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A HOMOTOPY DECOMPOSITION OF THE FIBRE OF THE SQUARING MAP ON $\Omega^3 S^{17}$

STEVEN AMELOTTE

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Abstract

We use Richter's 2-primary proof of Gray's conjecture to give a homotopy decomposition of the fibre $\Omega^3 S^{17}\{2\}$ of the H-space squaring map on the triple loop space of the 17-sphere. This induces a splitting of the mod-2 homotopy groups $\pi_*(S^{17}; \mathbb{Z}/2\mathbb{Z})$ in terms of the integral homotopy groups of the fibre of the double suspension $E^2 \colon S^{2n-1} \to \Omega^2 S^{2n+1}$ and refines a result of Cohen and Selick, who gave similar decompositions for S^5 and S^9. We relate these decompositions to various Whitehead products in the homotopy groups of mod-2 Moore spaces and Stiefel manifolds to show that the Whitehead square $[i_{2n}, i_{2n}]$ of the inclusion of the bottom cell of the Moore space $P^{2n+1}(2)$ is divisible by 2 if and only if $2n = 2, 4, 8$ or 16.

1. Introduction

For a based loop space ΩX, let $\Omega X\{k\}$ denote the homotopy fibre of the kth power map $k \colon \Omega X \to \Omega X$. In [**14**] and [**15**], Selick showed that after localizing at an odd prime p, there is a homotopy decomposition $\Omega^2 S^{2p+1}\{p\} \simeq \Omega^2 S^3\langle 3\rangle \times W_p$, where $S^3\langle 3\rangle$ is the 3-connected cover of S^3 and W_n is the homotopy fibre of the double suspension $E^2 \colon S^{2n-1} \to \Omega^2 S^{2n+1}$. Since $\Omega^2 S^{2p+1}\{p\}$ is homotopy equivalent to the pointed mapping space $\mathrm{Map}_*(P^3(p), S^{2p+1})$ and the degree p map on the Moore space $P^3(p)$ is nullhomotopic, an immediate consequence is that p annihilates the p-torsion in $\pi_*(S^3)$ when p is odd. In [**16**], Ravenel's solution to the odd primary Arf-Kervaire invariant problem [**12**] was used to show that, at least for $p \geqslant 5$, similar decompositions of $\Omega^2 S^{2n+1}\{p\}$ are not possible if $n \neq 1$ or p.

The 2-primary analogue of Selick's decomposition, namely that there is a 2-local homotopy equivalence $\Omega^2 S^5\{2\} \simeq \Omega^2 S^3\langle 3\rangle \times W_2$, was later proved by Cohen [**4**]. Similarly, since $\Omega^2 S^5\{2\}$ is homotopy equivalent to $\mathrm{Map}_*(P^3(2), S^5)$ and the degree 4 map on $P^3(2) \simeq \Sigma\mathbb{R}P^2$ is nullhomotopic, this product decomposition gives a "geometric" proof of James' classical result that 4 annihilates the 2-torsion in $\pi_*(S^3)$. Unlike the odd primary case, however, for reasons related to the divisibility of the Whitehead square $[\iota_{2n-1}, \iota_{2n-1}] \in \pi_{4n-3}(S^{2n-1})$, the fibre of the squaring map on $\Omega^2 S^{2n+1}$ admits nontrivial product decompositions for some other values of n.

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First, in their investigation of the homology of spaces of maps from mod-2 Moore spaces to spheres, Campbell, Cohen, Peterson and Selick [**1**] found that if $2n+1 \neq 3, 5, 9$ or 17, then $\Omega^2 S^{2n+1}\{2\}$ is atomic and hence indecomposable. Following this, it was shown in [**5**] that after localization at the prime 2, there is a homotopy decomposition $\Omega^2 S^9\{2\} \simeq BW_2 \times W_4$ and W_4 is a retract of $\Omega^3 S^{17}\{2\}$. Here BW_n denotes the classifying space of W_n first constructed by Gray [**6**]. Since BW_1 is known to be homotopy equivalent to $\Omega^2 S^3\langle 3\rangle$, the pattern suggested by the decompositions of $\Omega^2 S^5\{2\}$ and $\Omega^2 S^9\{2\}$ led Cohen and Selick to conjecture that $\Omega^2 S^{17}\{2\} \simeq BW_4 \times W_8$. In this note we prove this is true after looping once. (This weaker statement was also conjectured in [**3**].)

Theorem 1.1. *There is a* 2*-local homotopy equivalence* $\Omega^3 S^{17}\{2\} \simeq W_4 \times \Omega W_8$.

