B_p(n-1, r-2) - B'_p(n,r) - B'_p(n, r-1).$$

Corollary 1.2. *We have*

$$\dim H^r(C_n(S^2), \mathbb{Z}/2\mathbb{Z}) = B_2(n,r) + B_2(n-1, r-2).$$

Our main tool is a cellular decomposition of $C_n(S^2)$ due to Napolitano [**18**]. It is an extension of the Fox-Neuwirth cell structure for $C_n(\mathbb{R}^2)$ [**13**] used by Fuks [**14**] and Vainshtein [**25**] to compute the mod p cohomology of $C_n(\mathbb{R}^2)$.

Theorem 1.1 could also be deduced from [**21**, Th. 18.3]. However, our approach is more elementary and allows to determine the integral cohomology:

Theorem 1.3. *The first cohomology groups $H^r(C_n(S_2), \mathbb{Z})$ are*

$$H^0(C_n(S_2), \mathbb{Z}) = \mathbb{Z}, \qquad H^1(C_n(S_2), \mathbb{Z}) = 0,$$

$$H^2(C_n(S_2), \mathbb{Z}) = \mathbb{Z}/(2n-2)\mathbb{Z}, \qquad H^3(C_n(S_2), \mathbb{Z}) = \begin{cases} 0 & n = 1, 2, \\ \mathbb{Z} & n = 3, \\ \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} & n \geqslant 4. \end{cases}$$

For $r \geqslant 4$, the cohomology groups $H^r(C_n(S^2), \mathbb{Z})$ are finite and contain no elements of order p^2.

Hence we can reconstruct all integral cohomology groups by Theorem 1.1 and the universal coefficient Theorem. Previously, the cohomology of $C_n(S^2)$ was known with rational coefficients [**23, 19, 21**], for low degree cases [**23, 18**], for mod 2 coefficients [**4**] and for mod p coefficients [**21**]. The description of $H^r(C_n(S^2), \mathbb{Z})$ seems to be new.

We will first explain the computations of the cohomology of $C_n(\mathbb{C})$ with $\mathbb{Z}/p\mathbb{Z}$-coefficients by Fuks [**14**] and Vainshtein [**25**] and discuss the Fox-Neuwirth cell structure. Afterwards, we present the extension of this cell complex due to Napolitano [**18**] used to calculate $H^*(C_n(S^2),\mathbb{Z})$ for $n \leqslant 9$. The main idea of this paper is the construction of a very specific chain homotopy that simplifies Napolitano's complex.

1.1. Conventions

We write

$$\mathrm{Part}(n,s) = \{\,[n_1,\ldots,n_s] \in \mathbb{Z}^q_{>0} \,|\, n_1+\cdots+n_s = n\}$$

for partitions of n into s positive summands, for example

$$\mathrm{Part}(5,3) = \{[3,1,1],\ [1,3,1],\ [1,1,3],\ [2,2,1],\ [2,1,2],\ [1,2,2]\,\}.$$

We call s the length and n the size of the partition.

The residue ring $\mathbb{Z}/m\mathbb{Z}$ is from now on abbreviated by $\mathbb{Z}_m$. For any abelian group G and prime p, we write $G_{T_p} = \{g \in G|\, p^n g = 0 \text{ for some } n\}$ for the p-torsion subgroup.

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2. Configuration spaces of the plane

2.1. Cellular decomposition of $C_n(\mathbb{C})^+$

The following construction comes from [**14**] and [**25**]. The projection

$$\mathbb{C} \to \mathbb{R}, x+iy \mapsto x$$

to the real line maps any configuration in $C_n(\mathbb{C})$ to a finite sets of points in $\mathbb{R}$. Counting the number of preimages of each of these points, we get a partition of n. Here we use that the one-dimensional line is ordered. The union of all points in $C_n(\mathbb{C})$ mapping to the same partition $n = n_1+\cdots+n_s$ and the point ∞ is an $n+s$-dimensional cell in the one point compactification $C_n(\mathbb{C})^+$. We denote this cell by $[n_1,\ldots,n_s]$. All such cells together with the point ∞ are a cellular decomposition of $C_n(\mathbb{C})^+$. Using Poincaré-Lefschetz duality for Borel-Moore homology [**6**], [**26**, Chapter 13.2]

$$H^i(C_n(\mathbb{C})) = \tilde{H}_{2n-i}(C_n(\mathbb{C})^+),$$

this cell complex can be used to compute the cohomology of $C_n(\mathbb{C})$.

The (co)-chains of the resulting (cochain)-complex $A_n^\bullet = (A_n^r)$ with the property

$$H^*(C_n(\mathbb{C}),\mathbb{Z}) = H^*(A_n^\bullet)$$

are the free $\mathbb{Z}$-modules

$$A_n^r = \mathbb{Z}\,\mathrm{Part}(n, n-r).$$

The basis elements are the partitions $[n_1,\ldots,n_s] \in \mathrm{Part}(n,s)$ with $s = n-r$. The boundary maps $\delta\colon A_n^r \to A_n^{r+1}$ are

$$\delta[n_1,\dots,n_s] = \sum_{l=1}^{s-1}(-1)^{l-1}P(n_l,n_{l+1})[n_1,\dots,n_{l-1},n_l+n_{l+1},n_{l+2},\dots,n_s],$$

with coefficients

$$P(x,y) = \begin{cases} 0 & \text{if } x\equiv y\equiv 1 \mod 2,\\ \binom{\lfloor x/2+y/2\rfloor}{\lfloor x/2\rfloor} & \text{otherwise.}\end{cases}$$

Intuitively, in the boundary of a cell, the points lying on two neighbouring vertical lines come together onto the same vertical line. The coefficient $P(x,y)$ gives a signed count of the different ways to combine two sets of x and y points on a line.

2.2. Subcomplexes of $A_n^\bullet$

As $P(x,y)=0$ for odd x and y, the complex $A_n^\bullet$ can be written as a direct sum

$$A_n^\bullet = A_{n,0}^\bullet \oplus \cdots \oplus A_{n,n}^\bullet$$

of subcomplexes $A_{n,t}^\bullet$ generated by partitions with t odd entries.

Take any $I\subset\{1,\dots,s+t\}$ with t elements, say $I=\{i_1,\dots,i_t\}$ where $i_1<\cdots< i_t$. Then we insert $1's$ at the positions i_1 to i_t with alternating signs:

$$\mathrm{Ins}_I[a_1,\dots,a_s] = (-1)^{\sum_j i_j}[a_1,\dots,a_{i_1-1},1,a_{i_1},\dots,a_{i_2-2},1,a_{i_2-1},\dots].$$

The map

$$\mathrm{Ins}_t = (-1)^{st}\sum_{I\subset\{1,\dots,s+t\},|I|=t}\mathrm{Ins}_I$$

is actually a chain map

$$\mathrm{Ins}_t\colon A_{n,0}^\bullet \to A_{n+t,t}^\bullet$$

that induces isomorphisms [**25**, Prop. 1]

$$H^r(A_{n-t,0}^\bullet)\simeq H^r(A_{n,t}^\bullet).$$

Hence we get

$$H^*(A_n^\bullet) = H^*(A_{n,0}^\bullet)\oplus H^*(A_{n-1,0})\oplus\cdots\oplus H^*(A_{0,0}^\bullet).$$

As $A_{n,0}^r=0$ if $n>2r$, we can immediately deduce that the cohomology groups stabilize

$$H^r(A_n^\bullet) = H^r(A_{n+1}^\bullet)$$

if $n>2r$. Later, we will use the notation

$$H^r(C_\infty(\mathbb{C})) = H^r(C_n(\mathbb{C}))$$

for any $n>2r$.

Example 2.1. The cohomology group $H^0(C_n(\mathbb{C}),\mathbb{Z})=\mathbb{Z}$ is generated by the class of

$$(-1)^{n(n-1)/2}[1,\dots,1] = \mathrm{Ins}_n([\,]).$$

For $n\geqslant 2$, the cohomology group $H^1(C_n(\mathbb{C}),\mathbb{Z})=\mathbb{Z}$ is generated by the class of

$$[2,1,\dots,1]-[1,2,1,\dots,1]+\cdots = (-1)^{(n-2)(n-3)/2+n}\,\mathrm{Ins}_{n-2}[2].$$

2.3. Explicit basis of $H^*(A^\bullet_{n,0}, \mathbb{Z}_p)$

We will now present the description of the group $H^r(A^\bullet_{n,0}, \mathbb{Z}_p)$ by Vainshtein and work out some of the details and proofs omitted in [25].

