

EULER CHARACTERISTICS FOR SPACES OF STRING LINKS AND THE MODULAR ENVELOPE OF \mathcal{L}_∞

PAUL ARNAUD SONGHAFUO TSOPMÉNÉ AND VICTOR TURCHIN

(communicated by Dev P. Sinha)

Abstract

We make calculations in graph homology which further understanding of the topology of spaces of string links, in particular, calculating the Euler characteristics of finite-dimensional summands in their homology and homotopy. In doing so, we also determine the supercharacter of the symmetric group action on the positive arity components of the modular envelope of \mathcal{L}_∞ .

1. Introduction

Let $\text{Emb}_c(\coprod_{i=1}^r \mathbb{R}^{m_i}, \mathbb{R}^d)$ be the space of smooth embeddings $f: \coprod_{i=1}^r \mathbb{R}^{m_i} \hookrightarrow \mathbb{R}^d$ that coincide outside a compact set with a fixed affine embedding ι . Such embeddings are called *string links*. Let $\text{Imm}_c(\coprod_{i=1}^r \mathbb{R}^{m_i}, \mathbb{R}^d)$ be the space of smooth immersions with the same behavior at infinity. In this paper and in [33] we study the homotopy fiber over ι of the obvious inclusion $\text{Emb}_c(\coprod_{i=1}^r \mathbb{R}^{m_i}, \mathbb{R}^d) \subset \text{Imm}_c(\coprod_{i=1}^r \mathbb{R}^{m_i}, \mathbb{R}^d)$, which we denote $\overline{\text{Emb}}_c(\coprod_{i=1}^r \mathbb{R}^{m_i}, \mathbb{R}^d)$. In [33] we built on Goodwillie-Weiss calculus [15] to define two types of complexes computing the rational homology and homotopy groups of $\overline{\text{Emb}}_c(\coprod_{i=1}^r \mathbb{R}^{m_i}, \mathbb{R}^d)$, in the case when $d > 2 \max\{m_i \mid 1 \leq i \leq r\} + 1$, which split into a direct sum of finite complexes. In Theorems 1.1 and 1.2 below we compute the generating functions for the Euler characteristic of these summands, giving, to our knowledge, the first known “lower-bounds” for the homology and homotopy of these spaces.

The first type of complexes we study, which we call *Koszul complexes*, are built up from the (co)homology of configuration spaces of points in \mathbb{R}^{m_i} and in \mathbb{R}^d . These complexes are more amenable to computation. The second type of complexes, which we call *hairly graph-complexes*, are similar to Kontsevich’s commutative operad graph-complexes [24] except that our graphs are allowed to have external univalent vertices, called *hairs*, and thus are more similar to graph-complexes studied by Conant, Kassabov and Vogtmann [9]. We show that these complexes are built up from the positive

This work has been supported by Fonds de la Recherche Scientifique–FNRS (F.R.S.–FNRS), that the authors acknowledge. It has also been supported by the Kansas State University (KSU), where this paper was partially written during the stay of the first author, and which he thanks for hospitality. The second author is partially supported by the Simons Foundation “Collaboration grant for mathematicians,” award ID: 519474.

Received September 8, 2016, revised November 10, 2017; published on May 16, 2018.

2010 Mathematics Subject Classification: 57Q45, 18D50.

Key words and phrases: string link, Euler characteristic, modular operad.

Article available at <http://dx.doi.org/10.4310/HHA.2018.v20.n2.a7>

Copyright © 2018, Paul Arnaud Songhafouo Tsopméné and Victor Turchin. Permission to copy for private use granted.

arity components of the modular envelope $\mathbf{Mod}(\mathcal{L}_\infty)$ of the \mathcal{L}_∞ operad in case of odd d , and from their twisted version $\mathbf{Mod}_{\text{Det}}(\mathcal{L}_\infty)$ for even d . As a consequence we determine in Theorem 1.3 the *supercharacter* of the symmetric group action on the positive arity components of these modular envelopes, which up to regrading are the Feynman transforms $F_{\text{Det}}\text{Com}$ and $F\text{Com}$, respectively, defined by Getzler and Kapranov in [14].

Spaces of embeddings are central objects of study in differential topology, with for example their components corresponding to isotopy classes and their higher homotopy groups useful for generating interesting phenomena [6]. The main applications of modular operads are in mathematical physics, in particular, in Chern-Simons theory, open and closed string field theories, Batalin-Vilkovisky formalism, and formal geometry [3, 7, 18, 21, 24, 26, 27]. Graph-complexes and modular operads also have applications in topology and geometry, in particular, in the study of the moduli spaces of curves with marked points, automorphisms of free groups, and Vassiliev invariants of knots and string links [9, 14, 23, 24, 34, 19]. In fact, $\mathbf{Mod}(\mathcal{L}ie) = H_0\mathbf{Mod}(\mathcal{L}_\infty)$, the modular envelope of $\mathcal{L}ie$, is related to Bar-Natan's space of univalent graphs modulo AS and IHX relations from Vassiliev theory in exactly the same way as the modular envelope of \mathcal{L}_∞ is related to our hairy graph-complexes.

Direct computations of the supercharacter of the symmetric group action on $\mathbf{Mod}(\mathcal{L}_\infty)$ and $\mathbf{Mod}_{\text{Det}}(\mathcal{L}_\infty)$ using graph complexes have previously been much less tractable than we find here. Such were first attempted in case of arity zero, where these correspond to the well known Kontsevich graph-complexes associated to the commutative operad [24], by Willwacher and Živković in [37]. The original paper of Getzler and Kapranov [14], where the notion of a modular operad is introduced, gives a universal method to compute the supercharacter of a Feynman transform of any modular operad, but explicit calculations were done in [14] only for the associative operad. Thus our work deepens understanding of modular operads and graph-complexes as well as embedding spaces.

1.1. Basic definitions and previous results

We recall the main results of [33]. To define complexes arising from Goodwillie-Weiss calculus and computing the rational homology and homotopy groups of $\overline{\text{Emb}}_c(\coprod_{i=1}^r \mathbb{R}^{m_i}, \mathbb{R}^d)$, we first define two categories: Rmod_Ω and Rmod_Γ . Let Ω denote the category of finite unpointed sets $\underline{n} = \{1 \dots n\}$, $n \geq 0$, and surjections. Define a *right Ω -module* as a contravariant functor from Ω to any given category. The category of right Ω -modules in chain complexes over the rationals is denoted Rmod_Ω . This category can be endowed with several model structures. We choose the one called *projective model structure*. In that model, weak equivalences are quasi-isomorphisms, fibrations are degreewise surjective maps [20]. Given two objects P and Q in Rmod_Ω , we write $\text{Rmod}_\Omega(P, Q)$ for the space (chain complex) of morphisms between them, and we write $h\text{Rmod}_\Omega(P, Q)$ for the derived mapping space. For specific computations we will need to apply this construction only to Ω -modules with zero differential, in which case $h\text{Rmod}_\Omega(-, -)$ can be expressed as a product of Ext groups.

For a pointed topological space X , define the functor $X^{\wedge \bullet}: \Omega \rightarrow \text{Top}$ from Ω to topological spaces by $X^{\wedge \bullet}(\underline{n}) = X^{\wedge n}$. Here $\wedge n$ is the n -fold smash product operation, with $X^{\wedge 0} = S^0$ the two-point space. For a morphism f in Ω , $X^{\wedge \bullet}(f)$ is induced by

the diagonal maps. Let $\widetilde{C}_*(-)$ denote the reduced singular chain complex functor, so the functors $\widetilde{C}_*(X^\wedge^\bullet)$ and $\widetilde{H}_*(X^\wedge^\bullet)$ (with zero differential) are objects of Rmod_Ω . Note that in case X is a suspension, any *strict surjection* (that is, a surjection which is not a bijection) acts on $\widetilde{H}_*(X^\wedge^\bullet)$ as a zero map.

Let Γ be the category whose objects are finite pointed sets $n_+ = \{0, 1, \dots, n\}$, with 0 as the basepoint, and whose morphisms are pointed maps.¹ The category of contravariant functors from Γ to chain complexes is denoted Rmod_Γ . Objects of that category are called *right Γ -modules*. As an example of a right Γ -module, we have the homology $H_*(C(\bullet, \mathbb{R}^d), \mathbb{Q})$, $d \geq 2$, where $C(k, \mathbb{R}^d)$ denotes the configuration space of k labeled points in \mathbb{R}^d . One can see that $H_*(C(\bullet, \mathbb{R}^d), \mathbb{Q})$ is indeed a right Γ -module as follows. First, since there is a morphism $\text{Com} \rightarrow H_*(C(\bullet, \mathbb{R}^d), \mathbb{Q})$ of operads from the commutative operad $\text{Com} = H_0(C(\bullet, \mathbb{R}^d), \mathbb{Q})$ to the homology $H_*(C(\bullet, \mathbb{R}^d), \mathbb{Q}) = H_*(B_d(\bullet), \mathbb{Q})$ of the little d -disks operad, it follows that $H_*(C(\bullet, \mathbb{R}^d), \mathbb{Q})$ is an *infinitesimal bimodule* (see [1, Definition 3.8] or [35, Definition 4.1]) over Com . Secondly, the category of infinitesimal bimodules over Com is equivalent to the category of right Γ -modules ([1, Corollary 4.10] or [35, Lemma 4.3]). One can also show that the sequence $\mathbb{Q} \otimes \pi_* C(\bullet, \mathbb{R}^d)$, $d \geq 3$, has a natural structure of a right Γ -module.

The two categories Rmod_Ω and Rmod_Γ we just defined are equivalent. To prove it, Pirashvili [29] constructed a functor $\text{cr}: \text{Rmod}_\Gamma \rightarrow \text{Rmod}_\Omega$, called *cross effect*, and showed that it is actually an equivalence of categories. Let $\widehat{H}_*(C(\bullet, \mathbb{R}^d), \mathbb{Q})$ (or, respectively, $\mathbb{Q} \otimes \widehat{\pi}_* C(\bullet, \mathbb{R}^d)$) denote the cross effect of $H_*(C(\bullet, \mathbb{R}^d), \mathbb{Q})$ (respectively of $\mathbb{Q} \otimes \pi_* C(\bullet, \mathbb{R}^d)$).

Theorems 0.1 and 0.2 in [33] express the rational homology and homotopy groups of $\mathcal{L} := \overline{\text{Emb}}_c(\prod_{i=1}^r \mathbb{R}^{m_i}, \mathbb{R}^d)$ as the homology groups of derived mapping complexes of right Ω -modules. More precisely, for $d > 2 \max\{m_i \mid 1 \leq i \leq r\} + 1$, there are isomorphisms:²

$$H_*(\mathcal{L}, \mathbb{Q}) \cong H\left(\text{hRmod}_\Omega\left(\widetilde{H}_*((\bigvee_{i=1}^r S^{m_i})^\wedge^\bullet, \mathbb{Q}), \widehat{H}_*(C(\bullet, \mathbb{R}^d), \mathbb{Q})\right)\right), \quad (1.1)$$

$$\mathbb{Q} \otimes \pi_* \mathcal{L} \cong H\left(\text{hRmod}_\Omega\left(\widetilde{H}_*((\bigvee_{i=1}^r S^{m_i})^\wedge^\bullet, \mathbb{Q}), \mathbb{Q} \otimes \widehat{\pi}_* C(\bullet, \mathbb{R}^d)\right)\right). \quad (1.2)$$

For our purposes we need to split the right-hand sides of (1.1) and (1.2). In the sequel a sequence of r integers s_1, \dots, s_r will be written as \vec{s} . Also we will write $|\vec{s}|$ for $s_1 + \dots + s_r$, and $\Sigma_{\vec{s}}$ for $\Sigma_{s_1} \times \dots \times \Sigma_{s_r}$. If x_1, \dots, x_r is another sequence, we will write $\vec{s} \cdot \vec{x}$ for $s_1 x_1 + \dots + s_r x_r$, and $\vec{x}^{\vec{s}}$ for $\prod_i x_i^{s_i}$. We will also write $\vec{s} \geq 0$ if $s_i \geq 0$, $i = 1 \dots r$.

Since $\bigvee_{i=1}^r S^{m_i}$ is a suspension, any strict surjection acts on $\widetilde{H}_*((\bigvee_{i=1}^r S^{m_i})^\wedge^\bullet, \mathbb{Q})$ as zero. This implies that the Ω -module $\widetilde{H}_*((\bigvee_{i=1}^r S^{m_i})^\wedge^\bullet, \mathbb{Q})$ splits as follows:

$$\widetilde{H}_*((\bigvee_{i=1}^r S^{m_i})^\wedge^\bullet, \mathbb{Q}) \cong \bigoplus_{\vec{s} \geq 0} Q_{\vec{s}}^{\vec{m}}, \quad (1.3)$$

¹We follow Pirashvili's notation for Ω and Γ , see [29, 30]. In the literature following Segal [31], one often denotes by Γ the opposite category. We choose Pirashvili's notation as in the sequel we use the higher order Hochschild-Pirashvili homology defined in [30] by means of these categories Ω and Γ .

