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PARTIAL EULER CHARACTERISTIC, NORMAL GENERATIONS AND THE STABLE $D(2)$ PROBLEM

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Abstract

We study the interplay among Wall's $D(2)$ problem, the normal generation conjecture (the Wiegold Conjecture) of perfect groups and Swan's problem on partial Euler characteristic and deficiency of groups. In particular, for a 3-dimensional complex X of cohomological dimension 2 with finite fundamental group, assuming the Wiegold conjecture holds, we prove that X is homotopy equivalent to a finite 2-complex after wedging a copy of sphere S^2.

1. Introduction

In this article, we study several classical problems in low-dimensional homotopy theory and group theory, focusing on the interplay among these problems.

Let us first recall Swan's problem. Let G be a group and $\mathbb{Z}G$ the group ring. Swan [**16**] defines the partial Euler characteristic $\mu_n(G)$ as follows. Let F be a resolution

$$\cdots \to F_2 \to F_1 \to F_0 \to \mathbb{Z} \to 0$$

of the trivial $\mathbb{Z}G$-module $\mathbb{Z}$, in which each F_i is $\mathbb{Z}G$-free on f_i generators. For an integer $n \geqslant 0$, if

$$f_0, f_1, f_2, \ldots, f_n$$

are finite, define

$$\mu_n(F) = f_n - f_{n-1} + f_{n-2} - \cdots + (-1)^n f_0.$$

If there exists a resolution F such that $\mu_n(F)$ is defined, we let $\mu_n(G)$ be the infimum of $\mu_n(F)$ over all such resolutions F. We call the truncated free resolution

$$F_n \to \cdots \to F_1 \to F_0 \to \mathbb{Z} \to 0$$

an algebraic n-complex if each F_i is finitely generated as a $\mathbb{Z}G$-module (following the terminology of Johnson [**8**]).

For a finitely presentable group G, the deficiency $\operatorname{def}(G)$ is the maximum of $d-k$ over all presentations $\langle g_1, g_2, \ldots, g_d \mid r_1, r_2, \ldots, r_k\rangle$ of G. It is not hard to see that

$$\operatorname{def}(G) \leqslant 1 - \mu_2(G) \tag{1}$$

[**16**, Proposition 1]. However, Swan mentions in [**16**] that "the problem of determining

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when $\operatorname{def}(G) = 1 - \mu_2(G)$ seems very difficult even if G is a finite p-group".

Next, we consider Wall's $D(2)$ problem (*cf.* [**17**]). The cohomological dimension $\operatorname{cd}(X)$ of a CW complex X is defined as the largest integer n such that $H^n(X, M) \neq 0$ for some $\mathbb{Z}[\pi_1(X)]$-module M. For a 3-dimensional CW complex X of cohomological dimension $\operatorname{cd}(X) = 2$, Wall's $D(2)$ problem asks whether X is homotopy equivalent to a 2-dimensional CW complex. A positive answer to this problem will imply the Eilenberg-Ganea conjecture, which says that a group of cohomological dimension two has a 2-dimensional classifying space. A finitely presentable group G is said to have $D(2)$ property if any finite 3-dimensional CW complex X, of cohomological dimension 2 with fundamental group G, is homotopy equivalent to a 2-dimensional CW complex. For the status of $D(2)$ problem, see Johnson [**8, 9**] (see also [**5, 7**] for some recent work).

It is well-known that a finite perfect group G is normally generated by one element [**11**, 4.2]. The Wiegold conjecture (*cf.* [**1**, FP14] and [**15**, 5.52]) asserts that the same holds for any finitely generated perfect group:

Conjecture 1.1 (Wiegold conjecture)**.** *Let G be any finitely generated perfect group, i.e. $G = [G, G]$, the commutator subgroup of G. Then G can be normally generated by a single element.*

Our main result is the following, which gives a relaxed lower bound of $\operatorname{def}(G)$ assuming the Wiegold conjecture.