In addition to the exponent results mentioned above, decompositions of $\Omega^m S^{2n+1}\{p\}$ also give decompositions of homotopy groups of spheres with $\mathbb{Z}/p\mathbb{Z}$ coefficients. Recall that the mod-p homotopy groups of X are defined by $\pi_k(X;\mathbb{Z}/p\mathbb{Z}) = [P^k(p), X]$.

Corollary 1.2. $\pi_k(S^{17};\mathbb{Z}/2\mathbb{Z}) \cong \pi_{k-4}(W_4) \oplus \pi_{k-3}(W_8)$ *for all* $k \geqslant 4$.

In Section 3 we relate the problem of decomposing $\Omega^2 S^{2n+1}\{2\}$ to a problem considered by Mukai and Skopenkov in [**11**] of computing a certain summand in a homotopy group of the mod-2 Moore space $P^{2n+1}(2)$—more specifically, the problem of determining when the Whitehead square $[i_{2n}, i_{2n}] \in \pi_{4n-1}(P^{2n+1}(2))$ of the inclusion of the bottom cell $i_{2n}\colon S^{2n} \to P^{2n+1}(2)$ is divisible by 2. The indecomposability result for $\Omega^2 S^{2n+1}\{2\}$ in [**1**] (see also [**2**]) was proved by showing that for $n > 1$ the existence of a spherical homology class in $H_{4n-3}(\Omega^2 S^{2n+1}\{2\})$ imposed by a nontrivial product decomposition implies the existence of an element $\theta \in \pi^S_{2n-2}$ of Kervaire invariant one such that $\theta\eta$ is divisible by 2, where η is the generator of the stable 1-stem π^S_1. Such elements are known to exist only for $2n = 4, 8$ or 16. We show that the divisibility of the Whitehead square $[i_{2n}, i_{2n}]$ similarly implies the existence of such Kervaire invariant elements to obtain the following.

Theorem 1.3. *The Whitehead square* $[i_{2n}, i_{2n}] \in \pi_{4n-1}(P^{2n+1}(2))$ *is divisible by* 2 *if and only if* $2n = 2, 4, 8$ *or* 16.

This will follow from a preliminary result (Proposition 3.1) equating the divisibility of $[i_{2n}, i_{2n}]$ with the vanishing of a Whitehead product in the mod-2 homotopy of the Stiefel manifold $V_{2n+1,2}$, i.e., the unit tangent bundle over S^{2n}. It is shown in [**17**] that there do not exist maps $S^{2n-1} \times P^{2n}(2) \to V_{2n+1,2}$ extending the inclusions of the bottom cell S^{2n-1} and bottom Moore space $P^{2n}(2)$ if $2n \neq 2, 4, 8$ or 16. When $2n = 2, 4$ or 8, the Whitehead product obstructing an extension is known to vanish for reasons related to Hopf invariant one, leaving only the boundary case $2n = 16$ unresolved. We find that the Whitehead product is also trivial in this case.

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2. The decomposition of $\Omega^3 S^{17}\{2\}$

The proof of Theorem 1.1 will make use of the 2-primary version of Richter's proof of Gray's conjecture, so we begin by reviewing this conjecture and spelling out some of its consequences. In his construction of a classifying space of the fibre W_n of the double suspension, Gray [**6**] introduced two p-local homotopy fibrations

$$S^{2n-1} \xrightarrow{E^2} \Omega^2 S^{2n+1} \xrightarrow{\nu} BW_n,$$

$$BW_n \xrightarrow{j} \Omega^2 S^{2np+1} \xrightarrow{\phi} S^{2np-1},$$

with the property that $j \circ \nu \simeq \Omega H$, where $H\colon \Omega S^{2n+1} \to \Omega S^{2np+1}$ is the pth James-Hopf invariant. In addition, Gray showed that the composite $BW_n \xrightarrow{j} \Omega^2 S^{2np+1} \xrightarrow{p} \Omega^2 S^{2np+1}$ is nullhomotopic and conjectured that the composite $\Omega^2 S^{2np+1} \xrightarrow{\phi} S^{2np-1} \xrightarrow{E^2} \Omega^2 S^{2np+1}$ is homotopic to the pth power map on $\Omega^2 S^{2np+1}$. This was recently proved by Richter in [**13**].

Theorem 2.1 ([**13**]). *For any prime p, there is a homotopy fibration*

$$BW_n \xrightarrow{j} \Omega^2 S^{2np+1} \xrightarrow{\phi_n} S^{2np-1}$$

such that $E^2 \circ \phi_n \simeq p$.

For odd primes, it was shown in [**21**] that there is a homotopy fibration $\Omega W_{np} \to BW_n \to \Omega^2 S^{2np+1}\{p\}$ based on the fact that a lift $\overline{S}\colon BW_n \to \Omega^2 S^{2np+1}\{p\}$ of j can be chosen to be an H-map when p is odd. One consequence of Theorem 2.1 is that this homotopy fibration exists for all primes and can be extended one step to the right by a map $\Omega^2 S^{2np+1}\{p\} \to W_{np}$.