²In other words, these formulae express the rational homology and homotopy of $\overline{\text{Emb}}_c(\prod_{i=1}^r \mathbb{R}^{m_i}, \mathbb{R}^d)$ as the higher order Hochschild homology over the space $\bigvee_{i=1}^r S^{m_i}$ with coefficients in the Γ -modules $H_*(C(\bullet, \mathbb{R}^d), \mathbb{Q})$ and $\mathbb{Q} \otimes \pi_* C(\bullet, \mathbb{R}^d)$, respectively, see [33].

where $Q_{\vec{s}}^{\vec{m}}$ is the right Ω -module defined by

$$Q_{\vec{s}}^{\vec{m}}(k) = \begin{cases} 0 & \text{if } k \neq |\vec{s}|; \\ \text{Ind}_{\Sigma_{\vec{s}}}^{\Sigma_k} \tilde{H}_*(S^{\vec{s} \cdot \vec{m}}; \mathbb{Q}) & \text{if } k = |\vec{s}|. \end{cases}$$

Consider now $\widehat{H}_*(C(\bullet, \mathbb{R}^d), \mathbb{Q})$ and $\mathbb{Q} \otimes \widehat{\pi}_* C(\bullet, \mathbb{R}^d)$ that appear in (1.1) and (1.2) respectively. One has the splittings of Ω -modules:

$$\widehat{H}_*(C(\bullet, \mathbb{R}^d), \mathbb{Q}) = \prod_{t \geq 0} \widehat{H}_{t(d-1)}(C(\bullet, \mathbb{R}^d), \mathbb{Q}), \quad (1.4)$$

and

$$\mathbb{Q} \otimes \widehat{\pi}_* C(\bullet, \mathbb{R}^d) = \prod_{t \geq 0} \mathbb{Q} \otimes \widehat{\pi}_{t(d-2)+1} C(\bullet, \mathbb{R}^d). \quad (1.5)$$

Combining (1.1), (1.2), (1.3), (1.4), and (1.5), we get the following splittings

$$\begin{aligned} H_*(\overline{\text{Emb}}_c(\prod_{i=1}^r \mathbb{R}^{m_i}, \mathbb{R}^d), \mathbb{Q}) &\cong \prod_{\vec{s}, t} \text{hRmod}_{\Omega} \left(Q_{\vec{s}}^{\vec{m}}, \widehat{H}_{t(d-1)}(C(\bullet, \mathbb{R}^d), \mathbb{Q}) \right) \\ &\cong \bigoplus_{\vec{s}, t} \text{hRmod}_{\Omega} \left(Q_{\vec{s}}^{\vec{m}}, \widehat{H}_{t(d-1)}(C(\bullet, \mathbb{R}^d), \mathbb{Q}) \right), \end{aligned} \quad (1.6)$$

$$\begin{aligned} \mathbb{Q} \otimes \pi_*(\overline{\text{Emb}}_c(\prod_{i=1}^r \mathbb{R}^{m_i}, \mathbb{R}^d)) &\cong \prod_{\vec{s}, t} \text{hRmod}_{\Omega} \left(Q_{\vec{s}}^{\vec{m}}, \mathbb{Q} \otimes \widehat{\pi}_{t(d-2)+1} C(\bullet, \mathbb{R}^d) \right) \\ &\cong \bigoplus_{\vec{s}, t} \text{hRmod}_{\Omega} \left(Q_{\vec{s}}^{\vec{m}}, \mathbb{Q} \otimes \widehat{\pi}_{t(d-2)+1} C(\bullet, \mathbb{R}^d) \right). \end{aligned} \quad (1.7)$$

The product is replaced by the direct sum because only finitely many factors contribute for any given degree, as $d > 2 \max\{m_i \mid 1 \leq i \leq r\} + 1$, using the graph-complexes we explicitly described in [33, Remark 2.4]. (For (1.7) this is true even for a weaker constraint $d > \max\{m_i \mid 1 \leq i \leq r\} + 2$. Moreover, we conjecture in [33, Section 3] that (1.7) holds always in that range for $* \geq 0$.)

1.2. Statements of main results

We can now state the main results of this paper. For $\vec{s} \geq 0$ and $t \geq 0$, let $\mathcal{X}_{\vec{s}, t}$ be the Euler characteristic of the summand of (1.6) indexed by \vec{s}, t . The associated generating function is $F_{\vec{m}, d}^H(x_1, \dots, x_r, u) = \sum_{\vec{s}, t \geq 0} \mathcal{X}_{\vec{s}, t} \cdot u^t \vec{x}^{\vec{s}}$.

Let $\Gamma(-)$ denote the gamma function, and let $\mu(-)$ denote the standard Möbius function. Given a variable x and an integer $l \geq 1$, let $E_l(x)$ denote the sum

$$E_l(x) = \frac{1}{l} \sum_{p|l} \mu(p) x^{\frac{l}{p}}. \quad (1.8)$$

The following result computes the generating function above.

Theorem 1.1. *Assume that $d > 2 \max\{m_i \mid 1 \leq i \leq r\} + 1$. The generating function $F_{\vec{m},d}^H(x_1, \dots, x_r, u)$ is given by the formula*

$$F_{\vec{m},d}^H(x_1, \dots, x_r, u) = \prod_{l=1}^{+\infty} \frac{\Gamma((-1)^{d-1} E_l(\frac{1}{u}) - \sum_{i=1}^r (-1)^{m_i-1} E_l(x_i))}{((-1)^{d-1} l u^l)^{\sum_{i=1}^r (-1)^{m_i-1} E_l(x_i)} \Gamma((-1)^{d-1} E_l(\frac{1}{u}))}, \quad (1.9)$$

where each factor is understood as the asymptotic expansion of the underlying function when u is complex and $(-1)^{d-1} u^l \rightarrow +0$ and x_1, \dots, x_r are considered as fixed parameters.

When $r = 1$, the formula (1.9) coincides with that of [2, Theorem 6.1].

Using (1.9) we also compute the generating function of the Hodge splitting in the rational homotopy, which is our second main result. First we need a couple more definitions. Let B_p denote the p th Bernoulli number, so that $\sum_{p \geq 0} \frac{B_p x^p}{p!} = \frac{x}{e^x - 1}$. Recall that $B_{2n+1} = 0, n \geq 1$. Bernoulli's summation formula equates $1^j + 2^j + \dots + n^j$ with $S_j(n)$ where

$$S_j(x) = \frac{1}{j+1} \sum_{p=0}^j (-1)^p \binom{j+1}{p} B_p x^{j+1-p}, \quad j \geq 1. \quad (1.10)$$

Define also $F_l(u)$ by

$$F_l(u) = l u^l E_l\left(\frac{1}{u}\right) = \sum_{t|l} \mu(t) u^{l-\frac{l}{t}} = 1 - u^{l-\frac{l}{p_1}} - u^{l-\frac{l}{p_2}} + u^{l-\frac{l}{p_1 p_2}} + \dots + \mu(l) u^{l-1}, \quad (1.11)$$

where p_1 and p_2 are the first prime factors of l . Notice that one always has $F_l(0) = 1$.

Similarly, consider $F_{\vec{m},d}^\pi(x_1, \dots, x_r, u)$ associated to the splitting (1.7).

Theorem 1.2. *The generating function $F_{\vec{m},d}^\pi(x_1, \dots, x_r, u)$ is given by the formula*

$$F_{\vec{m},d}^\pi(x_1, \dots, x_r, u) = \sum_{k,l,j \geq 1} \frac{\mu(k)}{k^j} S_j \left(\sum_{i=1}^r (-1)^{m_i-1} E_l(x_i^k) \right) \left(\frac{(-1)^{d-1} l u^{kl}}{F_l(u^k)} \right)^j - \sum_{k,l \geq 1} \sum_{i=1}^r \frac{\mu(k)}{k} (-1)^{m_i-1} E_l(x_i^k) \ln(F_l(u^k)),$$

where the polynomials E_l, F_l, S_j are respectively defined by (1.8), (1.11), (1.10).

The latter result essentially encodes the same information as the *supercharacter* (which will be defined in Subsection 4.1) of the symmetric group action on the positive arity components of $\mathbf{Mod}(\mathcal{L}_\infty)$ and $\mathbf{Mod}_{\text{Det}}(\mathcal{L}_\infty)$.

Theorem 1.3. *The supercharacters of the symmetric group action on the modular envelope $\{\mathbf{Mod}(\mathcal{L}_\infty)((k))\}_{k \geq 0}$ of \mathcal{L}_∞ and on the Det-twisted modular envelope $\{\mathbf{Mod}_{\text{Det}}(\mathcal{L}_\infty)((k))\}_{k \geq 0}$ of \mathcal{L}_∞ are described by the cycle index sums as follows:*

$$\begin{aligned}
\hbar^{-1} \left(\frac{p_1^2 + p_2}{2} \right) + \text{Ch}(\mathbf{Mod}(\mathcal{L}_\infty)) &= \hbar^{-1} Z_{\mathcal{X}\mathbf{Mod}(\mathcal{L}_\infty)}(\hbar; p_1, p_2, p_3, \dots) \\
&= \Psi^{odd}(\hbar) + \sum_{k, l, j \geq 1} \frac{\mu(k)}{kj} S_j \left(\frac{1}{l} \sum_{a|l} \mu \left(\frac{l}{a} \right) \frac{p_{ak}}{\hbar^{ak}} \right) \left(\frac{l\hbar^{kl}}{F_l(\hbar^k)} \right)^j - \\
&\quad \sum_{k, l \geq 1} \frac{\mu(k)}{kl} \left(\sum_{a|l} \mu \left(\frac{l}{a} \right) \frac{p_{ak}}{\hbar^{ak}} \right) \ln(F_l(\hbar^k)), \quad (1.12)
\end{aligned}$$

$$\begin{aligned}
\hbar^{-1} \left(\frac{p_1^2 + p_2}{2} \right) + \text{Ch}(\mathbf{Mod}_{\text{Det}}(\mathcal{L}_\infty)) &= \hbar^{-1} Z_{\mathcal{X}\mathbf{Mod}_{\text{Det}}(\mathcal{L}_\infty)}(\hbar; p_1, p_2, p_3, \dots) \\
&= \Psi^{even}(\hbar) - \sum_{k, l, j \geq 1} \frac{\mu(k)}{kj} S_j \left(-\frac{1}{l} \sum_{a|l} \mu \left(\frac{l}{a} \right) \frac{p_{ak}}{\hbar^{ak}} \right) \left(\frac{-l\hbar^{kl}}{F_l(\hbar^k)} \right)^j - \\
&\quad \sum_{k, l \geq 1} \frac{\mu(k)}{kl} \left(\sum_{a|l} \mu \left(\frac{l}{a} \right) \frac{p_{ak}}{\hbar^{ak}} \right) \ln(F_l(\hbar^k)), \quad (1.13)
\end{aligned}$$

where the variable \hbar is responsible for the genus. The functions $\Psi^{odd}(\hbar)$ and $\Psi^{even}(\hbar)$ are expressed in terms of Willwacher-Živković' functions $P^{odd}(s, t)$, $P^{even}(s, t)$ from [37, Theorem 1] as follows

$$\Psi^{odd/even}(\hbar) = \sum_{l \geq 1} \frac{\mu(l)}{l} \ln \left(1 + P^{odd/even}(\mp \hbar^{-l}, \pm \hbar^l) \right). \quad (1.14)$$

The first line in (1.12) (and also in (1.13)) is to account the difference of the Getzler-Kapranov notation [14] (left-hand side) with ours (right-hand side). The factor \hbar^{-1} in front of $Z_{\mathcal{X}\mathbf{Mod}(\mathcal{L}_\infty)}$ appears because we consider the grading by the genus (first Betti number) of the graphs, while in [14] the additional grading is minus the Euler characteristics. The summand $\hbar^{-1} \left(\frac{p_1^2 + p_2}{2} \right)$ in the left-hand side is because Getzler-Kapranov consider non-unital modular operads, while we follow Hinich-Vaintrob's conventions [19], and allow operads have a unit.

Some readers might be more familiar with the Feynman transform rather than with the modular envelope. They are related one to another through a regrading $F_{\text{Det}}\text{Com} = \Sigma_5\mathbf{Mod}(\mathcal{L}_\infty)$, $F\text{Com} = \Sigma_5\mathbf{Mod}_{\text{Det}}(\mathcal{L}_\infty)$, see Lemma 4.4, which implies

$$\begin{aligned}
\text{Ch}(F_{\text{Det}}\text{Com})(\hbar; p_1, p_2, p_3, \dots) &= -\text{Ch}(\mathbf{Mod}(\mathcal{L}_\infty))(\hbar; -p_1, -p_2, -p_3, \dots), \\
\text{Ch}(F\text{Com})(\hbar; p_1, p_2, p_3, \dots) &= -\text{Ch}(\mathbf{Mod}_{\text{Det}}(\mathcal{L}_\infty))(\hbar; -p_1, -p_2, -p_3, \dots).
\end{aligned}$$

Comparing our answer with $\text{Ch}(F_{\text{Det}}\text{Ass})$ computed in [14, Theorem 9.18], we see that the positive arity part (which we computed) is the “easy part” of the expression. The arity zero part, expressed by $\Psi^{odd}(\hbar)$, $\Psi^{even}(\hbar)$, is the difficult one, which is still not explicitly computed. To recall Willwacher and Živković [37] expressed the generating functions $P^{odd}(s, t)$, $P^{even}(s, t)$ of the dimensions of the graph-complexes fGC_d

(where d is either *odd* or *even*)³ as an infinite sum of rather complicated expressions, which do not make sense when one passes to the generating function of the Euler characteristics by taking $s = \mp \hbar^{-1}$, $t = \pm \hbar$. The right-hand side of (1.14) is the plethystic logarithm [14, Equations (8.6), (8.1.3')], which works in any arity and, in particular, in arity zero, and which determines the Euler characters (supertraces for higher arities) of the connected part of the graph-complexes.⁴ We added 1 inside the logarithm to take into account the empty graph.

1.3. Outline of the paper

In Section 2 we prove Theorem 1.1, which presents a formula for the generating function of Euler characteristics of summands in the homological splitting (1.6). We end with Subsection 2.4, which gives formulas obtained from (1.9) by taking $m_i = 1$, $x_i = \pm 1$, $1 \leq i \leq r$.

In Section 3 we prove three results: Theorem 1.2, Theorem 3.5 and Theorem 3.8. The first one computes the generating function of Euler characteristics of summands in the homotopical splitting (1.7). The key point is Lemma 3.1, which presents the generating function in homotopy as a sum of logarithms of the generating function in homology. The second result (respectively the third result) computes the generating function for the homology ranks of the summands in (1.7) of genus zero (respectively of genus one). Both from Theorem 3.5 and Theorem 3.8 one can see the exponential growth of the Betti numbers of the rational homotopy of $\overline{\text{Emb}}_c(\coprod_{i=1}^r \mathbb{R}^{m_i}, \mathbb{R}^d)$, $r \geq 2$.

In Section 4 we prove Theorem 1.3, which determines the supercharacter of the symmetric group action on the positive arity components of the modular envelope of \mathcal{L}_∞ . This result is an interesting byproduct for the theory of operads, obtained from the computations of the previous section.

In the appendix we give a link to and explain how we produced, using the generating function from Theorem 1.2, tables of Euler characteristics of the summands in the homotopical splitting (1.7).

Acknowledgments

The authors are grateful to Dev Sinha for his thoughtful comments and numerous suggestions to improve the manuscript.

2. Generating functions of Euler characteristics in homology

The goal of this section is to prove Theorem 1.1, which gives a formula for the generating function $F_{\vec{m},d}^H(x_1, \dots, x_r, u)$ of the Euler characteristics of the summands in the homological splitting of the space of string links.

We first recall the definition of Koszul complexes computing (1.6). In short these complexes were obtained in [33] by taking the projective resolution of the source Ω -modules.

³The graph-complexes fGC_d are defined similarly as our hairy graph-complexes except that their graphs do not have hairs (univalent vertices) and can be disconnected.

⁴Compare with our Lemma 3.1 and also with [37, Lemma 3].

2.1. Koszul complexes

Definition 2.1. Let $V = \{V(n)\}_{n \geq 0}$ and $W = \{W(n)\}_{n \geq 0}$ be symmetric sequences. Define a new symmetric sequence $V \widehat{\otimes} W$ by $V \widehat{\otimes} W(n) = \bigoplus_{p+q=n} \text{Ind}_{\Sigma_p \times \Sigma_q}^{\Sigma_n} V(p) \otimes W(q)$.

In practice we need to deal with multigraded vector spaces. Besides the usual homological degree, they will have the Hodge multi-grading (s_1, \dots, s_r) and grading by complexity t . As usual when we take tensor product all the degrees get added.