Theorem 1.2. *Assume that Conjecture 1.1 is true. Let X be a finite 3-dimensional CW complex of cohomological dimension 2 with finite fundamental group. We have the following:*

(i) the complex X is homotopy equivalent to a finite 3-dimensional complex with just one 3-cell;

(ii) the wedge $X \vee S^2$ is homotopy equivalent to a finite 2-dimensional complex;

(iii) $1 - \mu_2(G) \geqslant \operatorname{def}(G) \geqslant -\mu_2(G)$ for any finite group G.

Our discussions are based on the study of a stable version of the $D(2)$ problem (for details, see Section 3). For a group G having a finite classifying space $\mathrm{B}G$ of dimension at most 2, we have $\operatorname{def}(G) = 1 - \mu_2(G)$, which confirms the equality of partial Euler characteristic and deficiency (cf. Theorem 4.3 (i)). A famous conjecture of Whitehead says that any subcomplex of an aspherical 2-dimensional CW complex is aspherical (*cf.* [**2**]). As an application of the results proved, we reprove the following (*cf.* Bogley [**2**]).

Corollary 1.3. *A subcomplex X of a finite aspherical 2-dimensional CW complex is aspherical if and only if the fundamental group $\pi_1(X)$ has a finite classifying space $\mathrm{B}\pi_1(X)$ of dimension at most 2.*

The article is organized as follows. In Section 2, we discuss the Quillen plus construction of 2-dimensional CW complexes. This motivates the stable Wall's $D(2)$ property being discussed in Section 3. In the last section, the Euler characteristics are studied for groups of low geometric dimensions.

2. Quillen's plus construction of 2-dimensional CW complexes

Let X be a CW complex with fundamental group G and P a perfect normal subgroup of G, *i.e.* $P = [P, P]$. Quillen shows that there exists a CW complex X_P^+, whose fundamental group is G/P; and an inclusion $f\colon X \to X_P^+$ such that

$$H_n(X; f^*M) \cong H_n(X_P^+; M),$$

for any integer n and local coefficient system M over X_P^+. Here X_P^+ is called the plus-construction of X with respect to P. It is unique up to homotopy equivalence. One of the main applications of the plus construction is to define higher algebraic K-theory. In general, the space X_P^+ is obtained from X by attaching 2-cells and 3-cells. We need the following definition.

Definition 2.1. The cohomological dimension $\mathrm{cd}(X)$ of a CW complex X is defined as the smallest integer n such that $H^m(X, M) = 0$ for any integer $m > n$ and any $\mathbb{Z}[\pi_1(X)]$-module M. If no such n exists, the cohomological dimension $\mathrm{cd}(X)$ is defined to be ∞.

It is obvious that an n-dimensional CW complex is of cohomological dimension at most n. The following well-known lemma gives a property enjoyed by any 3-dimensional CW complex with cohomological dimension 2.

Lemma 2.2. *Suppose that X is a 3-dimensional CW complex and $\tilde{X}$ is the universal cover of X. Let $C_*(\tilde{X})$ be the cellular chain complex of $\tilde{X}$. Then X is of cohomological dimension 2 if and only if the image of $C_3(\tilde{X})$ is a direct summand of $C_2(\tilde{X})$ as $\mathbb{Z}[\pi_1(X)]$-modules.*

The following result shows that for certain 2-dimensional CW complexes, the Quillen plus construction is homotopy equivalent to a 2-dimensional CW complex. Let X be a finite 2-dimensional CW complex. Suppose that a perfect normal subgroup P in $\pi_1(X)$ is normally generated by n elements. With respect these normal generators, there is a canonical construction Y for X^+ that attaches a 2-cell bounded by each generator and a 3-cell to kill the resulting homology. Moreover, the number of attached 3-cells and the number of attached 2-cells are both n (*cf.* the proof of Theorem 1 in [**18**]). The cellular chain complex $C_*(\tilde{Y})$ of the universal cover $\tilde{Y}$ is

