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ON THE MOTIVIC PETERSON CONJECTURE

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Abstract

We show that the analogue of the Peterson conjecture on the action of Steenrod squares does not hold in motivic cohomology.

1. Introduction

The mod 2 cohomology $H^*(BV_n)$ of the classifying space of the elementary abelian 2-group $V_n = (\mathbb{Z}/2)^n$ of rank n is a polynomial algebra of n-variables

$$\mathbb{Z}/2[x_1, \dots, x_n],$$

where $\deg x_i = 1$. We denote it by P_n. Also, we denote by P_n^d the subspace of P_n spanned by monomials of degree d. Then, we have

$$P_n = \bigoplus_{d=0}^{\infty} P_n^d.$$

Throughout this paper, an element means a homogeneous element. The action of the Steenrod squares $\mathrm{Sq}^1, \mathrm{Sq}^2, \mathrm{Sq}^3, \dots$ on P_n is given by the unstable condition

$$\mathrm{Sq}^1(x_i) = x_i^2,$$
$$\mathrm{Sq}^a(x_i) = 0 \quad \text{for } a > 1,$$

and the Cartan formula

$$\mathrm{Sq}^a(x \cdot y) = \sum_{b=0}^{a} \mathrm{Sq}^{a-b}(x) \cdot \mathrm{Sq}^b(y),$$

where $x, y \in P_n$ and Sq^0 is the identity map, that is, $\mathrm{Sq}^0(x) = x$, $\mathrm{Sq}^0(y) = y$.

Finding a minimal set of generators of P_n as a module over the mod 2 Steenrod algebra $\mathcal{A}$ is known as the hit problem. One may consider the quotient space

$$QP_n^d = P_n^d / (\mathcal{A}_+(P_n) \cap P_n^d),$$

where $\mathcal{A}_+$ is the set of positive degree elements in the mod 2 Steenrod algebra $\mathcal{A}$ and

$$\mathcal{A}_+(P_n) = \{ax \mid a \in \mathcal{A}_+,\ x \in P_n\}.$$

Then, the hit problem is the problem of finding a basis for QP_n^d. Since its formulation in mid-1980 by Peterson through the computation of QP_2^d, the hit problem has been

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and is studied by many mathematicians. Among recently published papers and books are Ault [**1**], Pengelley and Williams [**4**], and Sum [**6**] and Walker and Wood [**8**].

The most fundamental result on the hit problem is Wood's theorem. It was known as Peterson's conjecture before Wood proved it. The Peterson conjecture was formulated in terms of the number $\alpha(d+n)$ of 1's in the binary expression of $d+n$. However, we may state it in terms of the function $\beta(d)$ defined by

$$\beta(d) = \min\{\ s \in \mathbb{N} \mid d = (2^{i_1}-1) + \cdots + (2^{i_s}-1),\ i_1, \ldots, i_s \in \mathbb{N}\ \}.$$

Theorem 1.1 (Wood [**9**])**.** *If* $\beta(d) > n$, *then* $\dim QP_n^d = 0$.

Wood's theorem gives us a sufficient condition for $QP_n^d = \{0\}$ in terms of d and n. The condition $\beta(d) > n$ is also a necessary condition for $QP_n^d = \{0\}$ since a monomial of the form

$$x_1^{2^{i_1}-1} \cdots x_n^{2^{i_n}-1}$$

is not in $\mathcal{A}_+(P_n)$. Furthermore, Wood's theorem is the foundation for various results on the hit problem, for example, the computation of $\dim QP_3^d$ and so on. Singer's transfer homomorphism [**5**] relates the hit problem to the cohomology of the mod 2 Steenrod algebra. It is the E_2-term of the classical Adams spectral sequence hence the hit problem is related to the stable homotopy theory. Notably, Minami's new doomsday conjecture [**3**] is inspired by the Peterson conjecture and its consequences.

On the other hand, in the 21st century, motivic cohomology theory is studied in both algebraic geometry and algebraic topology. In particular, a motivic analogue of the Adams spectral sequence and its E_2-term, that is, the cohomology of the mod 2 motivic Steenrod algebra, is studied by Dugger and Isaksen [**2**]. So, it is reasonable to think of motivic counterparts of the hit problem, Singer transfer, new doomsday conjecture and so on.

In this paper, we consider the hit problem in motivic cohomology and an analogue of the Peterson conjecture. To be precise, we disprove the motivic version of the Peterson conjecture. Our result seems to indicate a significant difference between the classical stable homotopy theory and the motivic stable homotopy theory. We hope our result sheds some light on both classical and motivic stable homotopy theory.

For simplicity, we assume that the base field is the complex number field. Then, $H^{*,*}(\mathrm{Spec}(\mathbb{C})) = \mathbb{Z}/2[\tau]$, where $\deg \tau = (0,1)$. We refer the reader to [**7**, Section 9] and [**10**, Section 2] for the details of the mod 2 motivic Steenrod algebra $\mathcal{A}^{*,*}$ and the mod 2 motivic cohomology $H^{*,*}(BV_n)$ of BV_n. The mod 2 motivic cohomology of the classifying space of elementary abelian 2-group V_n of rank n is given by

$$M_n = \mathbb{Z}/2[\tau, x_1, \ldots, x_n, y_1, \ldots, y_n]/(x_1^2 + \tau y_1, \ldots, x_n^2 + \tau y_n),$$

where $\deg \tau = (0,1)$, $\deg x_i = (1,1)$, $\deg y_i = (2,1)$. The mod 2 motivic Steenrod algebra is generated by $Q_0, \wp^1, \wp^2, \wp^3, \ldots$. Its action on the M_n is given by the unstable condition

$$\begin{aligned}
&Q_0(\tau) = 0, && \wp^a(\tau) = 0 \quad \text{for } a \geqslant 1, && \\
&Q_0(x_i) = y_i, && \wp^a(x_i) = 0 \quad \text{for } a \geqslant 1, && \\
&Q_0(y_i) = 0 && \wp^1(y_i) = y_i^2, && \wp^a(y_i) = 0 \quad \text{for } a \geqslant 2,
\end{aligned}$$

and the Cartan formula

$$Q_0(xy) = Q_0(x)y + xQ_0(y),$$
$$\wp^a(xy) = \sum_{b=0}^{a} \wp^{a-b}(x)\wp^b(y) + \tau \sum_{b=0}^{a-1} Q_0\wp^{a-1-b}(x)Q_0\wp^b(y).$$

