ON THE PATH HOMOLOGY THEORY OF DIGRAPHS AND EILENBERG-STEENROD AXIOMS

ALEXANDER GRIGOR'YAN, ROLANDO JIMENEZ, YURI MURANOV
AND SHING-TUNG YAU

(*communicated by Alexander Mishchenko*)

Abstract

In the paper we continue the investigation of the path homology theory of digraphs that was constructed in our previous papers. We prove basic theorems that are similar to the theorems of classical algebraic topology and introduce several natural constructions of digraphs which are very helpful to investigate the path homology theory. We describe relation of our results to the Eilenberg-Steenrod axiomatic of homology theory.

1. Introduction

The homology and homotopy theory of digraphs considered in this paper were introduced in [**16, 17, 18, 19**]. Our approach is closely related to geometric and algebraic topology [**14, 13, 29, 24, 21**], to physical applications of graph theory [**32, 11, 10, 4, 9**], to Atkins homotopy theory [**1, 2, 3, 6**], to the theory of quivers [**8, 22, 15, 19**], and to various discrete (co)homology and homotopy theories [**5, 3, 6, 31, 12, 28, 23, 7**].

In the present paper we introduce a path homology theory on the category of digraph pairs (Section 3). We verify that this homology theory satisfies the properties that are analogous to Eilenberg-Steenrod axioms (Section 5). In particular, we prove that the path homologies satisfy a digraph version of the excision axiom of Eilenberg-Steenrod (Theorem 3.29).

Another important result is construction of the Mayer-Vietoris exact sequence for certain triads of digraphs (Theorem 3.25). This result is used, in particular, to obtain the homology groups of suspensions and wedge sums (Theorems 4.13, 4.14, Corollary 3.28).

We construct new exact sequences and a braid of exact sequence for various pairs and triples of digraphs (Theorems 3.15, 3.18, 3.21).

Note that many results of this paper can be transferred to the case of non-directed graphs, using the natural identification of the category of non-directed graphs with a full subcategory of digraphs (see [**16**]).

The paper is organized as follows. In Section 2, we give some preliminary material concerning the homotopy theory for digraphs (see also [**16, 18**]) and prove several technical results.

In Section 3, we introduce and study the path homology theory for the category of digraph pairs. In particular, we construct new exact sequences of the homology groups and describe new relations between cones, suspensions, and cylinders of digraphs.

In Section 4, we introduce the natural homotopical generalizations of cylinder, cone, and suspension for digraphs and prove their properties.

In Section 5, we discuss basic properties of path homology in connection with the Eilenberg-Steenrod axioms of homology theory on topological spaces. It seems that the discrete analogue of this system of axioms does not possess completeness, and an interesting question arises of what additional axioms are needed.

In Section 6, we provide several examples which illustrate the distinction between path homology theory and homology theory for topological spaces.

Acknowledgments

2. On homotopy theory for digraphs

In this section we give necessary definitions and preliminary material about the homotopy theory for digraphs constructed in [**16**]. We also prove several technical results.

Definition 2.1. A *directed graph (digraph)* X is a couple (V_X, E_X) where V_X is an arbitrary set whose elements are called *vertices*, and $E_X \subset \{V_X \times V_X \setminus \text{diag}\}$. The elements of E_X are called *arrows* (or directed edges). The fact that $(v, w) \in E_X$ will be denoted by $v \rightarrow w$. We write $v \xrightarrow{\rightrightarrows} w$ if either $v = w$ or $v \rightarrow w$.

A *based digraph* X^* is a digraph X together with a fixed vertex $* \in V_X$.

Definition 2.2. A *digraph map* (or simply *map*) from a digraph X to a digraph Y is a map $f \colon V_X \rightarrow V_Y$ such that, for any arrow $v \rightarrow w$ in X we have $f(v) \xrightarrow{\rightrightarrows} f(w)$ on Y.

A *digraph map of based digraphs* $f \colon X^v \rightarrow Y^w$ is a digraph map of digraphs such that $f(v) = w$.

The image $f(X)$ of a digraph map $f \colon X \rightarrow Y$ is a digraph whose vertices are $f(v)$, $v \in V_X$ and whose arrows are $f(v) \rightarrow f(w)$ provided $v \rightarrow w$ and $f(v) \neq f(w)$.

A digraph map $f \colon X \rightarrow Y$ is called an isomorphism if f is a bijection and $v \rightarrow w$ in X is equivalent to $f(v) \rightarrow f(w)$ in Y. If there is an isomorphism $f \colon X \rightarrow Y$ then X and Y are called isomorphic. We write in this case $X \cong Y$.

Denote by $\mathcal{D}$ the category of digraphs in which the morphisms are digraph maps and by $\mathcal{D}^*$ the category of based digraphs and based digraph maps.

Definition 2.3. A *Cartesian product* $\Pi = X \square Y$ of two digraphs $X = (V_X, E_X)$ and $Y = (V_Y, E_Y)$ is a digraph Π with the set of vertices $V_\Pi = V_X \times V_Y$ and the set of arrows E_Π that is defined as follows: for all $x, x' \in V_X$ and $y, y' \in V_y$

$$(x, y) \to (x', y') \text{ in } \Pi \iff x' = x, \ y \to y' \quad \text{or} \quad x \to x', y' = y.$$

Let I_n $(n \geqslant 0)$ denote a digraph with the set of vertices $V_{I_n} = \{0, 1, \ldots, n\}$ and the set of arrows E_{I_n} containing exactly one of the arrows $i \to (i+1)$, $(i+1) \to i$ for each $i = 0, 1, \ldots, n-1$, and no other edges. Such a digraph I_n is called a *line digraph*. The number n is called the height of I_n. If I is a line digraph, then its height will be denoted by $|I|$. Denote by I_n^* the based line digraph with base vertex $* = 0$.

Denote by $\mathcal{I}_n$ the set of all line digraphs of height n, and by $\mathcal{I}_n^*$ the set of all based line digraphs of height n. Set also

$$\mathcal{I} = \bigcup_{n \geqslant 0} \mathcal{I}_n, \qquad \mathcal{I}^* = \bigcup_{n \geqslant 0} \mathcal{I}_n^*.$$

Note that there is only one line digraph of height 0, that is $\{0\}$, and two line digraphs of height 1: $I = (0 \to 1)$ and $I^- = (0 \leftarrow 1)$.

Definition 2.4. i) We call digraph maps $f, g \colon X \to Y$ *homotopic* and write $f \simeq g$, if there exists a line digraph $I_n \in \mathcal{I}$ and a digraph map

$$F \colon X \square I_n \to Y \text{ such that } F|_{X \square \{0\}} = f, \ \ F|_{X \square \{n\}} = g,$$

where we identify X with $X \square \{0\}$ and with $X \square \{n\}$ by means of the natural inclusions. If $n = 1$ then we refer to F as *a one-step* homotopy from f to g, and the maps f and g will be called *one-step homotopic*.

ii) Two based digraph maps $f, g \colon X^v \to Y^*$ are called homotopic if there is a homotopy F as in i) with the additional property $F(\{v\} \square I_n) = * \in Y$.

Now the *homotopy equivalence* $X \simeq Y$ of two digraphs and a *contractible* digraph are defined by a standard way.

Denote by $\mathrm{Ho}\,\mathcal{D}$ the category of digraphs with classes of homotopic maps and by $\mathrm{Ho}\,\mathcal{D}^*$ the category of based digraphs with classes of based homotopic maps.

Definition 2.5. i) A digraph Y is called a *sub-digraph* of a digraph X if $V_Y \subset V_X$ and $E_Y \subset E_X$. In this case we shall write $Y \sqsubset X$.

ii) If in addition

$$v, w \in V_Y \text{ and } v \to w \text{ in } X \implies v \to w \text{ in } Y,$$

then we say that Y is an *induced sub-digraph* of X and write $Y \subset X$.

Clearly $Y \subset X$ implies $Y \sqsubset X$. A pair (X, Y) consisting of a digraph X with a sub-digraph $Y \sqsubset X$ will be called a *digraph pair*.

Let (X, Y) be a digraph pair with $Y \subset X$. Then we have the standard definition of a *retraction* $r \colon X \to Y$ and a *(strong) deformation retraction* $F \colon X \square I_n \to Y$ of X onto Y.

Definition 2.6. If we have a deformation retraction F with $n = 1$, then we refer to F as a *one-step* retraction.

Note that if $Y \sqsubset X$ but $Y \not\subset X$, then there is no retraction of X onto Y.
The following results follows directly from Definition 2.6.

Proposition 2.7. *Consider a digraph X and two sub-digraphs $Y_i \in X$ $(i = 1, 2)$ such that $X = Y_1 \cup Y_2$ and there is a strong deformation retraction $F \colon Y_2 \square I_n \to Y_1 \cap Y_2$. Then there is a strong deformation retraction $\Phi \colon X \square I_n \to Y_1$ such that $\Phi|_{Y_2 \square I_n} = F$.*

Proposition 2.8. *Consider a triple of digraphs $Z \subset Y \subset X$. Let Z be a retract (deformation retract, strong deformation retract) of Y, and let Y be a retract (deformation retract, strong deformation retract) of X. Then Z is a retract (deformation retract, strong deformation retract) of X.*

Definition 2.9. i) Let (X, Y) and (X', Y') be digraph pairs. A digraph map $f \colon X \to X'$ is called a *digraph map of pairs* if $f(Y) \sqsubset Y'$. In this case we write

$$f \colon (X, Y) \to (X', Y').$$

ii) Two digraph maps of pairs

$$f, g \colon (X, Y) \to (X', Y')$$

are called *homotopic* if there exists a map of digraph pairs

$$F \colon (X \square I_n, Y \square I_n) \to (X', Y'),$$

such that

$$F|_{(X \square \{0\}, Y \square \{0\})} = f, \quad F|_{(X \square \{n\}, Y \square \{n\})} = g.$$

It is easy to check that we obtain a well defined category $\mathcal{D}^2$ of digraph pairs and digraph maps of pairs, and a well defined category $\mathrm{Ho}\,\mathcal{D}^2$ of digraph pairs and homotopy classes of digraph maps of pairs. Note that a based digraph X^* can be considered as a digraph pair $(X, *)$. The based digraph map $f \colon X^v \to Y^w$ can be considered as a digraph map of pairs $f \colon (X, v) \to (Y, w)$. Thus, the category $\mathcal{D}^*$ is a full subcategory of $\mathcal{D}^2$, and similarly $\mathrm{Ho}\,\mathcal{D}^*$ is a full subcategory of $\mathrm{Ho}\,\mathcal{D}^2$.

Definition 2.10. Let $f \colon X \to Y$ be a digraph map. The *cylinder* C_f of f is a digraph with the set of vertices $V_{C_f} = V_X \cup V_Y$ and with the set of arrows that consists of all arrows of X, all arrows of Y, and of all arrows of the form $v \to f(v)$, $v \in V_X$.

The *inverse cylinder* C_f^- has the same set of vertices as C_f and in the definition of the set of arrows $v \to f(v)$ should be replaced by $v \leftarrow f(v)$.