Recalling the notation $\widehat{H}_*(-)$ from the introduction, we have the following result.

Proposition 2.2 ([33, Proposition 4.5]). *For $d > 2 \max\{m_i \mid 1 \leq i \leq r\} + 1$, there is a quasi-isomorphism*

$$C_*(\overline{\text{Emb}}_c(\prod_{i=1}^r \mathbb{R}^{m_i}, \mathbb{R}^d)) \otimes \mathbb{Q} \simeq \left(\bigoplus_{k \geq 0} \text{hom}_{\Sigma_k} \left(\widehat{\otimes}_{1 \leq i \leq r} \overline{H}_*(C(\bullet, \mathbb{R}^{m_i}), \mathbb{Q})(k), \widehat{H}_*(C(k, \mathbb{R}^d), \mathbb{Q}) \right), \partial \right). \quad (2.1)$$

Here $\overline{H}_*(-)$ is the Borel-Moore homology functor.

The differential ∂ here is defined similarly to the $r = 1$ case [2, Section 5]. We will not describe it explicitly here. However, in [33, Subsection 4.3] we explicitly describe the dual complex computing the cohomology of the space of links.

From now on let us assume that all $m_i > 1$. We can do so because the summands in (1.6) up to a regrading depend only on the parities of m_i , $i = 1 \dots r$, and that of d . The right hand side of (2.1) can be rewritten as

$$\bigoplus_{\vec{s}, t \geq 0} \left(\bigoplus_{k \geq 0} \text{hom}_{\Sigma_k} \left(A_{\vec{s}}^k, \widehat{H}_{t(d-1)}(C(k, \mathbb{R}^d), \mathbb{Q}) \right), \partial \right),$$

where

$$A_{\vec{s}}^k = \bigoplus_{k=|\vec{k}|} \left(\text{Ind}_{\Sigma_{\vec{k}}}^{\Sigma_k} \bigotimes_{i=1}^r \overline{H}_{s_i(m_i-1)+k_i}(C(k_i, \mathbb{R}^{m_i}), \mathbb{Q}) \right).$$

The \vec{s}, t summand in the complex above computes exactly the corresponding summand in (1.6). Thus the Euler characteristic $\mathcal{X}_{\vec{s}, t}$ used in $F_{\vec{m}, d}^H(x_1, \dots, x_r, u) = \sum_{\vec{s}, t \geq 0} \mathcal{X}_{\vec{s}, t} \cdot u^t \vec{x}^{\vec{s}}$ can be defined as

$$\mathcal{X}_{\vec{s}, t} = \sum_{k \geq 0} (-1)^{t(d-1) - \sum_{i=1}^r s_i(m_i-1) - k} \dim X_{\vec{s}, t}^k, \quad (2.2)$$

where

$$X_{\vec{s}, t}^k = \text{hom}_{\Sigma_k} \left(A_{\vec{s}}^k, \widehat{H}_{t(d-1)}(C(k, \mathbb{R}^d), \mathbb{Q}) \right). \quad (2.3)$$

2.2. Cycle index sum and proof of Theorem 1.1

Before starting the proof of Theorem 1.1 we will state two lemmas. For a permutation $\sigma \in \Sigma_k$, let $j_l(\sigma)$ denote the number of its cycles of length l . Let $\text{tr}(-)$ denote the trace function from the space of matrices to the ground field.

Definition 2.3. Let $\{p_1, p_2, \dots\}$ be a family of commuting variables.

- For a representation $\rho^V : \Sigma_k \rightarrow GL(V)$ of the symmetric group Σ_k , the *cycle index sum* of V , denoted $Z_V(p_1, p_2, \dots)$, is defined by

$$Z_V(p_1, p_2, \dots) = \frac{1}{|\Sigma_k|} \sum_{\sigma \in \Sigma_k} \text{tr}(\rho^V(\sigma)) \prod_l p_l^{j_l(\sigma)}. \tag{2.4}$$

- If $V = \{V(k)\}_{k \geq 0}$ is a symmetric sequence, we define $Z_V(p_1, p_2, \dots) = \sum_{k \geq 0} Z_{V(k)}(p_1, p_2, \dots)$.

The vector spaces in the symmetric sequences below will be always multigraded. The trace in (2.4) will be a graded trace, i.e. it will be a generating function of traces on each component. For example, if $V = \bigoplus_{i, \vec{s}} V_{i, \vec{s}}$, where i is the homological degree, and \vec{s} is the Hodge multigrading, then

$$\text{tr}(\rho^V(\sigma)) = \sum_{i, \vec{s}} \text{tr}(\rho^{V_{i, \vec{s}}}(\sigma)) z^i \vec{x}^{\vec{s}}.$$

With such definition, the cycle index sum Z_V of a symmetric sequence will also depend on z, x_1, \dots, x_r , and also possibly on u , where the last variable will be responsible for the grading by complexity.

Given two families $\{p_1, p_2, \dots\}$ and $\{b_1, b_2, \dots\}$ of commuting variables, we will write $Z_V(p_l \leftarrow b_l; l \in \mathbb{N})$ for $Z_V(b_1, b_2, \dots)$. This notation means substituting the value p_l with the value b_l in Z_V . For a function $f = f(p_1, p_2, \dots)$ on variables p_1, p_2, \dots , the notation $\left\{ Z_V\left(\frac{\partial}{\partial p_l}; l \in \mathbb{N}\right) f(p_1, p_2, \dots) \right\} \Big|_{p_l=0}$ means that we apply the differential operator $Z_V\left(\frac{\partial}{\partial p_l}; l \in \mathbb{N}\right)$ to f , and at the end we take $p_l = 0$ for all $l \geq 0$.

The following lemma is well known, see for example [35, Corollary 15.5].

Lemma 2.4. *Let $V = \bigoplus_i V_i$ and $W = \bigoplus_j W_j$ be two graded vector spaces admitting an action of the symmetric group Σ_k , that respects the grading, on each of them. Consider the series*

$$\begin{aligned} \dim \text{hom}_{\Sigma_k}(V, W) &= \sum_{i, j} \dim \text{hom}_{\Sigma_k}(V_i, W_j) z^{j-i}, \\ Z_V(z; p_1, p_2, \dots) &= \sum_{i, j} Z_{V_i}(p_1, p_2, \dots) z^i, \\ \text{and} \quad Z_W(z, p_1, p_2, \dots) &= \sum_j Z_{W_j}(p_1, p_2, \dots) z^j. \end{aligned}$$

Then

$$\begin{aligned} \dim \text{hom}_{\Sigma_k}(V, W) &= \left\{ Z_V\left(\frac{1}{z}; p_l \leftarrow \frac{\partial}{\partial p_l}, l \in \mathbb{N}\right) Z_W(z; p_l \leftarrow l p_l, l \in \mathbb{N}) \right\} \Big|_{p_l=0} \\ &= \left\{ Z_V\left(\frac{1}{z}; p_l \leftarrow l \frac{\partial}{\partial p_l}, l \in \mathbb{N}\right) Z_W(z; p_1, p_2, \dots) \right\} \Big|_{p_l=0}. \end{aligned} \tag{2.5}$$

For the application below, V will also be Hodge multigraded and W will also be graded by complexity. This will add up variables in the expression.

The second result we need to prove Theorem 1.1 is the following well known lemma, see for example [35, Proposition 15.3] or [12].

Lemma 2.5. *Let $V = \{V(k)\}_{k \geq 0}$ and $W = \{W(k)\}_{k \geq 0}$ be two finite symmetric sequences (that is, for all $k \geq 0$, $V(k)$ and $W(k)$ are of finite dimensions). Then*

$$Z_{V \widehat{\otimes} W}(p_1, p_2, \dots) = Z_V(p_1, p_2, \dots) Z_W(p_1, p_2, \dots).$$

Again we will be using a multigraded version of this lemma.

We are now ready to prove the main result of this section.

Proof of Theorem 1.1. Consider the following generating function

$$\Psi_{\vec{m}, d}(x_1, \dots, x_r, u, z) = \sum_{\vec{s}, t, k} (\dim X_{\vec{s}, t}^k) \vec{x}^{\vec{s}} u^t z^{t(d-1) - \sum_{i=1}^r s_i(m_i-1) - k},$$

where $X_{\vec{s}, t}^k$ is the space from (2.3). Define two symmetric sequences V and W by

$$V(k) = \bigoplus_{k=\vec{k}} \left(\text{Ind}_{\Sigma_{\vec{k}}}^{\Sigma_k} \otimes_{i=1}^r \overline{H}_*(C(k_i, \mathbb{R}^{m_i}), \mathbb{Q}) \right) \quad \text{and} \quad W(k) = \widehat{H}_*(C(k, \mathbb{R}^d), \mathbb{Q}).$$

Then, by Lemma 2.4, the generating function

$$\begin{aligned} & \Psi_{\vec{m}, d}(x_1, \dots, x_r, u, z) \\ &= \left\{ Z_V\left(\frac{1}{z}, x_1, \dots, x_r; p_l \leftarrow \frac{\partial}{\partial p_l}, l \geq 1\right) Z_W(z, u; p_l \leftarrow l p_l, l \geq 1) \right\} \Big|_{p_l=0}. \end{aligned}$$

Since (recalling the definition of $E_l(-)$ from the introduction, just before Theorem 1.1):

- $Z_W(z, u; p_l \leftarrow l p_l, l \geq 1) = \prod_{l=1}^{+\infty} e^{-p_l} (1 + (-1)^{dl} ((-z)^{d-1} u)^l p_l)^{(-1)^d E_l(\frac{1}{(-z)^{d-1} u})}$ by [2, Proposition 6.4],
- $Z_V(\frac{1}{z}, x_1, \dots, x_r; p_l \leftarrow \frac{\partial}{\partial p_l}, l \geq 1) = \prod_{i=1}^r Z_{\overline{H}_*(C(\bullet, \mathbb{R}^{m_i}), \mathbb{Q})}(\frac{1}{z}, x_i; p_l \leftarrow \frac{\partial}{\partial p_l}, l \geq 1)$ by Lemma 2.5 and the fact that $V = \widehat{\otimes}_{i=1}^r \overline{H}_*(C(\bullet, \mathbb{R}^{m_i}), \mathbb{Q})$,

and since

$$Z_{\overline{H}_*(C(\bullet, \mathbb{R}^{m_i}), \mathbb{Q})}(\frac{1}{z}, x_i; p_l \leftarrow \frac{\partial}{\partial p_l}, l \geq 1) = \prod_{l=1}^{+\infty} \left(1 + \left(-\frac{1}{z}\right)^l \frac{\partial}{\partial p_l} \right)^{(-1)^{m_i} E_l(\frac{x_i}{(-z)^{m_i-1}})},$$

by [2, Proposition 6.5], it follows that

$$\begin{aligned} \Psi_{\vec{m}, d}(x_1, \dots, x_r, u, z) &= \left\{ \prod_{i=1}^r \prod_{l=1}^{+\infty} \left(1 + \left(-\frac{1}{z}\right)^l \frac{\partial}{\partial p_l} \right)^{(-1)^{m_i} E_l(\frac{x_i}{(-z)^{m_i-1}})} Y_l \right\} \Big|_{p_l=0} \\ &= \left\{ \prod_{l=1}^{+\infty} \left(1 + \left(-\frac{1}{z}\right)^l \frac{\partial}{\partial p_l} \right)^{\sum_{i=1}^r (-1)^{m_i} E_l(\frac{x_i}{(-z)^{m_i-1}})} Y_l \right\} \Big|_{p_l=0}, \end{aligned}$$

where $Y_l = e^{-p_l} (1 + (-1)^{dl} ((-z)^{d-1} u)^l p_l)^{(-1)^d E_l(\frac{1}{(-z)^{d-1} u})}$. Notice that the l -th factor uses p_l and not any other p_i , $i \neq l$, which is then taken to be zero. For this reason

in the formula below we replace each p_l by a . By looking at (2.2), and the definition of $\Psi_{\bar{m},d}(x_1, \dots, x_r, u, z)$, we have the equality

$$F_{\bar{m},d}^H(x_1, \dots, x_r, u) = \Psi_{\bar{m},d}(x_1, \dots, x_r, u, -1),$$

and this implies

$$\begin{aligned} F_{\bar{m},d}^H(x_1, \dots, x_r, u) &= \left\{ \prod_{l=1}^{+\infty} \left(1 + \frac{\partial}{\partial a} \right)^{-\sum_{i=1}^r (-1)^{m_i-1} E_l(x_i)} Z_l \right\} \Big|_{a=0} \\ &= \prod_{l=1}^{+\infty} \frac{\Gamma \left((-1)^{d-1} E_l \left(\frac{1}{u} \right) - \sum_{i=1}^r (-1)^{m_i-1} E_l(x_i) \right)}{((-1)^{d-1} l u^l)^{\sum_{i=1}^r (-1)^{m_i-1} E_l(x_i)} \Gamma \left((-1)^{d-1} E_l \left(\frac{1}{u} \right) \right)}, \end{aligned}$$

where $Z_l = e^{-a} (1 + (-1)^d l u^l a)^{(-1)^d E_l(\frac{1}{u})}$. The last equality follows from the identity

$$\left(1 + \frac{\partial}{\partial a} \right)^{-X} e^{-a} (1 + (-1)^d l u^l a)^{(-1)^d E_l(\frac{1}{u})} \Big|_{a=0} = \frac{\Gamma \left((-1)^{d-1} E_l \left(\frac{1}{u} \right) - X \right)}{((-1)^{d-1} l u^l)^X \Gamma \left((-1)^{d-1} E_l \left(\frac{1}{u} \right) \right)},$$

whose proof is identical to that of [35, Proposition 15.7].

We thus obtain the desired result. □

2.3. Understanding generating function of the Euler characteristics of the homology splitting

The formula (1.9) might appear confusing at first as it is defined in terms of asymptotic expansions that still need to be deciphered. Let us first fix some notation. For two variables x, u , let $\Gamma(x, u)$ denote the asymptotic expansion of $\frac{\Gamma(\frac{1}{u}-x)}{u^x \Gamma(\frac{1}{u})}$ when u goes to $+0$. For instance, in the case $x = n$ is a positive integer, one has

$$\Gamma(n, u) = \frac{1}{(1-u)(1-2u) \cdots (1-nu)}. \tag{2.6}$$

In the case x is a negative integer: $x = -n$, one has

$$\Gamma(-n, u) = (1+u)(1+2u) \cdots (1+(n-1)u). \tag{2.7}$$

(In particular, $\Gamma(-1, u) = \Gamma(0, u) = 1$.) The series $\Gamma(x, u)$ can be seen as a generating function in u of a sequence of polynomials in x , see [35, Subsection 14.1]. It has rather bad convergence properties: for any $x \in \mathbb{C} \setminus \mathbb{Z}$ its radius of convergence in u is zero [35, Proposition 14.5].

