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SAMELSON PRODUCTS IN QUASI-p-REGULAR EXCEPTIONAL LIE GROUPS

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Abstract

There is a product decomposition of a compact connected Lie group G at the prime p, called the mod p decomposition, when G has no p-torsion in homology. Then in studying the multiplicative structure of the p-localization of G, the Samelson products of the factor space inclusions of the mod p decomposition are fundamental. This paper determines the (non-)triviality of these fundamental Samelson products in the p-localized exceptional Lie groups when the factor spaces are of rank $\leqslant 2$, that is, G is quasi-p-regular.

1. Introduction

Let G be a compact connected Lie group. Recall from [10] that if G has no p-torsion in integral homology, then there is a p-local homotopy equivalence

$$G \simeq_{(p)} B_1 \times \cdots \times B_{p-1} \tag{1}$$

such that B_i is resolvable by spheres of dimension $2i-1 \mod 2(p-1)$, where each B_i is indecomposable if G is simple except for type D. This is called the mod p decomposition of G. For maps $\alpha\colon A \to X, \beta\colon B \to X$ into a homotopy associative H-space with inverse X, the composite

$$A \wedge B \xrightarrow{\alpha\wedge\beta} X \wedge X \to X$$

is called the Samelson product of α, β and is denoted by $\langle\alpha,\beta\rangle$, where the last arrow is the reduced commutator map. Then in studying the standard multiplication of the p-localization $G_{(p)}$, the Samelson products of the inclusions $B_i \to G_{(p)}$ are fundamental, and there are applications of these Samelson products as in [9, 5, 6]. In this paper, we aim to determine (non-)triviality of these fundamental Samelson products in $G_{(p)}$ when G is the quasi-p-regular exceptional Lie group, which is a continuation of the previous work [4] on p-regular exceptional Lie groups.

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Let us recall the result of the previous work [4]. We say that G is p-regular if G is p-locally homotopy equivalent to a product of spheres. By the classical result of Hopf, G is rationally homotopy equivalent to a product of spheres of dimension $2n_1-1,\dots,2n_\ell-1$ for $n_1\leqslant\cdots\leqslant n_\ell$. The sequence $n_1,\dots,n_\ell$ is called the type of G and is denoted by $t(G)$. There is a list of types of simple Lie groups in [5]. It is known that when G is simply connected, G is p-regular if and only if p is no less than the maximum of $t(G)$ (cf. [10]). Obviously, if G is p-regular, G is p-locally homotopy equivalent to a product of spheres of dimension $2i-1$ for $i\in t(G)$. Let $\epsilon_i\colon S^{2i-1}\to G_{(p)}$ denote the inclusion for $i\in t(G)$ when G is p-regular.

Theorem 1.1 (Hasui, Kishimoto, and Ohsita [4])**.** *Let G be a p-regular exceptional Lie group. The Samelson product $\langle\epsilon_i,\epsilon_j\rangle$ in $G_{(p)}$ is non-trivial if and only if there is $k\in t(G)$ such that $i+j=k+p-1$.*

Let $B(2i-1,2i+2p-3)$ be the S^{2i-1}-bundle over $S^{2i+2p-3}$ classified by an element $\frac{1}{2}\alpha_1\in\pi_{2i+2p-4}(S^{2i-1})\cong\mathbb{Z}/p$ as in [10, 11], where α_1 is a generator of the p-component of $\pi_{2i+2p-4}(S^{2i-1})$ which is isomorphic with $\mathbb{Z}/p$. Recall that G is quasi-p-regular if G is p-locally homotopy equivalent to the product of $B(2i-1,2i+2p-3)$'s and spheres. When G is exceptional, it is shown in [11] that G is quasi-p-regular if and only if $p\geqslant 5$ for $G=\mathrm{G}_2,\mathrm{F}_4,\mathrm{E}_6$ and $p\geqslant 11$ for $G=\mathrm{E}_7,\mathrm{E}_8$. In these cases, the specific mod p decomposition is:

G_2	$p=5$	$B(3,11)$
	$p>5$	$S^3\times S^{11}$
F_4	$p=5$	$B(3,11)\times B(15,23)$
	$p=7$	$B(3,15)\times B(11,23)$
	$p=11$	$B(3,23)\times S^{11}\times S^{15}$
	$p>11$	$S^3\times S^{11}\times S^{15}\times S^{23}$
E_6	$p=5$	$\mathrm{F}_4\times B(9,17)$
	$p>5$	$\mathrm{F}_4\times S^9\times S^{17}$
E_7	$p=11$	$B(3,23)\times B(15,35)\times S^{11}\times S^{19}\times S^{27}$
	$p=13$	$B(3,27)\times B(11,35)\times S^{15}\times S^{19}\times S^{23}$
	$p=17$	$B(3,35)\times S^{11}\times S^{15}\times S^{19}\times S^{23}\times S^{27}$
	$p>17$	$S^3\times S^{11}\times S^{15}\times S^{19}\times S^{23}\times S^{27}\times S^{35}$
E_8	$p=11$	$B(3,23)\times B(15,35)\times B(27,47)\times B(39,59)$
	$p=13$	$B(3,27)\times B(15,39)\times B(23,47)\times B(35,59)$
	$p=17$	$B(3,35)\times B(15,47)\times B(27,59)\times S^{23}\times S^{39}$
	$p=19$	$B(3,39)\times B(23,59)\times S^{15}\times S^{27}\times S^{35}\times S^{47}$
	$p=23$	$B(3,47)\times B(15,59)\times S^{23}\times S^{27}\times S^{35}\times S^{39}$
	$p=29$	$B(3,59)\times S^{15}\times S^{23}\times S^{27}\times S^{35}\times S^{39}\times S^{47}$
	$p>29$	$S^3\times S^{15}\times S^{23}\times S^{27}\times S^{35}\times S^{39}\times S^{47}\times S^{59}$

Let $t_p(G)$ be the subset of $t(G)$ consisting of $i\in t(G)$ such that $2i-1$ is the dimension of the bottom cell of some B_j in the mod p decomposition of $G_{(p)}$, where $t_p(G)$ is possibly not a subset of $\{1,\dots,p-1\}$. Since there is a one-to-one correspondence between B_i's and $t_p(G)$, we ambiguously denote the factor space of $G_{(p)}$ corresponding to $i \in t_p(G)$ by B_i. In our case, the set $t_p(G)$ can easily be deduced from the above table as:

$t_p(\mathrm{G}_2)$	$p=5$	2
$t_p(\mathrm{F}_4)$	$p=5$	$2,8$
	$p=7$	$2,6$
	$p=11$	$2,6,8$
$t_p(\mathrm{E}_6)$	$p=5$	$2,5,8$
	$p=7$	$2,5,6,9$
	$p=11$	$2,5,6,8,9$
$t_p(\mathrm{E}_7)$	$p=11$	$2,6,8,10,14$
	$p=13$	$2,6,8,10,12$
	$p=17$	$2,6,8,10,12,14$
$t_p(\mathrm{E}_8)$	$p=11$	$2,8,14,20$
	$p=13$	$2,8,12,18$
	$p=17$	$2,8,12,14,20$
	$p=19$	$2,8,12,14,18,24$
	$p=23$	$2,8,12,14,18,20$
	$p=29$	$2,8,12,14,18,20,24$

We now state our main result, where we owe the p-regular case to Theorem 1.1. Let $\epsilon_i\colon B_i \to G_{(p)}$ denote the inclusion for $i \in t_p(G)$, and put $r_i = \operatorname{rank} B_i$.

Theorem 1.2. *Let G be a quasi-p-regular exceptional Lie group. Then for $i,j \in t_p(G)$, the Samelson product $\langle \epsilon_i, \epsilon_j \rangle$ in $G_{(p)}$ is trivial if and only if one of the following conditions holds:*

1. *there is no $k \in t_p(G)$ such that $i+j \equiv k \mod (p-1)$ and $i+j+(r_i+r_j-1)(p-1) > k + r_k(p-1)$;*
2. *$r_i + r_j \geqslant 3$ and there is $k \in t_p(G)$ such that $k = i+j+(r_i+r_j-3)(p-1)$;*
3. *$i+j = p+1$ and $r_i + r_j = 3$;*
4. *$(G,p,\{i,j\}) = (\mathrm{E}_6, 7, \{2,6\})$, $(\mathrm{E}_7, 11, \{2,8\})$, $(\mathrm{E}_7, 11, \{8,10\})$, $(\mathrm{E}_8, 19, \{2,12\})$, $(\mathrm{E}_8, 19, \{12,12\})$.*

Remark 1.3. This theorem includes the result of McGibbon [9] that G_2 at the prime 5 is homotopy commutative.

The proof of Theorem 1.2 consists of three parts. The first part shows triviality of the Samelson products by looking at the homotopy groups of G. The second part applies a criterion for non-triviality of the Samelson products by the Steenrod operations on the mod p cohomology of the classifying space of G which is a generalization of the criterion used to prove Theorem 1.1 in [4]. The third part determines (non-)triviality of the remaining Samelson products by considering a homotopy fibration $\mathrm{hofib}(\rho) \to G \xrightarrow{\rho} \mathrm{SU}(\infty)$ for a stabilized representation ρ, where the easiest case that ρ is the inclusion of $\mathrm{SU}(n)$ is studied in [3]. Since $\mathrm{SU}(\infty)$ is homotopy commutative, Samelson products lift to $\mathrm{hofib}(\rho)$. Then the important point is to identify the homotopy fiber $\mathrm{hofib}(\rho)$, and to this end, we decompose ρ with respect to the mod p decompositions of G and $\mathrm{SU}(\infty)$, which is not needed in [3]. We then describe lifts of the Samelson products through the identification of $\mathrm{hofib}(\rho)$ and to determine (non-)triviality of the Samelson products.

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2. Triviality of Samelson products

Hereafter we localize everything at the prime p. Suppose that (G,p) is as in Table 1.

Table 1:

$\mathrm{SU}(n)$	$n \leqslant (p-1)(p-2)+1$
$\mathrm{Sp}(n), \mathrm{Spin}(2n+1)$	$2n \leqslant (p-1)(p-2)$
$\mathrm{Spin}(2\mathrm{n})$	$2(n-1) \leqslant (p-1)(p-2)$
$\mathrm{G}_2, \mathrm{F}_4, \mathrm{E}_6$	$p \geqslant 5$
$\mathrm{E}_7, \mathrm{E}_8$	$p \geqslant 7$

We first fix a homotopy equivalence (1). It is shown in [**14**] that there is a subcomplex A_i of B_i such that the inclusion $A_i \to B_i$ induces an isomorphism

$$\Lambda(\widetilde{H}_*(A_i)) \cong H_*(B_i),$$

where B_i is as in (1). Put $A = A_1 \vee \cdots \vee A_{p-1}$ and $B = B_1 \times \cdots \times B_{p-1}$.

Theorem 2.1 (Kishimoto [**6**] and Theriault [**13, 14**]). *Suppose that (G,p) is in Table 1. The subcomplex A has the following properties:*

1. *there is a map $j\colon A \to G$ such that Σj has a left homotopy inverse, say $t\colon \Sigma G \to \Sigma A$;*
2. *the inclusion $\Sigma G \to BG$ is homotopic to the composite*

$$\Sigma G \xrightarrow{t} \Sigma A \xrightarrow{j'} BG,$$

where $j'\colon \Sigma A \to BG$ is the adjoint of the map $j\colon A \to G$.

