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CONSTRUCTING EQUIVARIANT HOMOTOPY EQUIVALENCES VIA EXTENSION OF SCALARS

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Abstract

Given a finite group G, we present a new algorithm to construct a $\mathbb{Z}G$-linear strong homotopy equivalence of $\mathbb{Z}G$-complexes from a $\mathbb{Z}$-linear one.

1. Motivation

A classical topological problem is to determine whether or not a given simplicial complex K can be embedded into $\mathbb{R}^d$. Algorithmic aspects of this problem were studied in the paper [**5**]. Nevertheless, the question of whether or not the problem is algorithmically solvable remained open in the metastable range, i.e. $\dim K \leqslant \frac{2}{3}d - 1$. In this range the problem is equivalent to the existence of a $\mathbb{Z}/2$-equivariant map from the deleted product $(K \times K) \smallsetminus \Delta_K$ into S^{d-1} (see [**8**]). A more general problem has been solved in [**1**]. The authors considered two finite simplicial sets X and Y with a free action of a finite group G such that $\dim X \leqslant 2 \operatorname{conn} Y + 1$ and $\operatorname{conn} Y \geqslant 1$. In this case the set of homotopy classes of G-equivariant maps is an empty set or can be equipped with the structure of a finitely generated abelian group. The isomorphism type of this set can be determined algorithmically. The algorithm works with the equivariant Moore-Postnikov tower for a certain map $Y \to EG$ where EG is an equivariant analogue of a point among free G-spaces. The stages of the tower are twisted products of EG and Eilenberg-MacLane spaces which are generally infinite simplicial sets. To work with them algorithmically their associated non-degenerate chain complexes have to be strongly homotopy equivalent to locally finitely generated chain complexes equivariantly. In the so-called effective homological algebra introduced in [**6**] it is well-known that chain complexes associated to Eilenberg-MacLane spaces $K(\pi, n)$ (here π is a finitely generated abelian group) and their twisted products with EG are strongly homotopy equivalent to locally finite chain complexes but not equivariantly (see [**2**]). The problem of construction of a suitable $\mathbb{Z}G$-homotopy equivalence has already been solved in [**7**] using homotopy algebras. Here we give a different algorithm based only on perturbation lemmas.

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2. The main theorem

Notation

The integral group algebra of a finite group G will be denoted by $\mathbb{Z}G$. By abuse of notation a symbol $\mathbb{Z}$ will stand for a differential graded algebra $(\mathbb{Z}, 0)$ concentrated in dimension 0. A chain complex of free $\mathbb{Z}G$-modules will be called a $\mathbb{Z}G$-complex. The pair $(\mathcal{R}, d)$ will denote a differential non-negatively graded algebra (dga) over $\mathbb{Z}$ whose underlying chain complex is a $\mathbb{Z}$-complex with 1 as a generator. We say that a $\mathbb{Z}G$-complex is *locally finite* if it consists of finitely generated $\mathbb{Z}G$-modules.

We will work with differential graded modules over the dga $\mathcal{R}$ and we follow the approach of the paper [**4**, p. 832].

Definition 2.1. A (left) module over the dga $(\mathcal{R}, d)$ is a chain complex (M, d_M) together with a linear map of degree zero $\cdot\colon \mathcal{R} \otimes_{\mathbb{Z}} M \to M$ defined as $r \otimes m \mapsto r \cdot m$ such that the following conditions hold for $1, r, r' \in \mathcal{R}$, $m \in M$:

- associativity $(rr') \cdot m = r \cdot (r' \cdot m)$,
- identity element $1 \cdot m = m$,
- the Leibniz rule $d_M(r \cdot m) = d(r) \cdot m + (-1)^{|r|} r \cdot d_M(m)$.

An $\mathcal{R}$-linear map between $(\mathcal{R}, d)$-modules (M, d_M) and (N, d_N) of degree k is a $\mathbb{Z}$-linear map $\varphi\colon M \to N$ of degree k satisfying $\varphi(r \cdot m) = (-1)^{k|r|} r \cdot \varphi(m)$ for $r \in \mathcal{R}$ and $m \in M$. We say that the $(\mathcal{R}, d)$-module (M, d_M) is *free* if it is free as a graded module over the graded algebra $\mathcal{R}$ i.e. $M \cong \mathcal{R} \otimes_{\mathbb{Z}} V$ with V a free graded $\mathbb{Z}$-module. A free $(\mathcal{R}, d)$-module (M, d_M) ($M \cong \mathcal{R} \otimes_{\mathbb{Z}} V$) is *locally finite* if the $\mathbb{Z}$-modules V_n are finitely generated for all $n \in \mathbb{Z}$.