Proposition 2.6.

$$F_{\bar{m},d}^H(x_1, \dots, x_r, u) = \frac{\prod_{l \geq 1} \Gamma \left(\sum_{i=1}^r (-1)^{m_i-1} E_l(x_i), (-1)^{d-1} \frac{l u^l}{F_l(u)} \right)}{\prod_{l \geq 1} (F_l(u))^{\sum_{i=1}^r (-1)^{m_i-1} E_l(x_i)}}, \tag{2.8}$$

where polynomials $F_l(u)$ are defined in (1.11).

Proof. This is deduced from Theorem 1.1 and from the formula

$$\frac{\Gamma \left((-1)^{d-1} E_l \left(\frac{1}{u} \right) - X \right)}{((-1)^{d-1} l u^l)^X \Gamma \left((-1)^{d-1} E_l \left(\frac{1}{u} \right) \right)} = \frac{\Gamma \left(X, (-1)^{d-1} \frac{l u^l}{F_l(u)} \right)}{F_l(u)^X},$$

see [35, Lemma 14.6] for similar computations for the case of long knots. □

2.4. Some special cases of the generating function in homology

The generating function $F_{\vec{m},d}^H(x_1, \dots, x_r, u)$ from (1.9) can be used to estimate the ranks of the homology and homotopy groups of $\overline{\text{Emb}}_c(\prod_{i=1}^r \mathbb{R}^{m_i}, \mathbb{R}^d)$. However, the formula itself is hard to use to estimate the asymptotics of the growth since it has infinitely many rather complicated factors. However, sometimes it is possible to choose specific values for x_i that would kill all but finitely many factors. This has been done in the case of long knots $r = 1$, $m_1 = 1$ in [35], where choosing $x_1 = -1$ allows one to prove the exponential growth of the ranks of the rational homology and homotopy groups of the space of long knots. Notice that choosing $x_1 = 1$ (all $x_i = 1$ in the general case) corresponds to forgetting the Hodge splitting. In the case of long knots the latter choice does not prove exponential growth, see [22, 35] or computations below, which showed that the Hodge splitting was essential.

In this subsection we produce similar computations for the spaces of classical string links: all $m_i = 1$, $i \in \{1, \dots, r\}$. Recall that it has been shown that the homology ranks of $\overline{\text{Emb}}_c(\prod_{i=1}^r \mathbb{R}^1, \mathbb{R}^d)$ grow exponentially for $r \geq 2$ without using the Hodge splitting (this being obtained as a combination of the main results of [22] and [32]).⁵ The computations that we produce below are rather disappointing: we compare the choice of all $x_i = -1$ versus all $x_i = 1$ and see that only for $r \leq 2$ the first choice may give a better estimate.⁶

- Assume d odd, and consider the function $F_{\text{odd}}(1, u)$ defined by $F_{\text{odd}}(1, u) = F_{1,d}^H(1, \dots, 1, u)$. We want to compute $F_{\text{odd}}(1, u)$. By substituting m_i by 1, and x_i by 1, $1 \leq i \leq r$, in the formula (2.8), and by noticing that $E_l(1) = \begin{cases} 1 & \text{if } l = 1 \\ 0 & \text{if } l \geq 2 \end{cases}$, we obtain

$$F_{\text{odd}}(1, u) = \Gamma(r, u) = \frac{1}{(1-u)(1-2u) \cdots (1-ru)},$$

where the last equality comes from (2.6). The radius of convergence in this case is $\frac{1}{r}$ which implies exponential growth of rational homology ranks in case $r \geq 2$.

- For d even, similar computations give

$$F_{\text{even}}(1, u) = F_{1,d}^H(1, \dots, 1, u) = \frac{1}{(1+u)(1+2u) \cdots (1+ru)}.$$

The radius of convergence is $\frac{1}{r}$.

- Assume d odd, and consider the function $F_{\text{odd}}(-1, u)$ defined by $F_{\text{odd}}(-1, u) = F_{1,d}^H(-1, \dots, -1, u)$. By substituting m_i by 1, and x_i by -1 , $1 \leq i \leq r$, in the

⁵In general the homology ranks of $\overline{\text{Emb}}_c(\prod_{i=1}^r \mathbb{R}^{m_i}, \mathbb{R}^d)$ grow exponentially for $r \geq 2$. One can see that both from Theorem 3.5 and Theorem 3.8. For $r = 1$ and m_1 odd one still gets the exponential growth [1]. But for $r = 1$ and m_1 even the question about the exponential growth is open.

⁶As we said earlier it does give a better estimate for $r = 1$. In case $r = 2$ and d odd it also gives a better estimate, but the case of $r = 2$ and d even is more subtle as both generating functions happen to have the same radius of convergence.

formula (2.8), and by noticing that $E_l(-1) = \begin{cases} (-1)^l & \text{if } l \in \{1, 2\} \\ 0 & \text{if } l \geq 3 \end{cases}$, we obtain

$$F_{\text{odd}}(-1, u) = \Gamma(-r, u) \times \frac{\Gamma\left(r, \frac{2u^2}{F_2(u)}\right)}{F_2(u)^2}. \tag{2.9}$$

Since $F_2(u) = 1 - u$ and applying (2.6) and (2.7), the formula (2.9) produces

$$F_{\text{odd}}(-1, u) = \frac{(1+u)(1+2u)\cdots(1+(r-1)u)}{(1-u-2u^2)(1-u-4u^2)\cdots(1-u-2ru^2)}. \tag{2.10}$$

It is easy to see that the smallest root (in absolute value) of the denominator of (2.10) is $\frac{-1+\sqrt{1+8r}}{4r}$. Therefore the radius of convergence is equal to $\frac{-1+\sqrt{1+8r}}{4r}$. This gives a better estimate of the homology ranks growth for $r \leq 2$. In case $r \geq 4$, the produced estimate is weaker than the one obtained from $F_{\text{odd}}(1, u)$.

- For d even, similar computations give

$$F_{\text{even}}(-1, u) = \frac{(1-u)(1-2u)\cdots(1-(r-1)u)}{(1-u+2u^2)(1-u+4u^2)\cdots(1-u+2ru^2)}. \tag{2.11}$$

As before, the radius of convergence of (2.11) is equal to $\left|\frac{1+\sqrt{1-8r}}{4r}\right| = \frac{1}{\sqrt{2r}}$. In particular, for $r = 1$ this implies exponential growth of the homology ranks. In case $r \geq 3$, the produced estimate is weaker than the one obtained from $F_{\text{even}}(1, u)$.

3. Generating functions of Euler characteristics in homotopy

In this section we prove Theorem 1.2, which computes the generating function of Euler characteristics of summands in the homotopical splitting (1.7). We also compute, using a different complex, the generating function of the ranks of the summands in (1.7) of genus zero and one.⁷

3.1. The generating function for $\mathbb{Q} \otimes \pi_*(\overline{\text{Emb}}_c(\prod_{i=1}^r \mathbb{R}^{m_i}, \mathbb{R}^d))$

In Section 2 we have computed the generating function $F_{\vec{m}, u}^H(x_1, \dots, x_r, u)$ of Euler characteristics of summands in the homological splitting of $H_*(\overline{\text{Emb}}_c(\prod_{i=1}^r \mathbb{R}^{m_i}, \mathbb{R}^d), \mathbb{Q})$. The aim of this section is to compute the generating function $F_{\vec{m}, u}^\pi(x_1, \dots, x_r, u)$ of the Euler characteristics of summands, but now in the homotopy groups $\mathbb{Q} \otimes \pi_*(\overline{\text{Emb}}_c(\prod_{i=1}^r \mathbb{R}^{m_i}, \mathbb{R}^d))$. Proposition 2.6 will be a key ingredient in our computations. If there is no confusion, the first generating function will be denoted by $F^H(x_1, \dots, x_r, u)$, and the second one just by $F^\pi(x_1, \dots, x_r, u)$. Recalling the equation (1.7) from the introduction, for a sequence $\vec{s}, t \geq 0$, let $\mathcal{X}_{\vec{s}, t}^\pi$ denote the Euler characteristic of the summand $\text{hRmod}_\Omega(Q_{\vec{s}}^{\vec{m}}, \mathbb{Q} \otimes \widehat{\pi}_{t(d-2)+1}(C(\bullet, \mathbb{R}^d)))$. We will also

⁷Genus g is defined as the complexity minus total Hodge degree plus one: $g = t - (s_1 + \dots + s_r) + 1$.

write $\vec{x}^{\vec{s}}$ for $\prod_{i=1}^r x_i^{s_i}$. The generating function we look at here is defined by

$$F_{\vec{m},d}^{\pi}(x_1, \dots, x_r, u) = \sum_{\vec{s},t} \mathcal{X}_{\vec{s},t}^{\pi} \vec{x}^{\vec{s}} u^t. \quad (3.1)$$

In order to compute $F_{\vec{m},d}^{\pi}(x_1, \dots, x_r, u)$, we need the following lemma, which connects $F^H(x_1, \dots, x_r, u)$ and $F^{\pi}(x_1, \dots, x_r, u)$.

Lemma 3.1. *We have*

$$F_{\vec{m},d}^{\pi}(x_1, \dots, x_r, u) = \sum_{l=1}^{+\infty} \frac{\mu(l)}{l} \ln(F^H(x_i \leftarrow x_i^l, u \leftarrow u^l)). \quad (3.2)$$

Here $\mu(-)$ is the standard Möbius function. The notation $F^H(x_i \leftarrow x_i^l, u \leftarrow u^l)$ means that in the expression $F^H(x_1, \dots, x_r, u)$ the variable u is replaced by u^l , and x_i is replaced by x_i^l , $1 \leq i \leq r$.

Proof. The plan of the proof is to compute the right hand side of (3.2), and compare the obtained result with the right hand side of (3.1). From [35, Lemma 16.1], it is straightforward to get the following

$$F^H(x_1, \dots, x_r, u) = \prod_{\vec{s},t} \frac{1}{(1 - \vec{x}^{\vec{s}} u^t)^{\mathcal{X}_{\vec{s},t}^{\pi}}},$$

which gives (by replacing u by u^l and x_i by x_i^l)

$$F^H(x_i \leftarrow x_i^l, u \leftarrow u^l) = \prod_{\vec{s},t} \frac{1}{(1 - (\vec{x}^{\vec{s}} u^t)^l)^{\mathcal{X}_{\vec{s},t}^{\pi}}}. \quad (3.3)$$

Taking now the logarithm of (3.3), and using the well known series expansion $\ln(1-x) = -\sum_{p \geq 1} \frac{x^p}{p}$, we obtain

$$\ln(F^H(x_i \leftarrow x_i^l, u \leftarrow u^l)) = \sum_{\vec{s},t} \sum_{p \geq 1} \mathcal{X}_{\vec{s},t}^{\pi} \frac{(\vec{x}^{\vec{s}} u^t)^{pl}}{p}. \quad (3.4)$$

By putting (3.4) in the right hand side of (3.2), we get

$$\begin{aligned} \sum_{l=1}^{+\infty} \frac{\mu(l)}{l} \ln(F^H(x_i \leftarrow x_i^l, u \leftarrow u^l)) &= \sum_{l \geq 1} \sum_{\vec{s},t} \sum_{p \geq 1} \mathcal{X}_{\vec{s},t}^{\pi} \mu(l) \frac{(\vec{x}^{\vec{s}} u^t)^{pl}}{pl} \\ &= \sum_{\vec{s},t} \sum_{q \geq 1} \mathcal{X}_{\vec{s},t}^{\pi} \frac{1}{q} (\vec{x}^{\vec{s}} u^t)^q \left(\sum_{l|q} \mu(l) \right). \end{aligned}$$

Since $\sum_{l|q} \mu(l) = \begin{cases} 1 & \text{if } q = 1 \\ 0 & \text{if } q \geq 2 \end{cases}$, it follows that

$$\sum_{l=1}^{+\infty} \frac{\mu(l)}{l} \ln(F^H(x_i \leftarrow x_i^l, u \leftarrow u^l)) = \sum_{\vec{s},t} \mathcal{X}_{\vec{s},t}^{\pi} \vec{x}^{\vec{s}} u^t = F^{\pi}(x_1, \dots, x_r, u) \text{ by (3.1).} \quad \square$$

The last thing we need is an explicit expansion of $\ln(\Gamma(x, u))$, where $\Gamma(x, u)$ is defined at the beginning of Subsection 2.3.

Lemma 3.2. $\ln(\Gamma(x, u)) = \sum_{j \geq 1} S_j(x) \frac{u^j}{j}$.

Proof. This identity easily follows from (2.6) and from Bernoulli’s summation formula $S_j(n) = 1^j + \dots + n^j$ when $x = n$ is a positive integer. For other values of x , it follows from the fact that $\ln(\Gamma(x, u))$ is a generating function of u of a sequence of polynomials in x [35, Lemma 14.2]. \square

Now we are ready to prove Theorem 1.2, which is the main result of this subsection.

Proof of Theorem 1.2. The proof is a direct application of Proposition 2.6 and Lemmas 3.1–3.2. \square

3.2. The generating function in homotopy for genus ≤ 1

In [33, Subsection 2.1] we explicitly described a complex $\mathcal{E}_\pi^{\vec{m},d}$ of hairy graphs that computes the rational homotopy of the space $\overline{\text{Emb}}_c(\prod_{i=1}^r \mathbb{R}^{m_i}, \mathbb{R}^d)$ of high dimensional string links. In short, this complex is obtained in [33] by taking an injective resolution of the target Ω -modules in (1.7). The graph-complex $\mathcal{E}_\pi^{\vec{m},d}$ is a differential graded vector space spanned by so called *hairy graphs*, i.e. connected graphs having a finite non-empty set of external vertices (of valence 1 and called *hairs*), a finite set of non-labeled internal vertices (of valence ≥ 3). The edges are oriented. One allows both multiple edges and tadpoles. The external vertices are colored with $\{1, \dots, r\}$ as the set of colors. As a part of the data, each graph comes with the ordering of its *orientation set* that consists of the following elements: edges (of degree $d - 1$), internal vertices (of degree $-d$), external vertices (of degree $-m_i$ if it is colored by i). Changing orientation of an edge in a graph produces sign $(-1)^d$. Changing the order of the orientation set produces the Koszul sign of permutation that takes into account the degree of the elements. The differential is defined as the sum of expansion of internal vertices.

For a hairy graph $G \in \mathcal{E}_\pi^{\vec{m},d}$, its homological degree is the sum of the degrees of the elements in its orientation set. Its corresponding *Hodge multi-degree* is the tuple (\vec{s}) , where s_i is the number of external vertices colored by i . Its *complexity* is the first Betti number of the graph obtained from G by gluing together all its external vertices. For hairy graphs it is also natural to define *genus* g as their first Betti number.

As an example of the complexity, the graph in Figure 1 is of complexity 2, while the complexity is three in Figure 2, and five in Figure 3. Concerning the genus, intuitively, a graph is of genus g if it contains g “loops”. See Figure 1, Figure 2 and Figure 3 for some examples of graphs of genus 0, 1 and 2 respectively.