Consider the composite

$$G \xrightarrow{E} \Omega\Sigma G \xrightarrow{\Omega t} \Omega\Sigma A \xrightarrow{\bar{\jmath}} G$$

which we denote by g, where $\bar{\jmath}$ is the extension of the map $j\colon A \to G$. Since $g \circ j \simeq j$, the map g is an isomorphism in cohomology since $H^*(G)$ is an exterior algebra and $j^*\colon H^*(G) \to H^*(A)$ is identified with the projection onto the module of indecomposables. Hence g is a homotopy equivalence by the J.H.C. Whitehead theorem.

Theorem 2.2 (Theriault [**13, 14**]). *Suppose that (G,p) is in Table 1. There is a map $r\colon \Omega\Sigma A \to B$ satisfying the following properties:*

1. *the inclusion $A \to B$ is homotopic to the composite*

$$A \xrightarrow{E} \Omega\Sigma A \xrightarrow{r} B;$$

2. *the composite*

$$G \xrightarrow{E} \Omega\Sigma G \xrightarrow{\Omega t} \Omega\Sigma A \xrightarrow{r} B$$

is a homotopy equivalence.

We denote the homotopy equivalence in Theorem 2.2 by $h\colon G \to B$ and put $e = g \circ h^{-1}$. Let $\epsilon_i\colon B_i \to G$ be the composite $B_i \xrightarrow{\mathrm{incl}} B \xrightarrow{e} G$ and $\bar{\epsilon}_i\colon A_i \to G$ be its restriction. By Theorem 2.2, $h \circ j \simeq k$, where $k\colon A \to B$ is the inclusion. Then we have

$$\bar{\epsilon}_i = \epsilon_i \circ k|_{A_i} = e \circ k|_{A_i} \simeq g \circ h^{-1} \circ h \circ j|_{A_i} \simeq g \circ j|_{A_i} \simeq j|_{A_i}.$$

Corollary 2.3 (Kishimoto [6]). *Suppose that (G,p) is in Table 1. The Samelson product $\langle \epsilon_i, \epsilon_j \rangle$ is trivial if and only if $\langle \bar{\epsilon}_i, \bar{\epsilon}_j \rangle$ is trivial.*

We then consider (non-)triviality of the Samelson products $\langle \bar{\epsilon}_i, \bar{\epsilon}_j \rangle$ instead of $\langle \epsilon_i, \epsilon_j \rangle$. We show triviality of the Samelson products by looking at the homotopy groups of spheres and B_i.

Proposition 2.4 (Toda [15]). *For $i \geqslant 2$ and $* \leqslant 2i + 2p(p-1) - 4$, we have*

$$\pi_*(S^{2i-1}) \cong \begin{cases} \mathbb{Z}_{(p)} & * = 2i-1, \\ \mathbb{Z}/p & * = 2i-2+2j(p-1) \quad (j = 1, \ldots, p-1), \\ \mathbb{Z}/p & * = 2i-3+2j(p-1) \quad (j = i, \ldots, p-1), \\ 0 & \textit{otherwise.} \end{cases}$$

Proposition 2.5 (Mimura and Toda [11], and Kishimoto [6]). *For $* \leqslant 2p(p-1)$, we have*

$$\pi_*(B(3, 2p+1)) \cong \begin{cases} \mathbb{Z}_{(p)} & * = 3, 3+2(p-1), \\ \mathbb{Z}/p & * = 2j(p-1)+2 \quad (j = 2, \ldots, p-1), \\ 0 & \textit{otherwise} \end{cases}$$

and for $i \geqslant 3$ and $* \leqslant 2i - 4 + 2p(p-1)$, we have

$$\pi_*(B(2i-1, 2i-1+2(p-1))) \cong \begin{cases} \mathbb{Z}_{(p)} & * = 2i-1, 2i-1+2(p-1), \\ \mathbb{Z}/p^2 & * = 2i-2+2j(p-1) \quad (j = 2, \ldots, p-1), \\ \mathbb{Z}/p & * = 2i-3+2j(p-1) \quad (j = i, \ldots, p-1), \\ 0 & \textit{otherwise.} \end{cases}$$

When G is a quasi-p-regular simple Lie group except for $\mathrm{Spin}(4n)$, there is a one-to-one correspondence between $t_p(G)$ and non-trivial B_i, and we have

$$A_i \simeq \begin{cases} S^{2i-1} & r_i = 1, \\ S^{2i-1} \cup_{\alpha_1} e^{2i-1+2(p-1)} & r_i = 2 \end{cases}$$

for $i \in t_p(G)$.

Corollary 2.6. *Let G be a quasi-p-regular simple Lie group except for* $\mathrm{Spin}(4n)$*. For $i, j \in t_p(G)$, we have:*

1. *if there is no $k \in t_p(G)$ such that $i + j \equiv k \mod (p-1)$ and $i + j + (r_i + r_j - 1)(p-1) > k + r_k(p-1)$, then the homotopy set $[A_i \wedge A_j, G]$ is trivial;*
2. *if there is $k \in t_p(G)$ such that $i + j \equiv k \mod (p-1)$ and $i + j + (r_i + r_j - 1)(p-1) > k + r_k(p-1)$, then k is unique and $[A_i \wedge A_j, G] \cong [A_i \wedge A_j, B_k]$.*

Proof. Since $A_i \wedge A_j$ for $i, j \in t_p(G)$ has cells in dimension $2i + 2j - 2 + 2r(p-1)$ for $0 \leqslant r \leqslant r_i + r_j - 2$, the corollary follows from Propositions 2.4 and 2.5. ∎

Corollary 2.7. *Let G be a quasi-p-regular simple Lie group except for* $\mathrm{Spin}(4n)$*. If for $i, j \in t_p(G)$ there is no $k \in t_p(G)$ such that*

$$i + j \equiv k \mod (p-1) \quad and \quad i + j + (r_i + r_j - 1)(p-1) > k + r_k(p-1),$$

then the Samelson product $\langle \bar{\epsilon}_i, \bar{\epsilon}_j \rangle$ is trivial.

We further prove triviality of the Samelson products $\langle \bar{\epsilon}_i, \bar{\epsilon}_j \rangle$ in the special cases.

Proposition 2.8. *Suppose that G is a quasi-p-regular simple Lie group except for* $\mathrm{Spin}(4n)$*. If for $i, j \in t_p(G)$, there is $k \in t_p(G)$ such that*

$$r_i + r_j \geqslant 3 \quad and \quad k = i + j + (r_i + r_j - 3)(p-1),$$

then the Samelson product $\langle \bar{\epsilon}_i, \bar{\epsilon}_j \rangle$ is trivial.

Proof. By Corollary 2.6, we have $[A_i \wedge A_j, G] \cong [A_i \wedge A_j, B_k]$, so it is sufficient to show that $[A_i \wedge A_j, S^{2k-1}]$ is trivial since $r_k = 1$ for degree reasons. We first consider the case $r_i + r_j = 3$. In this case, we have $A_i \wedge A_j \simeq S^{2(i+j-1)} \cup_{\alpha_1} e^{2(i+j+p-2)}$, where $i + j = k$. Then there is a homotopy cofibration $S^{2(i+j+p-2)-1} \xrightarrow{\alpha_1} S^{2(i+j-1)} \to A_i \wedge A_j$ which induces an exact sequence

$$\pi_{2k-1}(S^{2k-1}) \xrightarrow{\alpha_1^*} \pi_{2k+2p-4}(S^{2k-1}) \to [A_i \wedge A_j, S^{2k-1}] \to \pi_{2k-2}(S^{2k-1}).$$

Since $\pi_{2k+2p-4}(S^{2k-1})$ is generated by α_1, the second arrow is trivial, and so for $\pi_{2k-2}(S^{2k-1}) = 0$, we get $[A_i \wedge A_j, S^{2k-1}] = *$.

We next consider the case $r_i = r_j = 2$. In this case, we have $[A_i \wedge A_j, S^{2k-1}] \cong [A_i \wedge A_j / S^{2(i+j-1)}, S^{2k-1}]$ and $A_i \wedge A_j / S^{2(i+j-1)} \simeq S^{2(i+j+p-2)} \vee (S^{2(i+j+p-2)} \cup_{\alpha_1} e^{2(i+j+2p-3)})$, where $k = i + j + p - 1$. Then we get $[A_i \wedge A_j, S^{2k-1}] = *$ in the same way as above. □

By Corollary 2.7 and Proposition 2.8, it remains to check (non-)triviality of the Samelson products $\langle \bar{\epsilon}_i, \bar{\epsilon}_j \rangle$ for $(G, p, \{i, j\})$ in Table 2. Note that the G_2 case and the p-regular case are done in the previous section and Theorem 1.1. Further, since the inclusion $\mathrm{F}_4 \to \mathrm{E}_6$ has a right homotopy inverse at the prime $p \geqslant 3$ as in the table of the mod p decomposition, we only have to consider E_6, E_7 and E_8.

Table 2:

E_6	$p = 5$	$\{2,8\}, \{5,5\}, \{5,8\}, \{8,8\}$
	$p = 7$	$\{2,6\}, \{2,9\}, \{5,6\}, \{5,9\}, \{6,6\}, \{6,9\}, \{9,9\}$
	$p = 11$	$\{6,9\}, \{8,8\}, \{9,9\}$
E_7	$p = 11$	$\{2,8\}, \{2,10\}, \{2,14\}, \{6,10\}, \{6,14\}, \{8,8\}, \{8,10\}, \{8,14\}$, $\{10,10\}, \{10,14\}, \{14,14\}$
	$p = 13$	$\{2,6\}, \{2,12\}, \{6,6\}, \{6,8\}, \{6,12\}, \{8,12\}, \{10,10\}, \{10,12\}$, $\{12,12\}$
	$p = 17$	$\{8,14\}, \{10,12\}, \{10,14\}, \{12,12\}, \{12,14\}, \{14,14\}$

E_8	$p = 11$	$\{2, 20\}, \{8, 14\}, \{8, 20\}, \{14, 14\}, \{14, 20\}, \{20, 20\}$
	$p = 13$	$\{2, 12\}, \{2, 18\}, \{8, 12\}, \{8, 18\}, \{12, 12\}, \{12, 18\}, \{18, 18\}$
	$p = 17$	$\{8, 20\}, \{14, 14\}, \{14, 20\}, \{20, 20\}$
	$p = 19$	$\{2, 12\}, \{2, 18\}, \{2, 24\}, \{8, 12\}, \{8, 18\}, \{8, 24\}, \{12, 12\}, \{12, 14\}, \{12, 18\}, \{12, 24\}, \{14, 18\}, \{14, 24\}, \{18, 18\}, \{18, 24\}, \{24, 24\}$
	$p = 23$	$\{14, 20\}, \{18, 18\}, \{20, 20\}$
	$p = 29$	$\{12, 24\}, \{18, 18\}, \{18, 24\}, \{20, 20\}, \{24, 24\}$

3. Cohomology of BG

The coefficient of cohomology will be $\mathbb{Z}_{(p)}$ unless otherwise specified. Suppose that $H_*(G)$ has no p-torsion and $t(G) = \{n_1, \dots, n_\ell\}$. Then the cohomology of the classifying space BG is given by

$$H^*(BG) = \mathbb{Z}_{(p)}[x_{2n_1}, \dots, x_{2n_\ell}], \quad |x_i| = i.$$

We recall from [**17, 18, 4**] a choice of generators x_i when G is an exceptional Lie group. As in [**1**], there is a commutative diagram of subgroup inclusions

$$\begin{tikzcd}
\mathrm{Spin}(9) \arrow[r, "i_0"] \arrow[d, "j_0"] & \mathrm{Spin}(10) \arrow[r, "i_1"] \arrow[d, "j_1"] & \mathrm{Spin}(11) \arrow[r, "i_2"] \arrow[d, "j_2"] & \mathrm{Spin}(15) \arrow[d, "j_3"] \\
\mathrm{F}_4 \arrow[r, "k_0"] & \mathrm{E}_6 \arrow[r, "k_1"] & \mathrm{E}_7 \arrow[r, "k_2"] & \mathrm{E}_8.
\end{tikzcd}$$

The choice of generators x_i is made through these inclusions. Recall that we have

$$H^*(B\mathrm{Spin}(2n+1)) = \mathbb{Z}_{(p)}[p_1, \dots, p_n], \quad H^*(B\mathrm{Spin}(2n)) = \mathbb{Z}_{(p)}[p_1, \dots, p_{n-1}, c_n],$$

where p_i and c_n are the Pontrjagin class and the Euler class of the universal bundle respectively. If a polynomial P is a sum of a polynomial Q and other terms, then we write $P \rhd Q$.