Remark 2.2. We note that a $\mathbb{Z}G$-complex is a free differential graded $(\mathbb{Z}G, 0)$-module.

Effective homological algebra

To state the main theorem we need to recall some basic notions of effective homological algebra (see [**6**]) for $(\mathcal{R}, d)$-modules.

Definition 2.3 ([**6**, Definition 47])**.** Let (C, d_C) and (D, d_D) be $(\mathcal{R}, d)$-modules. A triple of mappings $(f\colon C \to D, g\colon D \to C, h\colon C \to C)$ is called a reduction if the following conditions hold:

- f and g are $\mathcal{R}$-linear chain maps of degree 0,
- h is an $\mathcal{R}$-linear map of degree 1,
- $fg = \mathrm{id}_D$ and $\mathrm{id}_C - gf = [d_C, h] = d_C h - h d_C$,
- $fh = 0$, $hg = 0$ and $hh = 0$.

The reductions are denoted by $(f, g, h)\colon (C, d_C) \Rightarrow (D, d_D)$.

Definition 2.4 ([**6**, Definition 53])**.** A strong homotopy equivalence $C \Leftrightarrow D$ between $(\mathcal{R}, d)$-modules C, D is an $(\mathcal{R}, d)$-module E and a pair of reductions $C \Leftarrow E \Rightarrow D$.

Definition 2.5 ([**7**, pp. 3–4])**.** An $(\mathcal{R}, d)$-module is *locally effective* if

- elements of the graded module can be represented in a computer,
- the operations of zero, addition, differential and multiplication by elements of $\mathcal{R}$ are computable.

A locally effective $(\mathcal{R}, d)$-module is free if an arbitrary element of the underlying $(\mathcal{R}, d)$-module is represented in a computer as an $\mathcal{R}$-linear combination of its distinguished basis. A map is *locally effective* if there is an algorithm computing its values.

The previous definition can be illustrated by the following example.

Example 2.6. Consider a simplicial set X with a free action of a finite group G and its associated normalized chain complex $C_*(X)$. The distinguished $\mathbb{Z}$-basis of $C_*(X)$ consists of non-degenerate simplices. If there is a free simplicial G-action defined on the simplicial set X then the chain complex $C_*(X)$ has induced free G-action by chain maps. Thus the distinguished $\mathbb{Z}G$-basis of $C_*(X)$ is formed by representatives of G-orbits.

More detailed descriptions of the above terminology can be found in [**1**, p. 9].

Convention 2.7. We assume that all $(\mathcal{R}, d)$-modules and maps are locally effective.

The aim of this paper is to prove the following statement.

Theorem 2.8. *Let M be a locally effective $\mathbb{Z}G$-complex and let N be a locally effective and locally finite $\mathbb{Z}$-complex. Given a locally effective strong homotopy equivalence of $\mathbb{Z}$-complexes $M \Leftrightarrow N$, there is an algorithm which constructs a locally effective and locally finite $\mathbb{Z}G$-complex N' and a locally effective $\mathbb{Z}G$-linear strong homotopy equivalence $M \Leftrightarrow N'$.*

So the main statement will be the consequence of the following more general theorem if we put $\mathcal{R} = \mathbb{Z}G$.

Theorem 2.9. *Let $\mathcal{R}$ be a differential non-negatively graded algebra over $\mathbb{Z}$ whose underlying chain complex is a $\mathbb{Z}$-complex with 1 as a generator. Let M be a free locally effective $(\mathcal{R}, d)$-module and let N be a locally effective and locally finite $\mathbb{Z}$-complex. Given a locally effective strong homotopy equivalence of $\mathbb{Z}$-complexes $M \Leftrightarrow N$, there is an algorithm which constructs a free locally effective and locally finite $(\mathcal{R}, d)$-module N' and a locally effective $\mathcal{R}$-linear strong homotopy equivalence $M \Leftrightarrow N'$.*