Lemma 2.2. *For any prime p, there is a homotopy fibration*

$$BW_n \longrightarrow \Omega^2 S^{2np+1}\{p\} \longrightarrow W_{np}.$$

Proof. The homotopy pullback of ϕ_n and the fibre inclusion $W_{np} \to S^{2np-1}$ of the double suspension defines a map $\Omega^2 S^{2np+1}\{p\} \to W_{np}$ with homotopy fibre BW_n, which can be seen by comparing fibres in the homotopy pullback diagram

$$\begin{array}{ccccc}
BW_n & \longrightarrow & \Omega^2 S^{2np+1}\{p\} & \longrightarrow & W_{np} \\
\| & & \downarrow & & \downarrow \\
BW_n & \xrightarrow{j} & \Omega^2 S^{2np+1} & \xrightarrow{\phi_n} & S^{2np-1} \\
 & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle E^2} \\
 & & \Omega^2 S^{2np+1} & = & \Omega^2 S^{2np+1}.
\end{array} \tag{1}$$

□

Looping once, we obtain a homotopy fibration

$$W_n \longrightarrow \Omega^3 S^{2np+1}\{p\} \longrightarrow \Omega W_{np},$$

which we will show is split when $p = 2$ and $n = 4$. We now fix $p = 2$ and localize all spaces and maps at the prime 2. Homology will be taken with mod-2 coefficients unless otherwise stated.

The next lemma describes a factorization of the looped second James-Hopf invariant, an odd primary version of which appears in [**21**]. By a well-known result due to Barratt, $\Omega H\colon \Omega^2 S^{2n+1} \to \Omega^2 S^{4n+1}$ has order 2 in the group $[\Omega^2 S^{2n+1}, \Omega^2 S^{4n+1}]$ and hence lifts to a map $\Omega^2 S^{2n+1} \to \Omega^2 S^{4n+1}\{2\}$. Improving on this, a feature of Richter's construction of the map ϕ_n is that the composite $\Omega^2 S^{2n+1} \xrightarrow{\Omega H} \Omega^2 S^{4n+1} \xrightarrow{\phi_n} S^{4n-1}$ is nullhomotopic [**13**, Lemma 4.2]. This recovers Gray's fibration $S^{2n-1} \xrightarrow{E^2} \Omega^2 S^{2n+1} \xrightarrow{\nu} BW_n$ and the relation $j \circ \nu \simeq \Omega H$ since there then exists a lift $\nu\colon \Omega^2 S^{2n+1} \to BW_n$ making the diagram

$$\begin{array}{ccc} & & BW_n \\ & \overset{\nu}{\nearrow} & \big\downarrow j \\ \Omega^2 S^{2n+1} & \xrightarrow{\Omega H} & \Omega^2 S^{4n+1} \end{array}$$

commute up to homotopy. Since j factors through $\Omega^2 S^{4n+1}\{2\}$, by composing the lift ν with the map $BW_n \to \Omega^2 S^{4n+1}\{2\}$ we obtain a choice of lift $S\colon \Omega^2 S^{2n+1} \to \Omega^2 S^{4n+1}\{2\}$ of the looped James-Hopf invariant. Hence we have the following consequence of Richter's theorem.

Lemma 2.3. *There is a homotopy commutative diagram*

$$\begin{array}{ccc} \Omega^2 S^{2n+1} & \xrightarrow{S} & \Omega^2 S^{4n+1}\{2\} \\ \big\downarrow \nu & & \big\| \\ BW_n & \longrightarrow & \Omega^2 S^{4n+1}\{2\}, \end{array}$$

where S is a lift of the looped second James-Hopf invariant $\Omega H\colon \Omega^2 S^{2n+1} \to \Omega^2 S^{4n+1}$ and the map $BW_n \to \Omega^2 S^{4n+1}\{2\}$ has homotopy fibre ΩW_{2n}.

The following homological result was proved in [**1**] and used to obtain the homotopy decompositions of [**4**] and [**5**].