Remark 2.2. In particular, the explicit formula for the base elements seems to be stated in a misleading way in [25] (definition of morphism Φ, top of page 136). There, the operator Perm is defined via transpositions and does not create a cycle even for partitions of length 3, because different permutations show up with different coefficients. Our definition of Perm seems to be the intended one.

Let $[n_1, \dots, n_s]$ be any partition of n. Then the alternating sum of its permutations

$$\sum_{\sigma \in S_s} \operatorname{sign}(\sigma)[n_{\sigma(1)}, \dots, n_{\sigma(s)}]$$

is a cycle in $A^\bullet_n$. With $\mathbb{Z}_p$-coefficients, the following subset of permutations

$$\operatorname{Perm}[n_1, \dots, n_s] = \sum_{\substack{\sigma \in S_s \text{ where } \sigma(i)<\sigma(j) \\ \text{if } i<j \text{ and } n_i = n_j \text{ or} \\ \text{if } i<j \text{ and } P(n_i,n_j)=0 \bmod p}} \operatorname{sign}(\sigma)[n_{\sigma(1)}, \dots, n_{\sigma(s)}]$$

will be used in the next paragraph to create special cycles in $A^\bullet_n \otimes \mathbb{Z}_p$.

Take integers $1 \leqslant i_1 \leqslant \dots \leqslant i_k$ and $0 \leqslant j_1 < \dots < j_l$ such that

$$t = n - 2(p^{i_1} + \dots + p^{i_k} + p^{j_1} + \dots + p^{j_l}) \geqslant 0$$

and let

$$r = (2p^{i_1} - 2) + \dots + (2p^{i_k} - 2) + (2p^{j_1} - 1) + \dots + (2p^{j_l} - 1).$$

Then we give the chain

$$\operatorname{Ins}_t \operatorname{Perm}[2p^{i_1-1}, 2p^{i_1-1}(p-1), \dots, 2p^{i_k-1}, 2p^{i_k-1}(p-1), 2p^{j_1}, \dots, 2p^{j_l}]$$

the name $x_{i_1} \cdots x_{i_k} y_{j_1} \cdots y_{j_l}$. It is a cycle in $A^r_{n,t} \otimes \mathbb{Z}_p$ (but not in $A^\bullet_n$ if $k > 0$). Vainshtein showed that all such cycles form a basis of $H^r(A^\bullet_n, \mathbb{Z}_p)$. We call the quantity $n - t$ the size of the chain $x_{i_1} \cdots x_{i_k} y_{j_1} \cdots y_{j_l}$.

Theorem 2.3 ([25]). *The group $H^*(C_\infty, \mathbb{Z}_p)$ is the free graded commutative algebra over $\mathbb{Z}_p$ with generators*

x_i *for* $i \geqslant 1$	$\deg(x_i) = 2p^i - 2$	$\operatorname{size}(x_i) = 2p^i$,
y_i *for* $i \geqslant 0$	$\deg(y_i) = 2p^i - 1$	$\operatorname{size}(y_i) = 2p^i$.

There is a surjection $H^*(C_\infty(\mathbb{C}), \mathbb{Z}_p) \to H^*(C_n(\mathbb{C}), \mathbb{Z}_p)$ whose kernel is generated by the monomials $x_{i_1} \cdots x_{i_k} y_{j_1} \cdots y_{j_l}$ such that $\operatorname{size}(x_{i_1} \cdots x_{i_k} y_{j_1} \cdots y_{j_l}) > n$.

A equivalent formula was deduced by Cohen-Lada-May [8, Appendix to III].

Remark 2.4. For $p = 2$, the group $H^*(C_\infty, \mathbb{Z}_2)$ can be identified with a polynomial algebra with generators

$$z_i \text{ for } i \geqslant 1, \qquad \deg(z_i) = 2^i - 1$$

via $x_i \mapsto z_i^2$ and $y_i \mapsto z_{i-1}$. This is the form stated in [8].

Corollary 2.5. *Define*

$$B_p(n,r) = \left| \left\{ \begin{matrix} 1 \leqslant a_1 \leqslant a_2 \leqslant \cdots \leqslant a_g \\ 0 \leqslant b_1 < b_2 < \cdots < b_h \end{matrix} \middle| \begin{matrix} 2\sum_i p^{a_i} + 2\sum_j p^{b_j} - 2g - h = r \\ 2\sum_i p^{a_i} + 2\sum_j p^{b_j} \leqslant n \end{matrix} \right\} \right|.$$

Hence we have

$$\dim H^r(C_n(\mathbb{C}), \mathbb{Z}_p) = B_p(n,r).$$

Remark 2.6. Paolo Salvatore (private communication) gave this representation as a generating series:

$$\sum_{n,r \geqslant 0} B_p(n,r) w^r z^n = \frac{1+wz^2}{1-z} \prod_{i>0} \frac{1+w^{2p^i-1}z^{2p^i}}{1-w^{2p^i-2}z^{2p^i}}.$$

Remark 2.7. The notation suggests a product structure on $H^*(C_\infty(\mathbb{C}), \mathbb{Z}_p)$. It comes from the map

$$C_n(\mathbb{C}) \times C_m(\mathbb{C}) \to C_{n+m}(\mathbb{C})$$

by adding the points far apart. However, in this paper we will use it only as a convenient notation.

Remark 2.8. As

$$\binom{p^a + p^b}{p^a} \equiv \begin{cases} 1 & a \neq b \\ 2 & a = b \end{cases} \mod p$$

and

$$\binom{p^a + p^b(p-1)}{p^a} \equiv \begin{cases} 1 & a \neq b \\ 0 & a = b \end{cases} \mod p$$

by Lucas's Theorem [**12**], the order of all entries of the form $2p^a$, $2p^a(p-1)$ in our basis elements is preserved by the operator Perm. All other entries are permuted.

Example 2.9. In order to give an example for all the constructions, we compute $H^*(C_{24}(\mathbb{C}), \mathbb{Z}/3\mathbb{Z})$. The generators have degrees:

generators	x_1	x_2	y_0	y_1	y_2	$\dots$
degree	4	16	1	5	17	$\dots$
size	6	18	2	6	18	$\dots$

In Table 1, we write down the basis elements and the corresponding chains, however, we will omit the application of the Ins_t-operators to lift the chains to sum 24.

2.4. Bockstein homomorphisms

The short exact sequences of coefficients

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{p\cdot} & \mathbb{Z} & \longrightarrow & \mathbb{Z}_p & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathbb{Z}_p & \xrightarrow{p\cdot} & \mathbb{Z}_{p^2} & \longrightarrow & \mathbb{Z}_p & \longrightarrow & 0 \end{array}$$

Table 1: The cohomology group $H^*(C_{24}(\mathbb{C}), \mathbb{Z}_3)$

r	basis of $H^r(C_{24}(\mathbb{C}), \mathbb{Z}_3)$
0	$1 = []$
1	$y_0 = [2]$
2	–
3	–
4	$x_1 = [2, 4]$
5	$y_1 = [6]$
	$x_1 y_0 = [2, 4, 2]$
6	$y_0 y_1 = [2, 6] - [6, 2]$
7	–
8	$x_1^2 = [2, 4, 2, 4]$
9	$x_1 y_1 = [2, 4, 6] - [2, 6, 4] + [6, 2, 4]$
	$x_1^2 y_0 = [2, 4, 2, 4, 2]$
10	$x_1 y_0 y_1 = [2, 4, 2, 6] - [2, 4, 6, 2] + [2, 6, 4, 2] - [6, 2, 4, 2]$
11	–
12	$x_1^3 = [2, 4, 2, 4, 2, 4]$
13	$x_1^2 y_1 = [2, 4, 2, 4, 6] - [2, 4, 2, 6, 4] + [2, 4, 6, 2, 4] - [2, 6, 4, 2, 4] + [6, 2, 4, 2, 4]$
	$x_1^3 y_0 = [2, 4, 2, 4, 2, 4, 2]$
14	$x_1^2 y_0 y_1 = [2, 4, 2, 4, 2, 6] - [2, 4, 2, 4, 6, 2] + [2, 4, 2, 6, 4, 2] - [2, 4, 6, 2, 4, 2,] + \cdots$
15	–
16	$x_2 = [6, 12]$
	$x_1^4 = [2, 4, 2, 4, 2, 4, 2, 4]$
17	$y_2 = [18]$
	$x_2 y_0 = [6, 12, 2] - [6, 2, 12] + [2, 6, 12]$
	$x_1^3 y_1 = [2, 4, 2, 4, 2, 4, 6] - [2, 4, 2, 4, 2, 6, 4] + \cdots$
18	$y_0 y_2 = [2, 18] - [18, 2]$
19	–
20	$x_1 x_2 = [2, 4, 6, 12] - [2, 6, 4, 12] + [6, 2, 4, 12] - [6, 2, 12, 4] + [2, 6, 12, 4] + [6, 12, 2, 4]$
21	$x_1 y_2 = [2, 4, 18] - [2, 18, 4] + [18, 2, 4]$
	$x_2 y_1 = [6, 12, 6]$
22	$y_1 y_2 = [6, 18] - [18, 6]$
$\geqslant 23$	–