The mod 2 Steenrod algebra is given as the quotient of the mod 2 motivic Steenrod algebra, that is, $\mathcal{A} = \mathcal{A}^{*,*}/(\tau+1)$. Similarly, we have $M_n/(\tau+1) = P_n$ and the projection

$$M_n \to P_n,$$

sending τ, x_i, y_i to $1, x_i, x_i^2$, respectively. This projection is nothing but the realization map

$$H^{*,*}(BV_n) \to H^*(BV_n).$$

We denote by $\mathcal{A}_+^{*,*}$ the set of elements a in $\mathcal{A}^{*,*}$ such that the sum of the first and second degrees of a is positive and let

$$\mathcal{A}_+^{*,*}(M_n) = \{ax \mid a \in \mathcal{A}_+^{*,*},\ x \in M_n\}.$$

Let $M_n^{d,*}$ be the subspace of M_n spanned by elements of degree $(d,*)$. We define $QM_n^{d,*}$ by

$$QM_n^{d,*} = M_n^{d,*}/(\mathcal{A}_+^{*,*}(M_n) \cap M_n^{d,*}).$$

We call the problem of finding a minimal set of generators of M_n as a module over the mod 2 motivic Steenrod algebra $\mathcal{A}^{*,*}$ the motivic hit problem. It is equivalent to the problem of finding a basis for $QM_n^{d,*}$. A monomial of the form

$$x_1 \cdots x_n y_1^{2^{i_1-1}-1} \cdots y_n^{2^{i_n-1}-1}$$

in $M_n^{d,*}$ is not in $\mathcal{A}_+^{*,*}(M_n)$. So, if $\beta(d) \leqslant n$, $\dim QM_n^{d,*} \neq 0$. Thus, it is reasonable to ask the following conjecture.

Conjecture 1.2. *If $\beta(d) > n$, then* $\dim QM_n^{d,*} = 0$.

This conjecture holds for some n and this is the motivic analogue of the Peterson conjecture. The purpose of this paper is to give counterexamples for this conjecture.

Theorem 1.3. *For n such that $\alpha(n-2) \geqslant 3$, let $k = n-3$, $d = (n-1)(2^{k+1}-2)+k$. Then, $\beta(d) > n$ but* $\dim QM_n^{d,*} \neq 0$.

For $n = 9$, the assumption of Theorem 1.3 holds. For $n = 9$, we have $k = 6$, $d = 1014$ and $\beta(1014) = 10 > 9$. We prove Theorem 1.3 by giving a family of monomials not in $\mathcal{A}_+^{*,*}(M_n)$. Let us denote by z_k the monomial

$$x_1 \cdots x_k y_1^{2^{k-1}-1} y_2^{2^k-2^{k-2}-1} \cdots y_k^{2^k-2^0-1} y_{k+1}^{2^k-1} \cdots y_n^{2^k-1}$$

of degree $d = (n-1)(2^{k+1}-2)+k$. Then, Theorem 1.3 could be divided to the following Propositions 1.4 and 1.5.

Proposition 1.4. *Suppose that $1 \leqslant k < n$. Then, the monomial z_k is not in $\mathcal{A}_+^{*,*}(M_n)$.*

Proposition 1.5. *Suppose that $\alpha(n-2) \geqslant 3$ and $k = n-3$. Then, we have $\beta(d) > n$.*

This paper is organized as follows. In Section 2, we prove Proposition 1.5. In Section 3, we recall some results on the classical hit problem. In Section 4, we give the details of the motivic hit problem. In Section 5, we prove Proposition 1.4.

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2. Proof of Proposition 1.5

In this section, we prove Proposition 1.5. First, we prove Proposition 2.1 below.

Proposition 2.1. *Let d, n be positive integers. Then, $\alpha(d+n) > n$ if and only if $\beta(d) > n$, where $\alpha(d+n)$ is the number of 1's in the binary expansion of $d+n$.*

Proof. We prove that $\alpha(d+n) \leqslant n$ if and only if $\beta(d) \leqslant n$. Recall that

$$\alpha(t) = \min\{\ s \mid t = 2^{i_1} + \cdots + 2^{i_s},\ i_1 \geqslant i_2 \geqslant \cdots \geqslant i_s \geqslant 0\ \}.$$

Therefore, we have $\alpha(s) \leqslant s$.

Suppose that $\beta(d) \leqslant n$. Then, we have

$$d = (2^{i_1} - 1) + \cdots + (2^{i_n} - 1),$$

for some $i_1 \geqslant \cdots \geqslant i_n \geqslant 0$. Hence, we have

$$d + n = 2^{i_1} + \cdots + 2^{i_n}.$$

Therefore, we have $\alpha(d+n) \leqslant n$.