Proposition 3.1. *For $p \geqslant 7$, generators x_i for E_8 can be chosen such that*

$$\begin{aligned}
j_3^*(x_4) &= p_1, \\
j_3^*(x_{16}) &= 12p_4 - \frac{18}{5}p_3p_1 + p_2^2 + \frac{1}{10}p_2p_1^2, \\
j_3^*(x_{24}) &= 60p_6 - 5p_5p_1 - 5p_4p_2 + 3p_3^2 - p_3p_2p_1 + \frac{5}{36}p_2^3, \\
j_3^*(x_{28}) &\equiv 480p_7 + 40p_5p_2 - 12p_4p_3 - p_3p_2^2 - 3p_4p_2p_1 + \frac{24}{5}p_3^2p_1 + \frac{11}{36}p_2^3p_1 \mod (p_1^2), \\
j_3^*(x_{36}) &\equiv 480p_7p_2 + 72p_6p_3 - 30p_5p_4 - \frac{25}{2}p_5p_2^2 + 9p_4p_3p_2 - \frac{18}{5}p_3^3 \\
&\quad - \frac{1}{4}p_3p_2^3 - 42p_6p_2p_1 + 9p_5p_3p_1 - \frac{3}{2}p_4p_2^2p_1 + \frac{9}{5}p_3^2p_2p_1 + \frac{1}{24}p_2^4p_1 \mod (p_1^2), \\
j_3^*(x_{40}) &\equiv 480p_7p_3 + 50p_6p_2^2 + 50p_5^2 - 10p_5p_3p_2 - \frac{25}{2}p_4^2p_2 + 9p_4p_3^2 \\
&\quad - \frac{25}{36}p_4p_2^3 + \frac{3}{4}p_3^2p_2^2 + \frac{25}{864}p_2^5 \mod (p_1),
\end{aligned}$$

$$\begin{aligned} j_3^*(x_{48}) \equiv & -200p_7p_5 - 60p_7p_3p_2 + 3p_6p_3^2 + \frac{25}{9}p_6p_2^3 + \frac{25}{3}p_5^2p_2 - \frac{5}{2}p_5p_4p_3 \\ & -\frac{25}{24}p_5p_3p_2^2 - \frac{25}{48}p_4^2p_2^2 + p_4p_3^2p_2 + \frac{25}{864}p_4p_2^4 - \frac{3}{10}p_3^4 - \frac{1}{36}p_3^2p_2^3 \\ & -\frac{25}{62208}p_2^6 \qquad \mod (p_1), \end{aligned}$$

$$\begin{aligned} j_3^*(x_{60}) \rhd\ & 144p_7p_5p_3 - 5p_5^3 + \frac{3}{2}p_5^2p_3p_2 - \frac{89}{1440}p_5p_4^2p_2 - \frac{229}{1600}p_5p_4p_3^2 - \frac{13}{320}p_5p_4p_2^3 \\ & -\frac{229}{3840}p_5p_3^2p_2^2 + \frac{29}{13824}p_5p_2^5 - \frac{43}{1920}p_4^3p_3 + \frac{1357}{38400}p_4^2p_3p_2^2 - \frac{59}{3200}p_4p_3^3p_2 \\ & -\frac{421}{153600}p_4p_3p_2^4 + \frac{177}{40000}p_3^5 + \frac{59}{115200}p_3^3p_2^3, \end{aligned}$$

where $j_3^*(x_{40}), j_3^*(x_{60})$ do not include multiples of $p_5p_4p_1, p_7^2p_1$, respectively.

Proof. The choice of x_i except for $i = 60$ is made in [4], where we subtract a multiple of x_4x_{36} from x_{40} if necessary so that $j^*(x_{40})$ does not include a multiple of $p_5p_4p_1$, and we can take x_{60} quite similarly. Further, by subtracting a multiple of $x_4x_{28}^2$ if necessary, we can take x_{60} so that $j_3^*(x_{60})$ does not include a multiple of $p_7^2p_1$, completing the proof. □

Proposition 3.2. *For $p \geqslant 7$, generators x_i for E_7 can be chosen such that:*

1. $k_2^*(x_i) = x_i \quad (i = 4, 16, 24, 28, 36) \quad j_2^*(x_{12}) = -6p_3 + p_2p_1 \quad j_2^*(x_{20}) = p_5$*;*
2. *modulo* $\widetilde{H}^*(B\mathrm{E}_7)^3$

$$\begin{aligned} k_2^*(x_{40}) &\equiv \frac{1}{24}x_{12}x_{28} + \frac{5}{24}x_{16}x_{24} + 50x_{20}^2, \\ k_2^*(x_{48}) &\equiv -\frac{1}{72}x_{12}x_{36} + \frac{5}{24}x_{20}x_{28} - \frac{1}{48}x_{24}^2, \\ k_2^*(x_{60}) &\equiv -\frac{131}{144000}x_{24}x_{36}. \end{aligned}$$

Proof. (1) is proved in [4], and (2) is obtained by Proposition 3.1 and (1). □

Proposition 3.3. *For $p \geqslant 5$, generators x_i for E_6 can be chosen such that*

$$\begin{array}{ll} j_1^*(x_4) = p_1, & j_1^*(x_{10}) = c_5, \\ j_1^*(x_{12}) = -6p_3 + p_2p_1, & j_1^*(x_{16}) = 12p_4 - 3p_3p_1 + p_2^2, \\ j_1^*(x_{18}) = p_2c_5, & j_1^*(x_{24}) = -72p_4p_2 + 27p_4p_1^2 + 27p_3^2 - 9p_3p_2p_1 + 2p_2^3. \end{array}$$

Proof. The argument on the choice of x_i $(i = 10, 18)$ for E_6 in [4] works also for $p \geqslant 5$, so we can choose x_i $(i = 10, 18)$ for E_6 as in the statement. On the other hand, Watanabe [17] chooses generators x_i for F_4 through the inclusion $j_0\colon \mathrm{Spin}(9) \to \mathrm{F}_4$. Then since $i_0^*(p_i) = p_i$ $(i = 1, 2, 3, 4)$ and $i_0^*(c_5) = 0$, a degree reason shows that the choice of x_i for F_4 implies the choice of x_i for E_6 $(i = 4, 12, 16, 24)$. □

Remark 3.4. We choose generators x_i for E_6 independently from $\mathrm{E}_7, \mathrm{E}_8$ since we have to consider the primes $5, 7$.

4. Steenrod operations and Samelson products

Suppose that (G, p) is in Table 1 except for $\mathrm{Spin}(4n)$, where we exclude $\mathrm{Spin}(4n)$ to make $t(G)$ consist of distinct integers. Define $y_{2j-1} \in H^{2j-1}(A_i)$ by $(\bar{\epsilon}_i')^*(x_{2j}) = \Sigma y_{2j-1}$ for $j \equiv i \mod (p-1)$, where $f' \colon \Sigma X \to Y$ denotes the adjoint of a map $f \colon X \to \Omega Y$. Then y_{2j-1} is non-trivial and satisfies

$$(\bar{\epsilon}_i')^*(x_{2k}) = \begin{cases} \Sigma y_{2k-1} & k \equiv i \mod (p-1), \\ 0 & k \not\equiv i \mod (p-1) \end{cases}$$

since $t(G)$ consists of distinct integers. We detect non-triviality of the Samelson products $\langle \bar{\epsilon}_i, \bar{\epsilon}_j \rangle$ by the following criterion (cf. [**8**, Proof of Theorem 1.1], [**7**]).

Proposition 4.1. *Suppose that (G, p) is in Table 1 except for* $\mathrm{Spin}(4n)$ *and that for $i, j \in t_p(G)$, there is $k \in t_p(G)$ such that $i + j > k$ and $\mathcal{P}^{r_k} x_{2k}$ includes the term $\lambda x_{2i+2s_i(p-1)} x_{2j+2s_j(p-1)}$ with $\lambda \neq 0$, $s_i \leqslant \min\{r_i - 1, r_k - 1\}$, $s_j \leqslant \min\{r_j - 1, r_k - 1\}$. Then $\langle \bar{\epsilon}_i, \bar{\epsilon}_j \rangle$ is non-trivial.*

Proof. Assume that $\langle \bar{\epsilon}_i, \bar{\epsilon}_j \rangle$ is trivial. Then by the adjointness of Samelson products and Whitehead products, the Whitehead product $[\bar{\epsilon}_i', \bar{\epsilon}_j']$ is trivial, implying $\bar{\epsilon}_i' \vee \bar{\epsilon}_j' \colon \Sigma A_i \vee \Sigma A_j \to BG$ extends to $\mu \colon \Sigma A_i \times \Sigma A_j \to BG$ up to homotopy. Let $\bar{\mu}$ be the restriction of μ to $\Sigma A^{(2i-1+2(r_k-1)(p-1))} \times \Sigma A^{(2j-1+2(r_k-1)(p-1))}$. Then we have $\mathcal{P}^{r_k} \bar{\mu}^*(x_{2k}) = 0$ since $\mathcal{P}^{r_k}$ is trivial on the mod p cohomology of $A^{(2n-1+2(r_k-1)(p-1))}$ for $n = i, j$. On the other hand, we have

$$\begin{aligned} \mathcal{P}^{r_k} \bar{\mu}^*(x_{2k}) &= \bar{\mu}^*(\mathcal{P}^{r_k} x_{2k}) \\ &= \bar{\mu}^*(\lambda x_{2i+2s_i(p-1)} x_{2j+2s_j(p-1)}) = \lambda \Sigma y_{2i+2s_i(p-1)} \otimes \Sigma y_{2j+2s_j(p-1)} \end{aligned}$$

since $\mathcal{P}^{r_k} x_{2k}$ has no linear part for a degree reason. This is a contradiction, so the Samelson product $\langle \bar{\epsilon}_i, \bar{\epsilon}_j \rangle$ is non-trivial. $\square$

In order to apply Proposition 4.1, we calculate the linear and the quadratic parts of $\mathcal{P}^1 x_{2k}$.