Figure 1: Graph of genus 0.

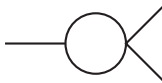


Figure 2: Graph of genus 1.

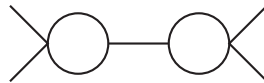


Figure 3: Graph of genus 2.

The graph-complex $\mathcal{E}_\pi^{\vec{m},d}$ can be split as follows. For $g \geq 0$, let $\mathcal{E}_{\pi g}^{\vec{m},d}$ denote the subcomplex of $\mathcal{E}_\pi^{\vec{m},d}$ generated by graphs of genus g . For integers \vec{s} , t , let $\mathcal{E}_{\pi g, \vec{s}, t}^{\vec{m},d}$ be the subcomplex of $\mathcal{E}_{\pi g}^{\vec{m},d}$ generated by graphs of Hodge multi-degree (\vec{s}) and complexity t .

All these subcomplexes are well defined since the differential in $\mathcal{E}_\pi^{\vec{m},d}$ preserves the Hodge multi-degree, the complexity, and the genus. In particular, one has the splitting

$$\mathcal{E}_\pi^{\vec{m},d} = \bigoplus_{g \geq 0} \mathcal{E}_{\pi g}^{\vec{m},d}. \quad (3.5)$$

Definition 3.3. Let $g \geq 0$.

- The *Hodge splitting* of the complex $\mathcal{E}_{\pi g}^{\vec{m},d}$ is the splitting

$$\mathcal{E}_{\pi g}^{\vec{m},d} = \bigoplus_{\vec{s}, t \geq 0} \mathcal{E}_{\pi g, \vec{s}, t}^{\vec{m},d}. \quad (3.6)$$

Each term of the right-hand side of (3.6) is a finite dimensional graph complex.

- The *generating function* associated to (3.6), and denoted $F_{\vec{m},d}^{\pi g}(x_1, \dots, x_r, u)$, is defined as

$$F_{\vec{m},d}^{\pi g}(x_1, \dots, x_r, u) = \sum_{\vec{s}, t \geq 0} \mathcal{X}_{g, \vec{s}, t}^\pi \bar{x}^{\vec{s}} u^t, \quad (3.7)$$

where $\mathcal{X}_{g, \vec{s}, t}^\pi$ is the Euler characteristic of $\mathcal{E}_{\pi g, \vec{s}, t}^{\vec{m},d}$.

It is easy to see the following relation between the genus and the complexity of a graph:

$$\text{genus} = \text{complexity} - \text{number of external vertices} + 1. \quad (3.8)$$

Thanks to (3.8), we can easily express the generating function of the Euler characteristics that takes into account the grading g as well:

$$F^\pi(x_1, \dots, x_r, u, \hbar) = \hbar F^\pi\left(\frac{x_1}{\hbar}, \dots, \frac{x_r}{\hbar}, u\hbar\right), \quad (3.9)$$

where \hbar is the variable responsible for the genus, and the right-hand side uses the generating function defined earlier (3.1). (Abusing notation we denote this function by F^π as well.) Note that $F^\pi = \sum_{g \geq 0} F^{\pi g} \hbar^g$. Applying the result from Theorem 1.2 one can get an explicit formula for $F^\pi(x_1, \dots, x_r, u, \hbar)$. However, even though this formula is very simple for explicit computer calculations it is not at all obvious that it produces zero for the negative genus. Indeed, there will be summands in which \hbar appears with a negative exponent, that must somehow cancel out with each other.

The aim of this subsection is to compute the generating function from (3.7) with $g = 0$ and 1 using hairy graph-complexes, which is done by Theorems 3.5 and 3.8 below. We need first to define a symmetric sequence of graph-complexes $\{M(P_d^k)\}_{k \geq 1}$ that will be used in our computations.

3.2.1. Graph-complexes $M(P_d^k)$

Recall that d denotes the dimension of the ambient space. For $k \geq 0$, $M(P_d^k)$ is the complex of graphs defined essentially in the same way as the hairy graph-complex with the only difference that its graphs have exactly k external vertices labeled bijectively by $1, \dots, k$. Also we exclude external vertices from the orientation set of such graphs. Thus the orientation set of any graph $G \in M(P_d^k)$ is the union of the set E_G of edges of G (each edge being of degree $d - 1$) and the set I_G of its internal vertices (each internal vertex being of degree $-d$). So that the total degree of G is $(d - 1)|E_G| - d|I_G|$.

The differential on $M(P_D^k)$ is defined in the same way as the sum of expansions of internal vertices. Since the differential preserves the genus (first Betti number of a graph), each of $M(P_d^k)$ can be split into a direct sum by genus g . That is, one has $M(P_d^k) = \bigoplus_{g \geq 0} M_g(P_d^k)$. Let V be the multi-graded r -dimensional vector space of colors⁸, whose basis is the set $\{v_1, \dots, v_r\}$, with each v_i being of degree m_i and of Hodge multi-degree $(\underbrace{0 \dots 0}_{i-1}, 1, \underbrace{0 \dots 0}_{r-i})$. Consider the symmetric sequence

$V^{\otimes \bullet} = \{V^{\otimes n}\}_{n \geq 0}$. One has an isomorphism

$$\mathcal{E}_\pi^{\vec{m}, d} \cong \text{hom}_\Sigma(V^{\otimes \bullet}, M(P_d^\bullet)).$$

This implies

$$\mathcal{E}_{\pi g}^{\vec{m}, d} \cong \text{hom}_\Sigma(V^{\otimes \bullet}, M_g(P_d^\bullet)). \quad (3.10)$$

Therefore the homology of each summand $\mathcal{E}_{\pi g}^{\vec{m}, d}$ in (3.5) is completely determined by the symmetric sequence of the homology of $M_g(P_d^\bullet)$. It turns out that for $g = 0$ and 1, the corresponding symmetric sequences can be easily described. Thus the plan to get Theorem 3.5 and Theorem 3.8 will be to compute the cycle index sums $Z_{H_* M_0(P_d^\bullet)}$, $Z_{H_* M_1(P_d^\bullet)}$ first, and then apply (4.4). The latter equation is a general formula that we obtain in the next section by first computing the cycle index sum of $V^{\otimes \bullet}$ (4.3) and then applying (2.5). We mention also at this point that for every given k both groups $H_* M_0(P_d^k)$ and $H_* M_1(P_d^k)$ are concentrated in a single homological degree. Thus computation of the homology ranks or of Euler characteristics carry essentially the same information.

3.2.2. Genus zero

By (3.10) the complex $\mathcal{E}_{\pi 0}$ of colored hairy graphs of genus zero has the form (take $g = 0$)

$$\mathcal{E}_{\pi 0} \cong \text{hom}_\Sigma(V^{\otimes \bullet}, M_0(P_d^\bullet)). \quad (3.11)$$

On the other hand the complexes $M_0(P_d^\bullet) = \{M_0(P_d^k)\}_{k \geq 1}$ are complexes of trees, whose homology is well known to be isomorphic up to a shift of degrees to the components $\mathcal{L}ie((k)) := \mathcal{L}ie(k-1)$ of the cyclic Lie operad [36]. More precisely one has an isomorphism of Σ_k -modules

$$H_*(M_0(P_d^k)) \cong_{\Sigma_k} \Sigma^{k(d-2)-d+3} \mathcal{L}ie((k)) \otimes (\text{sign})^{\otimes d}. \quad (3.12)$$

The homology is concentrated in the smallest possible degree of *uni-trivalent trees*.⁹ Such a tree with k external vertices must have $k-2$ internal vertices and $2k-3$ edges. Thus its degree is $(2k-3)(d-1) - (k-2)d = k(d-2) - d + 3$. Our hairy graph-complex specialized to the gradings when $d = 3$ and all $m_i = 1$, contains in the bottom degree homology the space of univalent graphs modulo AS and IHX relations, which encodes the finite type invariants of string links in \mathbb{R}^3 [4]. The tree-part of this space is well studied [16, 25, 17, 28]. We failed to find in the literature formulas similar to those given by Theorems 3.4–3.5, but they could be easily derived from the results of aforementioned papers and are known to specialists.

⁸A *color* for us is a component of links in $\overline{\text{Emb}}_c(\prod_{i=1}^r \mathbb{R}^{m_i}, \mathbb{R}^d)$.

⁹A tree is called *uni-trivalent* if all its internal vertices are trivalent.

Let $\mathbf{1}(\bullet)$ be the symmetric sequence defined by $\mathbf{1}(1) = \mathbb{Q}$, and $\mathbf{1}(n) = 0$ for all $n \neq 1$. One has

$$\mathcal{L}ie((n)) \cong_{\Sigma_n} \text{Ind}_{\Sigma_1 \times \Sigma_{n-1}}^{\Sigma_n} (\mathbf{1}(1) \otimes \mathcal{L}ie(n-1)) - \mathcal{L}ie(n), \quad n \geq 2.$$

Rationally this was proved in [25, 17] and integrally in [10]. Therefore $\mathcal{L}ie((\bullet)) \cong_{\Sigma} \mathbf{1}(\bullet) \widehat{\otimes} \mathcal{L}ie(\bullet) - \mathcal{L}ie(\bullet) + \mathbf{1}(\bullet)$. We add $\mathbf{1}(\bullet)$ to compensate subtraction of $\mathcal{L}ie(1)$, i.e. to have zero in arity one. This implies that (since $Z_{\mathbf{1}(\bullet)}(p_1, p_2, \dots) = p_1$)

$$Z_{\mathcal{L}ie((\bullet))} = (p_1 - 1)Z_{\mathcal{L}ie(\bullet)} + p_1.$$

Using now the following well known result (see for example [5]; another short and elegant proof is given in [11, Section 5.1])

$$Z_{\mathcal{L}ie(\bullet)}(p_1, p_2, \dots) = \sum_{l=1}^{+\infty} \frac{-\mu(l) \ln(1-p_l)}{l},$$

we get

$$Z_{\mathcal{L}ie((\bullet))}(p_1, p_2, \dots) = (1-p_1) \sum_{l=1}^{+\infty} \frac{\mu(l) \ln(1-p_l)}{l} + p_1. \quad (3.13)$$

From (3.12) and the fact that for genus zero graphs the complexity is the number of external vertices minus one, we get

$$Z_{H_* M_0(P_d^\bullet)}(u, z; p_1, p_2, \dots) = \frac{1}{z^{d-3}u} Z_{\mathcal{L}ie((\bullet))} \left(p_l \leftarrow (-1)^{(l-1)d} (z^{d-2}u)^l p_l, l \in \mathbb{N} \right). \quad (3.14)$$

To recall, the *Hodge splitting* is defined in (3.6).

Theorem 3.4. *The generating function of the dimensions of the Hodge summands of the complex $\mathcal{E}_{\pi_0}^{\overline{m},d}$ of hairy graphs of genus zero is*

$$\begin{aligned} & R_{\overline{m},d}^{\pi_0}(x_1, \dots, x_r, z, u) \\ &= z\alpha_1\left(\frac{1}{z}\right) + \frac{1 - z^{d-2}u\alpha_1\left(\frac{1}{z}\right)}{z^{d-3}u} \sum_{l=1}^{+\infty} \frac{\mu(l) \ln \left(1 - (-1)^{(l-1)d} (z^{d-2}u)^l \alpha_l\left(\frac{1}{z}\right) \right)}{l}, \end{aligned}$$

where $\alpha_l\left(\frac{1}{z}\right) = \sum_{i=1}^r (-1)^{m_i(l-1)} x_i^l \left(\frac{1}{z}\right)^{m_i l}$.

Proof. This formula is obtained from (3.13) by change of variables: first using (3.14) and then (4.4). \square

Theorem 3.5. *The generating function of the Euler characteristics of the Hodge summands of the complex $\mathcal{E}_{\pi_0}^{\overline{m},d}$ of hairy graphs of genus zero is*

$$\begin{aligned} & F_{\overline{m},d}^{\pi_0}(x_1, \dots, x_r, u) = \\ & - \sum_{i=1}^r (-1)^{m_i} x_i + \left(\sum_{i=1}^r (-1)^{m_i} x_i - \frac{(-1)^d}{u} \right) \sum_{l=1}^{+\infty} \frac{\mu(l) \ln \left(1 - (-1)^d u^l \sum_{i=1}^r (-1)^{m_i} x_i^l \right)}{l}. \end{aligned}$$

Proof. Take $z = -1$ in the previous theorem. \square

3.2.3. Genus one

In the previous section we have computed the generating function for graph-complexes of genus zero. Here we will make computations in the genus one case. Figure 4 and Figure 5 are examples of graphs of genus one.

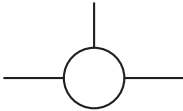


Figure 4: A graph of genus one.

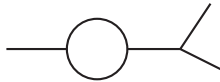


Figure 5: Another graph of genus one.

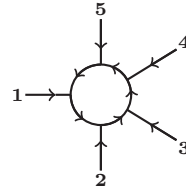


Figure 6: A graph in $H_*M_1(P_d^5)$.

The first goal will be to understand the Σ_n action on $H_*M_1(P_d^n)$. Notice that in this homology we can consider only hedgehogs: graphs whose external vertices are directly connected to the loop. This is because the other graphs are killed by the differential in homology (see [8]). A typical graph (or generator) in $H_*M_1(P_d^n)$ is the one of Figure 6.

A hedgehog with n external vertices has $2n$ edges and n internal vertices, thus the homology $H_*M_1(P_d^n)$ is concentrated in the only degree $2n(d - 1) - nd = n(d - 2)$. Let D_n denote the dihedral group of symmetries of the unit circle with n points marked $e^{\frac{2k\pi i}{n}}$, $k = 0 \dots n - 1$. One has an obvious homomorphism $D_n \rightarrow \Sigma_n$ corresponding to the permutation of the marked points. This map is an inclusion for $n \geq 3$. The hedgehog whose external vertices are marked in the cyclic order $1, \dots, n$ is sent to itself times certain sign when acted on by elements of D_n . Denote by λ_n the character of D_n corresponding to this sign. (This character depends on d as well.) As a Σ_n module in graded vector spaces $H_*M_1(P_d^n)$ is the $n(d - 2)$ -suspended induced representation

$$H_*M_1(P_d^n) \cong_{\Sigma_n} \Sigma^{n(d-2)} \text{Ind}_{D_n}^{\Sigma_n} \lambda_n. \tag{3.15}$$

Let us describe the character λ_n . Since D_n is generated by two elements: a $\frac{2\pi}{n}$ rotation and a reflection, it is enough to compute λ_n on those elements. For an element $\sigma \in D_n$, we will explicitly determine the sign $\lambda_n(\sigma)$ (using the Koszul sign of permutation taking into account the degrees of the elements in the orientation set) that appears in the equality $\sigma \cdot G = \lambda_n(\sigma)G$. Here G stands for the hedgehog whose external vertices are marked in the cyclic order $1, \dots, n$ (Figure 6 is an example of a hedgehog with 5 external vertices.) To recall the orientation set is the union of edges, of degree $d - 1$, and internal vertices, of degree $-d$. There are three possibilities:

- If σ is a rotation by $\frac{2\pi}{n}$, then one can easily see that $\sigma \cdot G = (-1)^{d(n-1)}G$;
- If n is odd and σ is a reflection whose one vertex is fixed (see Figure 7), then $\sigma \cdot G = (-1)^{\frac{n+1}{2}d}G$;
- If n is even and σ is a reflection whose two vertices are fixed (see Figure 8), then one has $\sigma \cdot G = (-1)^{\frac{nd}{2}-1}G$.