Suppose that $\alpha(d+n) \leqslant n$. Let s be the least positive integer such that

$$\alpha(d+s) \leqslant s.$$

By the definition of s, we have $s \leqslant n$. Let $r = \alpha(d+s)$. Then, again, by the definition of s, $r \leqslant s$ and

$$d + s = 2^{i_1} + \cdots + 2^{i_r},$$

for some $i_1 > \cdots > i_r \geqslant 0$. So,

$$d + s - 1 = 2^{i_1} + \cdots + 2^{i_r} - 1 = 2^{i_1} + \cdots + 2^{i_{r-1}} + 2^{i_r - 1} + \cdots + 2^0$$

and by the definition of s, we have $\alpha(d+s-1) = r - 1 + i_r > s - 1$. Therefore, we have $i_r > s - r \geqslant 0$. Let $j_a = i_a$ for $a \leqslant r-1$, $j_a = i_r - 1 - a + r$ for $r \leqslant a \leqslant s-1$ and $j_a = i_r - s + r$ for $a = s$. Then,

$$d + s = 2^{j_1} + \cdots + 2^{j_s}.$$

Hence, we have

$$d = (2^{j_1} - 1) + \cdots + (2^{j_s} - 1).$$

Therefore, we have $\beta(d) \leqslant s \leqslant n$. □

Remark 2.2. Suppose that $d = (n-1)(2^{k+1}-2)+k$. Then, we have

$$d + n = (n-1)\cdot 2^{k+1} + 2 - n + k.$$

For $k = n-1$, we have

$$d + n = (n-1)\cdot 2^{n} + 1.$$

Hence, we have

$$\alpha(d+n) = \alpha(n-1) + 1 \leqslant n.$$

Similarly, for $k = n-2$, we have

$$d + n = (n-1)\cdot 2^{n-1}$$

and

$$\alpha(d+n) = \alpha(n-1) \leqslant n.$$

Therefore, $\beta(d) \leqslant n$ for $k = n-1, n-2$.

Now, we prove Proposition 1.5.

Proof of Proposition 1.5. Let $d = (n-1)(2^{k+1}-2)+k$ and $k = n-3$. Then,

$$d + n = (n-1)\cdot 2^{n-2} - 1 = (n-2)\cdot 2^{n-2} + 2^{n-3} + \cdots + 2^{0}.$$

Hence, we have

$$\alpha(d+n) = \alpha(n-2) + n - 2.$$

Since we assumed that $\alpha(n-2) \geqslant 3$, we have

$$\alpha(d+n) > n.$$

Thus, by Proposition 2.1, we have $\beta(d) > n$. □

3. The classical hit problem

In this section, we recall some results on the classical hit problem.

For a monomial

$$v = x_1^{e_1}\cdots x_n^{e_n}$$

in P_n, let us define $\alpha_{ij}(v)$ in $\{0,1\}$ by

$$e_i = \sum_{j=0}^{\infty} \alpha_{ij}(v) 2^{j}.$$

We define non-negative integers $\alpha_i(v), \omega_j(v)$ by

$$\alpha_i(v) = \sum_{j=0}^{\infty} \alpha_{ij}(v),$$

$$\omega_j(v) = \sum_{i=1}^{n} \alpha_{ij}(v).$$

For finite sequences of non-negative integers of the same length c, say

$$\gamma = (\gamma_1, \dots, \gamma_c), \quad \text{and} \quad \delta = (\delta_1, \dots, \delta_c),$$

we consider the lexicographic order from the left, that is, we say

$$\gamma < \delta$$

if and only if there exists $a \geqslant 1$ such that $\gamma_b = \delta_b$ for $b < a$ and $\gamma_a < \delta_a$.

Let F_n be the subspace of P_n spanned by $\mathcal{A}_+(P_n)$ and monomials v in P_n such that

$$(\omega_0(v), \dots, \omega_{k-1}(v)) < (n-1, \dots, n-1).$$

We denote by F_{n-1} the subspace of P_{n-1} spanned by $\mathcal{A}_+(P_{n-1})$ and monomials v in P_{n-1} such that

$$(\omega_0(v), \dots, \omega_{k-1}(v)) < (n-1, \dots, n-1).$$

It is clear that F_n, F_{n-1} are closed under the action of Steenrod squares since, for each monomial v in P_n^d,

$$\mathrm{Sq}^a v$$

is a linear combination of monomials w such that

$$(\omega_0(w), \omega_1(w), \dots) < (\omega_0(v), \omega_1(v), \dots).$$

We denote the projections by the same symbol $\pi\colon P_n \to P_n/F_n$, $\pi\colon P_{n-1} \to P_{n-1}/F_{n-1}$.

Throughout the rest of this section, let k, n be fixed positive integers and $d_1 = (n-1)(2^k-1)$. We consider the quotient spaces

$$(P_n/F_n)^{d_1} = P_n^{d_1}/(F_n \cap P_n^{d_1})$$

and

$$(P_{n-1}/F_{n-1})^{d_1} = P_{n-1}^{d_1}/(F_{n-1} \cap P_{n-1}^{d_1}).$$

A monomial v in $P_{n-1}^{d_1}$ is $x_1^{2^{k-1}} \cdots x_{n-1}^{2^k-1}$ or $\omega(v) < (n-1, \dots, n-1)$, that is, $v \in F_{n-1}$. So, it is clear that $(P_{n-1}/F_{n-1})^{d_1} = \mathbb{Z}/2$ and spanned by the single element

$$x_1^{2^k-1} \cdots x_{n-1}^{2^k-1} = \pi(x_1^{2^k-1} \cdots x_{n-1}^{2^k-1}).$$

To describe a basis for the vector space $(P_n/F_n)^{d_1}$, we need the following definitions. For $1 \leqslant \ell \leqslant n$, let $\mathrm{Mono}(\ell)$ be the set of monotone increasing functions

$$\{1, \dots, \ell\} \to \{1, \dots, n\}.$$

We identify $\sigma \in \mathrm{Mono}(\ell)$ with the permutation σ on $\{1, \dots, n\}$ such that

$$\sigma(1) < \cdots < \sigma(\ell), \quad \sigma(\ell+1) < \cdots < \sigma(n).$$