Lemma 4.2. *The linear and the quadratic parts of $\mathcal{P}^1 x_i$ for* E_6 *are given by:*

	$i = 4$	$i = 10$	$i = 12$
	$i = 16$	$i = 18$	$i = 24$
$p = 5$	$-x_{12}$	$-x_{18}$	0
	x_{24}	$-x_{10}x_{16}$	x_{16}^2
$p = 7$	$-2x_{16} + 5x_4x_{12}$	$x_4x_{18} + 3x_{10}x_{12}$	$-2x_{24} - 4x_{12}^2$
	$-3x_{12}x_{16}$	$-x_{12}x_{18}$	$-2x_{18}^2$
$p = 11$	$-2x_{24} - 2x_{12}^2$	$-x_{12}x_{18}$	$-4x_{16}^2$
	$6x_{18}^2$	0	0

Proof. Recall from [12] that there is the mod p Wu formula

$$\mathcal{P}^1 p_n = \sum_{i_1+2i_2+\cdots+5i_5=n+\frac{p-1}{2}} (-1)^{i_1+\cdots+i_5+\frac{p+1}{2}} \frac{(i_1+\cdots+i_5-1)!}{i_1!\cdots i_5!} \times \left(2n-1-\frac{\sum_{j=1}^{n-1}(2n+p-1-2j)i_j}{i_1+\cdots+i_5-1}\right) p_1^{i_1}\cdots p_5^{i_5}$$

in $H^*(B\mathrm{Spin}(10);\mathbb{Z}/p)$, where $p_5 = c_5^2$. For example, consider the case $i = 16$ and $p = 11$. Then by Proposition 3.3 we have

$$j_1^*(\mathcal{P}^1 x_{16}) = \mathcal{P}^1 j^*(x_{16}) \triangleright 7p_5p_4 + 2p_5p_2^2,$$

so for a degree reason, we must have $\mathcal{P}^1 x_{16} \triangleright 6x_{18}^2 + 7x_{16}x_{10}^2$. The remaining calculations are done in the same way. ☐

Lemma 4.3. *The linear and the quadratic parts of $\mathcal{P}^1 x_i$ for E_8 are given by:*

	$i=4$	$i=16$
	$i=24$	$i=28$
	$i=36$	$i=40$
	$i=48$	$i=60$
$p=11$	$2x_{24}$	$6x_{36}$
	0	$3x_{48}$
	$3x_{16}x_{40}+9x_{28}^2$	$9x_{60}$
	$5x_{28}x_{40}$	$5x_{40}^2$
$p=13$	$-x_{28}+8x_4x_{24}$	$8x_{40}-2x_4x_{36}+4x_{16}x_{24}$
	$4x_{48}+5x_{24}^2$	$5x_{24}x_{28}$
	$5x_{60}+2x_{24}x_{36}$	$9x_4x_{60}-x_{16}x_{48}+x_{24}x_{40}+4x_{28}x_{36}$
	$-x_{24}x_{48}-x_{36}^2$	$8x_{24}x_{60}+7x_{36}x_{48}$
$p=17$	$4x_{36}$	$13x_{48}+2x_{24}^2$
	$11x_{16}x_{40}+7x_{28}^2$	$12x_{60}+5x_{24}x_{36}$
	0	$-x_{24}x_{48}+10x_{36}^2$
	$13x_{40}^2$	0
$p=19$	$4x_{40}+11x_4x_{36}+9x_{16}x_{24}$	$10x_4x_{48}+11x_{16}x_{36}+17x_{24}x_{28}$
	$10x_{60}+x_{24}x_{36}$	$-x_4x_{60}+10x_{16}x_{48}-3x_{24}x_{40}-4x_{28}x_{36}$
	$4x_{24}x_{48}+4x_{36}^2$	$5x_{16}x_{60}+x_{36}x_{40}$
	$11x_{24}x_{60}+9x_{36}x_{48}$	$-3x_{36}x_{60}-2x_{48}^2$
$p=23$	$10x_{48}+x_{24}^2$	$x_{60}+5x_{24}x_{36}$
	$-9x_{28}x_{40}$	$8x_{24}x_{48}-10x_{36}^2$
	$13x_{40}^2$	$3x_{24}x_{60}-8x_{36}x_{48}$
	0	0
$p=29$	$-2x_{60}-x_{24}x_{36}$	$-x_{24}x_{48}-5x_{36}^2$
	$11x_{40}^2$	$17x_{24}x_{60}-5x_{36}x_{48}$
	0	$14x_{36}x_{60}+13x_{48}^2$
	0	0

Proof. The proof is the same as Lemma 4.2. ☐

Lemma 4.4. *The linear and the quadratic parts of $\mathcal{P}^1 x_i$ for E_7 are given by:*

	$i = 4$	$i = 12$
	$i = 16$	$i = 20$
	$i = 24$	$i = 28$
	$i = 36$	
$p = 11$	$2x_{24}$	$-3x_4x_{28} - 5x_{12}x_{20} - 2x_{16}^2$
	$6x_{36}$	$-3x_4x_{36} - 5x_{12}x_{28} + 10x_{20}^2$
	0	$5x_{12}x_{36} + 2x_{20}x_{28} + 2x_{24}^2$
	$9x_{28}^2$	
$p = 13$	$-x_{28} + 8x_4x_{24}$	$-2x_{36} + 7x_{12}x_{24} + 2x_{16}x_{20}$
	$-2x_4x_{36} - 4x_{12}x_{28} - 3x_{16}x_{24} - 3x_{20}^2$	$7x_{16}x_{28} + 4x_{20}x_{24}$
	$5x_{12}x_{36} + 3x_{20}x_{28} + 6x_{24}^2$	$5x_{24}x_{28}$
	$-3x_{24}x_{36}$	
$p = 17$	$4x_{36}$	$5x_{16}x_{28} - 2x_{20}x_{24}$
	$-x_{12}x_{36} + 2x_{20}x_{28} + 6x_{24}^2$	$7x_{24}x_{28}$
	$7x_{28}^2$	$-3x_{24}x_{36}$
	0	

Proof. $\mathcal{P}^1 x_i$ for $i = 4, 16, 24, 28, 36$ can be calculated by Proposition 3.2 and Lemma 4.3, and $\mathcal{P}^1 x_i$ for $i = 12, 20$ can be calculated in the same way as Lemma 4.2. $\square$

We now prove:

Proposition 4.5. *The Samelson product $\langle \bar{\epsilon}_i, \bar{\epsilon}_j \rangle$ in G is non-trivial for $(G, p, \{i, j\})$ in the following table:*

E_6	$p = 5$	$\{5,8\}, \{8,8\}$
	$p = 7$	$\{2,9\}, \{5,6\}, \{6,6\}, \{6,9\}, \{9,9\}$
	$p = 11$	$\{6,9\}, \{8,8\}, \{9,9\}$
E_7	$p = 11$	$\{2,14\}, \{6,10\}, \{6,14\}, \{8,8\}, \{10,10\}, \{10,14\}, \{14,14\}$
	$p = 13$	$\{8,12\}, \{10,10\}, \{10,12\}, \{12,12\}$
	$p = 17$	$\{8,14\}, \{10,12\}, \{10,14\}, \{12,12\}, \{12,14\}, \{14,14\}$
E_8	$p = 11$	$\{8,20\}, \{14,14\}, \{14,20\}, \{20,20\}$
	$p = 13$	$\{2,18\}, \{8,12\}, \{12,12\}, \{12,18\}, \{18,18\}$
	$p = 17$	$\{8,20\}, \{14,14\}, \{20,20\}$
	$p = 19$	$\{2,24\}, \{8,12\}, \{8,18\}, \{8,24\}, \{12,14\}, \{12,18\}, \{12,24\}, \{14,18\}, \{18,18\}, \{18,24\}, \{24,24\}$
	$p = 23$	$\{14,20\}, \{18,18\}, \{20,20\}$
	$p = 29$	$\{12,24\}, \{18,18\}, \{18,24\}, \{20,20\}, \{24,24\}$

Proof. We can verify the conditions of Proposition 4.1 by Lemmas 4.2, 4.3 and 4.4, where we have $\mathcal{P}^1\mathcal{P}^1 = 2\mathcal{P}^2$ by the Adem relation. Thus the result follows from Proposition 4.1. $\square$

5. Chern classes

In order to determine (non-)triviality of the Samelson products that are not detected in the previous sections, we will use representations of the exceptional Lie groups and their Chern classes. Then we calculate these Chern classes. We refer to [1] for basic materials of representations that we consider in this section. For the canonical representation $\lambda_n\colon \mathrm{Spin}(n) \to \mathrm{SU}(n)$, we have

$$c_{2i-1}(\lambda_n) = 0, \quad c_{2i}(\lambda_n) = (-1)^i p_i \quad (i = 2, \dots, n), \tag{2}$$

where $p_{2k} = c_k^2$ if $n = 2k$. Then by Girard's formula on power sums and elementary symmetric polynomials

$$n!\mathrm{ch}_n = \sum_{i_1+2i_1+\cdots+ni_n=n} (-1)^{n+i_1+\cdots+i_n} \frac{n(i_1+\cdots+i_n-1)!}{i_1!\cdots i_n!} c_1^{i_1}\cdots c_n^{i_n}, \tag{3}$$

we can calculate the Chern character of λ, where ch_n denotes the $2n$-dimensional part of the Chern character. Let $\alpha\colon \mathrm{Spin}(11) \to \mathrm{SU}(55)$ be the adjoint representation of Spin(11), and let Δ^+, Δ be the positive half spin representation of Spin(10) and the spin representation of Spin(11). The weights of α are the roots of Spin(11) by definition. As in [1], the weights of Δ^+ are $\epsilon_1 t_1 + \cdots + \epsilon_5 t_5$ ($\epsilon_1 \cdots \epsilon_5 = 1$) and the weights of Δ are $\epsilon_1 t_1 + \cdots + \epsilon_5 t_5$ ($\epsilon_1 \cdots \epsilon_5 = \pm 1$). Then one can calculate $\mathrm{ch}(\alpha), \mathrm{ch}(\Delta), \mathrm{ch}(\Delta^+)$ with the assistance of a computer as follows.

Lemma 5.1.

1. $i!\mathrm{ch}_i(\lambda_{10} + \Delta^+ + 1)$ *includes the following terms:*

$i=2$	$6p_1$	$i=5$	$60c_5$
$i=6$	$18p_3$	$i=8$	$60p_4 + 24p_3p_1$
$i=9$	$126p_2c_5$	$i=10$	$135p_4p_1 + 630c_5^2$
$i=12$	$135p_4p_2 + 18p_3^2$	$i=14$	$231p_4p_3 + \frac{1337}{4}p_4p_2p_1 + 2233p_2c_5^2$
$i=20$	$\frac{6885}{4}p_4^2p_2$		

2. $i!\mathrm{ch}_i(2\lambda_{11} + \Delta + 2)$ *includes the following terms:*

$i=2$	$12p_1$
$i=6$	$36p_3$
$i=8$	$120p_4 + 48p_3p_1$
$i=10$	$1260p_5 + 270p_4p_1$
$i=12$	$270p_4p_2 + 36p_3^2$
$i=14$	$4466p_5p_2 + 462p_4p_3 + \frac{1337}{2}p_4p_2p_1$
$i=18$	$39672p_5p_4 + 6993p_5p_2^2 + 1134p_4p_3p_2$
$i=20$	$151300p_5^2 + 189190p_5p_4p_1 + 37570p_5p_3p_2 + \frac{50785}{2}p_5p_2^2p_1 + \frac{6885}{2}p_4^2p_2$
$i=22$	$179949p_5p_4p_2 + \frac{27797}{4}p_5p_2^3$
$i=24$	$950400p_5^2p_2 + 390390p_5p_4p_3 + \frac{19569}{2}p_4^3 + \frac{43887}{8}p_4^2p_2^2$
$i=30$	$\frac{27047655}{2}p_5^2p_3p_2 + \frac{3907395}{2}p_5p_4p_3^2 + \frac{2450295}{8}p_5p_3^2p_2^2$