Lemma 2.4 ([**1**])**.** *Let $n \geqslant 2$ and let $f\colon X \to \Omega^2 S^{2n+1}\{2\}$ be a map which induces an isomorphism on the module of primitives in degrees $2n-2$ and $4n-3$. If the mod-2 homology of X is isomorphic to that of $\Omega^2 S^{2n+1}\{2\}$ as a coalgebra over the Steenrod algebra, then f is a homology isomorphism.*

Theorem 2.5. *There is a homotopy equivalence $\Omega^3 S^{17}\{2\} \simeq W_4 \times \Omega W_8$.*

Proof. Let τ_n denote the map $BW_n \to \Omega^2 S^{4n+1}\{2\}$ appearing in Lemma 2.2. By (1), τ_n is a lift of j, implying that τ_n is nonzero in $H_{4n-2}(\ )$ by naturality of the Bockstein since j is nonzero in $H_{4n-1}(\ )$. We can therefore use the maps τ_n in place of the (potentially different) maps σ_n used in [**5**] to obtain product decompositions of $\Omega^2 S^{4n+1}\{2\}$ for $n = 1$ and 2, the advantage being that τ_n has fibre ΩW_{2n}. Explicitly, for $n = 2$ this is done as follows. By [**5**, Corollary 2.1], there exists a map $g\colon \Omega^3 S^{17}\{2\} \to \Omega^2 S^9\{2\}$ which is nonzero in $H_{13}(\ )$. Letting μ denote the loop multiplication on $\Omega^2 S^9\{2\}$, it follows that the composite

$$\psi\colon BW_2 \times W_4 \xrightarrow{\tau_2 \times (g \circ \Omega\tau_4)} \Omega^2 S^9\{2\} \times \Omega^2 S^9\{2\} \xrightarrow{\mu} \Omega^2 S^9\{2\}$$

induces an isomorphism on the module of primitives in degrees 6 and 13. Since

$H_*(BW_2 \times W_4)$ and $H_*(\Omega^2 S^9\{2\})$ are isomorphic as coalgebras over the Steenrod algebra, the map above is a homology isomorphism by Lemma 2.4 and hence a homotopy equivalence.

Now the map $\Omega\tau_4$ fits in the homotopy fibration

$$W_4 \xrightarrow{\Omega\tau_4} \Omega^3 S^{17}\{2\} \longrightarrow \Omega W_8$$

and has a left homotopy inverse given by $\pi_2 \circ \psi^{-1} \circ g$ where ψ^{-1} is a homotopy inverse of ψ and $\pi_2\colon BW_2 \times W_4 \to W_4$ is the projection onto the second factor. (Alternatively, composing $g\colon \Omega^3 S^{17}\{2\} \to \Omega^2 S^9\{2\}$ with the map $\Omega^2 S^9\{2\} \to W_4$ of Lemma 2.2 yields a left homotopy inverse of $\Omega\tau_4$.) It follows that the homotopy fibration above is fibre homotopy equivalent to the trivial fibration $W_4 \times \Omega W_8 \to \Omega W_8$. □

Corollary 2.6. $\pi_k(S^{17}; \mathbb{Z}/2\mathbb{Z}) \cong \pi_{k-4}(W_4) \oplus \pi_{k-3}(W_8)$ *for all* $k \geqslant 4$.

One consequence of the splitting of the fibration $W_n \to \Omega^3 S^{4n+1}\{p\} \to \Omega W_{2n}$ when $n \in \{1, 2, 4\}$ is a corresponding homotopy decomposition of the fibre of the map S appearing in Lemma 2.3. As in [**18**], we define the space Y and the map t by the homotopy fibration

$$Y \xrightarrow{t} \Omega^2 S^{2n+1} \xrightarrow{S} \Omega^2 S^{4n+1}\{2\}.$$

This space and its odd primary analogue play a central role in the construction of Anick's fibration in [**18, 21**] and the alternative proof given in [**20**] of Cohen, Moore and Neisendorfer's determination of the odd primary homotopy exponent of spheres. Unlike at odd primes, the lift S of ΩH cannot be chosen to be an H-map. Nevertheless, the corollary below shows that its fibre has the structure of an H-space in cases of Hopf invariant one.

Corollary 2.7. *There is a homotopy fibration* $S^{2n-1} \xrightarrow{f} Y \xrightarrow{g} \Omega W_{2n}$ *with the property that the composite* $S^{2n-1} \xrightarrow{f} Y \xrightarrow{t} \Omega^2 S^{2n+1}$ *is homotopic to the double suspension* E^2. *Moreover, if* $n = 1, 2$ *or* 4 *then the fibration splits, giving a homotopy equivalence*

$$Y \simeq S^{2n-1} \times \Omega W_{2n}.$$

Proof. By Lemma 2.3, the homotopy fibration defining Y fits in a homotopy pullback diagram

$$\begin{array}{ccccc}
S^{2n-1} & = & S^{2n-1} & & \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle E^2} & & \\
Y & \xrightarrow{t} & \Omega^2 S^{2n+1} & \xrightarrow{S} & \Omega^2 S^{4n+1}\{2\} \\
\downarrow{\scriptstyle g} & & \downarrow{\scriptstyle \nu} & & \| \\
\Omega W_{2n} & \longrightarrow & BW_n & \longrightarrow & \Omega^2 S^{4n+1}\{2\},
\end{array}$$

which proves the first statement. Note that when $n = 1, 2$ or 4, the map $\Omega W_{2n} \to BW_n$ is nullhomotopic by Theorem 1.1, hence t lifts through the double suspension. Since any choice of a lift $Y \to S^{2n-1}$ is degree one in $H_{2n-1}(\;)$, it also serves as a left homotopy inverse of f, which implies the asserted splitting. □