Then, the permutation σ of $\{1, \dots, n\}$ acts on P_n in the obvious manner, that is,

$$\sigma(x_1^{e_1} \cdots x_n^{e_n}) = x_{\sigma(1)}^{e_1} \cdots x_{\sigma(n)}^{e_n}.$$

For an integer ℓ such that $1 \leqslant \ell \leqslant \min\{k, n\}$, let us define the monomial v_ℓ in $P_n^{d_1}$ by

$$v_\ell = x_1^{2^{\ell-1}-1} x_2^{2^k-2^{\ell-2}-1} \cdots x_\ell^{2^k-2^0-1} x_{\ell+1}^{2^k-1} \cdots x_n^{2^k-1}.$$

First, we prove that the set of monomials

$$\{\pi(\sigma(v_\ell)) \mid 1 \leqslant \ell \leqslant \min\{k, n\}, \sigma \in \mathrm{Mono}(\ell)\}$$

spans the vectors space $(P_n/F_n)^{d_1}$. To this end, we prove the following Propositions 3.1 and 3.2.

Proposition 3.1. *Let v be a monomial in $P_n^{d_1}$. Then,*

$$(\omega_0(v), \dots, \omega_{k-1}(v)) \leqslant (n-1, \dots, n-1).$$

Proof. Proof by contradiction. Suppose that

$$(\omega_0(v), \dots, \omega_{\ell-1}(v)) = (n-1, \dots, n-1)$$

and

$$\omega_\ell(v) = n,$$

for $1 \leqslant \ell < k$. Then, on the one hand, since d_1 could be written as

$$\sum_{j=0}^{\infty} \omega_j(v) 2^{j-1} = (n-1)(2^\ell - 1) + n2^\ell + \sum_{j=\ell+1}^{\infty} \omega_j(v) 2^{j-1},$$

we have

$$d_1 - (n-1)(2^{\ell+1} - 1) = 2^\ell + \omega_{\ell+1}(v) 2^{\ell+1} + \cdots .$$

It is not divisible by $2^{\ell+1}$. On the other hand, since $d_1 = (n-1)(2^k - 1)$, we have

$$d_1 - (n-1)(2^{\ell+1} - 1) = (n-1)(2^k - 2^{\ell+1}).$$

It is divisible by $2^{\ell+1}$. It is a contradiction. □

Proposition 3.2. *For a monomial v in $P_n^{d_1}$ such that*

$$(\omega_0(v), \dots, \omega_{k-1}(v)) = (n-1, \dots, n-1),$$

there exists a unique pair (ℓ, σ) such that $1 \leqslant \ell \leqslant \min\{k, n\}$, $\sigma \in \mathrm{Mono}(\ell)$ and

$$v \equiv \sigma(v_\ell) \mod F_n.$$

Now, we prove Proposition 3.2. For each monomial v in P_n, let

$$u_j(v) = (\alpha_{1j}(v), \dots, \alpha_{nj}(v)).$$

Lemma 3.3. *Let v be a monomial such that*

$$(\omega_0(v), \dots, \omega_{k-1}(v)) = (n-1, \dots, n-1).$$

Suppose that $u_j(v) < u_{j+1}(v)$. Let v' be the unique monomial such that $u_a(v) = u_a(v')$ for $a \neq j, j+1$ and $u_j(v') = u_{j+1}(v)$, $u_{j+1}(v') = u_j(v)$. Then,

$$v \equiv v' \mod F_n.$$

Proof. Let w be the monomial such that

$$u_j(w) = (1, \ldots, 1),$$

$$u_{j+1}(w) = (\alpha_{1,j}(v)\alpha_{1,j+1}(v), \ldots, \alpha_{n,j}(v)\alpha_{n,j+1}(v)),$$

and

$$u_a(w) = u_a(v),$$

for $a \neq j, j+1$.

Let w_0, w_1 be monomials such that

$$(\omega_0(w_0), \ldots, \omega_{j-1}(w_0), \omega_j(w_0), \omega_{j+1}(w_0), \ldots) = (\omega_0(w), \ldots, \omega_{j-1}(w), 0, 0, \ldots)$$

and

$$(\omega_0(w_1), \omega_1(w_1), \ldots) = (\omega_j(w), \omega_{j+1}(w), \ldots).$$

Then, we have

$$w = w_0(w_1)^{2^j}$$

and, by the Cartan formula, we have

$$\mathrm{Sq}^{2^j}(w) = \sum_{a+b=2^j} \mathrm{Sq}^a(w_0)\mathrm{Sq}^b(w_1^{2^j}).$$

Furthermore, by the Cartan formula, for $0 < b < 2^j$, we have

$$\mathrm{Sq}^b(w_1^{2^j}) = 0$$

and for $b = 2^j$, we have

$$\mathrm{Sq}^b(w_1^{2^j}) = (\mathrm{Sq}^1(w_1))^{2^j}.$$

So, we have

$$\mathrm{Sq}^{2^j}(w) = w_0(\mathrm{Sq}^1(w_1))^{2^j} + (\mathrm{Sq}^{2^j}(w_0))w_1^{2^j}.$$

For $a > 0$ and a monomial w', $\mathrm{Sq}^a(w')$ is a linear combination of monomials w'' such that

$$(\omega_0(w''), \omega_1(w''), \ldots) < (\omega_0)w'), \omega_1(w'), \ldots).$$