In particular, we have natural inclusions $j \colon X \to C_f$ and $i \colon Y \to C_f$, and similarly for C_f^-. For example, for $f = \mathrm{Id}_X \colon X \to X$ we obtain digraphs

$$C_f \cong X \square I \cong C_f^- \cong X \square I^-,$$

that are called *cylinders* over the digraph X.

Definition 2.11. The cylinder of the map $X \to \{a\}$ of a digraph X to a single vertex digraph $\{a\}$ is called a *cone* over the digraph X and is denoted by $Cone_a(X)$ or simply by $Cone(X)$. The inverse cylinder of this map is called an *inverse cone* over the digraph X and is denoted $Cone_a^-(X)$ or simply by $Cone^-(X)$.

Definition 2.12. The *suspension* $S(X)$ of a digraph X is a digraph defined as the union

$$Cone_a(X) \bigcup_{X \sim X} Cone_b(X)$$

of two cones $Cone_a(X)$ and $Cone_b(X)$ where $a \neq b$, and we identify $X \subset Cone_a(X)$ with $X \subset Cone_b(X)$. Similarly, the *inverse suspension* $S^-(X)$ is the union of $Cone_a^-(X)$ and $Cone_b^-(X)$.

Definition 2.13. Let $f \colon X \to Y$ be a digraph map. The *cone* of f is a digraph $Cone_f$ whose set of vertices is $V_{Cone_f} = V_Y \cup \{*\}$ and whose set of arrows consists of all the arrows of Y and all arrows $* \to f(v)$, $v \in V_X$. Similarly one defines the *inverse cone* of f where $* \to f(v)$ is replaced by $* \leftarrow f(v)$.

For example, for an identity map $f = \mathrm{Id}_X \colon X \to X$, we have

$$Cone_f \cong Cone^-(X)$$

and

$$Cone_f^- \cong Cone(X).$$

Definition 2.14. i) For any $n \geqslant 0$ define a digraph Δ^n by

$$V_{\Delta^n} = \{0, 1, \ldots, n\}, \quad i \to j \text{ in } \Delta^n \Leftrightarrow i < j.$$

An *n-simplex* is any digraph that is isomorphic to Δ^n.

ii) For any $n \geqslant 0$, define a digraph I^n as

$$I^0 = \{*\}, \quad I^n = \underbrace{I \square \cdots \square I}_{n \text{ times}}.$$

An *n-cube* is any digraph that is isomorphic to I^n.

Proposition 2.15 ([**16**]). *i) For any digraph X and for any line digraph I_n there are homotopy equivalences:*

$$X \square I_n \simeq X, \quad Cone(X) \simeq Cone^-(X) \simeq \{*\}.$$

ii) For any digraph map $f \colon X \to Y$ we have homotopy equivalences

$$\mathrm{C}_f \simeq Y \simeq \mathrm{C}_f^-.$$

iii) For any $n \geqslant 0$, we have homotopy equivalences

$$I^n \simeq \{*\} \simeq \Delta^n.$$

Now we state and prove yet several results about homotopy properties of digraphs that will be needed in the following sections.

For an n-simplex Δ^n ($n \geqslant 1$) and for any $0 \leqslant k \leqslant n$, define its *k-face* as a sub-digraph Δ_k^{n-1} obtained from Δ^n by deleting one vertex $k \in V_{\Delta^n}$ and all arrows that are incident to k.

Recall that the n-cube I^n can be described as a digraph of 2^n vertices such that any vertex $\alpha \in V_{I^n}$ can be identified with a sequence $\alpha = (a_1, \ldots, a_n)$ of binary digits so that $\alpha \to \beta$ in I_n if and only if the sequence $\beta = (b_1, \ldots, b_n)$ is obtained from $\alpha = (a_1, \ldots, a_n)$ by replacing a digit 0 by 1 at exactly one position.

Let I^n be an n-cube digraph with $n \geqslant 1$. For any $1 \leqslant p \leqslant n$ we have two inclusions I^{n-1} to sub-digraphs I^{n-1}_{p0} and I^{n-1}_{p1} of I^n, that are defined in the following way. For $I^1 = 0 \to 1$, we have the inclusions

$$V^0_{10} \colon \{0\} \to I^0_{10} = \{0\} \subset I^1$$

and

$$V^0_{11} \colon \{0\} \to I^0_{11} = \{1\} \subset I^1.$$

For $n \geqslant 2$ the inclusions on the sub-digraphs

$$V^{n-1}_{p0} \colon I^{n-1} \to I^{n-1}_{p0} \subset I^n \quad \text{and} \quad V^{n-1}_{p1} \colon I^{n-1} \to I^{n-1}_{p1} \subset I^n$$

are given by

$$V^{n-1}_{p\epsilon}(i_1, \ldots, i_{n-1}) = (i_1, \ldots, i_{p-1}, \epsilon, i_p, \ldots, i_{n-1})$$

where $\epsilon = 0, 1$. We shall call the images $I^{n-1}_{p\epsilon} \subset I^n$ of the maps $V^{n-1}_{p\epsilon}$ for $\epsilon = 0, 1$ the *first* and *second* p-*faces* of the digraph I_n, respectively.

Proposition 2.16. *Let Y be a digraph.*

  i) Let $X = Y \cup \Delta^n$ and $Y \cap \Delta^n = \Delta^{n-1}_k$ where $n \geqslant 1, 0 \leqslant k \leqslant n$. Then there is a one-step strong deformation retraction $F \colon X \square I \to X$ of X onto Y.

  ii) Let $X = Y \cup I^n$ and $Y \cap I^n = I^{n-1}_{p\epsilon}$ where $n \geqslant 1, 0 \leqslant p \leqslant n$. Then there is a one-step strong deformation retraction $F \colon X \square I \to X$ of X onto Y.

Proof. i) Define a strong deformation retraction of Δ^n to Δ^{n-1}_k for two different cases of k. In the case $k \neq n$ define $F \colon \Delta^n \square I \to \Delta^n$ on the set of vertices as

$$F(v, i) = \begin{cases} v, & \text{for } i = 0, \\ v, & \text{for } i = 1, v \neq k, \\ k+1, & \text{for } i = 1, v = k. \end{cases}$$

In the case $k = n$ define $F \colon \Delta^n \square I^- \to \Delta^n$ on the set of vertices as

$$F(v, i) = \begin{cases} v, & \text{for } i = 0, \\ v, & \text{for } i = 1, v \neq n, \\ n-1, & \text{for } i = 1, v = n. \end{cases}$$

Now statement i) follows from Proposition 2.7.

  ii) Consider the case $\epsilon = 1$. Define a strong deformation retraction $F \colon I^n \square I \to I^n$ of I^n to I^{n-1}_{p1} on the set of vertices as

$$F((i_1, \ldots, i_n), i) = \begin{cases} (i_1, \ldots, i_n), & \text{for } i = 0, \\ (i_1, \ldots, i_n), & \text{for } i = 1, i_p = 1, \\ (i_1, \ldots, i_{p-1}, 1, i_p, \ldots, i_n), & \text{for } i = 1, i_p = 0. \end{cases}$$

Consider the case $\epsilon = 0$. Define a strong deformation retraction $F \colon I^n \square I^- \to I^n$ of

I^n to I^{n-1}_{p0} on the set of vertices as

$$F((i_1, \ldots, i_n), i) = \begin{cases} (i_1, \ldots, i_n), & \text{for } i = 0, \\ (i_1, \ldots, i_n), & \text{for } i = 1, i_p = 0, \\ (i_1, \ldots, i_{p-1}, 0, i_p, \ldots, i_n), & \text{for } i = 1, i_p = 1. \end{cases}$$

Now statement ii) follows from Proposition 2.7. $\qquad\square$

Proposition 2.17. *Let $Y_1 \subset Y$ be a digraph pair. Assume that there is a strong deformation retraction $F: Y \square I_n \to Y$ of Y to Y_1. Then, for any digraph X, there is a strong deformation retraction $\mathrm{Id} \square F: X \square (Y \square I_n) \to X \square Y$ of $X \square Y$ to $X \square Y_1$.*

Proof. The digraph homomorphism $\mathrm{Id} \square F$ is a deformation retraction. $\qquad\square$

Corollary 2.18. *Let $K = I^n \square \Delta^m$. Assume that K_1 is a sub-digraph of K that is equal to $I^n \square \Delta^{n-1}_m$ or $I^{n-1}_{p\epsilon}$, where $\Delta^{n-1}_m \subset \Delta^n$ and $I^{n-1}_{p\epsilon} \subset I^n$ are the faces of the cube and simplex, respectively. Then there is a strong deformation retraction $F: K \square I \to K$ of K to K_1.*

3. Path homology of digraphs and digraphs pairs

In this section we recall the definition of path homology for a finite digraph $X = (V, E)$ with coefficients in an arbitrary abelian group K (see [**16**] and [**18**]) and introduce the path homology theory for the category of digraph pairs. Then we construct new exact sequences of the homology groups and describe new relations between cones, suspensions, and cylinders of digraphs.

Let $G = (V, E)$ be a digraph. At first, we define a chain complex $\Lambda_*(V, K)$ of arbitrary paths on the set V of vertices. Then, using a digraph structure given by arrows, we define a chain complex of digraph G and the notion of homology of digraphs (see [**16**] and [**18**]). This approach is dual to the construction of the cochain complex and cohomology groups of digraphs that is based on the universal calculus on the algebra of functions on the vertices [**17**].

Definition 3.1. For any non-negative integer p, an *elementary p-path* on the set V of vertices is any sequence $\{i_k\}^p_{k=0}$ of $p+1$ vertices of V. For $p = -1$, an elementary p-path is the empty set $\emptyset$.

The elementary p-path $\{i_k\}^p_{k=0}$ will also be denoted simply by $i_0 \ldots i_p$, without delimiters between the vertices.

Denote by $\Lambda_p = \Lambda_p(V, K)$ the abelian group that consists of all formal K-linear combinations of all elementary p-paths.

Definition 3.2. The elements of Λ_p are called *p-paths* on V.

An elementary p-path $i_0 \ldots i_p$ as an element of Λ_p will be denoted by $e_{i_0 \ldots i_p}$. The empty set as an element of Λ_{-1} will be denoted by e. By definition, the set $\left\{ e_{i_0 \ldots i_p} \mid i_0, \ldots, i_p \in V \right\}$ is a basis of Λ_p. Each p-path v can be presented by a unique way as a finite sum

$$v = \sum_{i_0, \ldots, i_p \in V} v^{i_0 \ldots i_p} e_{i_0 \ldots i_p}, \tag{1}$$

where $v^{i_0 \ldots i_p} \in K$. Note that Λ_{-1} consists of all multiples of e, so that $\Lambda_{-1} \cong K$.

Definition 3.3. For any $p \geqslant 0$, define a homomorphism $\partial \colon \Lambda_p \to \Lambda_{p-1}$ on the elementary paths

$$\partial e_{i_0 \ldots i_p} = \sum_{q=0}^{p} (-1)^q e_{i_0 \ldots \widehat{i_q} \ldots i_p} \tag{2}$$

(where $\widehat{i_q}$ means omission of the index i_q) and then extend ∂ to Λ_p by K-linearity. We shall refer to ∂ as the *boundary operator*.