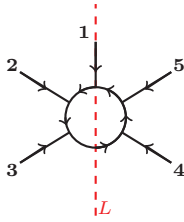


Figure 7: A reflection with respect to L ($n = 5$).

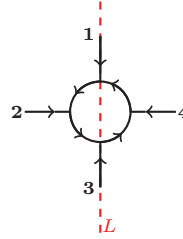


Figure 8: A reflection with respect to L ($n = 4$).

In order to simplify computations, we will write λ_n in another form. Let $\text{sign}: D_n \rightarrow \{-1, 1\}$ be the signature representation restricted from Σ_n , and let $\text{or}: D_n \rightarrow \{-1, 1\}$ be the orientation representation. Notice that the latter representation concerns only reflections in D_n , that is, “or” is equal to -1 on reflections, and to 1 on rotations. From our computations we obtain

$$\lambda_n = \text{sign}^{\otimes d} \otimes \text{or}^{\otimes(n+d+1)}. \tag{3.16}$$

We will need the following well known fact [12]. To recall the cycle index sum is defined in Definition 2.3.

Lemma 3.6. *Let $f: H \rightarrow \Sigma_n$ be a group homomorphism and let $\rho^V: H \rightarrow GL(V)$ be a representation of H , then the cycle index sum of the induced representation can be expressed as*

$$Z_{\text{Ind}_H^{\Sigma_n} V}(p_1, p_2, \dots) = \frac{1}{|H|} \sum_{h \in H} \text{tr}(\rho^V(h)) \prod_l p_l^{j_l(f(h))}.$$

Let $RO_n \subseteq D_n$ denote the subset of D_n formed by rotations, and let RE_n denote the subset formed by reflections. Using the above lemma we get

$$\begin{aligned} Z_{\text{Ind}_{D_n}^{\Sigma_n} \lambda_n}(p_1, p_2, \dots) &= \frac{1}{|D_n|} \sum_{\sigma \in D_n} \lambda_n(\sigma) \prod_l p_l^{j_l(\sigma)} \\ &= \frac{1}{2n} \sum_{\sigma \in RO_n} \lambda_n(\sigma) \prod_l p_l^{j_l(\sigma)} + \frac{1}{2n} \sum_{\sigma \in RE_n} \lambda_n(\sigma) \prod_l p_l^{j_l(\sigma)}. \end{aligned} \tag{3.17}$$

We will denote by $\overline{A}_n(p_1, p_2, \dots)$ and $\overline{B}_n(p_1, p_2, \dots)$ respectively the first and the second summands in (3.17).

Let $\mathbf{1}$ denote the trivial character of D_n . We compute first the cycle index sum $Z_{\text{Ind}_{D_n}^{\Sigma_n} \mathbf{1}}$. Then we will add some signs to describe $Z_{\text{Ind}_{D_n}^{\Sigma_n} \lambda_n}$:

$$Z_{\text{Ind}_{D_n}^{\Sigma_n} \mathbf{1}}(p_1, p_2, \dots) = A_n(p_1, p_2, \dots) + B_n(p_1, p_2, \dots),$$

where

$$A_n = \frac{1}{2n} \sum_{l|n} \varphi(l) p_l^{\frac{n}{l}} \quad \text{and} \quad B_n = \begin{cases} \frac{1}{2} p_1 p_2^{\frac{n-1}{2}} & \text{if } n \text{ odd,} \\ \frac{1}{4} (p_1^2 p_2^{\frac{n-2}{2}} + p_2^{\frac{n}{2}}) & \text{if } n \text{ even,} \end{cases}$$

where $\varphi(l)$ is the Euler's totient function that produces the number of positive integers less than or equal to l that are relatively prime to l . Explicitly $\varphi(l) = \sum_{a|l} \mu(a) \frac{l}{a}$.

Summing over $n \geq 1$, we get

$$Z_{\text{Ind}_{\mathbb{D}}^{\Sigma} \bullet} \mathbf{1}(p_1, p_2, \dots) = -\frac{1}{2} \sum_{l \geq 1} \frac{\varphi(l) \ln(1-p_l)}{l} + \frac{p_1^2 + p_2 + 2p_1}{4(1-p_2)}. \quad (3.18)$$

On the other hand, from (3.16) we get

$$\begin{aligned} \bar{A}_n(p_1, p_2, \dots) &= A_n(p_l \leftarrow (-1)^{d(l-1)} p_l), \\ \bar{B}_n(p_1, p_2, \dots) &= (-1)^{n+d+1} B_n(p_l \leftarrow (-1)^{d(l-1)} p_l) \\ &= (-1)^{d+1} B_n(p_l \leftarrow (-1)^{d(l-1)+l} p_l). \end{aligned}$$

Here the second equality is deduced from the explicit formula for B_n . Since $\sum_{n \geq 1} A_n$ and $\sum_{n \geq 1} B_n$ are respectively the first and second summands in (3.18), we get

$$\begin{aligned} Z_{\text{Ind}_{\mathbb{D}}^{\Sigma} \bullet} \lambda_{\bullet}(p_1, p_2, \dots) &= \\ &= -\frac{1}{2} \sum_{l \geq 1} \frac{\varphi(l) \ln(1 - (-1)^{d(l-1)} p_l)}{l} + (-1)^{d+1} \frac{p_1^2 + (-1)^d p_2 - 2p_1}{4(1 - (-1)^d p_2)}. \end{aligned} \quad (3.19)$$

From (3.15) and the fact that any genus 1 graph with k external vertices has complexity k ,

$$Z_{H_* M_1(P_d \bullet)}(z, u; p_1, p_2, \dots) = Z_{\text{Ind}_{\mathbb{D}}^{\Sigma} \bullet} \lambda_{\bullet} \left(p_l \leftarrow z^{(d-2)l} u^l p_l, l \in \mathbb{N} \right). \quad (3.20)$$

Recall the *Hodge splitting* from (3.6).

Theorem 3.7. *The generating function of the dimensions of the Hodge summands of the complex $\mathcal{E}_{\pi_1}^{\bar{m}, d}$ of hairy graphs of genus one is*

$$\begin{aligned} R_{\bar{m}, d}^{\pi_1}(x_1, \dots, x_r, z, u) &= -\frac{1}{2} \sum_{l \geq 1} \frac{\varphi(l) \ln(1 - (-1)^{d(l-1)} z^l (d-2) u^l \alpha_l(\frac{1}{z}))}{l} \\ &+ (-1)^{d+1} \frac{z^{2d-4} u^2 \alpha_1(\frac{1}{z})^2 + (-1)^d z^{2d-4} u^2 \alpha_2(\frac{1}{z}) - 2z^{d-2} u \alpha_1(\frac{1}{z})}{4(1 - (-1)^d z^{2d-4} u^2 \alpha_2(\frac{1}{z}))}, \end{aligned}$$

where $\alpha_l(\frac{1}{z}) = \sum_{i=1}^r (-1)^{m_i(l-1)} x_i^l (\frac{1}{z})^{m_i l}$.

Proof. This formula is obtained from (3.19) by change of variables: first using (3.20) and then (4.4) below. \square

Theorem 3.8. *The generating function of the Euler characteristics of the Hodge splitting of the complex $\mathcal{E}_{\pi_1}^{\bar{m}, d}$ of hairy graphs of genus one is*

$$\begin{aligned} F_{\bar{m}, d}^{\pi_1}(x_1, \dots, x_r, u) &= -\frac{1}{2} \sum_{l \geq 1} \frac{\varphi(l) \ln(1 - (-1)^d u^l \sum_{i=1}^r (-1)^{m_i} x_i^l)}{l} \\ &+ (-1)^{d+1} \frac{u^2 (\sum_{i=1}^r (-1)^{m_i} x_i)^2 + (-1)^d u^2 \sum_{i=1}^r (-1)^{m_i} x_i^2 - 2(-1)^d u \sum_{i=1}^r (-1)^{m_i} x_i}{4(1 - (-1)^d u^2 \sum_{i=1}^r (-1)^{m_i} x_i^2)}. \end{aligned}$$

Proof. Take $z = -1$ in the previous theorem. \square

4. Supercharacter of the symmetric group action on $\mathbf{Mod}(\mathcal{L}_\infty)$ and $\mathbf{Mod}_{\text{Det}}(\mathcal{L}_\infty)$

In Subsection 4.1 we compute the cycle index sum of the supercharacter of the symmetric group action on the sequence $M(P_d^\bullet)$ introduced in Subsection 3.2.1. In Subsection 4.2 we briefly recall basic facts about cyclic and modular operads and we explain how $M(P_d^\bullet)$ is related to $\mathbf{Mod}(\mathcal{L}_\infty)$ and $\mathbf{Mod}_{\text{Det}}(\mathcal{L}_\infty)$. At the end we prove Theorem 1.3.

4.1. Supercharacter for $M(P_d^\bullet)$

Let $M = (\oplus_i M_i, \partial)$ be a finite dimensional chain complex of Σ_k -modules over a ground field \mathbb{K} of characteristic 0. By the *supercharacter* we understand the character of the Σ_k action on the virtual representation $\mathcal{X}M$ defined as $\mathcal{X}M := \sum_i (-1)^i M_i$. The latter virtual representation is similar to the Euler characteristic in the sense that $\mathcal{X}M \simeq \mathcal{X}(H_*M)$, that is why we use this notation. The cycle index sum encoding the supercharacter of the Σ_k action on M can be defined as $Z_{\mathcal{X}M} = \sum_i (-1)^i Z_{M_i}$, or equivalently as $Z_M|_{z=-1}$.

For a symmetric sequence of chain complexes $M = \{M(k)\}_{k \geq 0}$, we similarly define $Z_{\mathcal{X}M} := \sum_{k \geq 0} Z_{\mathcal{X}M(k)}$.

For the rest of this section, V will denote the r -dimensional vector space whose basis is the set of colors v_1, \dots, v_r , with each v_i being of degree m_i and of Hodge multi-degree $(\underbrace{0 \dots 0}_{i-1}, 1, \underbrace{0 \dots 0}_{r-i})$. Consider the symmetric sequence $V^{\otimes \bullet} = \{V^{\otimes n}\}_{n \geq 0}$.

We will need to know the cycle index sum $Z_{V^{\otimes \bullet}}$. For each $1 \leq i \leq r$, consider the one dimensional vector space V_i spanned by v_i , and consider the symmetric sequence $V_i^{\otimes \bullet} = \{V_i^{\otimes n}\}_{n \geq 0}$. One can rewrite the vector space V in the form $V = V_1 \oplus \dots \oplus V_r$. Therefore,

$$V^{\otimes n} = \bigoplus_{|\vec{k}|=n} \text{Ind}_{\Sigma_{\vec{k}}}^{\Sigma_n} V_1^{\otimes k_1} \otimes \dots \otimes V_r^{\otimes k_r}.$$

Recalling Definition 2.1 we have $V^{\otimes \bullet} = V_1^{\otimes \bullet} \widehat{\otimes} \dots \widehat{\otimes} V_r^{\otimes \bullet}$, and by Lemma 2.5 one has

$$Z_{V^{\otimes \bullet}} = \prod_{i=1}^r Z_{V_i^{\otimes \bullet}}. \tag{4.1}$$

For $1 \leq i \leq r$ we will compute $Z_{V_i^{\otimes \bullet}}$. By noticing that the action of Σ_n on $V_i^{\otimes n}$ depends on the parity of m_i (for odd m_i the action is the sign representation, which means that if $\sigma \in \Sigma_n, x \in V_i^{\otimes n}$, then $\sigma x = \pm x$, and for even m_i it is the identity), by also noticing that V_i is a one dimensional vector space, it follows that $V_i^{\otimes \bullet}$ is the commutative unital operad “up to sign”. One has $Z_{\text{Com}}(p_1, p_2, \dots) = \exp\left(\sum_{l \geq 1} \frac{p_l}{l}\right)$, see for example [11, Section 5]. We deduce that

$$Z_{V_i^{\otimes \bullet}}(z, x_i; p_1, p_2, \dots) = Z_{\text{Com}}(p_l \leftarrow (-1)^{m_i(l-1)} x_i^l z^{m_i l} p_l) = \exp\left(\sum_{l \geq 1} (-1)^{m_i(l-1)} x_i^l z^{m_i l} \frac{p_l}{l}\right), \tag{4.2}$$

where the variable z is responsible for the usual homological degree and x_i is responsible for the i -th Hodge grading. The sign $(-1)^{m_i(l-1)}$ appears because a cycle of length l is an odd representation if and only if l is even. The factors x_i^l and $z^{m_i l}$ encode the fact that $V_i^{\otimes l}$ is concentrated in the Hodge multi-degree $(\underbrace{0 \dots 0}_{i-1}, l, \underbrace{0 \dots 0}_{r-i})$

and homological degree $m_i l$.

Combining (4.1) and (4.2), we have

$$Z_{V^{\otimes \bullet}}(z, x_1, \dots, x_r; p_1, p_2, \dots) = \exp \left(\sum_{l \geq 1} \alpha_l(z, x_1, \dots, x_r) \frac{p_l}{l} \right), \quad (4.3)$$

where

$$\alpha_l(z, x_1, \dots, x_r) = \sum_{i=1}^r (-1)^{m_i(l-1)} x_i^l z^{m_i l}.$$

For computations of the Euler characteristics we will need

$$\alpha_l(-1) = \sum_{i=1}^r (-1)^{m_i} x_i^l.$$

For any symmetric sequence $M(\bullet)$, we get

$$\begin{aligned} \dim \operatorname{hom}_\Sigma(V^{\otimes \bullet}, M(\bullet)) &= \left\{ Z_{V^{\otimes \bullet}}\left(\frac{1}{z}; p_l \leftarrow l \frac{\partial}{\partial p_l}, l \in \mathbb{N}\right) Z_{M(\bullet)}(z; p_1, p_2, \dots) \right\} \Big|_{p_l=0} \\ &= Z_{M(\bullet)}\left(z; p_l \leftarrow \alpha_l\left(\frac{1}{z}, x_1, \dots, x_r\right), l \in \mathbb{N}\right). \end{aligned} \quad (4.4)$$

Notation ‘‘dim’’ stays for the generating function of dimensions that takes into account both homological degree (with z responsible for it) and the Hodge degrees (x_1, \dots, x_r are the responsible variables).