Hence, we have

$$\mathrm{Sq}^{2^j}(w) \equiv w_0(\mathrm{Sq}^1(w_1))^{2^j} \equiv v + v' \mod F'_n,$$

where F'_n is the subspace spanned by monomials v'' such that

$$(\omega_0(v''), \ldots, \omega_{k-1}(v'')) < (n-1, \ldots, n-1).$$

Therefore, we have the desired result. □

Thus, for each monomial v in $P_n^{d_1}$ such that

$$(\omega_0(v), \dots, \omega_{k-1}(v)) = (n-1, \dots, n-1),$$

there exists a monomial v' such that

$$(\omega_0(v'), \dots, \omega_{k-1}(v')) = (n-1, \dots, n-1),$$

$$u_0(v') \geqslant u_1(v') \cdots \geqslant u_{k-1}(v')$$

and

$$v \equiv v' \mod F_n.$$

Lemma 3.4. *Let v be a monomial such that*

$$(\omega_0(v), \dots, \omega_{k-1}(v)) = (n-1, \dots, n-1).$$

Suppose that

$$u_j(v) = u_{j+1}(v) > u_{j+2}(v).$$

Let v' be the unique monomial such that $u_a(v) = u_a(v)$ for $a \neq j, j+1, j+2$ and $u_j(v') > u_{j+1}(v') = u_{j+2}(v')$, $u_j(v') = u_j(v)$, $u_{j+2}(v') = u_{j+2}(v)$. Then, $v \equiv v'$ modulo F_n.

Proof. Let w be the unique monomial such that

$$u_j(w) = u_{j+1}(w) = (1, \dots, 1),$$

$$u_{j+2}(w) = (\alpha_{1,j}(v)\alpha_{1,j+2}(v), \dots, \alpha_{n,j}(v)\alpha_{n,j+2}(v)),$$

and

$$u_a(w) = u_a(v),$$

for $a \neq j, j+1, j+2$. Let v'' be the unique monomial such that

$$u_j(v'') = u_{j+1}(v'') < u_{j+2}(v''),$$

$$u_j(v'') = u_{j+2}(v), u_{j+2}(v'') = u_j(v),$$

and

$$u_a(v'') = u_a(v),$$

for $a \neq j, j+1, j+2$. Then, as in the proof of Lemma 3.3, we have

$$\mathrm{Sq}^{2^j} w \equiv v + v'' \mod F'_n,$$

where F'_n is the subspace used in the proof of Lemma 3.3. Hence, we have

$$v \equiv v'' \mod F_n.$$

By applying Lemma 3.3 repeatedly, we have

$$v'' \equiv v' \mod F_n. \qquad \square$$

Thus, by Lemmas 3.3 and 3.4, for each monomial v in $P_n^{d_1}$ such that

$$(\omega_0(v), \dots, \omega_{k-1}(v)) = (n-1, \dots, n-1),$$

there exists a monomial v' such that

$$(\omega_0(v'), \dots, \omega_{k-1}(v')) = (n-1, \dots, n-1),$$

$$u_0(v') > u_1(v') \cdots > u_\ell(v') = \cdots = u_{k-1}(v')$$

and

$$v \equiv v' \mod F_n.$$

In other words, there exist $1 \leqslant \ell \leqslant \min\{k, n\}$ and $\sigma \in \mathrm{Mono}(\ell)$ such that

$$v \equiv \sigma(v_\ell) \mod F_n,$$

where $\alpha_i(v) < k$ for $i \in \{\sigma(1), \dots, \sigma(\ell)\}$ and $\alpha_i(v) = k$ for $i \notin \{\sigma(1), \dots, \sigma(\ell)\}$.

Next, we prove that

$$\{\pi(\sigma(v_\ell)) \mid 1 \leqslant \ell \leqslant \min\{k, n\}, \sigma \in \mathrm{Mono}(\ell)\}$$

is linearly independent. Let $\lambda(\sigma)\colon P_n \to P_{n-1}$ be a ring homomorphism defined by

$$\begin{aligned} &\lambda(\sigma)(x_i) = x_i && \text{for } i < \sigma(\ell),\\ &\lambda(\sigma)(x_i) = \sigma(x_1) + \cdots + \sigma(x_{\ell-1}) && \text{for } i = \sigma(\ell),\\ &\lambda(\sigma)(x_i) = x_{i-1} && \text{for } i > \sigma(\ell). \end{aligned}$$

Let us write

$$\tilde{u}_j(v) = x_1^{\alpha_{1j}(v)} \cdots x_n^{\alpha_{nj}(v)}.$$

Then, we have

$$v = \prod_{j=0}^{\infty} (\tilde{u}_j(v))^{2^j}.$$

It is clear that

$$\lambda(\sigma)(v) = \prod_{j=0}^{\infty} (\lambda(\sigma)(\tilde{u}_j(v)))^{2^j}$$

and

$$\lambda(\sigma)(x_1 \cdots \widehat{x}_a \cdots x_n) = \begin{cases} x_1 \cdots x_{n-1} + \sum v' & \text{if } a \in \{\sigma(1), \dots, \sigma(\ell)\},\\ \sum v' & \text{otherwise,} \end{cases}$$

where $x_1 \cdots \widehat{x}_a \cdots x_n$ is the monomial of degree $n-1$ obtained from $x_1 \cdots x_n$ by removing x_a and $\sum v'$ indicates a linear combination of monomials v' such that $\omega_0(v') < n-1$. Therefore, it is easy to see that the following Lemma 3.5 holds.