3. $i!\mathrm{ch}_i(\alpha + 4\Delta + 65)$ *includes the following terms:*

$$
\begin{array}{ll}
i=2 & 60p_1 \\
i=8 & 1440p_4 \\
i=10 & 0\cdot p_5 \\
i=12 & -7560p_4p_2 \\
i=14 & 92400p_5p_2-40110p_4p_2p_1 \\
i=18 & -982800p_5p_4 \\
i=20 & 4600200p_5^2-1748790p_4^2p_2 \\
i=22 & 34950300p_5^2p_1-6715170p_4^2p_3 \\
i=24 & -69872880p_5^2p_2+20077794p_4^3+\frac{22514031}{2}p_4^2p_2^2 \\
i=26 & -219540750p_5p_4^2-41441400p_4p_3^3-697554000p_5^2p_2p_1 \\
i=30 & -2289787500p_5^3+\frac{10586291625}{2}p_5p_4^2p_2-\frac{2479051575}{2}p_4^3p_3 \\
i=32 & -26808164160p_5^3p_1-29632951680p_5^2p_4p_2+1801812456p_5^2p_3^2 \\
 & -1601056128p_5p_3^3p_2 \\
i=34 & 66371012400p_5^3p_2 \\
i=38 & -952563046800p_5^3p_4-949011128850p_5^3p_2^2-46394357586p_5^2p_3^3 \\
i=42 & -1030173212250p_3^2p_5^3 \\
i=50 & -914425331875000p_5^5 \\
i=60 & -340771201982677620p_5^5p_3p_2-12363661137209454345p_5^4p_4^2p_2 \\
 & +776927112035890410p_5^4p_4p_3^2-\frac{20464209777645659655}{2}p_5^4p_4p_2^3 \\
 & +\frac{4154188924169320995}{4}p_5^4p_3^2p_2^2-\frac{15713809488581615145}{16}p_5^4p_2^5 \\
 & -\frac{12852298085402204835}{8}p_5^3p_4p_3^3p_2+\frac{446620868886161465}{2}p_5^3p_3^5 \\
 & -\frac{15212346544088595405}{32}p_5^3p_3^3p_2^3+\frac{16285084675436347155}{8}p_5^2p_4^5 \\
 & +\frac{634726068898356739815}{64}p_5^2p_4^4p_2^2-\frac{226812497505304105395}{32}p_5^2p_4^3p_3^2p_2 \\
 & +\frac{149364900987340422405}{32}p_5^2p_4^3p_2^4+\frac{5425603982540815815}{8}p_5^2p_4^2p_3^4 \\
 & -\frac{658004024293760076975}{128}p_5^2p_4^2p_3^2p_2^3+\frac{227646297744518937585}{512}p_5^2p_4^2p_2^6 \\
 & +\frac{16657727505432381165}{16}p_5^2p_4p_3^4p_2^2-\frac{236740660412585333865}{512}p_5^2p_4p_3^2p_2^5 \\
 & +\frac{19109048218262101365}{2048}p_5^2p_4p_2^8-32085165434973420p_5^2p_3^6p_2 \\
 & +\frac{10050849772724723655}{128}p_5^2p_3^4p_2^4-\frac{126872100695409424425}{512}p_4^7p_2 \\
 & -\frac{561833719261457328105}{2048}p_4^6p_2^3-\frac{417106897597216552845}{8192}p_4^5p_2^5
\end{array}
$$

Remark 5.2. For $(i,\ell)=(60,8)$, the twenty-four terms in the above list are chosen so that the following condition holds. For a monomial $m\in H^{120}(B\mathrm{Spin}(11))/(p_1)$, let a_m be the vector consisting of the coefficients of m in the monomials of $H^{120}(B\mathrm{E}_8)/(x_4)$. Then a_m for the above monomials form a square matrix, which is invertible. Note that $a_{p_5^6}$ and $a_{p_5^5p_3p_2}$ are linearly dependent.

Let ρ_ℓ be the irreducible 27, 56, 248 dimensional representation of E_ℓ for $\ell=6,7,8$ respectively. Then we have

$$\rho_6\circ j_1=\lambda_{10}+\Delta^++1,\quad \rho_7\circ j_2=2\lambda_{11}+\Delta+2,\quad \rho_8\circ j_3\circ i_2=\alpha+4\Delta+65.$$

Thus by Propositions 3.1, 3.2, 3.3 and Lemma 5.1, we can determine the linear and the quadratic parts of $\mathrm{ch}_i(\rho_\ell)$ except for the coefficient of $x_{36}x_{48}$ in $\mathrm{ch}_{42}(\rho_8)$. Then by the inductive use of (3), we obtain the following proposition, which gives the linear and the quadratic parts of $c_i(\rho_\ell)$ in each case except for $(i,\ell) = (42,8)$.

Proposition 5.3.

1. *The linear and the quadratic parts of the Chern classes $c_i(\rho_6)$ are:*

$i=2$	$-3x_4$	$i=5$	$12x_{10}$
$i=6$	$\frac{1}{2}x_{12}$	$i=8$	$-\frac{5}{8}x_{16} - \frac{11}{16}x_4x_{12}$
$i=9$	$14x_{18}$	$i=10$	$\frac{3}{4}x_4x_{16} + 9x_{10}^2$
$i=12$	$\frac{5}{32}x_{24} - \frac{13}{384}x_{12}^2$	$i=14$	$-\frac{19}{192}x_4x_{24} + \frac{17}{2}x_{10}x_{18} - \frac{1}{12}x_{12}x_{16}$
$i=20$	$\frac{1}{512}x_{16}x_{24}$		

2. *The linear and the quadratic parts of the Chern classes $c_i(\rho_7)$ are:*

$i=2$	$-6x_4$
$i=6$	x_{12}
$i=8$	$-\frac{5}{4}x_{16} - \frac{17}{4}x_4x_{12}$
$i=10$	$-126x_{20} + \frac{21}{4}x_4x_{16}$
$i=12$	$\frac{9}{2}x_{24} + \frac{907}{2}x_4x_{20} + \frac{1}{24}x_{12}^2$
$i=14$	$-\frac{319}{40}x_{28} - \frac{67}{8}x_4x_{24} + \frac{43}{80}x_{12}x_{16}$
$i=18$	$\frac{1229}{60}x_{36} - \frac{749}{200}x_{12}x_{24} + \frac{601}{24}x_{16}x_{20}$
$i=20$	$-\frac{1043}{24}x_4x_{36} - \frac{71}{480}x_{12}x_{28} - \frac{441}{160}x_{16}x_{24} + 373x_{20}^2$
$i=22$	$-\frac{137}{640}x_{16}x_{28} + \frac{1827}{20}x_{20}x_{24}$
$i=24$	$-\frac{1711}{1440}x_{12}x_{36} + \frac{297}{20}x_{20}x_{28} + \frac{4047}{800}x_{24}^2$
$i=30$	$\frac{963}{400}x_{24}x_{36}$

3. *The linear and the quadratic parts of the Chern classes $c_i(\rho_8)$ are:*

$i=2$	$-30x_4$
$i=8$	$-15x_{16}$
$i=12$	$-126x_{24}$
$i=14$	$-165x_{28} + 3306x_4x_{24}$
$i=18$	$-1820x_{36}$
$i=20$	$-\frac{23001}{5}x_{40} + \frac{408519}{10}x_4x_{36} + \frac{27821}{20}x_{16}x_{24}$
$i=22$	$106233x_4x_{40} + \frac{5685}{16}x_{16}x_{28}$
$i=24$	$\frac{1746822}{5}x_{48} + \frac{120512}{25}x_{24}^2$
$i=26$	$-7648542x_4x_{48} + \frac{92275}{24}x_{16}x_{36} + \frac{18645}{4}x_{24}x_{28}$
$i=30$	$-15265250x_{60} - \frac{24568999}{576}x_{24}x_{36}$
$i=32$	$\frac{1236701415}{4}x_4x_{60} - \frac{747928}{5}x_{16}x_{48} + \frac{40849521}{200}x_{24}x_{40} + \frac{29810159}{960}x_{28}x_{36}$
$i=34$	$-\frac{2170113}{10}x_{28}x_{40}$
$i=38$	$-\frac{230904475}{12}x_{16}x_{60} + \frac{16714837}{20}x_{28}x_{48} + \frac{95373791}{60}x_{36}x_{40}$
$i=42$	$68678214150x_{24}x_{60}$
$i=50$	$-2930823500x_{40}x_{60}$
$i=60$	$-\frac{23018190805225}{48}x_{60}^2$

Since we are computing $c_i(\rho_\ell)$ via Spin(10) and Spin(11) whose ranks are less than

E_ℓ, $c_i(\rho_\ell)$ might not be determined in some cases by the above direct computation. In these cases, we determine $c_i(\rho_\ell) \mod p$ by an indirect way as follows.

Proposition 5.4. *The quadratic parts of $c_i(\rho_\ell) \mod p$ are given by:*

$$(i,\ell,p)=(28,7,11) \quad 3x_{20}x_{36}-2x_{28}^2 \qquad\qquad (i,\ell,p)=(36,7,13) \quad -4x_{36}^2$$

Proof. We only calculate $c_{28}(\rho_7) \mod 11$ since the other case can be similarly calculated. Recall from [12] that there is the mod p Wu formula

$$\mathcal{P}^1 c_k = \sum_{i_1+2i_2+\cdots+ni_n=k+p-1} (-1)^{i_1+\cdots+i_n-1}\frac{(i_1+\cdots+i_n-1)!}{i_1!\cdots i_n!} \times\left(k-1-\frac{\sum_{j=2}^{k-1}(k+p-1-j)i_j}{i_1+\cdots+i_n-1}\right)c_1^{i_1}\cdots c_n^{i_n}$$

in $H^*(B\mathrm{U}(n);\mathbb{Z}/p)$. Then we get $\mathcal{P}^1c_{18} \rhd 6c_{28}+c_{10}c_{18}$. On the other hand, we have $6c_{28}(\rho_7)+c_{10}(\rho_7)c_{18}(\rho_7) \equiv 6c_{28}(\rho_7)-8x_{20}x_{36} \mod \widetilde{H}^*(B\mathrm{E}_7;\mathbb{Z}/11)^3$ by Proposition 5.3. On the other hand, by Lemma 4.4 and Proposition 5.3, we have $\mathcal{P}^1c_{18}(\rho_7) \equiv \mathcal{P}^1(\frac{1229}{60}x_{36}+\frac{601}{24}x_{16}x_{20}) \equiv -x_{28}^2-x_{20}x_{36} \mod \widetilde{H}^*(B\mathrm{E}_7;\mathbb{Z}/11)^3$. Then we obtain the desired result. □

6. Decomposition of representations

In order to calculate the Samelson products, we will need to identify the homotopy fiber of a stabilized representation of G. To this end, we decompose stabilized representations with respect to the mod p decomposition of G.

The following universality of the space B is proved by Theriault [13].

Theorem 6.1 (Theriault [13])**.** *If (G,p) is in Table 1, then B is a homotopy associative and homotopy commutative H-space satisfying the following properties:*

1. *the map $r\colon \Omega\Sigma A \to B$ in Theorem 2.2 is the H-map extending the inclusion $A \to B$;*
2. *for any map $f\colon A \to Z$ into a homotopy associative, homotopy commutative H-space Z, there is a unique, up to homotopy, H-map $f'\colon B \to Z$ such that $f'|_A \simeq f$.*

We compare the H-structure of B in Theorem 6.1 and the H-structure of G.