$$0 \to \mathbb{Z}[\pi_1(Y)]^n \to \mathbb{Z}[\pi_1(Y)]^n \oplus C_2(\tilde{X}) \otimes_{\mathbb{Z}[\pi_1(X)]} \mathbb{Z}[\pi_1(Y)] \\ \to C_1(\tilde{X}) \otimes_{\mathbb{Z}[\pi_1(X)]} \mathbb{Z}[\pi_1(Y)] \to C_0(\tilde{X}) \otimes_{\mathbb{Z}[\pi_1(X)]} \mathbb{Z}[\pi_1(Y)] \to 0,$$

with the first map the inclusion of a direct summand. This is homotopy equivalent to

$$0 \to C_2(\tilde{X}) \otimes_{\mathbb{Z}[\pi_1(X)]} \mathbb{Z}[\pi_1(Y)] \to C_1(\tilde{X}) \otimes_{\mathbb{Z}[\pi_1(X)]} \mathbb{Z}[\pi_1(Y)] \\ \to C_0(\tilde{X}) \otimes_{\mathbb{Z}[\pi_1(X)]} \mathbb{Z}[\pi_1(Y)] \to 0.$$

It follows that

Lemma 2.3. *The plus construction $(X \vee (S^2)^{\vee n})^+$ of the wedge of X and n copies of S^2, taken with respect to P, is homotopy equivalent to the 2-skeleton of Y.*

The following lemma is from Johnson [**8**, 59.4, p. 228]. Although the original version is stated for complexes with finite groups, it does hold for complexes with finitely presentable groups (*cf.* [**8**, appendix B] and Mannan [**14**]).

Lemma 2.4. *Let Y be a finite 3-dimensional CW complex of cohomological dimension 2. If the reduced chain complex of the universal cover*

$$0 \to C_2(\tilde{Y})/C_3(\tilde{Y}) \to C_1(\tilde{Y}) \to \mathbb{Z}\pi_1(Y) \to \mathbb{Z} \to 0$$

is homotopy equivalent to the chain complex of the universal cover of a 2-dimensional CW complex X, then Y is homotopy equivalent to X.

3. Wall's D(2) problem and its stable version

In this section, we apply the results obtained in the previous section to study the $D(2)$ problem. Let us recall the $D(2)$ problem raised in [**17**].

Conjecture 3.1 (The $D(2)$ problem)**.** *If X is a finite 3-dimensional CW complex of cohomological dimension at most 2, then X is homotopy equivalent to a 2-dimensional CW complex.*

In [**8**], Johnson proposes to systematically study the problem by parameterizing 3-dimensional CW complexes by their fundamental groups. For a finitely presentable group G, we say the $D(2)$ problem is true for G, if any finite 3-dimensional CW complex X, of cohomological dimension at most 2 with fundamental group $\pi_1(X) = G$, is homotopy equivalent to a 2-dimensional CW complex.

The $D(2)$ problem is very difficult in general. It is known to be true for a limited amount of groups (for an updated state, see [**4, 12**] and [**9**, p. 261]). We propose the following stable version by allowing taking wedge with copies of S^2.

Conjecture 3.2 (The $D(2,n)$ problem)**.** *Let $n \geqslant 0$ be an integer. If X is a finite 3-dimensional CW complex of cohomological dimension at most 2, then $X \vee (S^2)^{\vee n}$ is homotopy equivalent to a 2-dimensional CW complex.*

For a finitely presentable group G and an integer $n \geqslant 0$, we say that G has the $D(2,n)$ property (or the $D(2,n)$ problem holds for G) if Conjecture 3.2 is true for all those X with fundamental group G. The $D(2,0)$ problem is the original $D(2)$ problem. It is immediate that property $D(2)$ implies $D(2,n)$; and $D(2,n)$ implies $D(2,n+1)$ for any group G and any integer $n \geqslant 0$.

We now study the relation between the stabilization by wedging copies of S^2 with that by attaching 3-cells.