Lemma 3.5. *Suppose* $1 \leqslant \ell, m \leqslant \min\{k, n\}$, $\sigma \in \mathrm{Mono}(m)$ *and* $\tau \in \mathrm{Mono}(\ell)$. *Then,*

$$\lambda(\sigma)(\tau(v_\ell)) \equiv x_1^{2^k-1} \cdots x_{n-1}^{2^k-1} \not\equiv 0 \mod F_n$$

if and only if

$$\{\sigma(1), \dots, \sigma(m)\} \supseteq \{\tau(1), \dots, \tau(\ell)\},$$

where v_ℓ *is the monomial* v_ℓ *in Proposition 3.2.*

It follows from Lemma 3.5 that the linear map

$$\lambda\colon (P_n/F_n)^{d_1} \to \prod_{1\leqslant \ell\leqslant \min\{k,n\}} \left(\prod_{\sigma\in \mathrm{Mono}(\ell)} (P_{n-1}/F_{n-1})^{d_1} \right)$$

sending $\pi(v)$ to $(\pi(\lambda(\sigma)(v)))$ is an isomorphism. Thus,

$$\{\pi(\sigma(v_\ell)) \mid 1 \leqslant \ell \leqslant \min\{k,n\}, \sigma \in \mathrm{Mono}(\ell)\}$$

is a basis for $(P_n/F_n)^{d_1}$.

4. The motivic hit problem

In this section, we give the details of motivic hit problem. Let

$$N_n = M_n/(\tau) = \Lambda_n(x_1,\ldots,x_n) \otimes \mathbb{Z}/2[y_1,\ldots,y_n]$$

and

$$\mathcal{A}' = \mathcal{A}^{*,*}/(\tau).$$

Then, N_n is an $\mathcal{A}'$-module. Let $N_n^{d,*}$ be the subspace of N_n spanned by elements of degree $(d,*)$. From now on, for the sake of simplicity, we say an element is of degree d if its degree is $(d,*)$. Let $\mathcal{A}'_+$ be the subset of $\mathcal{A}'$ consisting of positive degree elements in $\mathcal{A}'$ and

$$\mathcal{A}'_+(N_n) = \{ax \mid a \in \mathcal{A}'_+,\ x \in N_n\}.$$

Then, it is easy to see that $QM_n^{d,*}$ in Section 1 is isomorphic to

$$N_n^{d,*}/(\mathcal{A}'_+(N_n) \cap N_n^{d,*}).$$

We consider the counterpart of F_n in N_n. For the sake of notational simplicity, we write Λ_n, Y_n for $\Lambda_n(x_1,\ldots,x_n)$, $\mathbb{Z}/2[y_1,\ldots,y_n]$, respectively. We denote by Λ_n^a, Y_n^{2b} the subspaces of Λ_n, Y_n spanned by elements of degree $a, 2b$, respectively. For a monomial

$$z = x_1^{\varepsilon_1}\cdots x_n^{\varepsilon_n} y_1^{e_1}\cdots y_n^{e_n}$$

in $N_n^{d,*}$, let us define $\alpha_{ij}(z) \in \{0,1\}$ by

$$\varepsilon_i + 2e_i = \sum_{j=0}^{\infty} \alpha_{ij}(z) 2^j.$$

We define non-negative integers $\alpha_i(z), \omega_j(z)$ by

$$\alpha_i(z) = \sum_{j=0}^{\infty} \alpha_{ij}(z),$$

$$\omega_j(z) = \sum_{i=1}^{n} \alpha_{ij}(z),$$

respectively. Let $\mathcal{P}$ be the subalgebra of $\mathcal{A}'$ generated by reduced power operations $\wp^1, \wp^2, \wp^3, \ldots$ of degree $2, 4, 6, \ldots$. Let $\mathcal{P}_+$ be the subset of $\mathcal{P}$ consisting of positive degree elements in $\mathcal{P}$. Let G_n be the subspace spanned by

$$\mathcal{P}_+(Y_n) = \{ax \mid a \in \mathcal{P}_+,\ x \in Y_n\}$$

and monomials z such that

$$(\omega_1(z), \dots, \omega_k(z)) < (n-1, \dots, n-1).$$

Then, $G_n \subset Y_n$ is the counterpart of F_n in Y_n. The ring isomorphism $\psi\colon Y_n \to P_n$ sending y_i to x_i commutes with the action of $\mathcal{P}$, $\mathcal{A}$ in the sense that $\psi(\wp^c y) = \mathrm{Sq}^c \psi(y)$. Thus, it induces an isomorphism

$$\psi\colon (Y_n/G_n)^{2b} \to (P_n/F_n)^b.$$

We use this isomorphism to identify $(Y_n/G_n)^{2b}$ with $(P_n/F_n)^b$, so that we can apply results on $(P_n/F_n)^{d_1}$ in Section 3 for $(Y_n/G_n)^{2d_1}$.

We denote the projection from $\Lambda_n^a \otimes Y_n^{2b}$ to $\Lambda_n^a \otimes (Y_n/G_n)^{2b}$ by

$$\pi\colon \Lambda_n^a \otimes Y_n^{2b} \to \Lambda_n^a \otimes (Y_n/G_n)^{2b}.$$

Let H_n be the subspace of N_n spanned by

$$\mathcal{A}'_+(N_n)$$

and monomials z such that

$$(\omega_0(z), \omega_1(z), \dots, \omega_k(z)) < (k, n-1, \dots, n-1).$$

Then, $H_n \subset N_n$ is the counterpart of $F_n \subset P_n$ in N_n. Since Q_0 maps $\Lambda_n^{a+1} \otimes Y_n^{2(b-1)}$ to $\Lambda_n^a \otimes Y_n^{2b}$ and, for $c > 0$, $\wp^c$ acts trivially on Λ_n, we have the following direct sum decomposition

$$(N_n/H_n)^{d,*} = \bigoplus_{a+2b=d} \Lambda_n^a \otimes (Y_n/G_n)^{2b} / \pi(Q_0(\Lambda_n^{a+1} \otimes Y_n^{2(b-1)})).$$

We prove the following proposition using Propositions 3.1 and 3.2.