Lemma 6.2. *Suppose that (G,p) is in Table 1. Given any H-map $f\colon G \to Z$ into a homotopy associative and homotopy commutative H-space Z, the map $f\circ e\colon B \to Z$ is too an H-map.*

Proof. Let $\bar{\jmath}\colon \Omega\Sigma A \to G$ be the extension of $j\colon A \to G$. Then since both $\bar{\jmath}$ and f are H-maps, the composite $f\circ\bar{\jmath}\colon \Omega\Sigma A \to Z$ is an H-map. There is an H-map $f'\colon B \to Z$

satisfying $f'|_A \simeq f \circ j$ and a homotopy commutative diagram

$$\begin{tikzcd} \Omega\Sigma A \arrow[r, "f\circ\bar{\jmath}"] \arrow[d, "r"] & Z \arrow[d, equal] \\ B \arrow[r, "f'"] & Z. \end{tikzcd}$$

Indeed, by Theorem 6.1, there is an H-map $f'\colon B \to Z$ such that $f'|_A \simeq f \circ j$, implying $(f' \circ r)|_A \simeq f \circ j$. Since $f' \circ r$ is an H-map, the universality of the loop-suspension guarantees that $f' \circ r$ is homotopic to $f \circ \bar{\jmath}$.

Next we prove the statement of the lemma. By the definition of the homotopy equivalences $g\colon G \to G$ and $h\colon G \to B$, we have

$$f \circ g = f \circ \bar{\jmath} \circ \Omega t \circ E \simeq f' \circ r \circ \Omega t \circ E \simeq f' \circ h.$$

Thus for $e = g \circ h^{-1}$, the proof is completed. □

Let $\mathrm{SU}(\infty) \simeq C_1 \times \cdots \times C_{p-1}$ be the mod p decomposition such that $\pi_*(C_k) = 0$ for $* \not\equiv 2k+1 \mod 2(p-1)$. Let $\widetilde{C}_k$ and $\widetilde{\mathrm{SU}}(\infty)$ be the 3-connective covers of C_k and $\mathrm{SU}(\infty)$, respectively. Then we have $\widetilde{SU}(\infty) \simeq \widetilde{C}_1 \times \cdots \times \widetilde{C}_{p-1}$ and there is a homotopy equivalence $\mathrm{SU}(\infty) \xrightarrow{\simeq} \Omega^2\widetilde{SU}(\infty)$ which is a loop map. We now decompose an H-map $\rho\colon G \to \mathrm{SU}(\infty)$ with respect to the mod p decompositions of G and $\mathrm{SU}(\infty)$. Define a map $\rho^k\colon B_i \to \Omega^2\widetilde{C}_k$ by the composite

$$B_k \xrightarrow{\epsilon_k} G \xrightarrow{\rho} \mathrm{SU}(\infty) \xrightarrow{\simeq} \Omega^2\widetilde{\mathrm{SU}}(\infty) \xrightarrow{\mathrm{proj}} \Omega^2\widetilde{C}_k.$$

Lemma 6.3. *Suppose that (G, p) is in Table 1. For an H-map $\rho\colon G \to \mathrm{SU}(\infty)$, the composite*

$$B = B_1 \times \cdots \times B_{p-1} \xrightarrow{\rho^1\times\cdots\times\rho^{p-1}} \Omega^2\widetilde{C}_1 \times \cdots \times \Omega^2\widetilde{C}_{p-1} \simeq \mathrm{SU}(\infty)$$

is an H-map too.

Proof. By Lemma 6.2, the map $\rho \circ e\colon B \to \mathrm{SU}(\infty)$ is an H-map. As in [**13, 14**], the H-structure of B is the product of certain H-structures of B_i, so the map $\rho^1 \times \cdots \times \rho^{p-1}$ is an H-map. Thus the proof is done. □

Theorem 6.4. *Suppose that (G, p) is in Table 1. If $\rho\colon G \to \mathrm{SU}(\infty)$ is an H-map, then $\rho \circ e \simeq \rho^1 \times \cdots \times \rho^{p-1}$.*

Proof. By Lemmas 6.2 and 6.3, the maps $\rho \circ e$ and $\rho^1 \times \cdots \times \rho^{p-1}$ are H-maps. Then for $(\rho \circ e)|_A \simeq (\rho^1|_{A_1}) \vee \cdots \vee (\rho^{p-1}|_{A_{p-1}}) \simeq (\rho^1 \times \cdots \times \rho^{p-1})|_A$, the proof is done by Theorem 6.1. □

Corollary 6.5. *Suppose that (G, p) is in Table 1. If $\rho\colon G \to \mathrm{SU}(\infty)$ is an H-map, then*

$$\mathrm{hofib}(\rho) \simeq \mathrm{hofib}(\rho^1) \times \cdots \times \mathrm{hofib}(\rho^{p-1}).$$

Put $d_k = k - 1 + r_k(p-1)$ for $k \in t_p(G)$. Similarly to B_k, we denote the factors of $\Omega^2\widetilde{\mathrm{SU}}(\infty)$ and ρ corresponding to $k \in t_p(G)$ by $\Omega^2\widetilde{C}_k$ and ρ^k, respectively. The following is immediate from the Serre spectral sequence for the homotopy fibration $\mathrm{hofib}(\rho^k) \to B_k \xrightarrow{\rho^k} \Omega^2\widetilde{C}_k$.

Proposition 6.6. *Suppose that $\rho^k \colon B_k \to \Omega^2 \widetilde{C}_k$ is an isomorphism in cohomology of dimension $< 2k-1+2r_k(p-1)$, then the cohomology of* $\mathrm{hofib}(\rho^k)$ *is given by*

$$\widetilde{H}^*(\mathrm{hofib}(\rho^k)) = \langle a_{2d_k}, a_{2d_k+2(p-1)} \rangle \quad \text{for} \quad * < 2d_k + 4(p-1)$$

such that a_{2n} transgresses to the suspension of c_{n+1} modulo decomposables.

Define a map

$$\Phi_k = a_{2d_k} \oplus a_{2d_k+2(p-1)} \colon [X, \mathrm{hofib}(\rho^k)] \to H^{2d_k}(X) \oplus H^{2d_k+2(p-1)}(X).$$

Corollary 6.7. *Assume the condition of Proposition 6.6. If* $\dim X < 2d_k + 4(p-1)$ *and* $[X, \mathrm{hofib}(\rho^k)]$ *is a free $\mathbb{Z}_{(p)}$-module, then the map Φ_k is an injective homomorphism.*

Proof. By Proposition 6.6, the map $a_{2d_k} \times a_{2d_k+2(p-1)} \colon \mathrm{hofib}(\rho^k) \to K(\mathbb{Z}_{(p)}, 2d_k) \times K(\mathbb{Z}_{(p)}, 2d_k + 2(p-1))$ is a rational $(2d_k + 4(p-1))$-equivalence. Then by $\dim X < 2d_k + 4(p-1)$, Φ_k is an isomorphism after tensoring with $\mathbb{Q}$, so since $[X, \mathrm{hofib}(\rho^k)]$ is a free $\mathbb{Z}_{(p)}$-module, the proof is completed. □

Suppose G is a quasi-p-regular exceptional Lie group. Then by Table 2 and Proposition 4.5, it remains to calculate the Samelson products $\langle \bar{\epsilon}_i, \bar{\epsilon}_j \rangle$ in G for $(G, p, \{i, j\})$ in Table 3.

Table 3:

E_6	$p=5$	$\{2,8\}, \{5,5\}$	$p=7$	$\{2,6\}, \{5,9\}$
E_7	$p=11$	$\{2,8\}, \{2,10\}, \{8,10\}, \{8,14\}$	$p=13$	$\{2,6\}, \{2,12\}, \{6,6\}, \{6,8\}, \{6,12\}$
E_8	$p=11$	$\{2,20\}, \{8,14\}$	$p=13$	$\{2,12\}, \{8,18\}$
	$p=17$	$\{14,20\}$	$p=19$	$\{2,12\}, \{2,18\}, \{12,12\}, \{14,24\}$

We denote the composite of the representation ρ_ℓ and the inclusion $\mathrm{SU}(N_\ell) \to \mathrm{SU}(\infty)$ by the same symbol ρ_ℓ, where $N_\ell = 27, 56, 248$ for $\ell = 6,7,8$. For $(G, p, \{i,j\}) = (\mathrm{E}_8, 19, \{12, 12\})$, it follows from Proposition 5.3 that the condition of Proposition 6.6 does not hold if $k = 24$. However, there is $a_{2i-2} \in H^{2i-2}(\mathrm{hofib}(\rho_8^{24}); \mathbb{Z}_{(p)})$ which transgresses to the suspension of c_i for $i = 42, 60$. Then we define

$$\Phi_{24} = a_{82} \oplus a_{118} \colon [X, \mathrm{hofib}(\rho_8^{24})] \to H^{82}(X; \mathbb{Z}_{(p)}) \oplus H^{118}(X; \mathbb{Z}_{(p)}).$$

Proposition 6.8. *For $i, j \in t_p(G)$, let X be the $(2d_{k(i,j)} + 2(p-1))$-skeleton of $A_i \wedge A_j$, where $k(i,j)$ is the integer k in Corollary 2.6. If $(G, p, \{i,j\})$ is in Table 3, then the map $\Phi_{k(i,j)}$ is an injective homomorphism.*

Proof. We first consider the case $(G, p, \{i,j\}) \neq (\mathrm{E}_8, 19, \{12,12\})$. By Proposition 2.4 and 6.6, we see that $\pi_i(\mathrm{hofib}(\rho_\ell^k)) \cong \mathbb{Z}_{(p)}$ and $\pi_{i+1}(\mathrm{hofib}(\rho_\ell^k)) = 0$ for $i = 2d_k, 2d_k + 2(p-1)$, where $k = k(i,j)$, since they are in the stable range. Then since X consists of cells in dimension $2d_k \bmod 2(p-1)$ and $\dim X \leqslant 2d_k + 2(p-1)$ by definition, we see that $[X, \mathrm{hofib}(\rho_\ell^k)]$ is a free $\mathbb{Z}_{(p)}$-module by skeletal induction. Thus the proof is done by Corollary 6.7.

We next consider the case $(G, p, \{i,j\}) = (\mathrm{E}_8, 19, \{12,12\})$. Since $c_{24}(\rho_8) = 19\lambda x_{48} + \cdots$ for $\lambda \in \mathbb{Z}_{(p)}^\times$, we have

$$H^*(\mathrm{hofib}(\rho_8^{24}); \mathbb{Z}/p) = \mathbb{Z}/p[a_{10}, \mathcal{P}^1 a_{10}, \mathcal{P}^2 a_{10}, \mathcal{P}^3 a_{10}] \otimes \Lambda(\beta\mathcal{P}^1 a_{10}), \quad |a_{10}| = 10$$

for $* < 154$. Let F be the 10-connective cover of $\mathrm{hofib}(\rho_8^{24})$. Then by the Serre spectral sequence of the homotopy fibration $K(\mathbb{Z}_{(p)}, 9) \to F \to \mathrm{hofib}(\rho_8^{24})$ and the Adem relation $\mathcal{P}^1\beta\mathcal{P}^2 = 2\beta\mathcal{P}^3 + \mathcal{P}^3\beta$, we get

$$H^*(F; \mathbb{Z}/p) = \langle b_{82}, \mathcal{P}^1 b_{82}\rangle$$

for $* < 154$. Then quite similarly to the above, we see that $[X, F]$ is a free $\mathbb{Z}_{(p)}$-module. Thus as in the proof of Corollary 6.7, we see that the map

$$\Phi' = b'_{82} \oplus b'_{118} \colon [X, F] \to H^{82}(X; \mathbb{Z}_{(p)}) \oplus H^{118}(X; \mathbb{Z}_{(p)})$$

is injective, where b'_{82}, b'_{118} are mapped to non-zero multiples of $b_{82}, \mathcal{P}^1 b_{82}$ respectively by the mod p reduction. Now by [**16**, Lemma 2.1], we may choose b'_{82}, b'_{118} such that the diagram

$$\begin{tikzcd}
{[X, F]} \arrow[r, "\Phi'"] \arrow[d] & H^{82}(X; \mathbb{Z}_{(p)}) \oplus H^{118}(X; \mathbb{Z}_{(p)}) \arrow[d, "\times 19"] \\
{[X, \mathrm{hofib}(\rho_8^{24})]} \arrow[r, "\Phi_{24}"] & H^{82}(X; \mathbb{Z}_{(p)}) \oplus H^{118}(X; \mathbb{Z}_{(p)})
\end{tikzcd}$$

commutes. Obviously, the left vertical arrow is an isomorphism. Then we obtain that the map Φ_{24} is injective, completing the proof. $\square$