Proposition 3.3. *Suppose that X is a finite 3-dimensional CW complex of cohomological dimension at most 2. Then $X \vee (S^2)^{\vee n}$ is homotopy equivalent to a finite 2-dimensional CW complex if and only if X is homotopy equivalent to a 3-dimensional CW complex with n 3-cells.*

Proof. Assume that X is homotopy equivalent to a 3-dimensional CW complex X' with n 3-cells. Denote by $X'^{(2)}$ the 2-skeleton of X' and let $Z = X' \vee (S^2)^n$. It is not hard to see that the reduced chain complex

$$0 \to C_2(\tilde{Z})/C_3(\tilde{Z}) \to C_1(\tilde{Z}) \to \mathbb{Z}\pi_1(Z) \to \mathbb{Z} \to 0$$

is homotopy equivalent to the chain complex of the universal cover of $X'^{(2)}$. By Lemma 2.4, $X \vee (S^2)^{\vee n}$ is homotopy equivalent to a 2-dimensional CW complex.

Conversely, suppose that $X \vee (S^2)^{\vee n}$ is homotopy equivalent to a finite 2-complex Y via a map $f\colon X \vee (S^2)^{\vee n} \to Y$. It is clear that

$$\pi_1(X) = \pi_1(X \vee (S^2)^{\vee n}) \cong \pi_1(Y).$$

Let $G = \pi_1(X)$ and $\tilde{X}, \tilde{Y}$ be the universal covering spaces of X, Y respectively. By the Hurewicz theorem, we have isomorphisms

$$\pi_2(Y) \cong \pi_2(\tilde{Y}) \cong H_2(\tilde{Y}) \cong \pi_2(\tilde{X}) \oplus \mathbb{Z}G^n.$$

Therefore, there are n maps $f_i \colon S^2 \to Y, 1 \leqslant i \leqslant n$, corresponding to the inclusion onto the second factor (for a fixed basis of $\mathbb{Z}G^n$)

$$\mathbb{Z}G^n \to H_2(\tilde{Y}) \cong \pi_2(\tilde{X}) \oplus \mathbb{Z}G^n.$$

Attaching 3-cells to Y along these f_i $(1 \leqslant i \leqslant n)$, we obtain a 3-dimensional CW complex $Y \cup_{i=1}^n e_i^3$. Let $i \colon X \xrightarrow{i} X \vee (S^2)^{\vee n}$ be the natural inclusion. By our construction, the canonical composition

$$f' \colon X \xrightarrow{i} X \vee (S^2)^{\vee n} \xrightarrow{f} Y \to Y \cup_{i=1}^n e_i^3$$

induces isomorphisms on both π_1 and π_2 (the same as the second homology groups of the universal covers). It is not hard to see that

$$H_3(\tilde{X}) = H_3(\widetilde{Y \cup_{i=1}^n e_i^3}) = 0.$$

Therefore, f' induces a homotopy equivalence between the chain complexes of the universal covering spaces. By the Whitehead theorem, f' is a homotopy equivalence. □

Proof of Theorem 1.2 (i) and (ii). By Proposition 3.3, (i) is equivalent to (ii). We prove (ii) as follows. By a result of Mannan [**13**], X is the plus construction of a finite 2-complex Y with respect to a perfect normal subgroup $P \leqslant \pi_1(Y)$. Therefore, we have a short exact sequence of groups

$$1 \to P \to \pi_1(Y) \to \pi_1(X) \to 1.$$

Since $\pi_1(Y)/P = \pi_1(X)$ is finite and Y is finite, the covering space of Y with fundamental group P is again a finite CW complex. Hence P is finitely generated. If the normal generation conjecture (Conjecture 1.1) holds, P is normally generated by a single element. Lemma 2.3 says that $X \vee S^2$ is homotopy equivalent to a 2-dimensional CW complex. □

Without the assumption of the Wiegold conjecture we only know that a finite group G has property $D(2, n)$ for $n = \max\{1, 1 - \operatorname{def}(G) - \mu_2(G)\}$, which follows the Swan-Jacobinski theorem in [**8**, 29.3, 29.4] and Browning's results [**3**].