For an arbitrary p-path (1) with $p \geqslant 0$, we have

$$\partial v = \sum_{i_0, \ldots, i_p} v^{i_0 \cdots i_p} \partial e_{i_0 \ldots i_p} = \sum_{i_0, \ldots, i_p} \sum_{q=0}^{p} (-1)^q v^{i_0 \cdots i_p} e_{i_0 \ldots \widehat{i_q} \ldots i_p},$$

whence

$$(\partial v)^{j_0 \cdots j_{p-1}} = \sum_{i_0, \ldots, i_p} \sum_{q=0}^{p} (-1)^q v^{i_0 \cdots i_p} \left(e_{i_0 \ldots \widehat{i_q} \ldots i_p} \right)^{j_0 \cdots j_{p-1}}$$

$$= \sum_{k \in V} \sum_{q=0}^{p} (-1)^q v^{j_0 \cdots j_{q-1} k\, j_q \cdots j_{p-1}},$$

where the index k is inserted in the path $j_0 \ldots j_{p-1}$ between j_{q-1} and j_q if $1 \leqslant q < p$, before j_0 if $q = 0$, and after j_{p-1} if $q = p$.

Set also $\Lambda_{-2} = \{0\}$ and define $\partial \colon \Lambda_{-1} \to \Lambda_{-2}$ to be zero.

Lemma 3.4. *We have* $\partial^2 = 0$.

Proof. The operator ∂^2 acts from Λ_p to Λ_{p-2}, so that the identity $\partial^2 = 0$ makes sense for all $p \geqslant 0$. In the case $p = 0$ the identity $\partial^2 = 0$ is trivial. For $p \geqslant 1$, we have by (1)

$$\partial^2 e_{i_0 \ldots i_p} = \sum_{q=0}^{p} (-1)^q \partial e_{i_0 \ldots \widehat{i_q} \ldots i_p}$$

$$= \sum_{q=0}^{p} (-1)^q \left(\sum_{r=0}^{q-1} (-1)^r e_{i_0 \ldots \widehat{i_r} \ldots \widehat{i_q} \ldots i_p} + \sum_{r=q+1}^{p} (-1)^{r-1} e_{i_0 \ldots \widehat{i_q} \ldots \widehat{i_r} \ldots i_p} \right)$$

$$= \sum_{0 \leqslant r < q \leqslant p} (-1)^{q+r} e_{i_0 \ldots \widehat{i_r} \ldots \widehat{i_q} \ldots i_p} - \sum_{0 \leqslant q < r \leqslant p} (-1)^{q+r} e_{i_0 \ldots \widehat{i_q} \ldots \widehat{i_r} \ldots i_p}.$$

After switching q and r in the last sum we see that the two sums cancel out, whence $\partial^2 e_{i_0 \ldots i_p} = 0$. This implies $\partial^2 v = 0$ for all $v \in \Lambda_p$. $\qquad \square$

Consequently, we have the following chain complex:

$$0 \leftarrow K \leftarrow \Lambda_0 \leftarrow \cdots \leftarrow \Lambda_{p-1} \leftarrow \Lambda_p \leftarrow \cdots,$$

where the arrows are given by the boundary operator ∂.

Definition 3.5. We say that an elementary path $e_{i_0 \ldots i_p}\,(p \geqslant 1)$ is *non-regular* if $i_{k-1} = i_k$ for some $k = 1, \ldots, p$, and *regular* otherwise.

For $p \geqslant 1$, consider the subgroup I_p of Λ_p spanned by non-regular elementary paths:

$$I_p = I_p(V) = \operatorname{span}\left\{e_{i_0 \ldots i_p} \mid i_0 \ldots i_p \text{ is non-regular}\right\}.$$

We set also $I_0 = I_{-1} = I_{-2} = 0$.

Lemma 3.6. *If* $e_{i_0 \ldots i_p} \in I_p$ $(p \geqslant 1)$ *then* $\partial e_{i_0 \ldots i_p} \in I_{p-1}$ *and, putting* $\mathcal{R}_p := \Lambda_p/I_p$, *we obtain the chain factor complex*

$$0 \leftarrow K \leftarrow \mathcal{R}_0 \leftarrow \cdots \leftarrow \mathcal{R}_{p-1} \leftarrow \mathcal{R}_p \leftarrow \cdots. \tag{3}$$

Proof. For an $e_{i_0 \ldots i_p} \in I_p$ there exists an index k such that $i_k = i_{k+1}$. Then we have

$$\begin{aligned}
\partial e_{i_0 \ldots i_p} = {} & e_{i_1 \ldots i_p} - e_{i_0 i_2 \ldots i_p} + \cdots \\
& + (-1)^k e_{i_0 \ldots i_{k-1} i_{k+1} i_{k+2} \ldots i_p} + (-1)^{k+1} e_{i_0 \ldots i_{k-1} i_k i_{k+2} \ldots i_p} \\
& + \cdots + (-1)^p e_{i_0 \ldots i_{p-1}}.
\end{aligned} \tag{4}$$

By $i_k = i_{k+1}$ the two terms in the middle line of (4) cancel out, whereas all other terms are non-regular, whence $\partial e_{i_0 \ldots i_p} \in I_{p-1}$. $\qquad\square$

Note that the chain complex $\mathcal{R}_*$ in (3) is defined for a finite set V. Thus we can denote it by $\mathcal{R}_*(V)$, and similarly $\Lambda_p(V)$ and $I_p(V)$ are defined. Any map of finite sets $f: V \to V'$ defines for any $p \geqslant 0$ the *induced homomorphism*

$$f_*: \Lambda_p(V) \to \Lambda_p(V')$$

by the rule $f_*(e_{i_0 \ldots i_p}) = e_{f(i_0) \ldots f(i_p)}$, extended to $\Lambda_p(V)$ by linearity. The map f_* can be considered as a morphism of chain complexes, because by (2) $\partial f_* = f_* \partial$. Since $f_*(I_p(V)) \subset I_p(V')$, we obtain the induced morphism of chain complexes (see [**16**])

$$\mathcal{R}_*(V) \to \mathcal{R}_*(V'), \tag{5}$$

which we also denote by f_*. We can define this homomorphism on basic elements by the rule

$$f_*(e_{i_0 \ldots i_p}) = \begin{cases} e_{f(i_0) \ldots f(i_p)}, & \text{if } e_{f(i_0) \ldots f(i_p)} \text{ is regular,} \\ 0, & \text{if } e_{f(i_0) \ldots f(i_p)} \text{ is non-regular.} \end{cases}$$

Now consider a digraph $X = (V, E)$.

Definition 3.7. An elementary regular p-path $e_{i_0 \ldots i_p}$ on X is called *allowed* if $i_k \to i_{k+1}$ for any $k = 0, \ldots, p-1$, and *non-allowed* otherwise.

For $p \geqslant 0$, let $\mathcal{A}_p(X)$ denote a subgroup of $\mathcal{R}_p(X) := \mathcal{R}_p(V)$ spanned by the allowed elementary p-paths. We put $\mathcal{A}_{-1}(X) = K$. The elements of $\mathcal{A}_p(X)$ are called *allowed p-paths*. Note that $\mathcal{A}_p(X) = \mathcal{R}_p(X)$ for $p = -1, 0, 1$.

Consider the following subgroup of $\mathcal{A}_p(X)$

$$\Omega_p \equiv \Omega_p(X) := \{v \in \mathcal{A}_p : \partial v \in \mathcal{A}_{p-1}\}$$

that is ∂-invariant. Indeed, $v \in \Omega_p$ implies $\partial v \in \mathcal{A}_{p-1}$ and $\partial(\partial v) = 0 \in \mathcal{A}_{p-2}$, whence $\partial v \in \Omega_{p-1}$. The elements of Ω_p are called *∂-invariant p-paths*.

Hence, for any digraph X we obtain a chain complex of abelian groups

$$0 \longleftarrow K \xleftarrow{\partial} \Omega_0 \xleftarrow{\partial} \Omega_1 \xleftarrow{\partial} \cdots \xleftarrow{\partial} \Omega_{p-1} \xleftarrow{\partial} \Omega_p \xleftarrow{\partial} \cdots, \qquad (6)$$

which we shall denote by $\widetilde{\Omega}_* = \widetilde{\Omega}_*(X) = \widetilde{\Omega}_*(X, K)$. By construction we have $\Omega_{-1} = K$, $\Omega_0 = \mathcal{R}_0(X)$ and $\Omega_1 = \mathcal{R}_1(X)$, while in general

$$\Omega_p \subset \mathcal{A}_p(X) \subset \mathcal{R}_p(X).$$

We shall call the chain complex (6) the *reduced path chain complex* of the digraph X. Its homology groups are denoted by $\widetilde{H}_*(X, K)$ and are referred to as the *reduced path homologies of the digraph X with coefficients from K*. We consider also the following part of the chain complex (6)

$$0 \leftarrow \Omega_0 \leftarrow \cdots \leftarrow \Omega_{n-1} \leftarrow \Omega_n \leftarrow \Omega_{n+1} \leftarrow \cdots, \qquad (7)$$

where the definition of the boundary operator ∂ on Ω_0 is modified by setting $\partial \equiv 0$. Note that (7) is a chain complex and our modification does not affect ∂ on Ω_n with $n \geqslant 1$. We shall denote this chain complex by $\Omega_*(X)$. The homology groups of $\Omega_*(X)$ (7) are referred to as the *path homology groups* of the digraph X and are denoted by $H_*(X) = H_*(X, K)$, $n \geqslant 0$. Let us note that the homology groups $H_p(X)$ (as well as the groups $\Omega_p(X)$) can be computed directly by definition using simple tools of linear algebra, in particular, those implemented in modern computational software.

Theorem 3.8 ([**16**]). *i) Let $f \colon X \to X'$ be a digraph map. Then the map $f_* \colon \mathcal{R}_*(V) \to \mathcal{R}_*(V')$ in (5) satisfies the condition*

$$f_*(\mathcal{A}_p(X)) \subset \mathcal{A}_p(X')$$

and its restriction $f_*|_{\Omega_p(X)}$ provides a morphism of reduced chain complexes

$$\widetilde{\Omega}_*(X, K) \to \widetilde{\Omega}_*(X', K)$$

and, consequently, a homomorphism of reduced homology groups

$$\widetilde{H}_*(X, K) \to \widetilde{H}_*(X', K)$$

that will also be denoted by f_*. The same is true for the non-reduced complex and its homology groups.

ii) The homotopy equivalent digraph maps $f, g \colon X \to X'$ induce the equal homomorphisms of (reduced) homology groups. Hence, the (reduced) homology groups introduced above are homotopy invariant.

Now we develop the path homology theory for the category of digraphs pairs. Let $Y \sqsubset X$ be a sub-digraph of a digraph X and denote by i the natural inclusion.

Proposition 3.9. *The induced by i morphism of reduced chain complexes*

$$i_* \colon \widetilde{\Omega}_*(Y) \to \widetilde{\Omega}_*(X)$$

is an inclusion.