induce long exact sequences

$$\begin{array}{ccccccccc}
H^{r-1}(A_n^\bullet, \mathbb{Z}_p) & \xrightarrow{\tilde{\beta}} & H^r(A_n^\bullet, \mathbb{Z}) & \xrightarrow{p\cdot} & H^r(A_n^\bullet, \mathbb{Z}) & \longrightarrow & H^r(A_n^\bullet, \mathbb{Z}_p) & \xrightarrow{\tilde{\beta}} & H^{r+1}(A_n^\bullet, \mathbb{Z}) \\
\| & & \downarrow & & \downarrow & & \| & & \downarrow \\
H^{r-1}(A_n^\bullet, \mathbb{Z}_p) & \xrightarrow{\beta} & H^r(A_n^\bullet, \mathbb{Z}_p) & \xrightarrow{p\cdot} & H^r(A_n^\bullet, \mathbb{Z}_{p^2}) & \longrightarrow & H^r(A_n^\bullet, \mathbb{Z}_p) & \xrightarrow{\beta} & H^{i+1}(A_n^\bullet, \mathbb{Z}_p)
\end{array}$$

where the connecting morphisms are the *Bockstein morphisms* β and $\tilde{\beta}$ (compare [**15**, Chap. 3.E]). The image of $\tilde{\beta}$ hence consists of all the elements of order p in $H^*(A_n^\bullet, \mathbb{Z})$.

Example 2.10. Let $i \neq j$. We determine the Bockstein on

$$x_i = [2p^{i-1}, 2p^{i-1}(p-1)]$$

and

$$x_i y_j = [2p^{i-1}, 2p^{i-1}(p-1), 2p^j] - [2p^{i-1}, 2p^j, 2p^{i-1}(p-1)] + [2p^j, 2p^{i-1}, 2p^{i-1}(p-1)].$$

In $A_n^\bullet$, we get

$$\delta(x_i) = \binom{p^i}{p^{i-1}}[2p^i] = \binom{p^i}{p^{i-1}} y_i,$$
$$\delta(x_i y_j) = \binom{p^i}{p^{i-1}}([2p^i, 2p^j] - [2p^j, 2p^i]) = \binom{p^i}{p^{i-1}} y_i y_j.$$

Hence we can conclude

$$\tilde{\beta}(x_i) = \frac{1}{p}\binom{p^i}{p^{i-1}} y_i, \qquad \tilde{\beta}(x_i y_j) = \frac{1}{p}\binom{p^i}{p^{i-1}} y_i y_j.$$

The coefficient

$$\frac{1}{p}\binom{p^i}{p^{i-1}} = \binom{p^i - 1}{p^{i-1} - 1}$$

is an integer congruent to 1 mod p by Lucas' Theorem [**12**].

Lemma 2.11. *The differential δ on $A_n^\bullet$ operates as follows:*

$$\delta(x_1^{a_1} \cdots x_k^{a_k} y_0^{b_1} \cdots y_l^{b_l}) = \sum_i \binom{p^i}{p^{i-1}} x_1^{a_1} \cdots x_i^{a_i - 1} \cdots x_k^{a_k} y_i y_0^{b_0} \cdots y_l^{b_l}.$$

Hence the Bocksteins are given by

$$\tilde{\beta}(x_1^{a_1} \cdots x_k^{a_k} y_0^{b_1} \cdots y_l^{b_l}) = \frac{1}{p}\sum_i \binom{p^i}{p^{i-1}} x_1^{a_1} \cdots x_i^{a_i - 1} \cdots x_k^{a_k} y_i y_0^{b_0} \cdots y_l^{b_l}$$

and

$$\beta(x_1^{a_1} \cdots x_k^{a_k} y_0^{b_1} \cdots y_l^{b_l}) = \sum_i x_1^{a_1} \cdots x_i^{a_i - 1} \cdots x_k^{a_k} y_i y_0^{b_0} \cdots y_l^{b_l}.$$

Proof. Let $m = x_1^{a_1} \cdots x_k^{a_k} y_0^{b_1} \cdots y_l^{b_l}$. Take any term $[\dots, n_1, n_2, \dots]$ in m. It only contributes $[\dots, n_1 + n_2, \dots]$ to $\delta(m)$ if $[n_1, n_2] = [2p^{i-1}, 2(p-1)p^{i-1}]$ or $[n_1, n_2] = [2(p-1)p^{i-1}, p^{i-1}]$. Otherwise, $[\dots, n_1 + n_2, \dots]$ is cancelled by $\delta([\dots, n_2, n_1, \dots])$ as $[\dots, n_2, n_1, \dots]$ shows up in m with opposite sign due to the definition of Perm. Now

$$\delta([\dots, 2p^{i-1}, 2(p-1)p^{i-1}, \dots]) = \pm\binom{p^i}{p^{i-1}}[\dots, 2p^i, \dots] + \cdots,$$

and a tedious calculation of signs proves the formula. □

As $\beta^2 = 0$, we can look at the *Bockstein cohomology groups*

$$BH^*(A_n^\bullet, \mathbb{Z}_p) = \operatorname{Ker}\beta / \operatorname{Im}\beta.$$

Lemma 2.12 ([**15**, Cor. 3E.4])**.** *The group $H^*(A_n^\bullet, \mathbb{Z})$ contains no element of order p^2 if and only if*

$$\dim_{\mathbb{Z}_p} BH^r(A_n^\bullet, \mathbb{Z}_p) = \operatorname{rk} H^r(A_n^\bullet, \mathbb{Z}).$$

In this case the map

$$H^*(A_n^\bullet, \mathbb{Z}) \to H^*(A_n^\bullet, \mathbb{Z}_p)$$

is injective on the p-torsion and its image is $\operatorname{Im}\beta$.

Vainshtein stated that $H^*(A_n^\bullet, \mathbb{Z})$ has no elements of order p^2:

Theorem 2.13 ([**25**]). *The integral cohomology is given by*

$$H^0(C_n(\mathbb{C}), \mathbb{Z}) = \mathbb{Z}, \qquad H^1(C_n(\mathbb{C}), \mathbb{Z}) = \mathbb{Z} \text{ if } n \geqslant 2$$

and

$$H^r(C_n(\mathbb{C}), \mathbb{Z}) = \bigoplus_p \tilde{\beta}_p H^{r-1}(C_n(\mathbb{C}), \mathbb{Z}_p) \text{ for } r \geqslant 2.$$

Proof. Take any $x \in \operatorname{Ker}\beta$ of the form

$$x = x_j^k f + x_j^{k-1} y_j g$$

for $k \geqslant 0$, $j > 0$ where f, g do not contain x_j or y_j. We compute

$$\beta(x) = x_j^{k-1} y_j f + x_j^k \beta(f) - x_j^{k-1} y_j \beta(g).$$

Hence we see $\beta(g) = f$ and $\beta(x_j^k g) = x$. So we have shown that

$$\operatorname{Ker}\beta / \operatorname{Im}\beta = \mathbb{Z}_p \oplus \mathbb{Z}_p y_0. \qquad \square$$