Remark 2.8. The first part of Corollary 2.7 and an odd primary version are proved by different means in [**18**] and [**20**], respectively (see Remark 6.2 of [**18**]). At odd primes, there is an analogous splitting for $n = 1$:

$$Y \simeq S^1 \times \Omega W_p \simeq S^1 \times \Omega^3 T^{2p^2+1}(p),$$

where $T^{2p^2+1}(p)$ is Anick's space (see [**19**]).

3. Relations to Whitehead products in Moore spaces and Stiefel manifolds

The special homotopy decompositions of $\Omega^3 S^{2n+1}\{2\}$ discussed in the previous section are made possible by the existence of special elements in the stable homotopy groups of spheres, namely elements of Arf-Kervaire invariant one $\theta \in \pi^S_{2n-2}$ such that $\theta\eta$ is divisible by 2. In this section, we give several reformulations of the existence of such elements in terms of mod-2 Moore spaces and Stiefel manifolds.

Let $i_{n-1}\colon S^{n-1} \to P^n(2)$ be the inclusion of the bottom cell and let $j_n\colon P^n(2) \to P^n(2)$ be the identity map. Similarly, let $i'_{2n-1}\colon S^{2n-1} \to V_{2n+1,2}$ and $j'_{2n}\colon P^{2n}(2) \to V_{2n+1,2}$ denote the inclusions of the bottom cell and bottom Moore space, respectively.[1]

Proposition 3.1. *The Whitehead product $[i'_{2n-1}, j'_{2n}] \in \pi_{4n-2}(V_{2n+1,2}; \mathbb{Z}/2\mathbb{Z})$ is trivial if and only if the Whitehead square $[i_{2n}, i_{2n}] \in \pi_{4n-1}(P^{2n+1}(2))$ is divisible by 2.*

Proof. Let $\lambda\colon S^{4n-2} \to P^{2n}(2)$ denote the attaching map of the top cell in $V_{2n+1,2} \simeq P^{2n}(2) \cup_\lambda e^{4n-1}$ and note that $[i'_{2n-1}, j'_{2n}] = j'_{2n} \circ [i_{2n-1}, j_{2n}]$ by naturality of the Whitehead product. The map $[i_{2n-1}, j_{2n}]\colon P^{4n-2}(2) \to P^{2n}(2)$ is essential since its adjoint is a Samelson product with nontrivial Hurewicz image $[u, v] \in H_{4n-3}(\Omega P^{2n}(2))$, where $H_*(\Omega P^{2n}(2))$ is isomorphic as an algebra to the tensor algebra $T(u, v)$ with $|u| = 2n-2$ and $|v| = 2n-1$ by the Bott-Samelson theorem. Since the homotopy fibre of the inclusion $j'_{2n}\colon P^{2n}(2) \to V_{2n+1,2}$ has $(4n-2)$-skeleton S^{4n-2} which maps into $P^{2n}(2)$ by the attaching map λ, it follows that $[i'_{2n-1}, j'_{2n}]$ is trivial if and only if $[i_{2n-1}, j_{2n}]$ is homotopic to the composite

$$P^{4n-2}(2) \xrightarrow{q} S^{4n-2} \xrightarrow{\lambda} P^{2n}(2),$$

where q is the pinch map.

To ease notation let P^n denote the mod-2 Moore space $P^n(2)$ and consider the morphism of EHP sequences

$$\begin{array}{ccccccc}
[S^{4n}, P^{2n+1}] & \xrightarrow{H} & [S^{4n}, \Sigma P^{2n} \wedge P^{2n}] & \xrightarrow{P} & [S^{4n-2}, P^{2n}] & \xrightarrow{E} & [S^{4n-1}, P^{2n+1}] \\
\downarrow{\scriptstyle q^*} & & \downarrow{\scriptstyle q^*} & & \downarrow{\scriptstyle q^*} & & \downarrow{\scriptstyle q^*} \\
[P^{4n}, P^{2n+1}] & \xrightarrow{H} & [P^{4n}, \Sigma P^{2n} \wedge P^{2n}] & \xrightarrow{P} & [P^{4n-2}, P^{2n}] & \xrightarrow{E} & [P^{4n-1}, P^{2n+1}]
\end{array}$$

induced by the pinch map. A homology calculation shows that the $(4n)$-skeleton of $\Sigma P^{2n} \wedge P^{2n}$ is homotopy equivalent to $P^{4n} \vee S^{4n}$. Let $k_1\colon P^{4n} \to \Sigma P^{2n} \wedge P^{2n}$ and

[1] Note that we index these maps by the dimension of their source rather than their target, so the element of $\pi_{4n-1}(P^{2n+1}(2))$ we call $[i_{2n}, i_{2n}]$ is called $[i_{2n+1}, i_{2n+1}]$ in [**11**].