Proof. As follows directly from the definition $\Lambda_*(Y) \subset \Lambda_*(X)$ and $\mathcal{R}_*(Y) \subset \mathcal{R}_*(X)$. We have $\mathcal{A}_{-1}(Y) = \mathcal{A}_{-1}(X) = K$ and

$$\mathcal{A}_0(Y) = \mathrm{span}\{v \mid v \in V_G\} \subset \mathrm{span}\{v \mid v \in V_X\} = \mathcal{A}_0(X).$$

For any $p \geqslant 1$ we have the natural inclusion $\mathcal{A}_p(Y) \subset \mathcal{A}_p(X)$ since $i_{k-1} \to i_k$ in Y implies $i_{k-1} \to i_k$ in X. Hence

$$[v \in \Omega_p(Y) \subset \mathcal{A}_p(Y) \subset \mathcal{A}_p(X)] \Rightarrow [\partial v \in \mathcal{A}_{p-1}(Y) \subset \mathcal{A}_{p-1}(X)]$$

and $\widetilde{\Omega}_*(Y) \subset \widetilde{\Omega}_*(X)$. □

Note that the inclusion map i_* in Proposition 3.9 is the identity map $K \to K$ in dimension -1, so that $\widetilde{\Omega}_{-1}(X)/\widetilde{\Omega}_{-1}(Y) = 0$. Now we define a chain complex $\Omega_*(X, Y)$ as factor-complex $\Omega_*(X)/\Omega_*(Y)$ that coincides with $\widetilde{\Omega}_*(X)/\widetilde{\Omega}_*(Y)$. The homology groups of this complex are denoted by $H_*(X, Y) = H_*(X, Y; K)$ and are called the *relative path homology groups*.

Let us give a graph interpretation of the factor-complex in a special case. If U is a subset of V_X then denote by $\mathcal{A}_p^U(X)$ the subgroup of $\mathcal{A}_p(X)$ spanned by all the elementary paths $e_{i_0\dots i_p}$ that intersect U (where the latter means that at least one of the vertices $i_0 \dots i_p$ belongs to U). Set also

$$\Omega_p^U(X) = \Omega_p(X) \cap \mathcal{A}_p^U(X).$$

Lemma 3.10. *Let $Y \sqsubset X$ and let $U = V_X \setminus V_Y$. If there are no arrows from U to V_Y then*

$$\Omega_p(X)/\Omega_p(Y) \cong \Omega_p^U(X). \tag{8}$$

Proof. Any $x \in \Omega_p(X)$ is a linear combination of elementary paths:

$$x = \sum_{i_0\dots i_p \in V_X} x^{i_0\dots i_p} e_{i_0\dots i_p}.$$

Each term $x^{i_0\dots i_p} e_{i_0\dots i_p}$ with non-zero $x^{i_0\dots i_p}$ will be called an elementary term of x. Denote by u the sum of all the elementary terms of x that intersect U, and by v the rest of x, that is the sum of the elementary terms of x that lie in Y. Hence, $x = u + v$. Since x is allowed, both u and v are allowed. Since x is ∂-invariant, the sum $\partial u + \partial v = \partial x$ is allowed. Let $ce_{i_0\dots i_p}$ be an elementary term of u. Since $e_{i_0\dots i_p}$ is allowed and there is no arrow from U to Y, we see that the sequence $i_0 \dots i_p$ looks as follows: some first part, say $i_0 \dots i_q$ lies in Y, and the rest $i_{q+1} \dots i_p$ lies in U, where the second part is non-empty, in particular, $i_p \in U$. Then all the elementary terms of $\partial e_{i_0\dots i_p}$ intersect U except for one case: if i_p is the only vertex of $e_{i_0\dots i_p}$ in U then the elementary term $e_{i_0\dots i_{p-1}}$ of $\partial e_{i_0\dots i_p}$ does not intersect U. Note that in the latter case $e_{i_0\dots i_{p-1}}$ is an allowed path in Y. Hence, we obtain

$$\partial u = (\text{allowed path in } Y) + (\text{sum of terms intersecting } U).$$

Since

$$\partial v = \text{sum of terms lying in } Y,$$

and $\partial u + \partial v$ is allowed, we see that the sum of the terms of ∂u intersecting U must

be allowed, that is,

$$\partial u = (\text{allowed path in } Y) + (\text{allowed path in } U).$$

Hence, ∂u is allowed. It follows that $\partial v = \partial x - \partial u$ is also allowed. Hence, both u and v are ∂-invariant, consequently, $u \in \Omega_p^U(X)$ and $v \in \Omega_p(Y)$.

Therefore, the decomposition $x = u + v$ determines a homomorphism

$$\Omega_p(X) \to \Omega_p^U(X),$$
$$x \mapsto u,$$

which is obviously an epimorphism. Clearly, its kernel is $\Omega_p(Y)$, which implies (8). $\square$

Now we transfer to path homology groups the standard results from algebraic topology (see, for example, [**20**, Chpt. 2.1] and [**27**, Chpt. 1.3]).

Theorem 3.11. *Let $Y \sqsubset X$ be a sub-digraph of a digraph X. Then there is a relative homology long exact sequence*

$$0 \leftarrow H_0(X,Y) \leftarrow H_0(X) \leftarrow H_0(Y) \leftarrow \cdots \leftarrow H_{n-1}(Y) \leftarrow H_n(X,Y) \leftarrow H_n(X) \leftarrow \cdots$$

in which the homomorphisms $H_n(Y) \to H_n(X)$ are induced by the inclusion $i \colon Y \to X$. In particular, if the inclusion $Y \to X$ is a homotopy equivalence, then $H_n(X,Y) = 0$ for all $n \geqslant 0$.

Proof. By definition, we have the short exact sequences of chain complexes

$$0 \to \Omega_*(Y) \to \Omega_*(X) \to \Omega_*(X,Y) \to 0$$

from which the result follows by standard arguments. $\square$

As follows from our construction, there is also a long exact sequence of reduced homology groups for a pair (X,Y) that has exactly the same form and the groups $\widetilde{H}_n(X,Y)$ are the same as $H_n(X,Y)$.

Proposition 3.12. *Let X be a connected digraph. Then $H_0(X) = K$. For any digraph X we have $H_0(X) \cong \bigoplus_n K$ where n is the number of connectivity components of X.*

Proof. Consider a connected digraph X. We can join any two vertices $v, w \in V_X$ by a sequence of edges $v = i_0 \sim v_1 \sim \cdots \sim i_n = w$ where $i_k \sim i_{k+1}$ means that at least one of the cases $i_k \to i_{k+1}$ or $i_k \leftarrow i_{k+1}$ is realized. In $\Omega_0(X)$ we have

$$e_{i_n} = e_{i_0} + \partial \left(\sum_{k=0,(i_k \to i_{k+1})}^{n-1} e_{i_k i_{k+1}} - \sum_{k=0,(i_k \leftarrow i_{k+1})}^{n-1} e_{i_{k+1} i_k} \right),$$

and hence any element from $\Omega_0(X)$ has the form $e_{i_0} + \operatorname{Im} \partial$. The element e_{i_0} does not lie in the image ∂, since for any $x = \sum_k a_k e_{i_k j_k} \in \Omega_1(X)$ we have

$$\partial \left(\sum_k a_k e_{i_k j_k} \right) = \sum_k a_k e_{j_k} - \sum_k a_k e_{e_k},$$

where the sum of coefficients in the right part is zero. The general case can be considered in a similar way. $\square$

Corollary 3.13. *Let X be a connected digraph, and $Y \sqsubset X$ a non-empty connected sub-digraph of X. Then $H_0(X,Y) = 0$.*

Proof. Follows from Theorem 3.11 and Proposition 3.12.   □

Proposition 3.14. *For a digraph X we have*

$$H_n(X) = \begin{cases} \widetilde{H}(X), & n \geqslant 1, \\ \widetilde{H}_0(X) \oplus K, & n = 0. \end{cases}$$

Proof. The proof is standard (see, for example, [**20**] and [**27**]).   □

Theorem 3.15. *A map $f \colon (X,Y) \to (X',Y')$ of digraph pairs induces a homomorphism $f_* \colon H_n(X,Y) \to H_n(X',Y')$ $(n \geqslant 0)$ fitting into a commutative diagram*

$$
\begin{array}{ccccccccc}
\cdots \longrightarrow & H_n(Y) & \longrightarrow & H_n(X) & \longrightarrow & H_n(X,Y) & \longrightarrow & H_{n-1}(Y) & \longrightarrow \cdots \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\cdots \longrightarrow & H_n(Y') & \longrightarrow & H_n(X') & \longrightarrow & H_n(X',Y') & \longrightarrow & H_{n-1}(Y') & \longrightarrow \cdots
\end{array}
$$

in which the rows are the relative long homology exact sequences.

Proof. Follows from [**25**, Chpt. 2, 4].   □

Theorem 3.16. *i) Let two digraph maps $f, g \colon (X,Y) \to (X',Y')$ be homotopic as digraph maps of pairs (see Definition 2.9). Then $f_* = g_* \colon H_n(X,Y) \to H_n(X',Y')$ for any $n \geqslant 0$.*

ii) Let a digraph map $f \colon (X,Y) \to (X',Y')$ be such that its restrictions $f|_X$ and $f|_Y$ are homotopy equivalences. Then $f_* \colon H_n(X,Y) \to H_n(X',Y')$ is an isomorphism for any $n \geqslant 0$.

Proof. Follows from the Five Lemma [**25**, Chpt. 1, 3] and Theorem 3.8.   □

Proposition 3.17. *Let $* \subset X$ be the one-vertex sub-digraph of a digraph X. Then*

$$H_n(*) = \begin{cases} K, & \text{for } n = 0, \\ 0, & \text{for } n \geqslant 1, \end{cases} \quad \text{and} \quad \widetilde{H}_n(*) = 0 \ \forall n \in \mathbb{Z}.$$

We have natural isomorphisms

$$H_*(X,*) \simeq \widetilde{H}_*(X)$$

and

$$H_n(X,*) \simeq H_n(X) \quad \text{for} \ \ n \geqslant 1.$$

Proof. The first statement follows directly from definition of path homology groups. The next statement follows from Proposition 3.14 and definition of relative homology groups (see also [**27**, §3.2]).   □

Now we give a digraph interpretation of the relative homology groups $H_*(X,Y)$ and introduce the relative homology groups $H_*(f)$ for a digraph map $f \colon X \to X'$.

Theorem 3.18. *For a digraph map $f\colon X \to X'$ there is a homology long exact sequence*

$$\cdots \longrightarrow H_n(X) \xrightarrow{f_*} H_n(X') \longrightarrow H_n(C_f, X) \longrightarrow H_{n-1}(X) \longrightarrow \cdots,$$

where we identify X with $j(X) \subset C_f$ by means of the natural inclusion $j\colon X \subset C_f$ from Definition 2.10.