Remark 2.14. The map β arises as the reduction mod p of the unique graded derivation β' on the free divided power algebra [**15**, Ex 3.5C] on generators

$$x_1, x_2, \ldots, y_0, y_1, \ldots \qquad \deg(x_i) = 2p^i - 2 \qquad \deg(y_i) = 2p^i - 1,$$

with multiplication $x_i^{j_1} \star x_i^{j_2} = \binom{j_1+j_2}{j_1} x_i^{j_1+j_2}$ where β' is given by the formulas

$$\beta'(X_i) = Y_i, \qquad \beta'(Y_i) = 0$$

and the rule (compare [**11**, Chap. 3])

$$\beta'(z_1 \star z_2) = \beta'(z_1) \star z_2 + (-1)^{\deg z_1} z_1 \star \beta'(z_2).$$

Corollary 2.15. *The p-Torsion of $H^{r+1}(C_\infty(\mathbb{C}), \mathbb{Z})$ is isomorphic to the degree r-part of the free graded algebra over $\mathbb{Z}_p$ with generators $x_1, x_2, \ldots, y_1, y_2, \ldots\rangle$ for $r > 0$.*

Proof. Write R for the free graded algebra over $\mathbb{Z}_p$ with generators x_1, x_2, $\ldots$, y_1, y_2, $\ldots$. Theorem 2.3 shows that

$$H^*(C_\infty(\mathbb{C}), \mathbb{Z}_p) = R \oplus y_0 R.$$

By Lemma 2.11 we know that $\beta(x y_0) = \beta(x) y_0$ and $\beta(R) \subset R$. This shows

$$\operatorname{Im}\beta = \beta(R) \oplus y_0 \beta(R).$$

Decompose $R = \beta(R) \oplus R'$. As $\operatorname{Ker}\beta = \operatorname{Im}\beta \oplus \mathbb{Z}_p \oplus \mathbb{Z}_p y_0$, the map

$$\beta(R) \oplus R' \to \beta(R) \oplus y_0 \beta(R) = \operatorname{Im}\beta, \; (z_1, z_2) \mapsto \beta(z_2) + y_0 z_1$$

is a bijective map between the degree r part of R and the degree $r+1$ part of $\operatorname{Im}\beta$ for $r > 0$.

However, it does not respect the size, so this isomorphism allows to describe the p-Torsion of $H^{r+1}(C_\infty(\mathbb{C}), \mathbb{Z})$, but not of $H^{r+1}(C_n(\mathbb{C}), \mathbb{Z})$ for $n < \infty$. $\square$

Remark 2.16. The description of the dimension of the p-torsion of $H^r(C_n(\mathbb{C}),\mathbb{Z})$ in [**8**, Appendix to III, Cor. A4] seems to be wrong. For example, we can compute that $H^{21}(C_\infty(\mathbb{C}),\mathbb{Z}_3)$ would be 5-dimensional, the 3-Torsion in $H^{20}(C_\infty(\mathbb{C}),\mathbb{Z})$ would be 2-dimensional and the 3-Torsion in $H^{21}(C_\infty(\mathbb{C}),\mathbb{Z}_3)$ would also be 2-dimensional. This contradicts the universal coefficient theorem.

The description of $H^*(C_n(\mathbb{C}),\mathbb{Z}_p)$ and of the Bockstein homomorphism in [**8**, Appendix to III] is correct, however, the image of the Bockstein is not given by the subalgebra described there in Corollary A4. A simple formula for the dimension of $H^r(C_n(\mathbb{C}),\mathbb{Z})$ probably does not exist.

Example 2.17. In Table 2, we compute $H^*(C_{24}(\mathbb{C}),\mathbb{Z}_3)_{T_3}$ by applying Theorem 2.13 and Formula 2.11 to our Example 2.9.

Table 2: The 3-torsion in the cohomology group $H^*(C_{24}(\mathbb{C}),\mathbb{Z})$

r	basis of $H^r(C_{24}(\mathbb{C}),\mathbb{Z})_{T_3}$ as $\mathbb{Z}_3$-module
0	–
1	–
2	–
3	–
4	–
5	$y_1 = [6]$
6	$y_0y_1 = [2,6] - [6,2]$
7	–
8	
9	$x_1y_1 = [2,4,6] - [2,6,4] + [6,2,4]$
10	$x_1y_0y_1 = [2,4,2,6] - [2,4,6,2] + [2,6,4,2] - [6,2,4,2]$
11	–
12	–
13	$x_1^2y_1 = [2,4,2,4,6] - [2,4,2,6,4] + [2,4,6,2,4] - [2,6,4,2,4] + [6,2,4,2,4]$
14	$x_1^2y_0y_1 = [2,4,2,4,2,6] - [2,4,2,4,6,2] + [2,4,2,6,4,2] - [2,4,6,2,4,2,] + \cdots$
15	–
16	–
17	$y_2 = [18]$ $x_1^3y_1 = [2,4,2,4,2,4,6] - \cdots$
18	$y_0y_2 = [2,18] - [18,2]$
19	–
20	–
21	$28x_1y_2 + x_2y_1 = 28([2,4,18] - [2,18,4] + [18,2,4]) + [6,12,6]$
22	$y_1y_2 = [6,18] - [18,6]$
$\geqslant 23$	–

3. Configuration spaces of the sphere

We will describe a cellular decomposition of $C_n(S^2)^+$ by Napolitano [**18**] and show how it can be used to compute the cohomology of $C_n(S^2)$.

3.1. Cellular decomposition of $C_n(S^2)^+$

The cellular decomposition of $C_n(\mathbb{C})^+$ can be extended to a cellular decomposition of $C_n(S^2)^+$. Using $S^2 = \mathbb{R}^2 \cup \infty$, we see that n points on S^2 are either n points on $\mathbb{R}^2$ or $n-1$ points on $\mathbb{R}^2$ and the point ∞. So the cells of $C_n(S^2)$ are the union of the cells of $C_n(\mathbb{R}^2)$ and $C_{n-1}(\mathbb{R}^2)$. The resulting complex $B_n^\bullet = (B_n^r)$ with $H^*(B_n^\bullet, \mathbb{Z}) = H^*(C_n(S^2), \mathbb{Z})$ has chains

$$B_n^r = A_n^r \oplus A_{n-1}^{r-2} = \mathbb{Z}\,\mathrm{Part}(n, n-r) \oplus \mathbb{Z}\,\mathrm{Part}(n-1, n-r+1).$$

The new boundary maps Δ were computed by Napolitano [**18**]. We define a new operator $D\colon A_n^r \to A_{n-1}^{r-1}$ by

$$D[n_1, \dots, n_s] = \sum_{i=1}^{s} Q(n_i)(-1)^{\sum_{j=1}^{i-1} n_j}[n_1, \dots, n_{i-1}, n_i - 1, n_{i+1}, \dots, n_s],$$

where

$$Q(n_i) = \begin{cases} 0 & \text{if } n_i \equiv 1 \mod 2, \\ 2 & \text{otherwise.} \end{cases}$$

The differential Δ of the complex $B_n^\bullet$ is then given by

$$\Delta\colon B_n^r \to B_n^{r+1}, (a, b) \mapsto (\delta(a), \delta(b) + (-1)^{n-r} D(a)).$$

Corollary 3.1. *We have* $D \equiv 0 \mod 2$ *and therefore* $B_n^\bullet \otimes \mathbb{Z}_2 = (A_n^\bullet \oplus A_{n-1}^\bullet) \otimes \mathbb{Z}_2$ *and*

$$H^r(C_n(S^2), \mathbb{Z}_2) = H^r(C_n(\mathbb{C}), \mathbb{Z}_2) \oplus H^{r-2}(C_{n-1}(\mathbb{C}), \mathbb{Z}_2).$$

The groups $H^r(C_n(S^2), \mathbb{Z}_2)$ have already been determined in [**4**].

$$\begin{array}{ccccccccc}
\cdots & \xrightarrow{\delta} & A_n^{r-1} & \xrightarrow{\delta} & A_n^{r} & \xrightarrow{\delta} & A_n^{r+1} & \xrightarrow{\delta} & \cdots \\
 & & \downarrow S & \searrow D & \downarrow S & \searrow D & \downarrow S & & \\
\cdots & \xrightarrow{\delta} & A_{n-1}^{r-3} & \xrightarrow{\delta} & A_{n-1}^{r-2} & \xrightarrow{\delta} & A_{n-1}^{r-1} & \xrightarrow{\delta} & \cdots
\end{array}$$

3.2. Mapping cone complex

Lemma 3.2. *We get a long exact sequence*

$$\cdots \to H^{r-1}(A_n^\bullet) \xrightarrow{D^*} H^{r-2}(A_{n-1}^\bullet) \to H^r(B_n^\bullet) \to H^r(A_n^\bullet) \xrightarrow{D^*} H^{r-1}(A_{n-1}^\bullet) \to \cdots.$$

Proof. The relation

$$D \circ \delta = \delta \circ D$$

is equivalent to $\Delta^2 = 0$. This means we can see D as a chain map

$$D\colon A_n^\bullet \to A_{n-1}^\bullet[1]$$

and the complex $B_n^\bullet$ can be interpreted as the mapping cone complex of the chain map D. The short exact sequence of chain complexes

$$0 \to A_{n-1}^\bullet[2] \to B_n^\bullet \to A_n^\bullet \to 0$$

given by $a_2 \mapsto (0, a_2)$ and $(a_1, a_2) \mapsto a_1$ induces a long exact sequence with a connecting homomorphism that can be identified with D^*. □

We can use this long exact sequence to compare the cohomology of $B_n^\bullet$, $A_n^\bullet$ and $A_{n-1}^\bullet$. Next we will construct a map

$$S\colon A_n^r \to A_{n-1}^{r-2},$$

which is almost a chain homotopy $D \approx 2\delta S + 2S\delta$ between D and the zero map. This allows us to compute the rank of D^*.