4. Partial Euler characteristic and the Whitehead conjecture

Recall definitions of $\mu_n(F)$ for an algebraic n-complex F_* and $\mu_n(G)$ from Introduction. For a finitely presentable group G, the following lemma follows from Swan [**16**] easily.

Lemma 4.1. *Assume that G is finitely presentable. The invariant $\mu_2(G)$ can be realized by an algebraic 2-complex. In other words, there exists an algebraic 2-complex*

$$F_2 \to F_1 \to F_0 \to \mathbb{Z} \to 0$$

such that

$$\mu_2(G) = \dim_{\mathbb{Z}G} F_2 + \dim_{\mathbb{Z}G} F_0 - \dim_{\mathbb{Z}G} F_1.$$

Proof. It is enough to notice that $\mu_2(G)$ is finite by Theorem 1.2 in [**16**]. □

Proof of Theorem 1.2 (iii). We prove a more general result: if a finitely presentable group G satisfies the $D(2, n)$ problem, then

$$\operatorname{def}(G) \geqslant (1 - n) - \mu_2(G).$$

By Lemma 4.1, we can choose an algebraic 2-complex

$$(F_*) \colon F_2 \to F_1 \to F_0 \to \mathbb{Z} \to 0$$

such that

$$\mu_2(G) = \dim_{\mathbb{Z}G} F_2 + \dim_{\mathbb{Z}G} F_0 - \dim_{\mathbb{Z}G} F_1.$$

Since every algebraic 2-complex is geometric realizable by a 3-dimensional CW complex (cf. Johnson [**8**, Theorem 60.2]), there is a finite 3-dimensional CW complex of cohomological dimension 2 such that the reduced chain complex

$$C_2(\tilde{Y})/C_3(\tilde{Y}) \to C_1(\tilde{Y}) \to \mathbb{Z}\pi_1(Y) \to \mathbb{Z} \to 0$$

is homotopy equivalent to (F_*). Assuming that G has the $D(2, n)$ property, the wedge $X \vee (S^2)^{\vee n}$ is homotopy equivalent to a 2-dimensional CW complex, which gives a presentation of G. This implies that $\mu_2(G) + n \geqslant 1 - \operatorname{def}(G)$, *i.e.* $\operatorname{def}(G) \geqslant (1 - n) - \mu_2(G)$. When Wiegold's Conjecture holds, the complex X has property $D(2, 1)$, which gives (iii). □

It is possible to place $\mu_2(G)$ in the broader setting of (G, n)-complexes, as follows (*cf.* [**6**]). Recall that a (G, n)-complex is a finite n-dimensional CW complex X with fundamental group G and vanishing homotopy group $\pi_i(X) = 0$ for $i = 2, 3, \ldots, n - 1$. In particular, a $(G, 2)$-complex is a usual finite 2-dimensional CW complex with fundamental group G.

Definition 4.2. Let G be a finitely presentable group. Define

$$\mu_n^g(G) = \min\{(-1)^n \chi(X) \mid X \text{ is a } (G, n)\text{-complex}\}.$$

If there is no such X exists, define $\mu_n^g(G) = +\infty$. We call that a (G, n)-complex X with $(-1)^n \chi(X) = \mu_n^g(G)$ is a complex realizing $\mu_n^g(G)$.

A few observations are immediate. It is clearly true that $\mu_n(G) \leqslant \mu_n^g(G)$. Therefore, $\mu_n^g(G) > -\infty$ since $\mu_n(G) > -\infty$ (*cf.* Swan [**16**]). Moreover, $\mu_2(G) = \mu_2^g(G)$ if and only if $\mu_2(G) = 1 - \operatorname{def}(G)$.

Now we study the partial Euler characteristic and deficiency for groups of low geometric dimensions. Recall that for a group G, the classifying space $\mathrm{B}G$ of G is defined as the connected CW complex with $\pi_1(\mathrm{B}G) = G$ and $\pi_i(\mathrm{B}G) = 0, i \geqslant 2$. It is unique up to homotopy.