Proof. Consider a commutative diagram of digraphs and digraph maps

$$\begin{array}{ccc} X & \xrightarrow{\ j\ } & C_f \\ \| & & \downarrow{\scriptstyle r} \\ X & \xrightarrow{\ f\ } & X' \end{array} \tag{9}$$

in which r is the deformation retraction. From (9) we obtain the commutative diagram of groups and homomorphisms

$$\begin{array}{ccccccc} \cdots \longrightarrow & H_n(X) & \xrightarrow{\ j_*\ } & H_n(C_f) & \longrightarrow & H_n(C_f, X) & \longrightarrow \cdots \\ & \| & & {\scriptstyle \cong}\downarrow{\scriptstyle r_*} & & & \\ & H_n(X) & \xrightarrow{\ f_*\ } & H_n(X') & & & \end{array}$$

and the statement of the theorem follows. $\qquad\square$

In what follows we shall denote by $H_*(f)$ the relative homology groups $H_*(C_f, X)$. Note that there is a result that is similar to Theorem 3.18 for reduced homology groups. It is an easy exercise to reformulate the result for the inverse cylinder C_f^-.

Proposition 3.19. *Let $Y \sqsubset X$ and assume that there is a deformation retraction $r\colon Y \to * \subset Y$. Then we have isomorphisms of relative homology groups*

$$H_*(X, Y) \cong H_*(X, *).$$

Proof. The natural map $(X, *) \to (X, Y)$ of digraph pairs induces the commutative diagram

$$\begin{array}{ccc} \Omega_*(*) & \longrightarrow & \Omega_*(X) \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle \mathrm{Id}} \\ \Omega_*(Y) & \longrightarrow & \Omega_*(X). \end{array}$$

Now the result follows from Theorem 3.15 and the Five Lemma, since the maps induced by i and the identity map Id are isomorphisms. $\qquad\square$

Corollary 3.20. *For any inclusion $i\colon Y \to X$ of digraphs we have isomorphisms of relative homology groups*

$$H_*(Cone_i, Cone(Y)) \cong H_*(Cone_i, *).$$

Proof. Any inclusion i induces the inclusion $Cone(Y) \subset Cone_i$. Now the result follows from Proposition 3.19, since there is a deformation retraction of $Cone(Y)$ onto its vertex $*$. $\qquad\square$

Now consider a triple of digraphs $Z \sqsubset Y \sqsubset X$. Denote the inclusions of the digraphs in the following way

$$i\colon Y \to X, \; j\colon Z \to Y, \; k = i \circ j\colon Z \to X.$$

Theorem 3.21. *For a triple of digraphs $Z \sqsubset Y \sqsubset X$ there is the commutative diagram of abelian groups and homomorphisms*

$$
\cdots \to H_{n+1}(X,Y) \longrightarrow H_n(Y,Z) \longrightarrow H_{n-1}(Z) \to \cdots
$$

$$
H_n(Y) \qquad\qquad H_n(X,Z) \tag{10}
$$

$$
\cdots \longrightarrow H_n(Z) \longrightarrow H_n(X) \longrightarrow H_n(X,Y) \to \cdots
$$

consisting of the following relative long exact sequences

$$
\cdots \longrightarrow H_{n+1}(X,Y) \longrightarrow H_n(Y) \xrightarrow{i_*} H_n(X) \longrightarrow H_n(X,Y) \to \cdots,
$$

$$
\cdots \longrightarrow H_{n+1}(X,Z) \longrightarrow H_n(Z) \xrightarrow{k_*} H_n(X) \longrightarrow H_n(X,Z) \to \cdots,
$$

$$
\cdots \longrightarrow H_{n+1}(Y,Z) \longrightarrow H_n(Z) \xrightarrow{j_*} H_n(Y) \longrightarrow H_n(Y,Z) \to \cdots,
$$

$$
\cdots \longrightarrow H_{n+1}(X,Y) \longrightarrow H_n(Y,Z) \longrightarrow H_n(X,Z) \longrightarrow H_n(X,Y) \to \cdots.
$$

Proof. By Proposition 3.9, there are the natural inclusions of chain complexes

$$\Omega_*(Z) \longrightarrow \Omega_*(Y) \longrightarrow \Omega_*(X).$$

By the Noether isomorphism theorem (see [**30**, Chpt. 4]) there is a short exact sequence of chain complexes

$$0 \longrightarrow \Omega_*(Y)/\Omega_*(Z) \longrightarrow \Omega_*(X)/\Omega_*(Z) \longrightarrow \Omega_*(X)/\Omega_*(Y) \longrightarrow 0.$$

Thus we can write down the commutative diagram of chain complexes and chain homomorphisms

$$
\begin{array}{ccccccccc}
& & 0 & & 0 & & \\
& & \downarrow & & \downarrow & & \\
0 \to & \Omega_*(Z) & \to & \Omega_*(Y) & \to & \Omega_*(Y)/\Omega_*(Z) & \to 0 \\
& \| & & \downarrow & & \downarrow & \\
0 \to & \Omega_*(Z) & \to & \Omega_*(X) & \to & \Omega_*(X)/\Omega_*(Z) & \to 0 \\
& & & \downarrow & & \downarrow & \\
& & \Omega_*(X)/\Omega_*(Y) & \xrightarrow{\cong} & \Omega_*(X)/\Omega_*(Y) & \\
& & \downarrow & & \downarrow & \\
& & 0 & & 0 &
\end{array}
$$

in which two rows and two columns are short exact sequences. Now, passing to the homology long exact sequences of these short exact sequences of chain complexes we obtain commutative diagram (10). $\qquad \square$

As follows from the proof, there exists the braid of exact sequence (10) also for reduced homology groups of the triple of digraphs.

Corollary 3.22. *Assume that, for a triple of digraphs $Z \sqsubset Y \sqsubset X$, one of the inclusions $i\colon Y \to X$ or $j\colon Z \to Y$ is a homotopy equivalence. Then in the first case we have an isomorphism $H_*(Y, Z) \cong H_*(X, Z)$, and in the second case $H_*(X, Z) \cong H_*(X, Y)$.*

Proof. Follows from Theorem 3.11 and diagram (10).   □

Note that the braid of exact sequences (10) is natural relative to digraph maps of triples of digraphs.

Now for the path homology groups, we formulate and prove a result that is similar to the classical Mayer-Vietoris exact sequence.

Let $X = Y_1 \cup Y_2$ and $Z = Y_1 \cap Y_2$ be, respectively, the union and intersection of two digraphs. Then we can write down the following commutative diagram of the natural inclusions of the digraphs:

$$
\begin{array}{ccc}
Z & \xrightarrow{\ i^1\ } & Y_1 \\
{\scriptstyle i^2}\big\downarrow & & \big\downarrow{\scriptstyle j^1} \\
Y_2 & \xrightarrow{\ j^2\ } & X.
\end{array}
\tag{11}
$$

For any $p \geqslant -1$ the commutative square (11) induces a commutative square of abelian groups

$$
\begin{array}{ccc}
\mathcal{A}_p(Z) & \xrightarrow{\ i^1_*\ } & \mathcal{A}_p(Y_1) \\
{\scriptstyle i^2_*}\big\downarrow & & \big\downarrow{\scriptstyle j^1_*} \\
\mathcal{A}_p(Y_2) & \xrightarrow{\ j^2_*\ } & \mathcal{A}_p(X),
\end{array}
\tag{12}
$$

in which all homomorphisms are injective.

Lemma 3.23. *Suppose that any allowed elementary path on X lies in Y_1 or Y_2. Then for $p \geqslant -1$ the square (12) induces the following short exact sequence of abelian groups:*

$$
0 \longrightarrow \mathcal{A}_p(Z) \xrightarrow{\ \delta\ } \mathcal{A}_p(Y_1) \oplus \mathcal{A}_p(Y_2) \xrightarrow{\ d\ } \mathcal{A}_p(X) \longrightarrow 0,
\tag{13}
$$

where $\delta = (i^1_*, i^2_*)$ and

$$
d(a, b) = j^1_*(a) - j^2_*(b).
$$

Proof. The map δ is evidently a monomorphism. The map d is an epimorphism by the assumptions. The condition $d \circ \delta = 0$ follows from the definition of these maps and commutativity of (12). The condition $d(a, b) = 0$ implies that the paths $j^1_*(a)$ and $j^2_*(b)$ coincide in $\mathcal{A}_p(X)$, and since the maps j^1_* and j^2_* are monomorphisms, this implies that elementary allowed p-paths fitting into basis decompositions of a and b are the same and hence these elementary p-paths lie in $\mathcal{A}_p(Y_1 \cap Y_2) = \mathcal{A}_p(Z)$. This means that $\operatorname{Ker} d \subset \operatorname{Im} \delta$, and the lemma is proved.   □

Lemma 3.24. *Let the assumption of Lemma 3.23 be satisfied. Assume in addition that the homomorphism*

$$\Omega_p(Y_1) \oplus \Omega_p(Y_2) \longrightarrow \Omega_p(X) \tag{14}$$

that is induced by d from Lemma 3.23 is an epimorphism for any p. Then there is a short exact sequence of chain complexes

$$0 \longrightarrow \widetilde{\Omega}_*(Z) \overset{\delta}{\longrightarrow} \widetilde{\Omega}_*(Y_1) \oplus \widetilde{\Omega}_*(Y_2) \overset{d}{\longrightarrow} \widetilde{\Omega}_*(X) \longrightarrow 0. \tag{15}$$

Proof. It follows from Proposition 3.9 that there is a commutative diagram of chain complexes

$$
\begin{array}{ccc}
\widetilde{\Omega}_*(Z) & \overset{i_*^1}{\longrightarrow} & \widetilde{\Omega}_*(Y_1) \\
{\scriptstyle i_*^2}\big\downarrow & & \big\downarrow{\scriptstyle j_*^1} \\
\widetilde{\Omega}_*(Y_2) & \overset{j_*^2}{\longrightarrow} & \widetilde{\Omega}_*(X)
\end{array}
\tag{16}
$$

in which all maps are inclusions. Hence the map δ in (15) is a monomorphism. The map d is an epimorphism by the assumptions of the lemma. From commutativity of (16) we obtain that $d \circ \delta = 0$ in (13).

Now let $d(a, b) = j_*^1(a) - j_*^2(b) = 0$ in (13), where

$$a \in \widetilde{\Omega}_n(Y_1) \subset \mathcal{A}_n(Y_1), \ b \in \widetilde{\Omega}_n(Y_2) \subset \mathcal{A}_n(Y_2).$$

By Lemma 3.23, there is an element $x \in \mathcal{A}_n(Z)$ such that

$$i_*^1(x) = a, \ i_*^2(x) = b, \ j_*^1 \circ i_*^1(x) = j_*^2 \circ i_*^2(x) \in \mathcal{A}_n(X).$$

Since all maps in (12) are inclusions, we conclude that $x \in \widetilde{\Omega}_n(Y_1)$ and $x \in \widetilde{\Omega}_n(Y_2)$. Hence $\partial x \in \mathcal{A}_{n-1}(Y_1)$ and $\partial x \in \mathcal{A}_{n-1}(Y_2)$, that is $\partial x \in \mathcal{A}_{n-1}(Z)$. Hence $x \in \widetilde{\Omega}_n(Z)$ and lemma is proved. □

Theorem 3.25. (Mayer-Vietoris exact sequence) *Under the assumptions of Lemma 3.24 the square* (10) *induces a long exact sequence of homology groups:*

$$\cdots \longrightarrow \widetilde{H}_n(Z) \overset{\delta}{\longrightarrow} \widetilde{H}_n(Y_1) \oplus \widetilde{H}_n(Y_2) \overset{d}{\longrightarrow} \widetilde{H}_n(X) \overset{\partial}{\longrightarrow} \widetilde{H}_{n-1}(Z) \longrightarrow \cdots,$$

where $\delta = (i_*^1, i_*^2)$, $d(a,b) = j_*^1(a) - j_*^2(b)$, and ∂ is a connecting homomorphisms.