4. Construction of (almost) a null homotopy

As a motivation we first look at the case $r = n-1$. We set $S[n] = [1, n-2]$. Then we have

$$2\delta S[n] = 2\delta[1, n-2] = 2[n-1] = D[n]$$

if n is even and

$$2\delta S[n] = 2\delta[1, n-2] = 0 = D[n]$$

otherwise.

In general, we define $S\colon A_n^r \to A_{n-1}^{r-2}$ by

$$S[n_1, \dots, n_s] = \sum_{1 \leqslant k \leqslant i \leqslant s} (-1)^{k+1+\sum_{m=1}^{k-1} n_m} [n_1, \dots, n_{k-1}, 1, n_k, \dots, n_{i-1}, n_i - 2, n_{i+1}, \dots, n_s].$$

If $n_i - 2 \leqslant 0$, we simply omit this summand. We remark, that all calculations in this chapter are done on the chain level.

Lemma 4.1. *Define $E = D - 2\delta \circ S - 2S \circ \delta$. For every partition $[n_1, \dots, n_s]$ with $n_s \neq 2$ we have*

$$E[n_1, \dots, n_s] = 0$$

and

$$E[n_1, \dots, n_{s-1}, 2] = 2 \sum_{1 \leqslant k \leqslant s} (-1)^{s+k+\sum_{m=1}^{k-1} n_m} [n_1, \dots, n_{k-1}, 1, n_k, \dots, n_{s-1}]$$

otherwise.

Proof. For convenience we introduce the operators δ_l by

$$\delta_l[m_1, \dots, m_t] = (-1)^{l-1} P(m_l, m_{l+1})[m_1, \dots, m_{l-1}, m_l + m_{l+1}, m_{l+2}, \dots, m_t]$$

and the abbreviations

$$n_{k,i} = (-1)^{k+1+\sum_{m=1}^{k-1} n_m} [n_1, \dots, n_{k-1}, 1, n_k, \dots, n_{i-1}, n_i - 2, n_{i+1}, \dots, n_s].$$

Let us first assume that all $n_i > 2$. We compute

$$\delta \circ S[n_1, \dots n_r] = \sum_{\substack{1 \leqslant l \leqslant s \\ k \leqslant i}} \delta_l(n_{k,i})$$

by splitting up the index set

$$I = \{1 \leqslant l \leqslant s, 1 \leqslant k \leqslant i \leqslant s\}$$

into

$$I = I_1 \sqcup \dots \sqcup I_8,$$

where

$$I_1 = \{1 \leqslant l < k-1, k \leqslant i\}, \qquad I_5 = \{l = i+1, k \leqslant i\},$$
$$I_2 = \{k+1 \leqslant l < i\}, \qquad I_6 = \{l = k-1, k \leqslant i\},$$
$$I_3 = \{i+2 \leqslant l \leqslant s, k \leqslant i\}, \qquad I_7 = \{l = k, k < i\},$$
$$I_4 = \{l = i, k < i\}, \qquad I_8 = \{l = k = i\}.$$

Now we look at the individual summands $T_j = \sum_{I_j} \delta_l(n_{k,i})$ and expand them after doing some index shifts. Write $\mathrm{ind} = k + l + \sum_{m=1}^{k-1} n_m$.

$$T_1 = \sum_{\substack{l<k-1\\ k\leqslant i}} (-1)^{\mathrm{ind}} P(n_l, n_{l+1})[\ldots, n_l + n_{l+1}, \ldots, n_{k-1}, 1, n_k, \ldots, n_{i-1}, n_i - 2, n_{i+1}, \ldots]$$

$$T_2 = \sum_{k\leqslant l<i-1} (-1)^{\mathrm{ind}+1} P(n_l, n_{l+1})[\ldots, n_{k-1}, 1, n_k, \ldots, n_l + n_{l+1}, \ldots, n_{i-1}, n_i - 2, n_{i+1}, \ldots]$$

$$T_3 = \sum_{k\leqslant i<l} (-1)^{\mathrm{ind}+1} P(n_l, n_{l+1})[\ldots, n_{k-1}, 1, n_k, \ldots, n_{i-1}, n_i - 2, n_{i+1}, \ldots, n_l + n_{l+1}, \ldots]$$

The next terms

$$T_4 = \sum_{k<i} (-1)^{k+i+\sum_{m=1}^{k-1} n_m} P(n_{i-1}, n_i - 2)[\ldots, n_{k-1}, 1, n_k, \ldots, n_{i-1} + n_i - 2, n_{i+1}, \ldots]$$

$$T_5 = \sum_{k\leqslant i} (-1)^{k+i+1+\sum_{m=1}^{k-1} n_m} P(n_i - 2, n_{i+1})[\ldots, n_{k-1}, 1, n_k, \ldots, n_{i-1}, n_i - 2 + n_{i+1}, \ldots]$$

sum up to

$$T_4 + T_5 = \sum_{k\leqslant i} (-1)^{k+i+1+\sum_{m=1}^{k-1} n_m} P(n_i, n_{i+1})[\ldots, n_{k-1}, 1, n_k, , \ldots, , n_{i-1}, n_i - 2 + n_{i+1}, \ldots]$$

where we use the identity $P(x-2, y) + P(x, y-2) = P(x, y)$. Altogether we have

$$T_1 + T_2 + T_3 + T_4 + T_5 = -S \circ \delta[n_1, \ldots, n_s].$$

The terms

$$T_6 = \sum_{k\leqslant i} (-1)^{2k-2+\sum_{m=1}^{k-1} n_m} P(n_{k-1}, 1)[\ldots, n_{k-2}, n_{k-1} + 1, n_k, \ldots, n_{i-1}, n_i - 2, n_{i+1}, \ldots],$$

$$T_7 = \sum_{k<i} (-1)^{2k-1+\sum_{m=1}^{k-1} n_m} P(1, n_k)[\ldots, n_{k-1}, 1 + n_k, n_{k+1}, \ldots, n_{i-1}, n_i - 2, n_{i+1}, \ldots]$$

contain the same summands with alternating signs and cancel each other. For the remaining summand

$$T_8 = \sum_{i} (-1)^{\sum_{m=1}^{i-1} n_m} P(1, n_i - 2)[\ldots, n_{i-1}, n_i - 1, n_{i+1}, \ldots],$$

the following equation holds

$$2T_8 = D[n_1, \ldots, n_s]$$

by the definition of D. Here we use $P(1, n_i - 2) = 1$ if n_i even and $P(1, n_i - 2) = 0$ if n_i odd. In the end we get

$$2\delta \circ S[n_1, \ldots, n_s] = -2S \circ \delta[n_1, \ldots, n_s] + D[n_1, \ldots, n_s].$$

In case that $n_j = 2$ with $j < s$, all contributions containing $n_j - 2$ in T_4, T_5 and