$k_2\colon S^{4n} \to \Sigma P^{2n} \wedge P^{2n}$ be the composites

$$P^{4n} \hookrightarrow P^{4n} \vee S^{4n} \simeq \mathrm{sk}_{4n}(\Sigma P^{2n} \wedge P^{2n}) \hookrightarrow \Sigma P^{2n} \wedge P^{2n}$$

and

$$S^{4n} \hookrightarrow P^{4n} \vee S^{4n} \simeq \mathrm{sk}_{4n}(\Sigma P^{2n} \wedge P^{2n}) \hookrightarrow \Sigma P^{2n} \wedge P^{2n}$$

defined by the left and right wedge summand inclusions, respectively. Then we have that $\pi_{4n}(\Sigma P^{2n} \wedge P^{2n}) = \mathbb{Z}/4\mathbb{Z}\{k_2\}$ and $P(k_2) = \pm 2\lambda$ by [**9**, Lemma 12]. It follows from the universal coefficient exact sequence

$$0 \longrightarrow \pi_{4n}(\Sigma P^{2n} \wedge P^{2n}) \otimes \mathbb{Z}/2\mathbb{Z} \longrightarrow \pi_{4n}(\Sigma P^{2n} \wedge P^{2n}; \mathbb{Z}/2\mathbb{Z}) \longrightarrow \mathrm{Tor}(\pi_{4n-1}(\Sigma P^{2n} \wedge P^{2n}), \mathbb{Z}/2\mathbb{Z}) \longrightarrow 0$$

that

$$\begin{aligned}\pi_{4n}(\Sigma P^{2n} \wedge P^{2n}; \mathbb{Z}/2\mathbb{Z}) &= [P^{4n}, \Sigma P^{2n} \wedge P^{2n}] \\ &= \mathbb{Z}/2\mathbb{Z}\{k_1\} \oplus \mathbb{Z}/2\mathbb{Z}\{k_2 \circ q\}\end{aligned}$$

and that the generator $k_2 \circ q$ is in the kernel of P since $P(k_2) = \pm 2\lambda$ implies

$$P(k_2 \circ q) = P(q^*(k_2)) = q^*(P(k_2)) = \pm \lambda \circ 2 \circ q = 0$$

by the commutativity of the above diagram and the fact that $q\colon P^{4n-2} \to S^{4n-2}$ and $2\colon S^{4n-2} \to S^{4n-2}$ are consecutive maps in a cofibration sequence. Therefore $[i_{2n-1}, j_{2n}] = P(k_1)$ since the suspension of a Whitehead product is trivial. On the other hand, $\Sigma\lambda$ is homotopic to the composite $S^{4n-1} \xrightarrow{[\iota_{2n},\iota_{2n}]} S^{2n} \xrightarrow{i_{2n}} P^{2n+1}$ by [**9**], which implies $E(\lambda \circ q) = i_{2n} \circ [\iota_{2n}, \iota_{2n}] \circ q = [i_{2n}, i_{2n}] \circ q$ is trivial in $[P^{4n-1}, P^{2n+1}]$ precisely when $[i_{2n}, i_{2n}]$ is divisible by 2. Hence $[i_{2n}, i_{2n}]$ is divisible by 2 if and only if $\lambda \circ q = P(k_1) = [i_{2n-1}, j_{2n}] \in [P^{4n-2}, P^{2n}]$, and the proposition follows. ☐

We use Proposition 3.1 in two ways. First, since the calculation of $\pi_{31}(P^{17}(2))$ in [**10**] shows that $[i_{16}, i_{16}] = 2\widetilde{\sigma}_{16}^2$ for a suitable choice of representative $\widetilde{\sigma}_{16}^2$ of the Toda bracket $\{\sigma_{16}^2, 2\iota_{16}, i_{16}\}$, it follows that the Whitehead product $[i'_{15}, j'_{16}]\colon P^{30}(2) \to V_{17,2}$ is nullhomotopic and hence there exists a map $S^{15} \times P^{16}(2) \to V_{17,2}$ extending the wedge of skeletal inclusions $S^{15} \vee P^{16}(2) \to V_{17,2}$. This resolves the only case left unsettled by Theorem 3.2 of [**17**].