Proof. Follows from Lemma 3.24 by the zig-zag lemma. □

Note that under the assumptions of Lemma 3.24 there is also the similar Mayer-Vietoris exact sequence of non-reduced homology groups.

Now we give yet one example in which the Mayer-Vietoris exact sequence exists.

Definition 3.26. Let (Y_1, y_1) and (Y_2, y_2) be based digraphs. The *wedge sum (or bouquet)*

$$X = Y_1 \bigvee_{y_1 \sim y_2} Y_2$$

of these digraphs is a digraph X with the set V_X of vertices that is obtained from the disjoint union $V_{Y_1} \bigcup V_{Y_2}$ by identification of based vertices y_1 and y_2, and with

the set of edges $E_X = E_{Y_1} \bigcup E_{Y_2}$. We shall denote by $*$ the vertex that is obtained by identification of y_1 and y_2.

Identifying Y_i with the sub-digraph of X and denoting $* = Z$, we have natural inclusions of digraphs as in commutative diagram (11).

Lemma 3.27. *Let* $X = Y_1 \bigvee_{y_1 \sim y_2} Y_2$ *and* $Z = * \subset X$. *Then there is a short exact sequence of chain complexes* (15) *and hence the Mayer-Vietoris exact sequence as in Theorem* 3.25.

Proof. We have the commutative square (16) in which all the maps are inclusions. Hence the map δ is a monomorphism and it follows from the commutativity that $d \circ \delta = 0$.

Let us show that $\operatorname{Ker} d \subset \operatorname{Im} \delta$. If $\omega \in \operatorname{Ker} d$ then $\omega = (a, b)$ where $a \in \widetilde{\Omega}_*(Y_1)$, $b \in \widetilde{\Omega}_*(Y_2)$ and $j^1_*(a) = j^2_*(b)$. However, the latter is possible only if a and b have dimensions 0 or -1. For these two cases the result follows directly from (16).

Finally, let us check that d in (15) is an epimorphism. In the dimensions $-1, 0, 1$ this is evidently. If $p \geqslant 2$ and $\omega \in \widetilde{\Omega}_*(X)$ then consider an elementary term $c e_{i_0 \ldots i_p}$ of ω $(c \neq 0)$. We need only to show that all the vertices $i_0 \ldots i_p$ lie either in Y_1 or in Y_2. Indeed, assume that this is not the case, that is, $*$ is one of the vertices $i_1, \ldots, i_{p-1}$. The path $\partial(c e_{i_0 \ldots i_p})$ contains the term $c e_{i_0 \ldots \widehat{*} \ldots i_p}$ that is not allowed. This term must be cancelled in $\partial \omega$ using another elementary term of ω. However, the latter term should then be $-c e_{i_0 \ldots i_p}$, which is not possible. $\square$

Corollary 3.28 (Cf. [**17**, Th.4.17])**.** *Under assumptions of Lemma* 3.27 *we have isomorphisms*

$$\widetilde{H}_n(Y_1 \bigvee Y_2, K) \cong \widetilde{H}_n(Y_1, K) \oplus \widetilde{H}_n(Y_2, K).$$

Proof. Follows from the Mayer-Vietoris exact sequence for $*, Y_1, Y_2, Y_1 \bigvee Y_2$ and the isomorphisms $\widetilde{H}_*(*, K) = 0$ in Proposition 3.17. $\square$

Now we give the path homology version of the *excision axiom of Eilenberg-Steenrod*.

Theorem 3.29. *Let* $X = (V_X, E_X)$ *be a digraph in which there are two subsets* U_1, U_2 *of* V_X *such that there are no outcoming arrows from* U_i *to* $V_X \setminus U_i$ *for* $i = 1, 2$. *Denote by* Y_i $(i = 1, 2)$ *the digraph that is obtained from* X *by deleting all the vertices from* U_k *with* $k \neq i$ *and all arrows adjoint to these vertices (see Fig.* 1*). Then the inclusion*

$$j \colon (Y_1, Y_1 \cap Y_2) \to (X, Y_2)$$

induces an isomorphism of chain complexes

$$j_* : \Omega_*(Y_1)/\Omega_*(Y_1 \cap Y_2) \to \Omega_*(X)/\Omega_*(Y_2)$$

and, hence, an isomorphism of homology groups

$$j_* \colon H_n(Y_1, Y_1 \cap Y_2) \to H_n(X, Y_2).$$
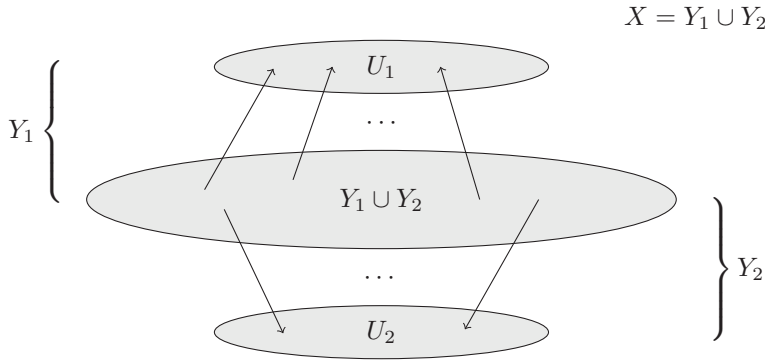
$$X = Y_1 \cup Y_2$$

Figure 1: Digraphs X, Y_1, Y_2.

Proof. By Lemma 3.10 we have isomorphisms

$$\Omega_n(X)/\Omega_n(Y_2) \cong \Omega_n^{U_1}(X)$$

and

$$\Omega_n(Y_1)/\Omega_n(Y_1 \cap Y_2) \cong \Omega_n^{U_1}(Y_1).$$

The inclusion j induces a morphism j_* of chain complexes, so that we have the following commutative diagram:

$$
\begin{array}{ccc}
\Omega_n(X)/\Omega_n(Y_2) & \xrightarrow{\cong} & \Omega_n^{U_1}(X) \\
{\scriptstyle j_*}\big\uparrow & & \big\uparrow{\scriptstyle =} \\
\Omega_n(Y_1)/\Omega_n(Y_1 \cap Y_2) & \xrightarrow{\cong} & \Omega_n^{U_1}(Y_1).
\end{array}
$$

Here the homomorphism $\Omega_n^{U_1}(X) \to \Omega_n^{U_1}(Y_1)$ is induced by the inclusion and the fact that it is the identity follows from the hypothesis that there are no outcoming arrows from $U_2 = V_X \setminus V_{Y_1}$ to V_{Y_1}. Hence, j_* is an isomorphism of chain complexes, which was to be proved. $\square$

Note that a similar result to Theorem 3.29 is true if we change in this theorem "outcoming arrows" to "incoming arrows".

4. Homotopy constructions: cylinder, cone and suspension

In this section we give several new constructions of digraphs which are very helpful for developing path homology theory. In particular, we describe several transformations of digraphs that preserve homology groups.

Definition 4.1. For any digraph X and any line digraph $I_n \in \mathcal{I}_n$ define a *direct cylinder of* X as the digraph

$$Cyl^d(X) = X \square I_n, \quad I_n \in \mathcal{I}_n, n \geqslant 1.$$

Note that there exist 2^n digraphs $X \square I_n$ and hence, the cylinder is not unique.

It is easy to check that there is a strong deformation retraction $Cyl^d(X) \to X$ (see [**16**]), and hence we have a homotopy equivalence $Cyl^d(X) \simeq X$.

Definition 4.2 ([**16**]). A digraph map $h\colon I_n \to I_m$ is called *shrinking* if $h(0) = 0$, $h(n) = m$, and $h(i) \leqslant h(j)$ whenever $i \leqslant j$.

Any shrinking map $h\colon I_n \to I_m$ is surjective and the preimage of any arrow of I_m consists of exactly one arrow of I_n. Furthermore, we have necessarily $m \leqslant n$, and if $n = m$ then h is a bijection.

Let $h\colon I_n \to I_m$ be a shrinking map. Then the cylinder C_h and inverse cylinder C_h^- of the map h were defined in Definition 2.10. Define a *double cylinder* C_{hh} *of the map* h as the following digraph:

$$C_{hh} = C_h \cup C_h^-, \quad \text{where } V_{C_{hh}} = V_{I_n} \cup V_{I_m}$$

and

$$E_{C_{hh}} = E_{I_n} \cup E_{I_m} \cup \{i \to h(i) \mid i \in V_{I_n}\} \cup \{i \leftarrow h(i) \mid i \in V_{I_n}\}.$$

Fix a digraph $X = (V_X, E_X)$. We write $v \sim w$ if v and w are adjacent vertices of X. In order to define a *homotopy cylinder* of X we need the following preliminary construction:

i) Choose a collection $C(V) = \{I(v)\}_{v \in V}$ where each $I(v)$ is a line digraph from $\mathcal{I}_{n(v)}$ with $n(v) \geqslant 0$.

ii) For any pair $v \sim w$ we choose either a shrinking map $h_{v,w}\colon I(v) \to I(w)$ or a shrinking map $h_{w,v}\colon I(w) \to I(v)$.

iii) For any pair $v \sim w$ we define a digraph $C_{v\sim w}$ as follows. In the case $(v \to w) \in E_X, (w \to v) \notin E_X$ we set

$$C_{v\sim w} = \begin{cases} C_{h_{v,w}} & \text{if } h_{v,w} \text{ was chosen in ii)}, \\ C_{h_{w,v}}^- & \text{if } h_{w,v} \text{ was chosen in ii).} \end{cases}$$

In the case $(v \to w) \in E_X, (w \to v) \in E_X$ set

$$C_{v\sim w} = \begin{cases} C_{h_{v,w}h_{v,w}} & \text{if } h_{v,w} \text{ was chosen in ii)}, \\ C_{h_{w,v}h_{w,v}} & \text{if } h_{w,v} \text{ was chosen in ii).} \end{cases}$$

Definition 4.3. Assume that X is a connected digraph and $E_X \neq \emptyset$. A *homotopy cylinder* $Cyl^h(X)$ of X is a digraph that is given by the union

$$\bigcup_{v \sim w} C_{v\sim w},$$

where we identify the sub-digraphs $I(v)$ lying in various digraphs $C_{v\sim w}$ in a natural way.

By definition, we have natural inclusions

$$i_v\colon I(v) \to Cyl^h(X) \quad \text{and} \quad j_{vw}\colon C_{v\sim w} \to Cyl^h(X),$$

for any vertex $v \in V_X$ and for any pair $v \sim w$.

Note that any direct cylinder $Cyl^d(X)$ has a natural structure of homotopy cylinder. For the next statements, we always assume that X is connected and $E_X \neq \emptyset$.