T_8 are missing in $\delta \circ S$, but not in $S \circ \delta$ and D. So we have to add

$$T_4' = \sum_{k<j} (-1)^{k+j+\sum_{m=1}^{k-1} n_m} P(n_{j-1}, 0)[\ldots, 1, n_k, \ldots, n_{j-2}, n_{j-1}, n_{j+1}, \ldots],$$
$$T_5' = \sum_{k\leqslant j} (-1)^{k+j+1+\sum_{m=1}^{k-1} n_m} P(0, n_{j+1})[\ldots, 1, n_k, \ldots, n_{j-1}, n_{j+1}, \ldots],$$
$$T_8' = (-1)^{\sum_{m=1}^{j-1} n_m} P(1,0)[\ldots, n_{j-1}, 1, n_{j+1}, \ldots],$$

which simplifies using $P(x,0) = 1$ and $P(0,y) = 1$ to:

$$T_4' + T_8' = \sum_{k\leqslant j} (-1)^{k+j+\sum_{m=1}^{k-1} n_m} [\ldots, n_{k-1}, 1, n_k, n_{j-2}, \ldots, n_{j-1}, n_{j+1}, \ldots],$$
$$T_5' = \sum_{k\leqslant j} (-1)^{k+j+1+\sum_{m=1}^{k-1} n_m} [\ldots, n_{k-1}, 1, n_k, \ldots, n_{j-1}, n_{j+1}, , \ldots].$$

Hence we have

$$(D - 2\delta \circ S - 2S \circ \delta)[n_1, \ldots, n_s] = 2T_4' + 2T_5' + 2T_5' = 0,$$

if $n_j = 2$ with $j < s$. In the case $n_s = 2$, the contributions containing $n_s - 2$ are missing in $\delta \circ S$, $S \circ \delta$ and D. So we get

$$\begin{aligned}(D - 2\delta \circ S - 2S \circ \delta)[n_1, \ldots, n_{s-1}, 2] &= 2T_4' + 2T_8' \\ &= 2 \sum_{1\leqslant k\leqslant s} (-1)^{s+k+\sum_{m=1}^{k-1} n_m} [n_1, \ldots, n_{k-1}, 1, n_k, \ldots, n_{s-1}].\end{aligned}$$

A similar argument deals with the case that some $n_j = 1$. □

Lemma 4.2. *For every partition $[n_1, \ldots, n_s]$ with all n_i even, we have*

$$(D - 2\delta \circ S - 2S \circ \delta) \operatorname{Ins}_t[n_1, \ldots, n_{s-1}, 2] = 2(t+1)(-1)^{t+1} \operatorname{Ins}_{t+1}[n_1, \ldots, n_{s-1}].$$

Proof. Take any $I \subset \{1, \ldots, s+t\}$ with $|I| = t+1$. The term $\operatorname{Ins}_I[n_1, \ldots, n_{s-1}]$ is created in $(D - 2\delta \circ S - 2S \circ \delta) \operatorname{Ins}_t[n_1, \ldots, n_{s-1}, 2]$ when the operator $D - 2\delta \circ S - 2S \circ \delta$ inserts a 1 into the summand

$$\operatorname{Ins}_{\{j|j\in I, j<i\}\cup\{j-1|j\in I, j>i\}}[n_1, \ldots, n_{s-1}, 2]$$

for any position $i \in I$. The coefficient of the summand $\operatorname{Ins}_I[n_1, \ldots, n_{s-1}]$ in $(D-2\delta \circ S - 2S \circ \delta) \operatorname{Ins}_t[n_1, \ldots, n_{s-1}, 2]$ is

$$\begin{aligned}2(-1)^{st+(s+t)} \sum_{i\in I} (-1)^{i+\sum_{j\in I, j<i} 1 + \sum_{j\in I, j<i} j + \sum_{j\in I, j>i} (j-1)} \\ = 2(-1)^{s(t+1)}(t+1)(-1)^{\sum_{j\in I} j}.\end{aligned}$$

The contributions in the exponent are an st from Ins_t, $(s+t) + i + \sum_{j\in I, j<i} 1$ from $(D - 2\delta \circ S - 2S \circ \delta)$ and $\sum_{j\in I, j<i} j + \sum_{j\in I, j>i} (j-1)$ from $\operatorname{Ins}_{\{j|j\in I, j<i\}\cup\{j-1|j\in I, j>i\}}$. Altogether, this is the coefficient of $\operatorname{Ins}_I[n_1, \ldots, n_{s-1}]$ in

$$2(t+1)(-1)^{t+1} \operatorname{Ins}_{t+1}[n_1, \ldots, n_{s-1}].$$ □

Corollary 4.3. *Let $p > 2$. Define the operator $E = D - 2\delta \circ S - 2S \circ \delta$. Take a monomial $x_1^{c_1} \cdots x_k^{c_k} y_1^{d_1} \cdots y_l^{d_l} y_0$ with size m. Then*

$$E(x_1^{c_1} \cdots x_k^{c_k} y_1^{d_1} \cdots y_l^{d_l}) = 0$$

and

$$E(x_1^{c_1}\cdots x_k^{c_k}y_1^{d_1}\cdots y_l^{d_l}y_0) = 2(-1)^{n-m+1}(n-m+1)x_1^{c_1}\cdots x_k^{c_k}y_1^{d_1}\cdots y_l^{d_l}.$$

Proof. All entries in all partitions of $x_1^{c_1}\cdots x_k^{c_k}y_1^{d_1}\cdots y_l^{d_l}$ are different from 2, so by Lemma 4.1 we have $E(x_1^{c_1}\cdots x_k^{c_k}y_1^{d_1}\cdots y_l^{d_l}) = 0$.

The chain $x_1^{c_1}\cdots x_k^{c_k}y_1^{d_1}\cdots y_l^{d_l}y_0$ can be written as

$$\mathrm{Ins}_{n-m}\,\mathrm{Perm}[2p^{i_1-1}, 2p^{i_1-1}(p-1), \ldots, 2p^{i_k-1}, 2p^{i_k-1}(p-1), 2p^{j_1}, \ldots, 2p^{j_l}, 2]$$

for some indices $i_1,\ldots,i_k,j_1,\ldots,j_l$. By Lemmas 4.1 and 4.2, the operator E cancels all partitions not having a 2 as last entry, otherwise it removes the last entry. So $E(x_1^{c_1}\cdots x_k^{c_k}y_1^{d_1}\cdots y_l^{d_l}y_0)$ is given by

$$2(n+m-1)(-1)^{n+m-1}\,\mathrm{Ins}_{n-m+1}\,\mathrm{Perm}[2p^{i_1-1}, 2p^{i_1-1}(p-1), \ldots, , 2p^{j_1}, \ldots, 2p^{j_l}]. \quad \square$$

A similar proof deals with the case $p = 2$.

Corollary 4.4. *Let $p = 2$. Take $x_1^{c_1}\cdots x_k^{c_k}y_1^{d_1}\cdots y_l^{d_l}y_0$ with size m. Then*

$$E(x_2^{c_2}\cdots x_k^{c_k}y_1^{d_1}\cdots y_l^{d_l}) = 0$$

and if $c_1 > 0$

$$E(x_1^{c_1}\cdots x_k^{c_k}y_1^{d_1}\cdots y_l^{d_l}) = 2(-1)^{n-m+3}(n-m+3)x_1^{c_1-1}\cdots x_k^{c_k}y_1^{d_1}\cdots y_l^{d_l}y_0.$$

Furthermore,

$$E(x_1^{c_1}\cdots x_k^{c_k}y_1^{d_1}\cdots y_l^{d_l}y_0) = 2(-1)^{n-m+1}(n-m+1)x_1^{c_1}\cdots x_k^{c_k}y_1^{d_1}\cdots y_l^{d_l}.$$

This allows us to compute the map $D^*\colon H^i(A_n^\bullet) \to H^{i-1}(A_{n-1}^\bullet)$ with both $\mathbb{Z}$ and $\mathbb{Z}_p$-coefficients.