The hairy graph-complex $\mathcal{E}_\pi^{\bar{m}, d}$ we recalled at the beginning of Subsection 3.2 has exactly this form:

$$\mathcal{E}_\pi^{\bar{m}, d} \cong \operatorname{hom}_\Sigma(V^{\otimes \bullet}, M(P_d^\bullet)),$$

where $M(P_d^k)$ is the graph-complex from Subsection 3.2.1.

Theorem 4.1. *The supercharacter of the symmetric group action on the graph-complexes $\{M(P_d^k)\}_{k \geq 1}$ is described by the cycle index sum*

$$\begin{aligned} Z_{\mathcal{X}M(P_d^\bullet)}(u; p_1, p_2, p_3, \dots) &= \sum_{k, l, j \geq 1} \frac{\mu(k)}{k^j} S_j \left(-\frac{1}{l} \sum_{a|l} \mu\left(\frac{l}{a}\right) p_{ak} \right) \left(\frac{(-1)^{d-1} l u^{kl}}{F_l(u^k)} \right)^j + \\ &\quad \sum_{k, l \geq 1} \frac{\mu(k)}{kl} \left(\sum_{a|l} \mu\left(\frac{l}{a}\right) p_{ak} \right) \ln(F_l(u^k)), \end{aligned}$$

where the variable u is as usual responsible for complexity.

Proof. Applying (4.4) to $M(\bullet) = M(P_d^\bullet)$ and taking $z = -1$, we get

$$F_{\bar{m}, d}^\pi(x_1, \dots, x_r, u) = Z_{\mathcal{X}(M(P_d^\bullet))}(u, p_l \leftarrow \alpha_l(-1)).$$

Thus to get $Z_{\mathcal{X}M(P_d^\bullet)}$ from $F_{\bar{m}, d}^\pi$ we need to replace each occurrence of $\alpha_l(-1) =$

$\sum_{i=1}^r (-1)^{m_i} x_i^l$ back to p_l . Using the result of Theorem 1.2 and the fact that $E_l(x) = \frac{1}{l} \sum_{a|l} \mu(\frac{l}{a}) x^a$ we get the result. \square

4.2. Proof of Theorem 1.3

4.2.1. Cyclic and modular operads

All the operads that we are going to consider are ones in chain complexes. Cyclic and modular operads were introduced by Getzler and Kapranov [13, 14]. In short a *cyclic operad* $O = \{O(n), n \geq 0\}$ is a usual symmetric operad for which the output of its elements has the same role as the inputs. In particular, each component $O(n)$ has an action of Σ_{n+1} . To distinguish the cyclic arity with the usual one, one writes $O((n + 1))$ for $O(n)$. For usual operads, the category that encodes the ways elements can be composed is the category of rooted trees, while for cyclic operads, it is the category of unrooted trees [13, Section 1].

A *modular operad* is a *stable collection* $M = \{M((g, n)); g \geq 0, n \geq 0\}$. *Stable* means $M((g, n)) = 0$ if $2g + n - 2 \leq 0$.¹⁰ The compositions for M are encoded by the categories of stable graphs $\Gamma((g, n))$, $g \geq 0, n \geq 0, 2g + n - 2 > 0$. An element in $\Gamma((g, n))$ is a connected graph G that has n *external vertices* of valence one and labeled bijectively by $\{1 \dots n\}$, and some set $V(G)$ of non-labeled *internal vertices*, together with a map $g: V(G) \rightarrow \mathbb{N}$ to the set of non-negative integers. We will denote by $|v|$ the valence of $v \in V(G)$. One also requires $\sum_{v \in V(G)} g(v) + \beta_1(G) = g$, where $\beta_1(G)$ is the first Betti number of G . One has a morphism $\rho: G_1 \rightarrow G_2$ in $\Gamma((g, n))$ if G_2 is obtained from G_1 by a contraction of some subset of internal edges also assuming that for any $v \in V(G_2)$, $g(v) = \sum_{v' \in V(\rho^{-1}(v))} g(v') + \beta_1(\rho^{-1}(v))$. Here $\rho^{-1}(v)$ is the preimage of the vertex $v \in V(G_2)$ under the edge contraction ρ taken together with its small neighborhood, so that $\rho^{-1}(v)$ can be viewed as a stable graph in $\Gamma((g(v), |v|))$. (This notation will be used below in the definition of a *cocycle* or *twist* of the modular operadic structure.)

The terminal element in $\Gamma((g, n))$ is $c_{g,n}$ – the n -corolla with the only vertex of genus g . The structure of a modular operad is determined by the composition maps

$$M((G)) := \bigotimes_{v \in V(G)} M((g(v), |v|)) \rightarrow M((g, n)) =: M((c_{g,n})),$$

corresponding to the morphisms $G \rightarrow c_{g,n}$ in $\Gamma((g, n))$ $g \geq 0, n \geq 0$, that should satisfy natural associativity properties. Here we implicitly assume that each $M((g, n))$ has a Σ_n action.

One has an adjunction **Mod**: CycOp \rightleftarrows ModOp: **Cyc** between the categories of cyclic and modular operads, where **Cyc** assigns to a modular operad its $g = 0$ part. Its left adjoint functor **Mod** assigns to a cyclic operad its modular envelope [19]. As relevant to us examples, *Lie* and its Koszul resolution \mathcal{L}_∞ are cyclic operads, for which one defines **Mod**(*Lie*) and **Mod**(\mathcal{L}_∞), cf. loc. cit.

The notion of a modular operad is more subtle than it might appear at the first sight. It comes in different *twisted* versions. Denote by $\text{Iso}\Gamma((g, n))$, $g \geq 0, n \geq 0$, the groupoids of isomorphisms of stable graphs. For any stable graph G , let Aut_G denote the endomorphism set of the graph G in one of these categories. It is the

¹⁰Below we also allow $M((0, 2))$ to be one-dimensional being spanned by the identity element. It appears in our graph-complexes, that is why we add it.

group of symmetries of G that can also permute its external vertices (if symmetries allow). A *cocycle* \mathfrak{D} is a family of functors $\mathfrak{D}: \text{Iso}\Gamma((g, n)) \rightarrow \text{grVect}$, $g \geq 0, n \geq 0$, to the category of graded vector spaces, that always assigns a one-dimensional vector space, and that has in addition the structure of a *hyperoperad*: to each morphism $\rho: G_1 \rightarrow G_2$ in $\Gamma((g, n))$, it is assigned a map $\nu_\rho: \mathfrak{D}(G_2) \otimes \bigotimes_{v \in V(G_2)} \mathfrak{D}(\rho^{-1}(v)) \rightarrow \mathfrak{D}(G_1)$ satisfying natural axioms [14, Section 4.1]. For any cocycle \mathfrak{D} , a \mathfrak{D} -twisted modular operad is a stable sequence $M = \{M((g, n)), g \geq 0, n \geq 0\}$ endowed with composition maps

$$\mathfrak{D}(G) \otimes M((G)) \rightarrow M((g, n)) = M((c_{g,n})),$$

for any $G \in \Gamma((g, n))$, $g \geq 0, n \geq 0$.

The cocycle Det from [14] is of a special interest to us. For a vector space W of dimension k define $\text{Det } W = \Sigma^{-k} \Lambda^k W$. It is a one-dimensional vector space in degree $-k$. One has

$$\text{Det}(W_1 \oplus W_2) = \text{Det } W_1 \otimes \text{Det } W_2. \tag{4.5}$$

Below, for a tensor product of one-dimensional vector spaces we will be using \cdot instead of \otimes . Also the dual of a one-dimensional space X will be denoted by X^{-1} .

The cocycle Det is defined as $\text{Det}(G) := \text{Det } H_1(G)$, for any stable graph $G \in \Gamma((g, n))$. A peculiar property of Det is that it restricts trivially on $\text{Iso}\Gamma((0, n))$. This means that Det -twisted cyclic operads are the usual cyclic ones. One gets a similar adjunction $\mathbf{Mod}_{\text{Det}}: \text{CycOp} \rightleftharpoons \text{ModOp}_{\text{Det}}: \mathbf{Cyc}$ between the usual cyclic operads and the Det -twisted modular ones. This, in particular, produces $\mathbf{Mod}_{\text{Det}}(\mathcal{L}_\infty)$ – the Det -twisted modular envelope of \mathcal{L}_∞ .

4.2.2. $M(P_d^\bullet)$ as $\mathbf{Mod}(\mathcal{L}_\infty)$ and $\mathbf{Mod}_{\text{Det}}(\mathcal{L}_\infty)$

For any stable collection $\{M((g, n))\}$ define a symmetric sequence $M((\bullet)) = \{\bigoplus_g M(g, n), n \geq 0\}$.

Consider now the modular envelope $\mathbf{Mod}(\mathcal{L}_\infty)$. It is easy to notice that graph-complexes $M(P_d^\bullet)$ are closely related to the components of $\mathbf{Mod}(\mathcal{L}_\infty)((\bullet))$. They are spanned by the same combinatorial graphs, where the grading genus corresponds to the first Betti number of the graphs. If we choose $d = 3$ and also tensor each component $M(P_d^k)$ with the sign representation and take a shift in degree (desuspension) by k we get $\mathbf{Mod}(\mathcal{L}_\infty)((k))$.

Lemma 4.2. *For any $k \geq 1$,*

$$\mathbf{Mod}(\mathcal{L}_\infty)((k)) \cong_{\Sigma_k} \Sigma^{-k} M(P_3^k) \otimes \text{sign}. \tag{4.6}$$

Proof. Combinatorially $\mathbf{Mod}(\mathcal{L}_\infty)((k))$ is a graph-complex consisting of exactly the same graphs as $M(P_3^k)$, so we only need to work out the signs and degrees properly. The operad \mathcal{L}_∞ is cyclic and is freely generated by operations of cyclic arity l , $l \geq 3$, which have degree $l - 3$ and the sign action of Σ_l . Graphically such operations correspond to vertices of arity l . As a conclusion, to orient a graph $G \in \mathbf{Mod}(\mathcal{L}_\infty)((k))$ we need to order its vertices, where a vertex v of valence $|v|$ is considered as element of degree $|v| - 3$, and for each vertex to order edges adjacent to it. Changing the order of adjacent edges at any vertex gives the sign of permutation; changing the order of vertices gives the *Koszul sign of permutation* (that is, the sign of permutation that takes

into account the degree of elements). Now when we look at a graph $G \in M(P_3^k)$, it is oriented by ordering the set of its vertices (considered as elements of degree -3) and edges (considered as elements of degree 2, therefore their placement in the orientation set can be ignored), and by orienting all edges. Changing orientation of an edge gives a negative sign. Now, we replace each edge in the orientation set by its two half-edges in the order – first source, second target. Then we change the order of the elements in the orientation set so that the vertices and adjacent to it half-edges come in one block – first the vertex than half-edges. The combined block corresponding to any vertex v has degree exactly $|v| - 3$. Notice, however, that k half-edges, corresponding to the external vertices, do not appear in any such block. These half-edges get annihilated with $\Sigma^{-k} \text{sign}$ in (4.6).

For this argument one should consider separately the case of the identity element $id \in \mathbf{Mod}(\mathcal{L}_\infty)((0, 2))$. It is described as a graph that has two external vertices connected by an edge (no internal vertices). The same graph in $M_0(P_3^2)$ has degree two and enjoys the sign action of Σ_2 . The degree and sign shift exactly correspond to the statement of the lemma.

The fact that the differentials agree, which in both cases are sums of expansions of vertices, is a tedious, but straightforward check. \square

One has a similar description for the components of $\mathbf{Mod}_{\text{Det}}(\mathcal{L}_\infty)((\bullet))$ in terms of $M(P_2^\bullet)$.

Lemma 4.3. *For any $k \geq 1$, $\mathbf{Mod}_{\text{Det}}(\mathcal{L}_\infty)((k)) \cong_{\Sigma_k} \Sigma^{-1}M(P_2^k)$.*

Proof. Similarly to the previous lemma, we need to check that the signs and gradings agree.

Forgetting the differential, \mathcal{L}_∞ is a free cyclic operad generated by a sequence of one-dimensional vector spaces. As a consequence, $\mathbf{Mod}(\mathcal{L}_\infty)$ and its twisted version $\mathbf{Mod}_{\text{Det}}(\mathcal{L}_\infty)$ are also free (twisted) modular operads generated by the same sequence viewed as a stable collection concentrated in genus $g = 0$. Thus for any stable graph G , for which $g|_{V(G)} \equiv 0$, there corresponds exactly one graph in $\mathbf{Mod}(\mathcal{L}_\infty)$ and $\mathbf{Mod}_{\text{Det}}(\mathcal{L}_\infty)$, that, abusing notation, we also denote by G . Recall that Aut_G is the group of symmetries of G . Denote by $\text{Aut}_G^{\text{int}}$ its subgroup of elements fixing external vertices of G pointwise.

Denote by $or(G)$, respectively $or_{\text{Det}}(G)$, the one dimensional sign representation of Aut_G that describes how the sign of G in $\mathbf{Mod}(\mathcal{L}_\infty)$, respectively $\mathbf{Mod}_{\text{Det}}(\mathcal{L}_\infty)$, changes when the symmetries get applied. One obviously has that $G = 0$ in $\mathbf{Mod}(\mathcal{L}_\infty)$, respectively $\mathbf{Mod}_{\text{Det}}(\mathcal{L}_\infty)$, if the restriction of this sign representation on $\text{Aut}_G^{\text{int}}$ is non-trivial. We consider $or(G)$ and $or_{\text{Det}}(G)$ as one-dimensional graded vector spaces concentrated in the degree of G , which is $|G| = \sum_{v \in V(G)} (|v| - 3)$ for $or(G)$ and $|G| = \sum_{v \in V(G)} (|v| - 3) - \beta_1(G)$ for $or_{\text{Det}}(G)$.

One can view G as an element of $M(P_2^k)$, respectively $M(P_3^k)$. One similarly defines $or_2(G)$ and $or_3(G)$ – the corresponding graded one-dimensional Aut_G -modules.

To prove the lemma, we should show that $or_{\text{Det}}(G) = \Sigma^{-1}or_2(G)$.

- Let $C_0^{\text{int}}(G)$ denote the vector space of cellular 0-chains spanned by the internal vertices of G .

- Let $C_1^{int}(G)$ denote the vector space of cellular 1-chains spanned by the internal edges of G . Here as usual, one considers oriented edges; changing orientation implies change in sign: $\overrightarrow{e} = -\overleftarrow{e}$.
- Let $C_1^{ext}(G)$ denote the vector space of cellular 1-chains spanned by the external edges of G .
- We also set $C_1(G) = C_1^{int}(G) \oplus C_1^{ext}(G)$.