Proposition 4.4. *For a homotopy cylinder $Cyl^h(X)$ there are natural inclusions*

$$i_0 \colon X \longrightarrow Cyl^h(X) \quad and \quad i_1 \colon X \longrightarrow Cyl^h(X)$$

and a retraction

$$r \colon Cyl^h(X) \longrightarrow X$$

such that $r \circ i_0 = \mathrm{Id}_X$, $r \circ i_1 = \mathrm{Id}_X$.

Proof. Define an inclusion $i_0 \colon X \to Cyl^h(X)$ on the set of vertices by the composition

$$v \xrightarrow{v \to 0} I(v) \xrightarrow{i_v} Cyl^h(X),$$

where the first map is the inclusion $v \to 0 \in I(v)$. Define an inclusion $i_1 \colon X \to Cyl^h(X)$ on the set of vertices by the composition

$$v \xrightarrow{v \to n(v)} I(v) \xrightarrow{i_v} Cyl^h(X),$$

where the first map is the inclusion $v \to n(v) \in I(v)$, where $n(v) = |I(v)|$. Define the map r on the set of vertices by $r(v, k) = v$ for $v \in V_X, k \in V_{I(v)}$. $\qquad\square$

Theorem 4.5. *The map r in Proposition 4.4 is a strong deformation retraction.*

Proof. Denote by C the digraph $Cyl^h(X)$. If all $n(v) = 0$ then there is nothing to prove. Consider a vertex $v \in V_X$ with $n(v) \geqslant 1$. Then $(v, n(v) - 1) \sim (v, n(v))$ in C.

Consider a set W of all vertices w of X that satisfy the following property: there is a sequence of vertices $v = v_0, v_1, \ldots, v_s = w$ on X such that $v_i \sim v_{i+1}$ in X for any $i = 0, \ldots, s - 1$, and

$$(v_i, n(v_i)) \sim (v_{i+1}, n(v_{i+1})), \ (v_i, n(v_i) - 1) \sim (v_{i+1}, n(v_{i+1}) - 1),$$

in C. Define a digraph map $\phi \colon C \to C$ on the set of vertices of C by

$$\phi(u, k) = \begin{cases} (u, k - 1), & \text{for } (u, k) = (w, n(w)), \quad w \in W \cup \{v\}, \\ (u, k), & \text{otherwise.} \end{cases}$$

The map ϕ is a strong deformation retraction (see [**16**, Corollary 3.7]) of C to a sub-digraph that is also a homotopical cylinder over X and that has $|V_C| - |W| - 1$ vertices. From now the result follows by induction using Proposition 2.8. $\qquad\square$

Corollary 4.6. *For any finite digraph X and any homotopy cylinder digraph $Cyl^h(X)$ we have an isomorphism*

$$\widetilde{H}_*(X) \cong \widetilde{H}_*(Cyl^h(X)).$$

Proof. Follows from Theorem 3.8. $\qquad\square$

Now we introduce a new definition of homotopy and prove that it is equivalent to our previous definition. As we shall see below this generalization is very natural.

Definition 4.7. Let $f, g \colon X \to Y$ be digraph maps. A digraph map $F \colon Cyl^h(X) \to Y$ is called a *generalized homotopy* of f and g if

$$F \circ i_0 = f, \quad F \circ i_1 = g,$$

where i_0, i_1 are the inclusions from Proposition 4.4. In this case the maps f and g are called *generalized homotopic.*

Theorem 4.8. *Let X be a finite digraph. Two digraph maps $f, g\colon X \to Y$ are generalized homotopic if and only if they are homotopic.*

Proof. In one direction the statement is trivial since a homotopy provides evidently a generalized homotopy.

Now by Theorem 4.5 there is a strong deformation retraction

$$R\colon Cyl^h(X)\square I_n \to Cyl^h(X)$$

such that

$$R|_{Cyl^h(X)\square\{0\}} = \mathrm{Id}, \quad R|_{Cyl^h(X)\square\{n\}}\colon Cyl^h(X)\square\{n\} \to i_0(X),$$

where the last map is the identity on $i_0(X)$, and as follows from the proof of Theorem 4.5 it maps isomorphically the sub-digraph $i_1(X)$ to $i_0(X)$ (for any vertex $v \in V_X$ we have $R(i_1(v)) = i_0(v)$).

Let F be a generalized homotopy of f and g. Consider a composition Ψ of digraph maps

$$X\square I_n \xrightarrow{i\square\,\mathrm{Id}} i_1(X)\square I_n \xrightarrow{R} Cyl^h(X) \xrightarrow{F} Y,$$

where $\Psi|_{X\square\{0\}} = F \circ i_1(X) = g$ and $\Psi|_{X\square\{n\}} = F \circ i_0(G) = f$. Clearly, Ψ is a homotopy of f, g. $\qquad\square$

Corollary 4.9. *Generalized homotopy digraph maps induce the same homomorphism of (reduced) path homology groups.*

Definition 4.10. i) Define a *homotopy cone* $Cone^h(X)$ of a digraph X as a digraph that is obtained from a homotopy cylinder $Cyl^h(X)$ by adding a new vertex a and the set of arrows

$$\{(v, n(v)) \to a \mid \forall v \in V_X\}.$$

ii) Define an *inverse homotopy cone* $Cone^{h^-}(X)$ of a digraph X as a digraph that is obtained from a homotopy cylinder $Cyl^h(X)$ by adding a new vertex a and the set of arrows

$$\{(v, n(v)) \leftarrow a \mid \forall v \in V_X\}.$$

The vertex a will be referred to as a *vertex of the (inverse) homotopy cone.*

Note, that the cone $Cone(X)$ and the inverse cone $Cone^-(X)$ of a digraph X (see Definition 2.9) are particular cases of a homotopy cone and inverse homotopy cone, respectively. It follows directly from Definition 4.10, that there are natural inclusions $i\colon X \to Cone^h(X)$ and $i^-\colon X \to Cone^{h^-}(X)$ that are induced by the inclusion $i_0\colon X \to Cyl^h(X)$. Using these inclusions we can consider the digraph X as a sub-digraph of $Cone^h(X)$ and $Cone^{h^-}(X)$.

Theorem 4.11. *Let X be a digraph and let $Cone^h(X)$ be a homotopy cone with a vertex $*$. Then there are strong deformation retractions*

$$r\colon Cone^h(X) \longrightarrow *, \ r^-\colon Cone^{h^-}(X) \longrightarrow *,$$

and hence, in particular, $Cone^h(X) \simeq * \simeq Cone^{h^-}(X)$.

Proof. Similar to Theorem 4.5. $\qquad\square$

Definition 4.12. The *homotopy suspension* $S^h(X)$ of a digraph X is a digraph defined as a union of two not necessarily equal digraphs of $Cone_1^h(X)$ and $Cone_2^h(X)$ that intersect in X:

$$S^h(X) = Cone_1^h(X) \bigcup_X Cone_2^h(X).$$

The *inverse homotopy suspension* $S^{h^-}(X)$ of a digraph X is a digraph defined as a union of two not necessarily equal digraphs of $Cone_1^{h^-}(X)$ and $Cone_2^{h^-}(X)$ that intersect in X:

$$S^{h^-}(X) = Cone_1^{h^-}(X) \bigcup_X Cone_2^{h^-}(X).$$

Note that the suspension $S(X)$ and inverse suspension $S^-(X)$ of a digraph X (see Definition 2.12) are particular cases of homotopy suspensions.

Now we would like to describe homology properties of homotopy suspensions.

Theorem 4.13. *Let X be a digraph. Then for any $n \geqslant 0$ there is an isomorphism $\widetilde{H}_{n+1}(S(X)) \cong \widetilde{H}_n(X)$.*

Proof. By Definition 2.12 we have

$$S(X) = Y_1 \bigcup_X Y_2,$$

where $Y_i = Cone_{a_i}(X)$, $i = 1, 2$ with distinct vertices a_1, a_2. Observe that any allowed elementary path x on $S(X)$ lies in Y_1 or Y_2, so that the hypothesis of Lemma 3.23 is satisfied. Let us verify also the hypothesis (14) of Lemma 3.24, that is, any ∂-invariant path on $S(X)$ is a sum of ∂-invariant paths on Y_1 and Y_2. Clearly, any path $x \in \mathcal{A}_n(S(X))$ can be represented in the form

$$x = w + w_1 + w_2,$$

where $w \in \mathcal{A}_n(X)$, $w_1 \in \mathcal{A}_n^{\{a_1\}}(Y_1)$, $w_2 \in \mathcal{A}_n^{\{a_2\}}(Y_2)$ (in the notation of Lemma 3.10). Let now $x \in \Omega_n(S(X))$ so that $\partial x \in \mathcal{A}_{n-1}(S(X))$. Since $\partial x = \partial w + \partial w_1 + \partial w_2$, arguing as in the proof of Lemma 3.10 we obtain that $\partial w \in \mathcal{A}_{n-1}(X)$ and $\partial w_i = u_i + v_i$ where $u_i \in \mathcal{A}_{n-1}(X)$ and $v_i \in \mathcal{A}_{n-1}^{(a_i)}(Y_i)$. Consequently, $w \in \Omega_n(X)$, $w_i \in \Omega_n(Y_i)$.

Hence, any $x \in \Omega_n(S(X))$ can be written in the form $x = (w + w_1) + w_2$ where

$$(w + w_1) \in \Omega_n(Y_1) \quad \text{and} \quad w_2 \in \Omega_n(Y_2).$$

By Lemma 3.24 we obtain the Mayer-Vietoris exact sequence of reduced homology groups

$$\cdots \to \widetilde{H}_n(X) \to \widetilde{H}_n(Y_1) \oplus \widetilde{H}_n(Y_2) \to \widetilde{H}_n(S(X)) \to \widetilde{H}_{n-1}(X) \to \cdots.$$

The cone Y_i is homotopy equivalent to a one-vertex digraph, hence all reduced homology groups of Y_i are trivial. Now from the exact sequence we obtain the claim. $\square$

Theorem 4.14. *For any finite digraph X and for $n \geqslant 0$ there are isomorphisms*

$$\widetilde{H}_{n+1}(S^h(X)) \cong \widetilde{H}_n(X),$$

where $S^h(X)$ is a homotopy suspension of the digraph X.

Proof. The proof is similar to Theorem 4.13. $\square$

5. Path homology theory and Eilenberg-Steenrod axioms

In this section we present an axiomatic approach to the definition of path homology theory on the category of digraph pairs. The situation here is very similar to the classical Eilenberg-Steenrod axioms of algebraic topology (see, for example, [**26**, §26]).

Denote by R a functor $\mathcal{D}^2 \to \mathcal{D}^2$ defined by the rule

$$R(X,Y) = (Y,\emptyset) \ \text{ and } \ R(f) = f|_Y \ (f\colon (X,Y) \to (X',Y')).$$

This functor evidently induces the functor $\operatorname{Ho} R\colon \operatorname{Ho}\mathcal{D}^2 \to \operatorname{Ho}\mathcal{D}^2$ by the rule

$$(\operatorname{Ho} R)(X,Y) = (Y,\emptyset) \ \text{ and } \ (\operatorname{Ho} R)[f] = [f|_Y] \ (f\colon (X,Y) \to (X',Y')),$$

where $[f]$ denote the homotopy class of the map f. Let $\mathcal{A}$ be the category of abelian groups and homomorphisms of abelian groups.