5. Proof of main theorem

Proof of Theorem 1.1. By Corollary 4.3 we can conclude that the rank of the map

$$D^*\colon H^r(A_n^\bullet, \mathbb{Z}_p) \to H^{r-1}(A_{n-1}^\bullet, \mathbb{Z}_p)$$

is given by the number of monomials

$$x_1^{c_1}\ldots x_k^{c_k}y_0y_1^{d_1}\ldots y_l^{d_l}$$

of degree r and size $m \leqslant n$ such that $p \nmid 2(n-m+1)$. Equivalently, the rank is $B'_p(n,r)$. By the long exact sequence of Lemma 3.2 we have determined

$$\dim H^r(C_n(S^2), \mathbb{Z}_p) = B_p(n,r) + B_p(n-1,r-2) - B'_p(n,r) - B'_p(n,r-1). \quad \square$$

Remark 5.1 (Paolo Salvatore, personal communication). Let

$$Q = \prod_{i>0} \frac{1 + w^{2p^i-1}z^{2p^i}}{1 - w^{2p^i-2}z^{2p^i}}.$$

Then we have for $p > 2$:

$$\sum_{r,n\geqslant 0} \dim H^r(C_n(S^2), \mathbb{Z}_p)\, w^r z^n = \left(\frac{1}{1-z} + \frac{wz^{p+1}}{1-z^p} + \frac{w^3z^3}{1-z} + \frac{w^2z}{1-z^p}\right) Q.$$

Corollary 5.2. *The groups $H^r(C_n(S^2), \mathbb{Z}_p)$ are eventually periodic:*

$$\dim H^r(C_{n+p}(S^2), \mathbb{Z}_p) = \dim H^r(C_n(S^2), \mathbb{Z}_p)$$

if $n \geqslant 2r + 2$.

This is a special case of the general results of [17].

Proof. As $\sum_{i=1}^{g} p^{a_i} + \sum_{j=1}^{h} p^{b_j} \geqslant 2g + h$, we get the inequalities $r \geqslant 2g + h$ and

$$2\sum_{i=1}^{g} p^{a_i} + 2\sum_{j=1}^{h} p^{b_j} \leqslant 2r.$$

Hence we have for $n \geqslant 2r + 2$ that

$$2\sum_{i=1}^{g} p^{a_i} + 2\sum_{j=1}^{h} p^{b_j} + 2 \leqslant n.$$

Hence

$$B_p(n, r) = B_p(n + 1, r), \qquad B'_p(n + p, r) = B'_p(n, r). \qquad \square$$

Proof of Theorem 1.3. For $n \leqslant 3$, we can easily check the theorem by direct computation with $A_n^\bullet$. Take $n \geqslant 4$. We look at the beginning of the long exact sequence of Lemma 3.2. We immediately read off

$$H^0(B_n^\bullet) \simeq H^0(A_n^\bullet) \simeq \mathbb{Z}$$

since both spaces are connected. As $H^2(A_n^\bullet) = H^2(A_{n-1}^\bullet) = 0$ by application of Theorem 2.13, we get the exact sequence

$$0 \to H^1(B_n^\bullet) \to H^1(A_n^\bullet) \xrightarrow{D^*} H^0(A_{n-1}) \to H^2(B_n^\bullet) \to 0.$$

The group $H^1(A_n^\bullet) = \mathbb{Z}$ is generated by the class of y_0 and the group $H^0(A_{n-1}^\bullet) = \mathbb{Z}$ is generated by the class 1 with the map $D^*(y_0) = (2n - 2) \cdot 1$ by Corollary 4.3. Hence we see

$$H^1(B_n^\bullet) = 0, \qquad H^2(B_n^\bullet) = \mathbb{Z}/(2n - 2)\mathbb{Z}.$$

If we had $D = 2\delta \circ S + 2S \circ \delta$, we would have a chain map

$$A_n^\bullet \to B_n^\bullet,\ a \mapsto (a, -2(-1)^{n-r}S(a)),$$

that would split the sequence

$$0 \to A_{n-1}^\bullet[2] \to B_n^\bullet \to A_n^\bullet \to 0,\ a_2 \mapsto (0, a_2),\ (a_1, a_2) \mapsto a_1$$

on the right.

In our case, the long exact sequence of Lemma 3.2 gives us short exact sequences

$$0 \to \operatorname{Coker} D^* \to H^r(B_n^\bullet) \to \operatorname{Ker} D^* \to 0.$$

We want to construct a right splitting $s : \operatorname{Ker} D^* \to H^r(B_n^\bullet)$. For $r \geqslant 2$, the cohomology group $H^r(A_n^\bullet)$ is finite and has no elements of order p^2. For every prime p, we can take a $\mathbb{Z}_p$-basis of the p-torsion in $\operatorname{Ker} D^*$ consisting of the classes $\overline{b_i}$ of the chains

$$b_i = \frac{1}{p}\delta(m_i)$$

such that $\overline{b_i} = \tilde{\beta}(m_i)$ for some monomials $m_i = x_1^{a_1} \dots x_k^{a_k} y_1^{b_1} \dots y_l^{b_l} y_0^{b_0} \in A_n^\bullet$. By Corollaries 4.3 and 4.4, there are integers k_i' and monomials m_i' such that

$$(D - 2S \circ \delta - 2\delta \circ S)(m_i) = k_i' m_i'.$$

As $b_i \in \operatorname{Ker} D^*$, we see that $p|p_i$. Write

$$(D - 2 \circ \delta - 2S\delta \circ S)(m_i) = k_i p m_i'.$$

Define $E = D - 2S \circ \delta - 2\delta \circ S$. Observe that $E \circ \delta = \delta \circ E$. Hence we get

$$E(m_i) = pk_i m_i', \qquad\qquad E(b_i) = k_i \delta(m_i').$$

Define a map

$$s\colon \operatorname{Ker} D^* \to H^r(B_n^\bullet, \mathbb{Z})$$

by setting

$$s(\bar{b_i}) = \left(b_i,\, -2(-1)^{n-r} S(b_i) - (-1)^{n-r} k_i m_i'\right).$$

We see that

$$\begin{aligned}\Delta \circ s(\bar{b_i}) &= \left(\delta(b_i),\, -2(-1)^{n-r}\delta \circ S(b_i) + (-1)^{n-r} D(b_i) - (-1)^{n-r} k_i \delta(m_i')\right) \\ &= \left(\delta(b_i),\, 2(-1)^{n-r} S \circ \delta(b_i) + (-1)^{n-r} E(b_i) - (-1)^{n-r} k_i \delta(m_i')\right) \\ &= 0\end{aligned}$$

and hence $s(\bar{b_i})$ is a cycle in $H^r(B_n^\bullet, \mathbb{Z})$. We have to show that $ps(\bar{b_i})$ is a boundary. We have $pb_i = \delta(m_i)$ and can compute

$$\begin{aligned}ps(\bar{b_i}) &= \left(pb_i,\, -2(-1)^{n-r} S(pb_i) - (-1)^{n-r} pk_i m_i'\right) \\ &= \left(\delta(m_i),\, -2(-1)^{n-r} S \circ \delta(m_i) - (-1)^{n-r} pk_i m_i'\right) \\ &= \left(\delta(m_i),\, (-1)^{n-r}(2\delta \circ S(m_i) - D(m_i) + E(m_i) - pk_i m_i')\right) \\ &= \left(\delta(m_i),\, 2(-1)^{n-r}\delta \circ S(m_i) - (-1)^{n-r} D(m_i)\right) \\ &= \Delta\left(m_i,\, S(m_i)\right).\end{aligned}$$

Hence s is a well-defined right splitting of the sequence

$$0 \to \operatorname{Coker} D^* \to H^r(B_n^\bullet) \to \operatorname{Ker} D^* \to 0.$$

For $r \geqslant 3$, both $\operatorname{Ker} D^*$ and $\operatorname{Coker} D^*$ have no elements of p^2, thus the same is true for $H^r(B_n^\bullet)$. □

Example 5.3. We want to compute the 3-torsion in the groups $H^6(C_9(S^2), \mathbb{Z})$ and $H^6(C_{10}(S^2), \mathbb{Z})$. We use the long exact sequence

$$\cdots \to H^5(A_n^\bullet) \xrightarrow{D^*} H^4(A_{n-1}^\bullet) \to H^6(B_n^\bullet) \to H^6(A_n^\bullet) \xrightarrow{D^*} H^5(A_{n-1}^\bullet) \to \cdots$$

for $n = 9$ and $n = 10$.