Proposition 4.1. *Suppose that $d = k + 2d_1$, $d_1 = (n-1)(2^k - 1)$, $1 \leqslant k < n$. For each monomial z in*

$$\Lambda_n^k \otimes Y_n^{2d_1},$$

$z \in \Lambda_n^k \otimes G_n^{2d_1}$ or there exist unique ℓ in $\{1, \dots, k\}$, $\sigma_1 \in \mathrm{Mono}(k)$ and $\sigma_2 \in \mathrm{Mono}(\ell)$ such that

$$z \equiv \sigma_1(x_1 \cdots x_k)\sigma_2(\psi^{-1}(v_\ell)) \mod \Lambda_n^k \otimes G_n^{2d_1},$$

where v_ℓ is the monomial $v_\ell \in P_n^{d_1}$ in Proposition 3.2.

Proof. Suppose that $z = x_{i_1} \cdots x_{i_k} \otimes v$ and that z is not in $\Lambda^k \otimes G_n^{2d_1}$. Then by Proposition 3.1, we have

$$(\omega_0(\psi(v)), \dots, \omega_{k-1}(\psi(v))) = (n-1, \dots, n-1).$$

So, by Proposition 3.2, there exists the unique pair (ℓ, σ_2) such that

$$\psi(v) \equiv \sigma_2(v_\ell) \mod F_n$$

in $P_n^{d_1}$. Let us define $\sigma_1 \in \mathrm{Mono}(k)$ by $\sigma_1(j) = i_j$. Then, we have

$$z \equiv \sigma_1(x_1 \cdots x_k)\sigma_2(\psi^{-1}(v_\ell)) \mod \Lambda_n^k \otimes G_n^{2d_1},$$

as desired. □

Let $\mathcal{M}_0$ be the set of monomials $z = \sigma_1(x_1 \cdots x_k)\sigma_2(\psi^{-1}(v_\ell))$ in $\Lambda_n^k \otimes Y_n^{2d_1}$ such that $\alpha_i(z) < k$ for some i and $\mathcal{M}_1$ the set of monomials $z = \sigma_1(x_1 \cdots x_k)\sigma_2(\psi^{-1}(v_\ell))$ in $\Lambda_n^k \otimes Y_n^{2d_1}$ such that $\alpha_i(z) = k$ for all i. In Section 3, we proved that

$$\{\pi(\sigma_2(v_\ell)) \mid 1 \leqslant \ell \leqslant k, \sigma_2 \in \mathrm{Mono}(\ell)\}$$

is a basis for $(P_n/F_n)^{d_1}$, Since

$$\{\sigma_1(x_1 \cdots x_k) \mid \sigma_1 \in \mathrm{Mono}(k)\}$$

is a basis for Λ_n^k, $\pi(\mathcal{M}_0 \cup \mathcal{M}_1)$ is a basis for $\Lambda_n^k \otimes (Y_n/G_n)^{2d_1}$. If $\ell \neq k$, then $\alpha_i(z) < k$ for some i in $\{1, \ldots, n\}$. If $\ell = k$ and $\sigma_1 \neq \sigma_2$, then $\alpha_i(z) < k$ for some i in $\{1, \ldots, n\}$. If $\ell = k$ and $\sigma_1 = \sigma_2$, then

$$v_k = x_1^{2^k - 2^{k-1} - 1} \cdots x_k^{2^k - 2^0 - 1} x_{k+1}^{2^k - 1} \cdots x_n^{2^k - 1}$$

and so

$$x_1 \cdots x_k \psi^{-1}(v_k) = x_1 \cdots x_k y_1^{2^k - 2^{k-1} - 1} \cdots y_k^{2^k - 2^0 - 1} y_{k+1}^{2^k - 1} \cdots y_n^{2^k - 1}$$

is the monomial z_k in Proposition 1.4. Therefore, we have

$$\begin{aligned}\mathcal{M}_0 = &\{\sigma_1(x_1 \cdots x_k)\sigma_2(\psi^{-1}(v_\ell)) \mid 1 \leqslant \ell < k, \sigma_1 \in \mathrm{Mono}(k), \sigma_2 \in \mathrm{Mono}(\ell)\} \\ &\cup \{\sigma_1(x_1 \cdots x_k)\sigma_2(\psi^{-1}(v_k)) \mid \sigma_1, \sigma_2 \in \mathrm{Mono}(k), \sigma_1 \neq \sigma_2\}, \\ \mathcal{M}_1 = &\{\sigma_1(x_1 \cdots x_k)\sigma_2(\psi^{-1}(v_k)) \mid \sigma_1, \sigma_2 \in \mathrm{Mono}(k), \sigma_1 = \sigma_2\} \\ = &\{\sigma(z_k) \mid \sigma \in \mathrm{Mono}(k)\}.\end{aligned}$$

5. Proof of Proposition 1.4

Throughout this section, we suppose that $d = k + 2d_1$, $d_1 = (n-1)(2^k - 1)$, $1 \leqslant k < n$. The algebra $\mathcal{A}'$ is generated by Q_0 and the reduced power operations $\wp^1, \wp^2, \ldots$. By definition, we have $\mathcal{P}_+(Y_n) \subset G_n$. Moreover, $\wp^a(x) = 0$ in N_n for $a > 0$, $x \in \Lambda_n$. Therefore, we have

$$\pi(\wp^a(x \otimes y)) = \pi(x \otimes \wp^a(y)) = 0,$$

for each monomial $x \otimes y$ in $N_n^{d-2a,*}$. So, we prove Proposition 1.4 by proving the following proposition.