All these spaces are viewed as Aut_G -modules concentrated in degree zero. One has an exact sequence of Aut_G -modules:

$$0 \longleftarrow H_0(G) \longleftarrow C_0^{int}(G) \xleftarrow{\partial} C_1^{int}(G) \longleftarrow H_1(G) \longleftarrow 0. \tag{4.7}$$

From Lemma 4.2, one has $or(G) = or_3(G) \cdot \text{Det } C_1^{ext}(G)$. From the definition of $M(P_d^k)$, one can easily get $or_2(G) = or_3(G) \cdot \text{Det } C_1(G) \cdot (\text{Det } C_0^{int}(G))^{-1}$. From the definition of the Det-cocycle, and using the fact that $\mathbf{Mod}(\mathcal{L}_\infty)$ and $\mathbf{Mod}_{\text{Det}}(\mathcal{L}_\infty)$ are free, one has

$$or_{\text{Det}}(G) = or(G) \cdot \text{Det } H_1(G) = or(G) \cdot \text{Det } C_1^{int}(G) \cdot (\text{Det } C_0^{int}(G))^{-1} \cdot \text{Det } H_0(G).$$

The last equation follows from (4.7) and (4.5).

From the three identities above, we get

$$\begin{aligned} or_{\text{Det}}(G) &= \Sigma^{-1} or(G) \cdot \text{Det } C_1^{int}(G) \cdot (\text{Det } C_0^{int}(G))^{-1} = \\ &= \Sigma^{-1} or_3(G) \cdot \text{Det } C_1(G) \cdot (\text{Det } C_0^{int}(G))^{-1} = \Sigma^{-1} or_2(G). \end{aligned}$$

The case of the identity element $id \in \mathbf{Mod}_{\text{Det}}(\mathcal{L}_\infty)((0, 2))$ should be considered separately. The corresponding graph in $M_0(P_2^2)$ has degree 1 and trivial action of Σ_2 , which is compatible with the statement of the lemma. \square

For $d > 3$, the components $M_g(P_d^k)$ can also be expressed in terms of $\mathbf{Mod}(\mathcal{L}_\infty)((g, k))$ and $\mathbf{Mod}_{\text{Det}}(\mathcal{L}_\infty)((g, k))$ using some regrading, which follows from the isomorphism $M_g(P_{d+2}^k) \simeq_{\Sigma_k} \Sigma^{2(g-1)+2k} M_g(P_d^k)$.

The lemma below is aimed to the readers more familiar with the Feynman transforms rather than with the modular envelope.

For a stable collection $M = \{M((g, n)), g \geq 0, n \geq 0\}$, one denotes by ΣM the objectwise suspension $\Sigma M = \{\Sigma M((g, n)), g \geq 0, n \geq 0\}$, and one denotes by $\mathfrak{s}M$ its modular operadic suspension

$$\mathfrak{s}M = \{\mathfrak{s}M((g, n)), g \geq 0, n \geq 0\} = \{\Sigma^{-2(g-1)-n} M((g, n)) \otimes \text{sign}, g \geq 0, n \geq 0\}.$$

Lemma 4.4. *The modular envelopes of Lie and the Feynman transforms of Com are related to each other by the following regrading: $\text{F}_{\text{Det}}\text{Com} = \Sigma \mathfrak{s}\mathbf{Mod}(\mathcal{L}_\infty)$, $\text{FCom} = \Sigma \mathfrak{s}\mathbf{Mod}_{\text{Det}}(\mathcal{L}_\infty)$.*

Proof. To the regratings Σ and \mathfrak{s} one assigns the twisting cocycles \mathfrak{D}_Σ and $\mathfrak{D}_\mathfrak{s}$, respectively, see [14]. One has that $\mathbf{Mod}_{\text{Det}}(\mathcal{L}_\infty)$ is a free Det-twisted modular operad generated by $\Sigma^{-1}\mathfrak{s}^{-1}\text{Com}$. Thus $\Sigma \mathfrak{s}\mathbf{Mod}_{\text{Det}}(\mathcal{L}_\infty)$ is a free $(\text{Det} \cdot \mathfrak{D}_\Sigma \cdot \mathfrak{D}_\mathfrak{s} =: \mathfrak{K})$ -twisted modular operad generated by $\text{Com}((\bullet))$, which is exactly the definition of FCom . Similarly, $\Sigma \mathfrak{s}\mathbf{Mod}(\mathcal{L}_\infty)$ is a free $(\mathfrak{D}_\Sigma \cdot \mathfrak{D}_\mathfrak{s} = \mathfrak{K} \cdot \text{Det}^{-1})$ -twisted modular operad generated by $\text{Com}((\bullet))$, which is the definition of $\text{F}_{\text{Det}}\text{Com}$. \square

4.2.3. Proof of the theorem

We concentrate on positive arities. The case of arity zero and a connection to Willwacher-Živković’ computations [37] have been explained in the introduction. For a symmetric sequence $\{M(n), n \geq 0\}$, we denote by $Z_M^{>0}$ the cycle index sum of $\{M(n), n \geq 1\}$.

Now we have all ingredients to prove Theorem 1.3: Lemmas 4.2–4.3 and Theorem 4.1. In the cycle index sum of the latter result, there is variable u , which is responsible for complexity. It follows from (4.6) that

$$Z_{\mathcal{X}\mathbf{Mod}(\mathcal{L}_\infty)}^{>0}(u; p_1, p_2, p_3, \dots) = Z_{\mathcal{X}M(P_3^\bullet)}(u; -p_1, -p_2, -p_3, \dots). \tag{4.8}$$

Since $\mathbf{Mod}(\mathcal{L}_\infty)$ is a modular operad, it is more natural to consider the splitting by genus rather than by complexity. We use the variable \hbar as the one responsible for the genus. It follows from (3.8) that we need to make the change of variables $u \leftarrow \hbar, p_l \leftarrow \frac{p_l}{\hbar^l}$, and in addition to it, multiply the result by \hbar . (Compare with (3.9).) Combining it with (4.8), we get

$$Z_{\mathcal{X}\mathbf{Mod}(\mathcal{L}_\infty)}^{>0}(\hbar; p_1, p_2, p_3, \dots) = \hbar Z_{\mathcal{X}M(P_3^\bullet)}(u \leftarrow \hbar; p_l \leftarrow \frac{p_l}{\hbar^l}, l \in \mathbb{N}).$$

To finish the proof of (1.12) we apply Theorem 4.1 for $d = 3$.

To prove (1.13), we similarly get

$$Z_{\mathcal{X}\mathbf{Mod}_{\text{Det}}(\mathcal{L}_\infty)}^{>0}(\hbar; p_1, p_2, p_3, \dots) = -\hbar Z_{\mathcal{X}M(P_2^\bullet)}(u \leftarrow \hbar; p_l \leftarrow \frac{p_l}{\hbar^l}, l \in \mathbb{N}).$$

And then apply Theorem 4.1 for $d = 2$.

Appendix A. Computation of Euler characteristics

In [2, Appendix A], Arone and the second author produced tables of computer calculations of the Euler characteristics of the Hodge summands both for the rational homology and rational homotopy of the spaces $\overline{\text{Emb}}_c(\mathbb{R}^m, \mathbb{R}^d)$, $d \geq 2m + 2$. Using Mathematica we produced similar computations in case of link spaces with two components ($r = 2$). Recall the splitting of the complex $\mathcal{E}_\pi^{\vec{m}, d}$ from (3.5). One has $\mathcal{E}_\pi^{\vec{m}, d} = \bigoplus_{g \geq 0} \bigoplus_{\vec{s}, t} \mathcal{E}_{\pi g, \vec{s}, t}^{\vec{m}, d}$. Let $\chi_{\vec{s}, t}^{\pi g}$ denote the Euler characteristic of each summand in that splitting. The tables (which can be found online at <http://dx.doi.org/10.4310/HHA.2018.v20.n2.a7>) furnish values of $\chi_{s_1, s_2, t}^{\pi g}$ for genus $g \in \{0, 1, 2, 3\}$ with m_1, m_2 and d odd. Recall also that complexity t is determined by the formula $g + s_1 + s_2 = t + 1$ from (3.8).

References

- [1] G. Arone and V. Turchin, On the rational homology of high dimensional analogues of spaces of long knots, *Geom. Topol.* **8** (2014), 1261–1322.
- [2] G. Arone and V. Turchin, Graph-complexes computing the rational homotopy of high dimensional analogues of spaces of long knots, *Ann. Inst. Fourier (Grenoble)* **65** (2015), no. 1, 1–62.
- [3] S. Barannikov, Solving the noncommutative Batalin-Vilkovisky equation. *Lett. Math. Phys.* **103** (2013), no. 6, 605–628.

- [4] D. Bar-Natan, Vassiliev homotopy string link invariants, *J. Knot Theory Ramifications* **4** (1995), no. 1, 13–32.
- [5] J. Brandt, The free Lie ring and Lie representations of the full linear group, *Trans. Amer. Math. Soc.* **56** (1944), 528–536.
- [6] R. Budney, A family of embedding spaces. *Groups, Homotopy and Configuration Spaces*, *Geom. Topol. Monogr.* **13**, Geom. Topol. Publ., Coventry, 2008, pages 41–83.
- [7] J. Chuang and A. Lazarev, Combinatorics and formal geometry of the Maurer-Cartan equation. *Lett. Math. Phys.* **103** (2013), no. 1, 79–112.
- [8] J. Conant, J. Costello, V. Turchin, and P. Weed, Two-loop part of the rational homotopy of spaces of long embeddings, *J. Knot Theory Ramifications* **23** (2014), no. 4, 1450018, 23 pp.
- [9] J. Conant, M. Kassabov, and K. Vogtmann, Hairy graphs and the unstable homology of $\text{Mod}(g, s)$, $\text{Out}(F_n)$ and $\text{Aut}(F_n)$, *J. Topol.* **6** (2013), no. 1, 119–153.
- [10] J. Conant, R. Schneiderman, and P. Teichner, Tree homology and a conjecture of Levine, *Geom. Topol.* **16** (2012), no. 1, 555–600.
- [11] V. Dotsenko and A. Khoroshkin, Character formulas for the operad of two compatible brackets and for the bihamiltonian operad, (*Russian*) *Funktsional. Anal. i Prilozhen.* **41** (2007), no. 1, 1–22, 96; translation in *Funct. Anal. Appl.* **41** (2007), no. 1, 1–17.
- [12] W. Feit, *Characters of Finite Groups*, W.A. Benjamin, New York 1967, vii+186pp.
- [13] E. Getzler and M. Kapranov, Cyclic operads and cyclic homology, *Geometry, Topology, & Physics, Conf. Proc. Lecture Notes Geom. Topology* **IV**, Int. Press, Cambridge, MA, 1995, pages 167–201.
- [14] E. Getzler and M. Kapranov, Modular operads, *Compos. Math.* **110** (1998), no. 1, 65–126.
- [15] T.G. Goodwillie and M. Weiss, Embeddings from the point of view of immersion theory: Part II, *Geom. Topol.* **3** (1999) 103–118.
- [16] N. Habegger and G. Masbaum, The Kontsevich integral and Milnor’s invariants, *Topology* **39** (2000), no. 6, 1253–1289.
- [17] N. Habegger and W. Pitsch, Tree level algebra structures of perturbative invariants, *J. Knot Theory Ramifications* **12** (2003), no. 3, 333–345.
- [18] A. Hamilton and A. Lazarev, Graph cohomology classes in the Batalin-Vilkovisky formalism. *J. Geom. Phys.* **59** (2009), no. 5, 555–575.
- [19] V. Hinich and A. Vaintrob, Cyclic operads and algebra of chord diagrams, *Selecta Math. (N.S.)* **8** (2002), no. 2, 237–282.
- [20] M. Hovey, *Model Categories*, *Math. Surveys Monogr.* **63**, AMS, 1999.
- [21] R.M. Kaufmann and B. Ward, *Feynman Categories*, *Astérisque* **387** (2017), vii+161 pp. ISBN: 978-2-85626-852-7.
- [22] G. Komawila Ngidioni and P. Lambrechts, Euler series, Stirling numbers and the growth of the homology of the space of long links, *Bull. Belg. Math. Soc. Simon Stevin* **20** (2013), no. 5, 843–857.

- [23] S. Kondo, Ch. Siegel, and J. Wolfson, Modular operads of embedded curves. *Geom. Topol.* **21** (2017), no. 2, 903–922.
- [24] M. Kontsevich, Formal non-commutative symplectic geometry, *The Gelfand Mathematical Seminars, 1990–1992*, Ed. L. Corwin, I. Gelfand, J. Lepowsky, 1993, pages 173–187.
- [25] J. Levine, Addendum and correction to: “Homology cylinders: An enlargement of the mapping class group”, *Algebr. Geom. Topol.* **2** (2002), 1197–1204.
- [26] M. Markl, Loop homotopy algebras in closed string field theory, *Comm. Math. Phys.* **221** (2001), no. 2, 367–384.
- [27] M. Markl, Modular envelopes, OSFT and nonsymmetric (non- Σ) modular operads. *J. Noncommut. Geom.* **10** (2016), no. 2, 775–809.
- [28] D. Moskovich, Acyclic Jacobi diagrams, *Kobe J. Math.* **23** (2006), nos. 1–2, 29–50.
- [29] T. Pirashvili, Dold-Kan type theorem for Γ -groups, *Math. Ann.* **318** (2000), no. 2, 277–298.
- [30] T. Pirashvili, Hodge decomposition for higher order Hochschild homology, *Ann. Sci. Éc. Norm. Supér. (4)* **33** (2000), no. 2, 151–179.
- [31] G. Segal, Categories and cohomology theories, *Topology* **13** (1974), 293–312.
- [32] P.A. Songhafou Tsopméné, The rational homology of spaces of long links, *Algebr. Geom. Topol.* **16** (2016), no. 2, 757–782.
- [33] P.A. Songhafou Tsopméné and Victor Turchin, Rational homology and homotopy of high dimensional string links, to appear in *Forum Math.* <https://doi.org/10.1515/forum-2016-0192>
- [34] J. Smillie, K. Vogtmann, A generating function for the Euler characteristic of $Out(F_n)$, in: *Proceedings of the Northwestern Conference on Cohomology of Groups, Evanston, Ill., 1985*, *J. Pure Appl. Algebra* **44** (1987), nos. 1–3, 329–348.
- [35] V. Turchin, Hodge-type decomposition in the homology of long knots, *J. Topol.* **3** (2010), no. 3, 487–534.
- [36] S. Whitehouse, *Gamma Homology of Commutative Algebras and Some Related Representations of the Symmetric Group*, PhD Thesis, Warwick University, 1994.
- [37] T. Willwacher and M. Živković, Multiple edges in M. Kontsevich’s graph complexes and computations of the dimensions and Euler characteristics, *Adv. Math.* **272** (2015), 553–578.

Paul Arnaud Songhafou Tsopméné pso748@uregina.ca

Department of Mathematics and Statistics, University of Regina, 3737 Wascana Pkwy, Regina, SK S4S 0A2, Canada

Victor Turchin turchin@ksu.edu

Department of Mathematics, Kansas State University, 138 Cardwell Hall, Manhattan, KS 66506, USA