For $p = 3$, the generators of $H^*(A_n^\bullet, \mathbb{Z}_3)$ are:

generator	x_1	x_2	y_0	y_1	y_2	$\dots$
degree	4	16	1	5	17	$\dots$
size	6	18	2	6	18	$\dots$

So

$$H^6(A_9^\bullet, \mathbb{Z}_3) = H^6(A_{10}^\bullet, \mathbb{Z}_3) = \mathbb{Z}_3 y_0 y_1.$$

and

$$H^4(A_9^\bullet, \mathbb{Z}_3) = H^4(A_{10}^\bullet, \mathbb{Z}_3) = \mathbb{Z}_3 x_1.$$

We have $D^*(y_0 y_1) = 2(n-7)y_1$ and $D^*(x_1 y_0) = 2(n-7)x_1$. Hence we get

$$H^6(B_9^\bullet, \mathbb{Z}_3) = 0, \qquad H^6(B_{10}^\bullet, \mathbb{Z}_3) = \mathbb{Z}_3^2.$$

The Bockstein $\tilde{\beta}(x_1 y_0) = y_0 y_1$ shows

$$H^6(A_9^\bullet, \mathbb{Z})_{T_3} = H^6(A_{10}^\bullet, \mathbb{Z})_{T_3} = \mathbb{Z}_3 y_0 y_1$$

and

$$H^4(A_9^\bullet, \mathbb{Z})_{T_3} = H^4(A_{10}^\bullet, \mathbb{Z})_{T_3} = 0.$$

We get

$$H^6(B_9^\bullet, \mathbb{Z})_{T_3} = 0, \qquad H^6(B_{10}^\bullet, \mathbb{Z})_{T_3} = \mathbb{Z}_3.$$

Example 5.4. Tables 3 and 4 were computed with the help of the computer algebra systems Sage [**20**] and Magma [**3**]. The integral cohomology groups $H^r(C_n(S^2), \mathbb{Z})$ have already been determined for $n \leqslant 9$ by Sevryuk [**23**] and Napolitano [**18**].

Remark 5.5. The whole argument of this paper is very specifically built for S^2. Similar cell structures exist for other surfaces [**18**]. However, a more conceptual argument might be useful for these, more complex cases.

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Table 3: Cohomology groups $H^i(C_n(\mathbb{C}),\mathbb{Z})$

$n \backslash i$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	$\mathbb{Z}$															
2, 3	$\mathbb{Z}$	$\mathbb{Z}$														
4, 5	$\mathbb{Z}$	$\mathbb{Z}$	0	$\mathbb{Z}_2$												
6, 7	$\mathbb{Z}$	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$										
8, 9	$\mathbb{Z}$	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_6$	$\mathbb{Z}_3$	$\mathbb{Z}_2$								
10, 11	$\mathbb{Z}$	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_6$	$\mathbb{Z}_6$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_5$						
12, 13	$\mathbb{Z}$	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_6$	$\mathbb{Z}_6$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2$	$\mathbb{Z}_2\times\mathbb{Z}_3\times\mathbb{Z}_5$	$\mathbb{Z}_2\times\mathbb{Z}_5$					
14, 15	$\mathbb{Z}$	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_6$	$\mathbb{Z}_6$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2\times\mathbb{Z}_3\times\mathbb{Z}_5$	$\mathbb{Z}_2^2\times\mathbb{Z}_3\times\mathbb{Z}_5$	$\mathbb{Z}_2$	0	$\mathbb{Z}_7$		
16, 17	$\mathbb{Z}$	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_6$	$\mathbb{Z}_6$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2\times\mathbb{Z}_3\times\mathbb{Z}_5$	$\mathbb{Z}_2^2\times\mathbb{Z}_3\times\mathbb{Z}_5$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2$	$\mathbb{Z}_2\times\mathbb{Z}_7$	$\mathbb{Z}_7$	$\mathbb{Z}_2$

Table 4: Cohomology groups $H^i(C_n(S^2),\mathbb{Z})$

$n \backslash i$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	$\mathbb{Z}$	0	$\mathbb{Z}$													
2	$\mathbb{Z}$	0	$\mathbb{Z}_2$													
3	$\mathbb{Z}$	0	$\mathbb{Z}_4$	$\mathbb{Z}$												
4	$\mathbb{Z}$	0	$\mathbb{Z}_6$	$\mathbb{Z}\times\mathbb{Z}_2$												
5	$\mathbb{Z}$	0	$\mathbb{Z}_8$	$\mathbb{Z}\times\mathbb{Z}_2$	0	$\mathbb{Z}_2$										
6	$\mathbb{Z}$	0	$\mathbb{Z}_{10}$	$\mathbb{Z}\times\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2\times\mathbb{Z}_3$										
7	$\mathbb{Z}$	0	$\mathbb{Z}_{12}$	$\mathbb{Z}\times\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2\times\mathbb{Z}_3$	$\mathbb{Z}_2$	$\mathbb{Z}_3$								
8	$\mathbb{Z}$	0	$\mathbb{Z}_{14}$	$\mathbb{Z}\times\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2\times\mathbb{Z}_3$	$\mathbb{Z}_2$	$\mathbb{Z}_2$								
9	$\mathbb{Z}$	0	$\mathbb{Z}_{16}$	$\mathbb{Z}\times\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2\times\mathbb{Z}_3$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_3$	$\mathbb{Z}_2$						
10	$\mathbb{Z}$	0	$\mathbb{Z}_{18}$	$\mathbb{Z}\times\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2\times\mathbb{Z}_3$	$\mathbb{Z}_2^2\times\mathbb{Z}_3$	$\mathbb{Z}_2^2\times\mathbb{Z}_3$	$\mathbb{Z}_2\times\mathbb{Z}_3$	$\mathbb{Z}_2\times\mathbb{Z}_5$						
11	$\mathbb{Z}$	0	$\mathbb{Z}_{20}$	$\mathbb{Z}\times\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2\times\mathbb{Z}_3$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2\times\mathbb{Z}_3$	$\mathbb{Z}_2\times\mathbb{Z}_5$	$\mathbb{Z}_2$	$\mathbb{Z}_5$				
12	$\mathbb{Z}$	0	$\mathbb{Z}_{22}$	$\mathbb{Z}\times\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2\times\mathbb{Z}_3$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^3$	$\mathbb{Z}_2^2\times\mathbb{Z}_3$	$\mathbb{Z}_2^2\times\mathbb{Z}_3\times\mathbb{Z}_5$	$\mathbb{Z}_2^2$	0				
13	$\mathbb{Z}$	0	$\mathbb{Z}_{24}$	$\mathbb{Z}\times\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2\times\mathbb{Z}_3$	$\mathbb{Z}_2^2\times\mathbb{Z}_3$	$\mathbb{Z}_2^3\times\mathbb{Z}_3$	$\mathbb{Z}_2^2\times\mathbb{Z}_3$	$\mathbb{Z}_2^3\times\mathbb{Z}_3\times\mathbb{Z}_5$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2\times\mathbb{Z}_3$	$\mathbb{Z}_2\times\mathbb{Z}_5$			
14	$\mathbb{Z}$	0	$\mathbb{Z}_{26}$	$\mathbb{Z}\times\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2\times\mathbb{Z}_3$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^3$	$\mathbb{Z}_2^3\times\mathbb{Z}_3$	$\mathbb{Z}_2^3\times\mathbb{Z}_3\times\mathbb{Z}_5$	$\mathbb{Z}_2^3$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2\times\mathbb{Z}_5$	$\mathbb{Z}_7$		
15	$\mathbb{Z}$	0	$\mathbb{Z}_{28}$	$\mathbb{Z}\times\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2\times\mathbb{Z}_3$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^3$	$\mathbb{Z}_2^3\times\mathbb{Z}_3$	$\mathbb{Z}_2^3\times\mathbb{Z}_3\times\mathbb{Z}_5$	$\mathbb{Z}_2^4$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2\times\mathbb{Z}_3\times\mathbb{Z}_5$	$\mathbb{Z}_2\times\mathbb{Z}_7$	0	$\mathbb{Z}_7$
16	$\mathbb{Z}$	0	$\mathbb{Z}_{30}$	$\mathbb{Z}\times\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2\times\mathbb{Z}_3$	$\mathbb{Z}_2^2\times\mathbb{Z}_3$	$\mathbb{Z}_2^3\times\mathbb{Z}_3$	$\mathbb{Z}_2^3\times\mathbb{Z}_3$	$\mathbb{Z}_2^4\times\mathbb{Z}_3\times\mathbb{Z}_5$	$\mathbb{Z}_2^4\times\mathbb{Z}_3\times\mathbb{Z}_5$	$\mathbb{Z}_2^3\times\mathbb{Z}_3\times\mathbb{Z}_5$	$\mathbb{Z}_2^3\times\mathbb{Z}_3\times\mathbb{Z}_5$	$\mathbb{Z}_2^2\times\mathbb{Z}_7$	0	$\mathbb{Z}_2$

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