7. Representations and Samelson products

Consider an H-map $\rho\colon G \to \mathrm{SU}(\infty)$. Then there is an exact sequence

$$\widetilde{K}(X) \cong [X, \Omega\mathrm{SU}(\infty)] \xrightarrow{\delta} [X, \mathrm{hofib}(\rho)] \to [X, G] \xrightarrow{\rho_*} [X, \mathrm{SU}(\infty)].$$

Suppose that $X = A \wedge B$ and consider the Samelson product $\langle\alpha, \beta\rangle$ in G of maps $\alpha\colon A \to G, \beta\colon B \to G$. Since $\mathrm{SU}(\infty)$ is homotopy commutative we have $\rho_*(\langle\alpha,\beta\rangle) = 0$, so there is $\gamma \in [X, \mathrm{hofib}(\rho)]$ which maps to $\langle\alpha,\beta\rangle$. Then we get:

Lemma 7.1. *The Samelson product $\langle\alpha,\beta\rangle$ is trivial if and only if $\gamma \in \mathrm{Im}\,\delta$.*

This simple criterion is considered by Hamanaka and Kono [**3**] in the case that ρ is the inclusion $\mathrm{SU}(n) \to \mathrm{SU}(\infty)$ for which $\mathrm{hofib}(\rho)$ is explicitly given by $\Omega\mathrm{SU}(\infty)/\mathrm{SU}(n)$. We apply Lemma 7.1 to determine (non-)triviality of the remaining Samelson products. Assume that $(G, p, \{i,j\})$ is in Table 3. Put X to be the $(2d_k + 2(p-1))$-skeleton of $A_i \wedge A_j$. Then by Corollary 2.6, there is only one $k(i,j)$ such that

$$[X, \mathrm{hofib}(\rho_\ell)] \cong [X, \mathrm{hofib}(\rho_\ell^{k(i,j)})]$$

and by Corollary 6.7 and Proposition 6.8, we have identified the homotopy set on the right hand side. Quite similarly to [**3**, Proposition 3.1], we can prove the following.

Proposition 7.2. *In the situation of Proposition 6.8, we have*

$$\Phi_{k(i,j)} \circ \delta = d_{k(i,j)}!\mathrm{ch}_{d_{k(i,j)}} \oplus (d_{k(i,j)} + p - 1)!\mathrm{ch}_{d_{k(i,j)}+p-1}.$$

Let us calculate the image of $\Phi_{k(i,j)}\circ\delta$ explicitly. To choose generators of $\widetilde{K}(\Sigma A_i)$, we calculate the Chern character of the restriction $\bar\rho_\ell\colon \Sigma A\to B\mathrm{SU}(\infty)$ of $B\rho_\ell\colon BG\to B\mathrm{SU}(\infty)$. Note that

$$\mathrm{ch}_n(\bar\rho_\ell)=\frac{(-1)^{n-1}}{(n-1)!}\iota^*(c_n(\rho_\ell))$$

by (3) where $\iota\colon \Sigma A\to BG$ denotes the inclusion. Then by Proposition 5.3, we have:

$$\begin{aligned}
\mathrm{ch}(\bar\rho_6)&=3\Sigma y_3+\frac{12}{4!}\Sigma y_9-\frac{1}{2\cdot 5!}\Sigma y_{11}+\frac{5}{8\cdot 7!}\Sigma y_{15}+\frac{14}{8!}\Sigma y_{17}-\frac{5}{32\cdot 11!}\Sigma y_{23},\\
\mathrm{ch}(\bar\rho_7)&=6\Sigma y_3-\frac{1}{5!}\Sigma y_{11}+\frac{5}{4\cdot 7!}\Sigma y_{15}+\frac{126}{9!}\Sigma y_{19}-\frac{9}{2\cdot 11!}\Sigma y_{23}+\frac{319}{40\cdot 13!}\Sigma y_{27}\\
&\quad-\frac{1229}{60\cdot 17!}\Sigma y_{35},\\
\mathrm{ch}(\bar\rho_8)&=30\Sigma y_3+\frac{15}{7!}\Sigma y_{15}+\frac{126}{11!}\Sigma y_{23}+\frac{165}{13!}\Sigma y_{27}+\frac{1820}{17!}\Sigma y_{35}+\frac{23001}{5\cdot 19!}\Sigma y_{39}\\
&\quad-\frac{1746822}{5\cdot 23!}\Sigma y_{47}+\frac{15265250}{29!}\Sigma y_{59}.
\end{aligned}$$

Remark 7.3. Our expression of the Chern character of $\bar\rho_6$ differs from that of [**18**] since our choice of generators of $H^*(B\mathrm{E}_6)$ differs from that of [**18**].

We now choose generators of $\widetilde{K}(\Sigma A_i)$. If $r_i=1$, then $\Sigma A_i=S^{2i}$, implying $\widetilde{K}(\Sigma A_i)$ is a free $\mathbb{Z}_{(p)}$-module generated by a single generator η_i such that

$$\mathrm{ch}(\eta_i)=u_{2i},$$

where u_m is a generator of $H^m(S^m)\cong\mathbb{Z}_{(p)}$. If $r_i=2$, then $\Sigma A_i=S^{2i}\cup e^{2i+2p-2}$, so there is a short exact sequence

$$0\to\widetilde{K}(S^{2i+2p-2})\to\widetilde{K}(\Sigma A_i)\to\widetilde{K}(S^{2i})\to 0,\tag{4}$$

where $\widetilde{K}(S^{2n})\cong\mathbb{Z}_{(p)}$. If we put $\xi_i=\bar\rho_\ell|_{\Sigma A_i}$ for i in Table 3, then it is easily verified that

$$\mathrm{ch}(\xi_i)=au_{2i}+\cdots\quad(a\in\mathbb{Z}_{(p)}^\times).$$

So it follows from (4) that $\widetilde{K}(\Sigma A_i)$ is a free $\mathbb{Z}_{(p)}$-module generated by ξ_i and η_i such that

$$\mathrm{ch}(\eta_i)=u_{2i+2(p-1)},$$

where η_i is explicitly given by the composite of the pinch map to the top cell $\Sigma A_i\to S^{2i+2p-2}$ and a generator of $\pi_{2i+2p-2}(B\mathrm{SU}(\infty))\cong\mathbb{Z}_{(p)}$.

Since $\widetilde{K}(A_i)$ is torsion free, we have

$$\widetilde{K}(A_i\wedge A_j)\cong\Sigma^{-2}\widetilde{K}(\Sigma A_i)\otimes\widetilde{K}(\Sigma A_j).$$

Thus we obtain the following by Proposition 7.2.

Lemma 7.4.

1. *The map* $\Phi_{k(i,j)}\circ\delta$ *is surjective for* $(G,p,\{i,j\})=(\mathrm{E}_7,11,\{2,10\})$, $(\mathrm{E}_7,13,\{2,12\})$, $(\mathrm{E}_7,13,\{6,8\})$, $(\mathrm{E}_8,19,\{2,18\})$.

2. *For* $(G,p,(i,j)) = (\mathrm{E}_6, 7, \{2,6\}), (\mathrm{E}_7, 11, \{2,8\}), (\mathrm{E}_7, 11, \{8,10\}), (\mathrm{E}_8, 19, \{2,12\})$, *we have*

$$\operatorname{Im}\Phi_{k(i,j)} \circ \delta \supset p \cdot (H^{2i+2j+2p-4}(A_i \wedge A_j; \mathbb{Z}_{(p)}) \oplus H^{2i+2j+4p-6}(A_i \wedge A_j; \mathbb{Z}_{(p)})).$$

3. $\operatorname{Im}\Phi_{k(i,j)} \circ \delta \mod p$ *is generated by:*

		(i,j)	
E_6	$p=5$	$(2,8)$	$(0,0)$
		$(5,5)$	$(0,*)$
	$p=7$	$(2,6)$	$(3y_3 \otimes y_{23} - y_{15} \otimes y_{11}, 4y_{15} \otimes y_{23})$
		$(5,9)$	$(0,0)$
E_7	$p=11$	$(2,8)$	$(5y_3 \otimes y_{35} - 2y_{23} \otimes y_{15}, 8y_{23} \otimes y_{35})$
		$(8,14)$	$(0,*)$
	$p=13$	$(2,6)$	$(5y_3 \otimes y_{35} - 3y_{27} \otimes y_{11}, *)$
		$(6,6)$	$(4y_{11} \otimes y_{35} + 4y_{35} \otimes y_{11}, -3y_{35} \otimes y_{35})$
		$(6,12)$	$(0,0)$
E_8	$p=11$	$(2,20)$	$(0,0)$
		$(8,14)$	$(0,0)$
	$p=13$	$(2,12)$	$(-y_3 \otimes y_{47} - 3y_{27} \otimes y_{23}, *)$
		$(8,18)$	$(0,0)$
	$p=17$	$(14,20)$	$(0,*)$
	$p=19$	$(2,12)$	$(13y_3 \otimes y_{59} + 9y_{39} \otimes y_{23}, 15y_{39} \otimes y_{59})$
		$(14,24)$	$(0,0)$

Corollary 7.5. *If $i+j=p+1$ and $r_i+r_j=3$ for $i,j \in t_p(G)$, the Samelson product $\langle \bar{\epsilon}_i, \bar{\epsilon}_j \rangle$ is trivial.*

Proof. For $i,j \in t_p(G)$, $i+j=p+1, r_i+r_j=3$ if and only if $(G,p,\{i,j\})$ is one of $(\mathrm{E}_7,11,\{2,10\}), (\mathrm{E}_7,13,\{2,12\}), (\mathrm{E}_7,13,\{6,8\}), (\mathrm{E}_8,19,\{2,18\})$. In these cases, we see from Lemma 7.4 that $\Phi_{k(i,j)}(\gamma)$ must be in $\operatorname{Im}\Phi_{k(i,j)} \circ \delta$, where γ is a lift of $\langle \bar{\epsilon}_i, \bar{\epsilon}_j \rangle$. Thus by Lemma 7.1, we obtain that the Samelson product $\langle \bar{\epsilon}_i, \bar{\epsilon}_j \rangle$ is trivial. □

We choose a lift γ explicitly and calculate $\Phi_{k(i,j)}(\gamma)$ by generalizing a calculation in [**3**]. We may set the map $B\rho_\ell \colon BG \to B\mathrm{SU}(\infty)$ to be a fibration such that its fiber is $B\mathrm{hofib}(\rho_\ell)$. Let $\hat{\gamma}$ be the composite

$$\Sigma A_i \times \Sigma A_j \xrightarrow{\bar{\epsilon}'_i \times \bar{\epsilon}'_j} BG \times BG \xrightarrow{B\rho_\ell \times B\rho_\ell} B\mathrm{SU}(\infty) \times B\mathrm{SU}(\infty) \to B\mathrm{SU}(\infty),$$

where the last arrow is the Whitney sum. Then we have

$$(\hat{\gamma}^*(c_n)) = \sum_{s+t=n} (\bar{\epsilon}'_i)^* \circ B\rho_\ell^*(c_s) \otimes (\bar{\epsilon}'_j)^* \circ B\rho_\ell^*(c_t). \tag{5}$$

Put $n = d_k + 1, d_k + p$ where $k = k(i,j)$. By Proposition 5.3, there exists $d_0 \geqslant 0$ such that $p^{d_0} x_i \in \operatorname{Im} B\rho_\ell^*$ for each generator x_i of $H^*(BG)$. Since $B\rho_\ell^*(c_n)$ has no linear part for degree reasons, there are a quadratic polynomial $Q \in \widetilde{H}^*(B\mathrm{SU}(\infty))^2$ and a higher degree polynomial $R \in \widetilde{H}^*(B\mathrm{SU}(\infty))^3$ such that $B\rho_\ell^*(p^d c_n - Q + R) = 0$ for some d. Note that

$$\hat{\gamma}^*(p^d c_n - Q + R) = \hat{\gamma}^*(p^d c_n - Q) \in \pi^*(H^*(\Sigma A_i \wedge \Sigma A_j)),$$

where $\pi\colon \Sigma A_i \times \Sigma A_j \to \Sigma A_i \wedge \Sigma A_j$ is the pinch map which has the canonical right inverse in cohomology.