Proposition 5.1. *For each monomial z in N_n^{d-1} with $\omega_0(z) = k + 1$,*

$$\pi(Q_0(z)) \in \Lambda_n^k \otimes (Y_n/G_n)^{2d_1}$$

is a linear combination of $\pi(z')$ $(z' \in \mathcal{M}_0)$ and $\pi(\sigma_1(z_k) + \sigma_2(z_k))$, where $\sigma_1, \sigma_2 \in$ $\mathrm{Mono}(k)$ and z_k is the monomial z_k in Proposition 1.4.

From Propositions 4.1 and 5.1, we have that $\pi(z_k)$ is not in

$$\begin{aligned}\pi(Q_0(\Lambda_n^{k+1} \otimes Y_n^{2(d_1-1)})) &= \pi(Q_0(\Lambda_n^{k+1} \otimes Y_n^{2(d_1-1)})) + \sum_{a>0} \pi(\wp^a(\Lambda_n^k \otimes Y_n^{2(d_1-a)}) \\ &= \pi(\mathcal{A}'_+(N_n)).\end{aligned}$$

Thus, once we prove Proposition 5.1, we complete the proof of Proposition 1.4.

Let z be a monomial in $\Lambda_n^{k+1} \otimes Y_n^{2(d_1-1)}$.

If $\omega_1(z) = n-1$, then

$$\begin{aligned}\omega_2(z)\cdot 2^2 + \omega_3(z)\cdot 2^3 + \cdots &= d-1-(k+1)-2(n-1)\\ &= 2d_1 - 2n\\ &= 2(n-1)(2^k-1)-2n\\ &= (n-1)2^{k+1}+2.\end{aligned}$$

Since $k \geqslant 1$, $(n-1)2^{k+1}+2$ is not divisible by 4 but $\omega_2(z)\cdot 2^2 + \omega_3(z)\cdot 2^3 + \cdots$ is divisible by 4. It is a contradiction. So, $\omega_1(z) \neq n-1$.

If $\omega_1(z) < n-2$, then $Q_0(z)$ is a linear combination of monomials z' such that $\omega_0(z') = k$ and $\omega_1(z') < n-1$. Hence, $Q_0(z) \in \Lambda_n^k \otimes G_n^{2d_1}$.

There remain two cases: $\omega_1(z) = n$ or $n-2$. First, we deal with the case $\omega_1(z) = n$.

Proposition 5.2. *Suppose that z is a monomial in N_n^{d-1} such that*

$$(\omega_0(z), \omega_1(z)) = (k+1, n).$$

Then, there exist monomials z' such that

$$(\omega_0(z'), \omega_1(z')) = (k+1, n-2)$$

and

$$Q_0(z) = Q_0(\sum z').$$

Proof. Without loss of generality, we may assume that

$$\begin{aligned}u_0(z) &= (1, \ldots, 1, 0, 0, \ldots, 0),\\ u_1(z) &= (1, \ldots, 1, 1, 1, \ldots, 1).\end{aligned}$$

Let z'' be a monomial such that

$$\begin{aligned}u_0(z'') &= (1, \ldots, 1, 1, 0, \ldots, 0),\\ u_1(z'') &= (1, \ldots, 1, 0, 1, \ldots, 1)\end{aligned}$$

and $u_a(z'') = u_a(z)$ for $a \geqslant 2$. Then,

$$Q_0(z'') = z + \sum z',$$

where $\sum z'$ is a linear combination of monomials z' such that $(\omega_0(z'), \omega_1(z')) = (k+1, n-2)$. Since $Q_0Q_0 = 0$, we have that

$$Q_0(z) = Q_0(\sum z')$$

as desired. □

So, Proposition 5.3 below completes the proof of Proposition 5.1.

Proposition 5.3. *Suppose that z is a monomial in N_n^{d-1} such that*

$$(\omega_0(z), \omega_1(z)) = (k+1, n-2).$$

If $\alpha_i(z) < k$ for some i, then $Q_0(z)$ is congruent to a linear combination of monomials in $\mathcal{M}_0$ modulo $\Lambda_n^k \otimes G_n^{2d_1}$. If $\alpha_i(z) = k$ for all $i \in \{1, \ldots, n\}$, then

$$Q_0(z) \equiv \sigma_1(z_k) + \sigma_2(z_k) \mod \Lambda_n^k \otimes G_n^{2d_1}.$$

Proof. In what follows, we consider everything modulo $\Lambda_n^k \otimes G_n^{2d_1}$. The element $Q_0(z)$

is congruent to a linear combination of monomials z' such that $\alpha_i(z') \leqslant \alpha_i(z)$ for all $i \in \{1, \ldots, n\}$. Hence, if $\alpha_i(z) < k$ for some i, then $Q_0(z)$ is congruent to a linear combination of z' such that $\alpha_i(z') < k$. Hence, it is congruent to a linear combination of elements in $\mathcal{M}_0$.

If $\omega_1(z) = n-2$ and $\alpha_i(z) = k$ for all $i \in \{1, \ldots, n\}$, then without loss of generality, we may assume that

$$u_0(z) = (1, 1, 1, \ldots, 1, 0, \ldots, 0),$$
$$u_1(z) = (0, 0, 1, \ldots, 1, 1, \ldots, 1).$$

Then, $Q_0(z)$ is congruent to $\sigma_1(z_k) + \sigma_2(z_k)$ where σ_1, σ_2 in $\mathrm{Mono}(k)$, $\sigma_1(1) = 1$, $\sigma_1(2) = 3, \ldots, \sigma_1(k) = k+1$, $\sigma_2(1) = 2, \ldots, \sigma_2(k) = k+1$. □

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