In the other direction, note that such maps $S^{2n-1} \times P^{2n}(2) \to V_{2n+1,2}$ restrict to maps $S^{2n-1} \times S^{2n-1} \to V_{2n+1,2}$ which exist only in cases of Kervaire invariant one by [**22**, Proposition 2.27], so Proposition 3.1 shows that when $2n \neq 2^k$ for some $k \geqslant 1$ the Whitehead square $[i_{2n}, i_{2n}]$ cannot be divisible by 2 for the same reasons that the Whitehead square $[\iota_{2n-1}, \iota_{2n-1}] \in \pi_{4n-3}(S^{2n-1})$ cannot be divisible by 2. Moreover, since maps $S^{2n-1} \times P^{2n}(2) \to V_{2n+1,2}$ extending the inclusions of S^{2n-1} and $P^{2n}(2)$ are shown not to exist for $2n > 16$ in [**17**], Proposition 3.1 implies that the Whitehead square $[i_{2n}, i_{2n}]$ is divisible by 2 if and only if $2n = 2, 4, 8$ or 16. In all other cases it generates a $\mathbb{Z}/2\mathbb{Z}$ summand in $\pi_{4n-1}(P^{2n+1}(2))$. This improves on the main theorem of [**11**] which shows by other means that $[i_{2n}, i_{2n}]$ is not divisible by 2 when $2n$ is not a power of 2.

These results are summarized in Theorem 3.3 below. First we recall the following well-known equivalent formulations of the Kervaire invariant problem.

Theorem 3.2 ([**2, 22**]). *The following are equivalent:*

(a) *The Whitehead square* $[\iota_{2n-1}, \iota_{2n-1}] \in \pi_{4n-3}(S^{2n-1})$ *is divisible by* 2;

(b) *There is a map* $P^{4n-2}(2) \to \Omega S^{2n}$ *which is nonzero in homology;*

(c) *There exists a space* X *with mod-2 cohomology* $\widetilde{H}^i(X) \cong \mathbb{Z}/2\mathbb{Z}$ *for* $i = 2n$, $4n-1$ *and* $4n$, *and zero otherwise, with* $Sq^{2n} \colon H^{2n}(X) \to H^{4n}(X)$ *and* $Sq^1 \colon H^{4n-1}(X) \to H^{4n}(X)$ *isomorphisms;*

(d) *There exists a map* $f \colon S^{2n-1} \times S^{2n-1} \to V_{2n+1,2}$ *such that* $f|_{S^{2n-1}\times *} = f|_{*\times S^{2n-1}}$ *is the inclusion of the bottom cell;*

(e) $n = 1$ *or there exists an element* $\theta \in \pi^S_{2n-2}$ *of Kervaire invariant one.*

The above conditions hold for $2n = 2, 4, 8, 16, 32$ and 64, and the recent solution to the Kervaire invariant problem by Hill, Hopkins and Ravenel [**8**] implies that, with the possible exception of $2n = 128$, these are the only values for which the conditions hold. Mimicking the reformulations above we obtain the following.

Theorem 3.3. *The following are equivalent:*

(a) *The Whitehead square* $[i_{2n}, i_{2n}] \in \pi_{4n-1}(P^{2n+1}(2))$ *is divisible by* 2;

(b) *There is a map* $P^{4n}(2) \to \Omega P^{2n+2}(2)$ *which is nonzero in homology;*

(c) *There exists a space* X *with mod-2 cohomology* $\widetilde{H}^i(X) \cong \mathbb{Z}/2\mathbb{Z}$ *for* $i = 2n+1$, $2n+2$, $4n+1$, $4n+2$ *and zero otherwise with* $Sq^{2n} \colon H^{2n+1}(X) \to H^{4n+1}(X)$, $Sq^1 \colon H^{2n+1}(X) \to H^{2n+2}(X)$ *and* $Sq^1 \colon H^{4n+1}(X) \to H^{4n+2}(X)$ *isomorphisms;*

(d) *There exists a map* $f \colon S^{2n-1} \times P^{2n}(2) \to V_{2n+1,2}$ *such that* $f|_{S^{2n-1}\times *}$ *and* $f|_{*\times P^{2n}(2)}$ *are the skeletal inclusions of* S^{2n-1} *and* $P^{2n}(2)$, *respectively;*

(e) $n = 1$ *or there exists an element* $\theta \in \pi^S_{2n-2}$ *of Kervaire invariant one such that* $\theta\eta$ *is divisible by* 2;

(f) $2n = 2, 4, 8$ *or* 16.