By definition, there is a strictly commutative diagram

$$\begin{tikzcd} \Sigma A_i \vee \Sigma A_j \arrow[r] \arrow[d, "\bar{\epsilon}'_i \vee \bar{\epsilon}'_j"] & \Sigma A_i \times \Sigma A_j \arrow[d, "\hat{\gamma}"] \\ BG \arrow[r, "B\rho_\ell"] & B\mathrm{SU}(\infty). \end{tikzcd}$$

Let $\omega\colon \Sigma A_i \wedge A_j \to \Sigma A_i \vee \Sigma A_j$ be the Whitehead product. Since $I_\omega \simeq \Sigma A_i \vee \Sigma A_j$ for the mapping cylinder I_ω of ω, we can apply a homotopy lifting property of the fibration $B\rho_\ell$ to get a commutative diagram

$$\begin{tikzcd} I_\omega \arrow[r] \arrow[d] & \Sigma A_i \times \Sigma A_j \arrow[d, "\hat{\gamma}"] \\ BG \arrow[r, "B\rho_\ell"] & B\mathrm{SU}(\infty), \end{tikzcd}$$

where the left and the upper arrows are equivalent to those of the above diagram and the upper one factors the pinch map $I_\omega \to C_\omega$ to the mapping cone. Then since $B\mathrm{hofib}(\rho_\ell)$ is a fiber of $B\rho_\ell$, we get a strictly commutative diagram

$$\begin{tikzcd} \Sigma A_i \wedge A_j \arrow[r, "\mathrm{incl}"] \arrow[d] & I_\omega \arrow[r] \arrow[d] & \Sigma A_i \times \Sigma A_j \arrow[d, "\hat{\gamma}"] \\ B\mathrm{hofib}(\rho_\ell) \arrow[r] & BG \arrow[r, "B\rho_\ell"] & B\mathrm{SU}(\infty). \end{tikzcd}$$

By adjointness of Whitehead products and Samelson products, we see that the adjoint of the left arrow is a lift of $\langle \bar{\epsilon}_i, \bar{\epsilon}_j \rangle$, so we fix γ to be this map.

The last commutative diagram induces a commutative diagram

$$\begin{tikzcd} H^{2n-1}(\Sigma A_i \wedge A_j) \arrow[r, "\partial"] & H^{2n}(I_\omega, \Sigma A_i \wedge A_j) & H^{2n}(\Sigma A_i \times \Sigma A_j) \arrow[l, "\cong"'] \\ H^{2n-1}(B\mathrm{hofib}(\rho_\ell)) \arrow[r, "\partial"] \arrow[u, "\bar{\gamma}^*"] & H^{2n}(BG, B\mathrm{hofib}(\rho_\ell)) \arrow[u] & H^{2n}(B\mathrm{SU}(\infty)), \arrow[l, "B\rho_\ell^*"'] \arrow[u, "\hat{\gamma}^*"'] \end{tikzcd}$$

where the upper ∂ is identified with the composite

$$H^{2n-1}(\Sigma A_i \wedge A_j) \xrightarrow{\cong} H^{2n}(\Sigma^2 A_i \wedge A_j) \xrightarrow{\pi^*} H^{2n}(\Sigma A_i \times \Sigma A_j)$$

for the projection $\pi\colon \Sigma A_i \times \Sigma A_j \to \Sigma^2 A_i \wedge A_j$. Since there is $e_n \in H^{2n-1}(B\mathrm{hofib}(\rho_\ell))$ which transgresses to $p^d c_n - Q + R$, we have

$$\bar{\gamma}^*(e_n) = \Sigma^{-1} \circ (\pi^*)^{-1}(\hat{\gamma}^*(p^d c_n - Q + R)) = \Sigma^{-1} \circ (\pi^*)^{-1}(\hat{\gamma}^*(p^d c_n - Q)).$$

Consider the Serre spectral sequence of the path-loop fibration of $B\mathrm{SU}(\infty)$. We have that the restriction of e_n transgresses to $p^d c_n$ by naturality. Since the transgression induces an isomorphism between the modules of indecomposables of $H^{2n-1}(\mathrm{SU}(\infty))$ and $H^{2n}(B\mathrm{SU}(\infty))$, the restriction of e_n coincides with $p^d\sigma(c_n)$ where $\sigma(c_n)$ denotes the suspension of c_n. Thus we see that, in the Serre spectral sequence of the path-loop fibration of $B\mathrm{hofib}(\rho_\ell)$, the suspension of e_n is equal to the pullback of $p^d a_{2n-2}$ through the projection $\mathrm{hofib}(\rho_\ell) \to \mathrm{hofib}(\rho_\ell^k)$. Since $H^{2n-2}(A_i \wedge A_j)$ is a free $\mathbb{Z}_{(p)}$-module, we have the following:

$$\gamma^*(a_{2n-2}) = \Sigma^{-2} \circ (\pi^*)^{-1}(\hat{\gamma}^*(p^d c_n - Q))/p^d.$$

Thus by combining this equation with (5), we obtain an explicit description of $\gamma^*(a_{2n-2})$.

We here give an example calculation of $\gamma^*(a_{2n-2})$ for $(G, p, \{i, j\}) = (\mathrm{E}_6, 5, \{5, 5\})$ and $n = 10$. By Proposition 5.3, we have $Q = \frac{1}{16}c_5^2, d = 0$, and then

$$(\pi^*)^{-1}(\hat{\gamma}^*(c_{10} - Q)) = (\pi^*)^{-1}((1 - \tfrac{1}{8})(\epsilon_5)^* \circ \rho_6^*(c_5) \otimes (\epsilon_5')^* \circ \rho_6^*(c_5)) = 126\Sigma^2 y_9 \otimes y_9,$$

so we get $\gamma^*(a_{18}) = 126 y_9 \otimes y_9$. Quite similarly to this calculation, we obtain:

Lemma 7.6. *$\Phi_{k(i,j)}(\gamma) \mod p$ for $(G, p, (i, j))$ is given by:*

		(i,j)	
E_6	$p=5$	$(2,8)$	$(3y_3 \otimes y_{15}, 2y_3 \otimes y_{23} + 3y_{11} \otimes y_{15})$
		$(5,5)$	$(y_9 \otimes y_9, *)$
	$p=7$	$(2,6)$	$(2y_3 \otimes y_{23} + 4y_{15} \otimes y_{11}, 5y_{15} \otimes y_{23})$
		$(5,9)$	$(2y_9 \otimes y_{17}, 0)$
E_7	$p=11$	$(2,8)$	$(7y_3 \otimes y_{35} + 6y_{23} \otimes y_{15}, 9y_{23} \otimes y_{35})$
		$(8,10)$	$(0,0)$
		$(8,14)$	$(-3y_{15} \otimes y_{27}, *)$
	$p=13$	$(2,6)$	$(3y_3 \otimes y_{35}, *)$
		$(6,6)$	$(2y_{11} \otimes y_{35} + 2y_{35} \otimes y_{11}, -2y_{35} \otimes y_{35})$
		$(6,12)$	$(4y_{35} \otimes y_{23}, 0)$
E_8	$p=11$	$(2,20)$	$(5y_3 \otimes y_{39}, -y_3 \otimes y_{59} + y_{23} \otimes y_{39})$
		$(8,14)$	$(7y_{15} \otimes y_{27}, y_{15} \otimes y_{47} + 2y_{35} \otimes y_{27})$
	$p=13$	$(2,12)$	$(-2y_3 \otimes y_{47} - y_{27} \otimes y_{23}, *)$
		$(8,18)$	$(-6y_{15} \otimes y_{35}, -9y_{15} \otimes y_{59} - 4y_{39} \otimes y_{35})$
	$p=17$	$(14,20)$	$(-9y_{27} \otimes y_{39}, *)$
	$p=19$	$(2,12)$	$(10y_3 \otimes y_{59} + 4y_{39} \otimes y_{23}, 13y_{39} \otimes y_{59})$
		$(14,24)$	$(-5y_{27} \otimes y_{47}, *)$

Proposition 7.7.

1. *The Samelson product $\langle \bar{\epsilon}_i, \bar{\epsilon}_j \rangle$ is trivial for $(G, p, \{i, j\}) = (\mathrm{E}_6, 7, \{2, 6\})$, $(\mathrm{E}_7, 11, \{2, 8\})$, $(\mathrm{E}_7, 11, \{8, 10\})$, $(\mathrm{E}_8, 19, \{2, 12\})$.*
2. *For the other $(G, p, \{i, j\})$ in the table of Lemma 7.6, $\langle \bar{\epsilon}_i, \bar{\epsilon}_j \rangle$ is non-trivial.*

Proof. By Lemma 7.4 and 7.6, we see that $\Phi_{k(i,j)}(\gamma) \in \operatorname{Im} \Phi_{k(i,j)} \circ \delta$ in the case of (1) and that $\Phi_{k(i,j)}(\gamma) \notin \operatorname{Im} \Phi_{k(i,j)} \circ \delta$ in the case of (2). Then by Proposition 6.8, the proof is completed. $\square$

Proposition 7.8. *The Samelson product $\langle \bar{\epsilon}_{12}, \bar{\epsilon}_{12} \rangle$ in E_8 at $p = 19$ is trivial.*

Proof. As above, we see that

$$\Phi_{24} \circ \delta(\xi_{12} \otimes \xi_{12}) \equiv 57(y_{23} \otimes y_{59} + y_{59} \otimes y_{23}) + 209 y_{59} \otimes y_{59},$$
$$\Phi_{24}(\gamma) \equiv 228(y_{23} \otimes y_{59} + y_{59} \otimes y_{23}) + 114 y_{59} \otimes y_{59}$$

modulo 19^2. Similarly to Lemma 7.4, we see that $\mathrm{Im}\,\Phi_{24} \circ \delta$ includes $19^2 y_{23} \otimes y_{59}$, $19^2 y_{59} \otimes y_{23}$, $19^2 y_{59} \otimes y_{59}$, so $\Phi_{24}(\gamma) \in \mathrm{Im}\,\Phi_{24} \circ \delta$. Thus the proof is completed by Proposition 6.8. □

Proof of Theorem 1.2. Combine Corollary 2.7 and Propositions 2.8, 4.5, Corollary 7.5, Propositions 7.7 and 7.8. □

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