Outline

We start in the same way as in [**7**] and we will construct $(\mathcal{R}, d)$-modules and an $\mathcal{R}$-linear strong homotopy equivalence $B(\mathcal{R}, \mathcal{R}) \otimes_{\mathbb{Z}} M \Leftrightarrow B(\mathcal{R}, \mathcal{R}) \otimes_{\mathbb{Z}} N$ where $B(\mathcal{R}, \mathcal{R})$ is the bar construction on the dga $\mathcal{R}$. Then the differential of the bar construction on the dga $\mathcal{R}$ with coefficients in M, denoted by $B(\mathcal{R}, \mathcal{R}, M)$, is obtained from the differential of $B(\mathcal{R}, \mathcal{R}) \otimes_{\mathbb{Z}} M$ by a perturbation. This perturbation can be transferred using the strong homotopy equivalence on the $(\mathcal{R}, d)$-module $B(\mathcal{R}, \mathcal{R}) \otimes_{\mathbb{Z}} N$. The resulting perturbated $(\mathcal{R}, d)$-module N' is strongly homotopy equivalent to $B(\mathcal{R}, \mathcal{R}, M)$ and using a natural reduction $B(\mathcal{R}, \mathcal{R}, M) \Rightarrow M$ we get

$$M \Leftarrow B(\mathcal{R}, \mathcal{R}, M) \Leftrightarrow N'.$$

3. Overview of basic notions

In this section we will summarize some fundamental results in homological algebra which will form the background of the last section. The results are adapted from the paper [**6**] and will be reformulated for $(\mathcal{R}, d)$-modules.

Definition 3.1 ([**6**, Definition 49])**.** Let (C, d_C) be an $(\mathcal{R}, d)$-module. An $\mathcal{R}$-linear mapping δ_C of degree -1 is a perturbation of the differential d_C if $(d_C + \delta_C)^2 = 0$. On the chain complex $(C, d_C + \delta_C)$ we consider the same multiplication by elements of $\mathcal{R}$ as in (C, d_C). Then $d_C + \delta_C$ satisfies the Leibniz rule since

$$\begin{aligned}(d_C + \delta_C)(r \cdot x) &= d_C(r \cdot x) + \delta_C(r \cdot x)\\ &= d(r) \cdot x + (-1)^{|r|} r \cdot d_C(x) + (-1)^{|r|} r \delta_C(x)\\ &= d(r) \cdot x + (-1)^{|r|} r \cdot (d_C + \delta_C)(x).\end{aligned}$$

Hence $(C, d_C + \delta_C)$ is again a module over dga $(\mathcal{R}, d)$.

Proposition 3.2 (Basic perturbation lemma)**.** *Let $(f, g, h)\colon (C, d_C) \Rightarrow (D, d_D)$ be a reduction between $(\mathcal{R}, d)$-modules and let δ_C be a perturbation of the differential d_C of the $(\mathcal{R}, d)$-module C. Assume that the following nilpotency condition is satisfied: for all $c \in C$, there exists $\nu \in \mathbb{N}_0$ satisfying $(h\delta_C)^\nu(c) = 0$. Then there exists a perturbation δ_D of the differential d_D and an $\mathcal{R}$-linear reduction $(f', g', h')\colon (C, d_C + \delta_C) \Rightarrow (D, d_D + \delta_D)$ where $(C, d_C + \delta_C)$ and $(D, d_D + \delta_D)$ have the canonical $(\mathcal{R}, d)$-module structure described in Definition 3.1.*

Proof. The $\mathcal{R}$-linearity of the maps (f', g', h') and δ_D directly follows from their definitions

$$\begin{aligned}f' &= f \circ \sum_{i=0}^{\infty} (-1)^i (\delta_C h)^i,\\ g' &= \sum_{i=0}^{\infty} (-1)^i (h\delta_C)^i \circ g,\\ h' &= \sum_{i=0}^{\infty} (-1)^i (h\delta_C)^i \circ h,\\ \delta_D &= f \circ \delta_C \circ g'.\end{aligned}$$

For the rest of this proof we refer to [**6**, Theorem 51]. □

The proof of the following "Easy perturbation lemma" is based on a similar statement for $(\mathbb{Z}, 0)$-modules (see [**6**, Proposition 50]).