Definition 5.1. A *non-reduced homology theory* on the category $\operatorname{Ho}\mathcal{D}^2$ of digraph pairs and homotopy classes of digraph maps of pairs consists of the sequence of functors

$$H_p\colon \operatorname{Ho}\mathcal{D}^2 \to \mathcal{A} \ \text{ for } p \in \mathbb{Z}$$

and natural transformations

$$\partial_n\colon H_n \longrightarrow H_{n-1} \circ R \ \text{ for } p \in \mathbb{Z},$$

such that the following axioms are satisfied.

i) *Exactness axiom:* for every pair $(X,Y) \in \operatorname{Ho}\mathcal{D}^2$ the sequence

$$\cdots \longrightarrow H_{n+1}(X,Y) \xrightarrow{\partial_{n+1}(X,Y)} H_n(Y,\emptyset) \xrightarrow{H_n(i)} H_n(X,\emptyset) \xrightarrow{H_n(j)} H_n(X,Y) \xrightarrow{\partial_n(X,Y)} \cdots$$

is exact, where

$$i\colon (Y,\emptyset) \to (X,\emptyset), \quad j\colon (X,\emptyset) \to (X,Y)$$

are the natural inclusions.

ii) *Excision axiom.* Consider two cases.

(a) Let X be a digraph in which there are two sub-digraphs A_1 and A_2 such that there are no outcoming arrows from V_{A_i} to $V_X \setminus V_{A_i}$ for $i = 1, 2$. Let Y_i $(i = 1, 2)$ be obtained from X by deleting the digraph A_i and all incoming to V_{A_i} arrows starting from $V_X \setminus V_{A_i}$. ("Outcoming arrows" can be replaced everywhere by "incoming arrows".)

(b) Let $X = Y_1 \cup Y_2$ and $Y_1 \cap Y_2 = *$ is a one-vertex digraph.

In both cases (a), (b) the inclusion

$$j\colon (Y_1, Y_1 \cap Y_2) \to (X, Y_2)$$

induces isomorphisms

$$H_n(j)\colon H_n(Y_1, Y_1 \cap Y_2) \to H_n(X, Y_2), \quad n \in \mathbb{Z}.$$

iii) *Additivity axiom:* if $X = \bigcup_\alpha X_\alpha$ is the disjoint union of digraphs then

$$H_n(X,\emptyset) = \bigoplus_\alpha H_n(X_\alpha,\emptyset).$$

iv) *Dimension axiom:* for a one-vertex digraph $*$ we have

$$H_n(*, \emptyset) = \begin{cases} K, & n = 0, \\ 0, & n \neq 0, \end{cases}$$

where K is fixed abelian group.

Theorem 5.2. *There exists a homology theory that satisfies this system of axioms.*

Proof. The proof follows from the results of Section 3. $\qquad\square$

Remark 5.3. The question remains, though, of what additional axioms are necessary to ensure that the path homology theory is a unique realization.

6.   Examples

In this section we consider several examples which illustrate the difference between the path homology theory and the homology theory for topological or simplicial spaces.

Example 6.1. i) Let $i\colon Y = * \cup \star \subset X$ where X is the following digraph $* \leftarrow \bullet \rightarrow \star$. Then $H_0(Y) = K \oplus K, H_0(X) = K$ and other homology groups of these digraphs are trivial. Hence $H_1(X, Y) = K$ and $H_1(X, Y) \cong H_1(Cone_i)$.

ii) Let $i\colon Y = * \cup \star \subset X$ where X is the following digraph $* \rightarrow \bullet \leftarrow \star$. Then $H_0(Y) = K \oplus K, H_0(X) = K$ and other homology groups of these digraphs are trivial. Hence $H_1(X, Y) = K$ and $H_1(Cone_i) = 0$.

Example 6.2. i) Consider a digraph X

$$
\begin{array}{ccc}
1 & \longrightarrow & 2 \\
\downarrow & & \uparrow \\
3 & \longleftarrow & 4.
\end{array}
$$
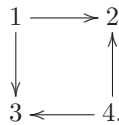
Taking one point sub-digraphs $A_1 = \{2\}, A_2 = \{3\}$ we can define sub-digraphs Y_1 and Y_2 and apply the excision axiom for computing $H_p(X)$. We obtain that $H_p(X) \cong K$ for $p = 0, 1$ and trivial for $p \geqslant 2$.

ii) Consider a digraph X

$$
\begin{array}{ccc}
1 & \longrightarrow & 2 \\
\uparrow & & \uparrow \\
3 & \longleftarrow & 4.
\end{array}
$$
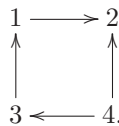
In this case we cannot cut X in two parts for applying the excision axiom. However, the homology groups of this digraph are the same as those in the previous example.

References

[1]   R.H. Atkin, An algebra for patterns on a complex, I. *Int. J. Man-Mach. Stud.* **6** (1974), 285–307.

[2]   R. Atkin, An algebra for patterns on a complex, II. *Int. J. Man-Mach. Stud.* **8** (1976), 483–498.

[3]   E. Babson, H. Barcelo, M. de Longueville and R. Laubenbacher, Homotopy theory of graphs. *J. Algebraic Combin.* **24** (2006), 31–44.

[4]   H.C. Baehr, A. Dimakis and F. Müller-Hoissen, Differential calculi on commutative algebras. *J. Phys. A* **28** (1995), 3197–3222.

[5]   H. Barcelo, V. Capraro and J.A. White, Discrete homology theory for metric spaces. *Bull. Lond. Math. Soc.* **46** (2014), 889–905.

[6]   H. Barcelo, X. Kramer, R. Laubenbacher and W. Christopher, Foundations of a connectivity theory for simplicial complexes. *Adv. Appl. Math.* **26** (2001), 97–128.

[7]   B. Chen, S.-T. Yau and Y.-N. Yeh, Graph homotopy and Graham homotopy. *Discrete Math.* **241** (2001), 153–170.

[8]   C. Cibils, Cohomology of incidence algebras and simplicial complexes. *J. Pure Appl. Algebra* **56** (1989), 221–232.

[9]   A. Connes, *Noncommutative Geometry*, Academic Press, San Diego, 1994.

[10] A. Dimakis and F. Müller-Hoissen, Differential calculus and gauge theory on finite sets. *J. Phys. A* **27** (1994), 3159–3178.

[11] A. Dimakis and F. Müller-Hoissen, Discrete differential calculus: graphs, topologies, and gauge theory. *J. Math. Phys.* **35** (1994), 6703–6735.

[12] A. Dochtermann, Homotopy groups of Hom complexes of graphs. *J. Combin. Theory Ser. A* **116** (2009), 180–194.

[13] N.P. Dolbilin, M.A. Shtan'ko and M.I. Shtogrin, Cubic manifolds in lattices (in Russian); translated from *Izv. Ross. Akad. Nauk Ser. Mat.* **58** (1994), 93–107; *Russian Acad. Sci. Izv. Math.* **44** (1995), 301–313.

[14] N.P. Dolbilin, Yu.M. Zinov'ev, A.S. Mishchenko, M.A. Shtan'ko and M.I. Shtogrin, Homological properties of two-dimensional coverings of lattices on surfaces (in Russian); translated from *Funktsional. Anal. i Prilozhen.* **30** (1996), 19–33, 95; *Funct. Anal. Appl.* **30** (1996), 163–173 (1997).

[15] M. Gerstenhaber and S.D. Schack, Simplicial cohomology is Hochschild cohomology. *J. Pure Appl. Algebra* **30** (1983), 143–156.

[16] A. Grigor'yan, Y. Lin, Y. Muranov and S.-T. Yau, Homotopy theory for digraphs. *Pure Appl. Math. Q.* **10** (2014), 619–674.

[17] A. Grigor'yan, Y. Lin, Y. Muranov and S.-T. Yau, Cohomology of digraphs and (undirected) graphs. *Asian J. Math.* **19** (2015), 887–932.

[18] A. Grigor'yan, Y. Muranov and S.-T. Yau, Graphs associated with simplicial complexes. *Homology Homotopy Appl.* **16** (2014), 295–311.

[19] A. Grigor'yan, Y. Muranov and S.-T. Yau, On a cohomology of digraphs and Hochschild cohomology. *J. Homotopy Relat. Struct.* **11** (2016), 209–230.

[20] A. Hatcher, *Algebraic Topology*, Cambridge University Press, Cambridge, 2002.

[21] M. Herlihy, D. Kozlov and S. Rajsbaum, *Distributed Computing through Combinatorial Topology*, Elsevier/Morgan Kaufmann, Waltham, 2014.

[22] G. Hochschild, On the cohomology groups of an associative algebra. *Ann. of Math. (2)* **46** (1945), 58–67.

[23] A.V. Ivashchenko, Contractible transformations do not change the homology groups of graphs. *Discrete Math.* **126** (1994), 159–170.

[24] D. Kozlov, *Combinatorial Algebraic Topology*, volume 21 of *Algorithms Comput. Math.*, Springer, Berlin, 2008.

[25] S. MacLane, *Homology*, volume 114 of *Grundlehren Math. Wiss.*, Springer-Verlag, Berlin, New York, 1967.

[26] J.R. Munkres, *Elements of Algebraic Topology*, Addison-Wesley Publishing Company, Menlo Park, 1984.

[27] V.V. Prasolov, *Elements of Homology Theory*, volume 81 of *Grad. Stud. Math.*, American Mathematical Society, Providence, 2007.

[28] P. Ribenboim, Algebraic structures on graphs, *Algebra Universalis* **16** (1983), 105–123.

[29] M.A. Shtan'ko and M.I. Shtogrin, Embedding cubic manifolds and complexes into a cubic lattice (in Russian); translated from *Uspekhi Mat. Nauk* **47** (1992), 219–220; *Russian Math. Surveys* **47** (1992), 267–268.

[30] E.H. Spanier, *Algebraic Topology*, Springer-Verlag, New-York, 1995.

[31] M.E. Talbi and D. Benayat, Homology theory of graphs. *Mediterr. J. Math.* **11** (2014), 813–828.

[32] R.R. Zapatrin, Polyhedral representations of discrete differential manifolds. *J. Math. Phys.* **38** (1997), 2741–2750.

Alexander Grigor'yan   grigor@math.uni-bielefeld.de

Mathematics Department, University of Bielefeld, Postfach 100131, Bielefeld, D-33501, Germany

Institute of Control Sciences of Russian Academy of Sciences, Moscow, Russia

Rolando Jimenez   rolando@matcuer.unam.mx

Instituto de Matematicas, UNAM, Unidad Oaxaca, Leon 2, Centro, Oaxaca, 68000, Mexico

Yuri Muranov   muranov@matman.uwm.edu.pl

Faculty of Mathematics and Computer Science, University of Warmia and Mazury, Sloneczna 54 Street, Olsztyn, 10-710, Poland

Shing-Tung Yau yau@math.harvard.edu

Mathematics Department, Harvard University, One Oxford Street, Cambridge, MA 02138, USA