Theorem 4.3. *Let G be a group having a finite n-dimensional classifying space $\mathrm{B}G$. We have the following:*

(i) $\mu_n(G) = \mu_n^g(G)$; *In particular,* $\mu_2(G) = 1 - \mathrm{def}(G)$ *if* G *has a finite* 2*-dimensional* BG;

(ii) *Any finite CW complex* X *with* $\pi_1(X) = G$ *satisfying the following properties:*

a) *the dimension is at most* $n + 1$;
b) *the cohomological dimension* $\mathrm{cd}(X)$ *is at most* n;
c) *if* $n \geqslant 3$, *the homotopy group* $\pi_i(X) = 0$ *for* $2 \leqslant i \leqslant n - 1$;
d) $(-1)^n\chi(X) = \mu_n^g(G)$,

is homotopy equivalent to BG.

Proof. Let EG be the universal cover of BG. Since EG is contractible, one obtains the exact cellular chain complex of EG:

$$C_*(\mathrm{E}G)\colon 0 \to C_n(\mathrm{E}G) \to C_{n-1}(\mathrm{E}G) \to \cdots \to \mathbb{Z}G \to 0.$$

This gives a (truncated) free resolution of G. In order to prove (i), it suffices to show that this resolution gives the minimal Euler characteristic $\mu_n(G)$ since we notice earlier that $\mu_n(G) \leqslant \mu_n^g(G)$.

Suppose that $\mu_n(G)$ is obtained from the following partial resolution of finitely generated free $\mathbb{Z}G$-modules:

$$F_*\colon F_n \xrightarrow{d} F_{n-1} \to \cdots \to F_1 \to \mathbb{Z}G \to 0.$$

We claim that F_* is exact at F_n, *i.e.* $\ker d = 0$. Once this is proved, $C_*(\mathrm{E}G)$ and F_* are chain homotopic to each other and hence have the same Euler characteristic.

To prove the claim, let J be the kernel of d. By Schanuel's lemma, there is an isomorphism

$$J \oplus C_n(\mathrm{E}G) \oplus F_{n-1} \oplus \cdots \cong F_n \oplus C_{n-1}(\mathrm{E}G) \oplus \cdots.$$

Applying the functor $- \otimes_{\mathbb{Z}G} \mathbb{Z}$ to both sides of this isomorphism, we see that $\mu_n(F) = (-1)^n\chi(\mathrm{B}G)$ and $J \otimes_{\mathbb{Z}G} \mathbb{Z} = 0$ by noticing the fact that the complex F_* attains minimal Euler characteristic after multiplying $(-1)^n$ among all the algebraic n-complexes. This implies that $C_n(\mathrm{E}G) \oplus F_{n-1} \oplus \cdots$ and $F_n \oplus C_{n-1}(\mathrm{E}G) \oplus \cdots$ have the same finite free $\mathbb{Z}G$-rank. By Kaplansky's theorem, J is the trivial $\mathbb{Z}G$-module(*cf.* [**10**], p. 328). This proves (i).

We now prove (ii). Let $C_*(\tilde{X})$ be the chain complex of the universal covering space of X. Since $\mathrm{cd}(X) \leqslant n$, $C_{n+1}(\tilde{X})$ is a direct summand of $C_n(\tilde{X})$, by the same argument given in Lemma 2.2. Let F^1 be the chain complex

$$F^1_*\colon C_n(\tilde{X})/C_{n+1}(\tilde{X}) \xrightarrow{d} C_{n-1}(\tilde{X}) \to \cdots \to C_1(\tilde{X}) \to \mathbb{Z}G \to 0.$$

It is not hard to see that $\pi_n(X) \cong \ker d$. Note that

$$\mu_n(F^1) = (-1)^n\chi(X) = \mu_n(G).$$

By the same argument as the first part of the proof, we get $\ker d = 0$. This implies that $\tilde{X}$ is n-connected. Since $H_{n+1}(\tilde{X}) = 0$, $\tilde{X}$ is contractible and X is homotopy equivalent to BG. □

Remark 4.4. Under the condition of Theorem 4.3, Harlander and Jensen [**6**] already prove that a (G, n)-complex realizing $\mu_n^g(G)$ is homotopy equivalent to BG. Note that a (G, n)-complex is a special case of X in Theorem 4.3.