Proposition 3.3 (Easy perturbation lemma)**.** *Let $(f, g, h)\colon (C, d_C) \Rightarrow (D, d_D)$ be a reduction between $(\mathcal{R}, d)$-modules and let δ_D be a perturbation of the differential d_D of the $(\mathcal{R}, d)$-module D. Then there is a perturbated reduction $(f, g, h)\colon (C, d_C + \delta_C) \Rightarrow (D, d_D + \delta_D)$ where the perturbation of the differential d_C is defined as $\delta_C = g\delta_D f$. The $(\mathcal{R}, d)$-module structures of $(C, d_C + \delta_C)$ and $(D, d_D + \delta_D)$ are canonical as described in Definition 3.1.*

Proposition 3.4 ([**6**, Proposition 60])**.** *Given two reductions $(f, g, h)\colon C \Rightarrow D$ and $(f', g', h')\colon D \Rightarrow E$ between $(\mathcal{R}, d)$-modules, there is a composition of the reductions $(f'f, gg', h + gh'f)\colon C \Rightarrow E$.*

Proposition 3.5 ([**6**, Proposition 61])**.** *Given two reductions $(f, g, h)\colon C \Rightarrow \widetilde{C}$ and $(f', g', h')\colon D \Rightarrow \widetilde{D}$ between $(\mathbb{Z}, 0)$-modules, there is a reduction $(f \otimes f', g \otimes g', h \otimes \mathrm{id}_D + (g \circ f) \otimes h')\colon C \otimes_{\mathbb{Z}} D \to \widetilde{C} \otimes_{\mathbb{Z}} \widetilde{D}$ which is called a tensor product of reductions.*

4. The bar construction

For the proof of our main result we utilize the bar construction $B(\mathcal{R},\mathcal{R})$ as described in [**4**, p. 845] and its differentials in [**7**, p. 12]. We recall that the suspension of the chain complex $(\mathcal{R},d)$ will be denoted by $(s\mathcal{R},-d)$ and defined by $(s\mathcal{R})_n = (\mathcal{R})_{n-1}$ for $n\in\mathbb{N}$. The tensor algebra of $s\mathcal{R}$ is defined as $T(s\mathcal{R}) = \bigoplus_{k=0}^{\infty}(s\mathcal{R})^{\otimes k}$ and we define $(s\mathcal{R})^{\otimes 0} = \mathbb{Z}$.

Firstly, we define a $(\mathbb{Z},0)$-module $B(\mathcal{R},\mathcal{R}) = \mathcal{R}\otimes_{\mathbb{Z}} T(s\mathcal{R})$ with a differential d_B. The generators of $B(\mathcal{R},\mathcal{R})$ will be denoted by $r_0|r_1|\cdots|r_k$. According to Eilenberg and Mac Lane's terminology [**3**, p. 73] the number k is called the *simplicial* degree and the sum $|r|_{0,1,\dots,k} = |r_0|+|r_1|+\cdots+|r_k|$ is the *tensor* degree of the generator $r_0|r_1|\cdots|r_k$. Moreover, we put $|r|_{-1}=0$. The differential d_B is defined as follows

$$
\begin{aligned}
d_B(r_0|r_1|\cdots|r_k) &= \sum_{i=0}^{k} d_i^{\otimes}(r_0|r_1|\cdots|r_k) + \sum_{m=1}^{k} d_m^{\mathrm{alg}}(r_0|r_1|\cdots|r_k), \qquad (1)\\
d_i^{\otimes}(r_0|r_1|\cdots|r_k) &= (-1)^{i+|r|_{0,1,\dots,i-1}} r_0|r_1|\cdots|d(r_i)|\cdots|r_k,\\
d_m^{\mathrm{alg}}(r_0|r_1|\cdots|r_k) &= (-1)^{m-1+|r|_{0,1,\dots,m-1}} r_0|r_1|\cdots|r_{m-1}r_m|\cdots|r_k.
\end{aligned}
$$

It is easy to verify that equation (1) defines a differential i.e. $d_B^2=0$. More details can be found in [**3**, Chapter 2]. The module $B(\mathcal{R},\mathcal{R})$ is usually called the bar construction on the dga $\mathcal{R}$. We note that $B(\mathcal{R},\mathcal{R})$ is an $(\mathcal{R},d)$-module with multiplication

$$r\cdot(r_0|r_1|\cdots|r_k) = (r\cdot r_0)|r_1|\cdots|r_k.$$

More generally, we define an $(\mathcal{R},d)$-module $(B(\mathcal{R},\mathcal{R},M),\partial_M+\delta_M)$ for a given $(\mathcal{R},d)$-module (M,d_M) in the following way (we use the Koszul sign convention)

$$
\begin{aligned}
B(\mathcal{R},\mathcal{R},M) &= \mathcal{R}\otimes_{\mathbb{Z}} T(s\mathcal{R})\otimes_{\mathbb{Z}} M,\\
\partial_M(r_0|r_1|\cdots|r_k\otimes x) &= (d_B\otimes_{\mathbb{Z}}\mathrm{id}_M)(r_0|r_1|\cdots|r_k\otimes x)+\\
&\quad + (\mathrm{id}_{B(\mathcal{R},\mathcal{R})}\otimes_{\mathbb{Z}} d_M)(r_0|r_1|\cdots|r_k\otimes x), \qquad (2)\\
\delta_M(r_0|r_1|\cdots|r_k\otimes x) &= \begin{cases} (-1)^{k+|r|_{0,1,\dots,k-1}} r_0|r_1|\cdots|r_{k-1}\otimes r_k x & \text{for } k>0,\\ 0 & \text{for } k=0,\end{cases} \qquad (3)
\end{aligned}
$$

where the differential d_B is defined in (1). Similarly as before, we need to show that $(\partial_M+\delta_M)^2=0$ which also follows from Lemma 9 in [**7**, p. 12]. The module $B(\mathcal{R},\mathcal{R},M)$ is called the bar construction on the dga $\mathcal{R}$ with coefficients in M. There is an augmentation map $\epsilon\colon B(\mathcal{R},\mathcal{R},M)\to M$ which is given by

$$
\begin{aligned}
\epsilon(r_0\otimes x) &= r_0\cdot x,\\
\epsilon(r_0|r_1|\cdots|r_k\otimes x) &= 0 \quad \text{for } k>0.
\end{aligned}
$$

Remark 4.1. The $(\mathcal{R},d)$-module $B(\mathcal{R},\mathcal{R},M)$ is exactly $B(\mathcal{R},\mathcal{R})\otimes_{\mathbb{Z}} M$ with the differential ∂_M perturbated by δ_M.

At the end of this section we recall a natural reduction which we will use in the last section.

Theorem 4.2. *Let $\mathcal{R}$ be a differential non-negatively graded algebra over $\mathbb{Z}$ whose underlying chain complex is a $\mathbb{Z}$-complex with 1 as a generator and let M be an*

$(\mathcal{R}, d)$-module which is free as a graded module. Then the augmentation map

$$\epsilon\colon (B(\mathcal{R}, \mathcal{R}, M), \partial_M + \delta_M) \to M$$

is a projection of an $\mathcal{R}$-linear reduction.

Proof. See Theorem 4 in [**7**, p. 7]. □

5. Construction of the $\mathcal{R}$-linear strong homotopy equivalence

In this section we construct the required strong homotopy equivalence i.e. we will give a proof of Theorem 2.9. We start with a $\mathbb{Z}$-linear pair of reductions

$$M \Leftarrow_{(f,g,h)} \widetilde{M} \Rightarrow_{(f',g',h')} N.$$

Here M is a free $(\mathcal{R}, d)$-module, N and $\widetilde{M}$ are $(\mathbb{Z}, 0)$-modules. Proposition 3.5 can be applied to get $\mathbb{Z}$-linear reductions

$$B(\mathcal{R}, \mathcal{R}) \otimes_{\mathbb{Z}} M \Leftarrow_{(\mathrm{id}\otimes f, \mathrm{id}\otimes g, \mathrm{id}\otimes h)} B(\mathcal{R}, \mathcal{R}) \otimes_{\mathbb{Z}} \widetilde{M} \Rightarrow_{(\mathrm{id}\otimes f', \mathrm{id}\otimes g', \mathrm{id}\otimes h')} B(\mathcal{R}, \mathcal{R}) \otimes_{\mathbb{Z}} N. \tag{4}$$

Since these modules are $(\mathcal{R}, d)$-modules and the identity on $B(\mathcal{R}, \mathcal{R})$ is $\mathcal{R}$-linear, the reductions in (4) are $\mathcal{R}$-linear.