We conclude with an application. Suppose that G is a finitely presentable group and

$$\mathbf{P} = \langle x_1, \ldots, x_n \mid r_1, \ldots, r_m \rangle$$

is a presentation of G. Denote by $G_{\mathbf{P}}$ the group given by the presentation $\mathbf{P}$. From each finite 2-dimensional CW complex X, one shrinks a spanning tree in the 1-skeleton to make X have only a single 0-cell and obtains a finite presentation of $\pi_1(X)$. Namely, the 1-cells correspond one-one to a set of generators while the 2-cells correspond one-one to a set of relators. Therefore, any counter-example to the Whitehead conjecture gives rise to a 2-complex with a single 0-cell. For a presentation $\mathbf{P}$, we will denote by $\chi(\mathbf{P}) = m - n + 1$. A sub-presentation of $\mathbf{P} = \langle x_1, \ldots, x_n \mid r_1, \ldots, r_m \rangle$ is a presentation $\langle y_1, \ldots, y_{n'} \mid s_1, \ldots, s_{m'} \rangle$ with each $y_i \in \{x_1, \ldots, x_n\}$ and each $s_i \in \{r_1, \ldots, r_m\}$ is only a word of $y_1, \ldots, y_{n'}$.

Lemma 4.5. *Suppose that $\mathbf{P}' = \langle y_1, \ldots, y_{n'} \mid s_1, \ldots, s_{m'} \rangle$ is a sub-presentation of $\mathbf{P} = \langle x_1, \ldots, x_n \mid r_1, \ldots, r_m \rangle$ of a group $G_{\mathbf{P}}$. If $\mathbf{P}''$ is another finite presentation of $G_{\mathbf{P}'}$, then one can obtain a presentation of $G_{\mathbf{P}}$ from $\mathbf{P}''$ by adding $n - n'$ generators and $m - m'$ relations. In particular, if $\mathbf{P}$ realizes $\mu_2^g(G_{\mathbf{P}})$, then $\mathbf{P}'$ realizes $\mu_2^g(G_{\mathbf{P}'})$.*

Proof. Re-indexing and re-naming if necessary, we assume that

$$y_1 = x_1, \ldots, y_{n'} = x_{n'}, n' \leqslant n$$

and

$$s_1 = r_1, \ldots, s_{m'} = r_{m'}, m' \leqslant m.$$

It is clear that the words corresponding to $s_1, \ldots, s_{m'}$ do not involve $x_{n'+1}, \ldots, x_n$. If

$$\mathbf{P}'' = \langle y_1', \ldots, y_u' \mid s_1', \ldots, s_v' \rangle$$

is another presentation of $G_{\mathbf{P}'}$, we form a group G'' with the presentation

$$\langle y_1', \ldots, y_u', x_{n'+1}, \ldots, x_n \mid s_1', \ldots, s_v' \rangle$$

by adding $n - n'$ free generators to $\mathbf{P}''$. For each $1 \leqslant i \leqslant n'$, the letter x_i, viewed as an element in $G_{\mathbf{P}'}$, has a lifting w_i in the free group $\langle y_1', \ldots, y_u' \rangle$. In other words, we choose w_i on the generators $y_1', \ldots, y_u'$ such that the bijection $x_i \mapsto w_i, 1 \leqslant i \leqslant n'$ induces an isomorphism $G_{\mathbf{P}'} \to G_{\mathbf{P}''}$.