Note that the differentials of $(\mathcal{R}, d)$-modules in (4) are given by the equation (2). We apply the Easy perturbation lemma (Proposition 3.3) to the reduction $(\mathrm{id}\otimes f, \mathrm{id}\otimes g, \mathrm{id}\otimes h)$ and the perturbation δ_M of the $(\mathcal{R}, d)$-module $B(\mathcal{R}, \mathcal{R}) \otimes_{\mathbb{Z}} M$ from (3). The resulting perturbation of the $(\mathcal{R}, d)$-module $B(\mathcal{R}, \mathcal{R}) \otimes_{\mathbb{Z}} \widetilde{M}$ is

$$\delta_{\widetilde{M}} = (\mathrm{id}\otimes g)\delta_M(\mathrm{id}\otimes f).$$

Now we are going to apply the Basic perturbation lemma (Proposition 3.2) to the reduction $(\mathrm{id}\otimes f', \mathrm{id}\otimes g', \mathrm{id}\otimes h')$ and the perturbation $\delta_{\widetilde{M}}$. We have to prove that the nilpotency condition is satisfied for all $c \in B(\mathcal{R}, \mathcal{R}) \otimes_{\mathbb{Z}} \widetilde{M}$. The main idea is based on the fact that the perturbation

$$\delta_{\widetilde{M}}\colon \mathcal{R} \otimes_{\mathbb{Z}} (s\mathcal{R})^k \otimes_{\mathbb{Z}} \widetilde{M} \to \mathcal{R} \otimes_{\mathbb{Z}} (s\mathcal{R})^{k-1} \otimes_{\mathbb{Z}} \widetilde{M}$$

decreases the simplicial degree k, if $k > 0$, and for $k = 0$

$$\delta_{\widetilde{M}}(r \otimes \widetilde{m}) = (id \otimes g)\delta_M(r \otimes f(\widetilde{m})) = 0$$

according to (3). We note that the map $\mathrm{id}\otimes h'$ does not have any impact on the simplicial degree. To demonstrate the previous statement we provide the explicit computation for $c = r_0|r_1|\cdots|r_k \otimes \widetilde{m}$. Using (3) we obtain:

$$\begin{aligned}
((\mathrm{id}\otimes h')\delta_{\widetilde{M}})^{k+1}(c) &= ((\mathrm{id}\otimes h')\delta_{\widetilde{M}})^{k}(\mathrm{id}\otimes (h'g))\delta_M(\mathrm{id}\otimes f)(r_0|r_1|\cdots|r_k \otimes \widetilde{m}) \\
&= ((\mathrm{id}\otimes h')\delta_{\widetilde{M}})^{k}(\mathrm{id}\otimes (h'g))\delta_M(r_0|r_1|\cdots|r_k \otimes f(\widetilde{m})) \\
&= \pm((\mathrm{id}\otimes h')\delta_{\widetilde{M}})^{k}(\mathrm{id}\otimes (h'g))(r_0|r_1|\cdots|r_{k-1} \otimes r_k \cdot f(\widetilde{m})) \\
&= \pm((\mathrm{id}\otimes h')\delta_{\widetilde{M}})^{k}(r_0|r_1|\cdots|r_{k-1} \otimes h'(g(r_k \cdot f(\widetilde{m})))) \\
&= \pm(\mathrm{id}\otimes h')\delta_{\widetilde{M}}(r_0 \otimes h'(g(r_1 \cdot f(\cdots(h'(g(r_k \cdot f(\widetilde{m})))) \cdots)))) \\
&= 0.
\end{aligned}$$

We denote the perturbated $(\mathcal{R}, d)$-module $(B(\mathcal{R}, \mathcal{R}) \otimes_{\mathbb{Z}} N, \partial_N + \delta_N)$ by N'. Now we use Theorem 4.2 to obtain

$$M \Leftarrow B(\mathcal{R}, \mathcal{R}, M) \Leftrightarrow N'.$$

Finally, the application of Proposition 3.4 yields the strong homotopy equivalence of $(\mathcal{R},d)$-modules

$$M \Leftrightarrow N'.$$

If N is a locally finite $\mathbb{Z}$-complex then N' is a free locally finite $(\mathcal{R}, d)$-module; this directly follows from its definition.

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