For each $1 \leqslant i \leqslant n$, define the word ω_i of $\{y_1', \ldots, y_u', x_{n'+1}, \ldots, x_n\}$ as

$$\omega_i = \begin{cases} w_i, & 1 \leqslant i \leqslant n'; \\ x_i, & n' < i \leqslant n. \end{cases}$$

Denote by ϕ the bijection

$$\phi \colon \{x_1, \ldots, x_n\} \to \{\omega_1, \ldots, \omega_n\}$$

given by $x_i \mapsto \omega_i$. For each $m' < i \leqslant m$, write $r_i = \Pi_{j=1}^{k_i} x_{ij}$ as a reduced word of $\{x_1, \ldots, x_n\}$, where $x_{ij} \in \{x_1^{\pm}, \ldots, x_n^{\pm}\}$. Let $r_i' = \Pi_{j=1}^{k_i} \phi(x_{ij})$ be the corresponding word of

$$\{y_1', \ldots, y_u', x_{n'+1}, \ldots, x_n\}.$$

Let K be the normal subgroup of G'' normally generated by the $m - m'$ elements $r_{m'+1}', \ldots, r_m'$. We obtain a short exact sequence of groups

$$1 \to K \to G'' \to G_{\mathbf{P}} \to 1,$$

where the third arrow is induced by the map $G_{\mathbf{P}'} \to G_{\mathbf{P}}$ from the natural inclusions of generators and relators. From this exact sequence, one obtains the desired presentation

$$\mathbf{P}_0 = \langle y'_1, \ldots, y'_u, x_{n'+1}, \ldots, x_n \mid s'_1, \ldots, s'_v, r'_{m'+1}, \ldots, r'_m \rangle$$

of $G_{\mathbf{P}}$.

Assume that $\mathbf{P}$ realizes $\mu_2^g(G_{\mathbf{P}})$, while a sub-presentation $\mathbf{P}'$ does not realize $\mu_2^g(G_{\mathbf{P}'})$. Suppose that $\mu_2^g(G_{\mathbf{P}'})$ is realized by a 2-dimensional complex X, which gives a presentation $\mathbf{P}''$. We obtain a new presentation $\mathbf{P}_0$ of $G_{\mathbf{P}}$ by adding relators and generators to $\mathbf{P}''$. However,

$$\chi(\mathbf{P}_0) = \chi(\mathbf{P}'') + m - m' - (n - n') = \chi(\mathbf{P}'') - \chi(\mathbf{P}') + \chi(\mathbf{P}) < \mu_2^g(G_{\mathbf{P}}).$$

This is a contradiction to the fact that $\mathbf{P}$ realizes $\mu_2^g(G_{\mathbf{P}})$. Therefore, $\mathbf{P}'$ realizes $\mu_2^g(G_{\mathbf{P}'})$. □

Recall that a CW complex X is aspherical if the universal cover $\tilde{X}$ is contractible. A famous conjecture of Whitehead says that any subcomplex Y of an aspherical 2-dimensional complex X is aspherical as well (for more details, see the survey article [**2**]). As an application of results proved above, we give an equivalent condition of the asphericity of Y, as follows.

Corollary 4.6. *Suppose that X is a finite aspherical 2-complex and Y is a subcomplex of X. We have the following:*

(i) The complex Y realizes $\mu_2^g(\pi_1(Y))$;

(ii) The complex Y is aspherical if and only if the fundamental group $\pi_1(Y)$ has a finite classifying space $\mathrm{B}\pi_1(Y)$ of dimension at most 2.

Proof. Since X is aspherical, it realizes $\mu_2^g(\pi_1(X))$ by Theorem 4.3. Notice that Y gives a presentation of $\pi_1(Y)$, which is a sub-presentation of the presentation given by X. Lemma 4.5 implies that Y realizes $\mu_2^g(\pi_1(Y))$. This proves part (i).

If Y is aspherical, it is $\mathrm{B}\pi_1(Y)$ and hence is of dimension at most 2. Conversely, assume that $\pi_1(Y)$ has a finite classifying space $\mathrm{B}\pi_1(Y)$ of dimension at most 2. By Theorem 4.3, all the $(\pi_1(Y), 2)$-complexes realizing $\mu_2^g(\pi_1(Y))$ are homotopic to $\mathrm{B}\pi_1(Y)$. Therefore, Y is aspherical by part (i). □

Corollary 1.3 is Corollary 4.6 (ii).

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