Proof. (a) is equivalent to (b): In the $n = 1$ case, $[\iota_2, \iota_2] = 2\eta_2$ implies $[i_2, i_2] = 0$, and since $\eta_3 \in \pi_4(S^3)$ has order 2 its adjoint $\widetilde{\eta}_3 \colon S^3 \to \Omega S^3$ extends to a map $P^4(2) \to \Omega S^3$. If this map desuspended, then $\widetilde{\eta}_3$ would be homotopic to a composite $S^3 \to P^4(2) \to S^2 \xrightarrow{E} \Omega S^3$, a contradiction since $\pi_3(S^2) \cong \mathbb{Z}$ implies that any map $S^3 \to S^2$ that factors through $P^4(2)$ is nullhomotopic. Hence the map $P^4(2) \to \Omega S^3$ has nontrivial Hopf invariant in $[P^4(2), \Omega S^5]$ from which it follows that $P^4(2) \to \Omega S^3$ is nonzero in $H_4(\ )$. Composing with the inclusion $\Omega S^3 \to \Omega P^4(2)$ gives a map $P^4(2) \to \Omega P^4(2)$ which is nonzero in $H_4(\ )$.

Now suppose $n > 1$ and $[i_{2n}, i_{2n}] = 2\alpha$ for some $\alpha \in \pi_{4n-1}(P^{2n+1}(2))$. Then $\Sigma\alpha$ has order 2 so there is an extension $P^{4n+1}(2) \to P^{2n+2}(2)$ whose adjoint $f \colon P^{4n}(2) \to \Omega P^{2n+2}(2)$ satisfies $f|_{S^{4n-1}} = E \circ \alpha$. Since $\Omega\Sigma(P^{2n+1}(2) \wedge P^{2n+1}(2))$ has $4n$-skeleton S^{4n}, to show that f_* is nonzero on $H_{4n}(P^{4n}(2))$ it suffices to show that $H_2 \circ f$ is nontrivial in $[P^{4n}(2), \Omega\Sigma(P^{2n+1}(2) \wedge P^{2n+1}(2))]$ where $H_2 \colon \Omega P^{2n+2}(2) \to \Omega\Sigma(P^{2n+1}(2) \wedge P^{2n+1}(2))$ is the second James-Hopf invariant. If $H_2 \circ f$ is nullhomotopic, then there is a map $g\colon P^{4n}(2) \to P^{2n+1}(2)$ making the diagram

$$\begin{array}{ccccc} P^{2n+1}(2) & \xrightarrow{E} & \Omega P^{2n+2}(2) & \xrightarrow{H_2} & \Omega\Sigma(P^{2n+1}(2) \wedge P^{2n+1}(2)) \\ \uparrow g & \nearrow f & & & \\ P^{4n}(2) & & & & \end{array}$$

commute. But then $\alpha - g|_{S^{4n-1}}$ is in the kernel of $E_*\colon \pi_{4n-1}(P^{2n+1}(2)) \to \pi_{4n}(P^{2n+2}(2))$ which is generated by $[i_{2n}, i_{2n}]$, so $\alpha - g|_{S^{4n-1}}$ is a multiple of $[i_{2n}, i_{2n}]$. Since $[i_{2n}, i_{2n}]$ has order 2 and clearly $2g|_{S^{4n-1}} = 0$, it follows that $[i_{2n}, i_{2n}] = 2\alpha = 0$, a contradiction. Therefore f_* is nonzero on $H_{4n}(P^{4n}(2))$.

Conversely, assume $n > 1$ and $f\colon P^{4n}(2) \to \Omega P^{2n+2}(2)$ is nonzero in $H_{4n}(\ )$. Since the restriction $f|_{S^{4n-1}}$ lifts through the $(4n-1)$-skeleton of $\Omega P^{2n+2}(2)$, there is a homotopy commutative diagram

$$\begin{array}{ccc} S^{4n-1} & \longrightarrow & P^{4n}(2) \\ \downarrow g & & \downarrow f \\ P^{2n+1}(2) & \xrightarrow{E} & \Omega P^{2n+2}(2), \end{array}$$

for some map $g\colon S^{4n-1} \to P^{2n+1}(2)$. Since $E \circ 2g$ is nullhomotopic, $2g$ is a multiple of $[i_{2n}, i_{2n}]$. But if $2g = 0$, then g admits an extension $e : P^{4n}(2) \to P^{2n+1}(2)$ and it follows that $f - E \circ e$ factors through the pinch map $q\colon P^{4n}(2) \to S^{4n}$. This makes the Pontrjagin square $u^2 \in H_{4n}(\Omega P^{2n+2}(2))$ a spherical homology class, and this is a contradiction which can be seen as follows. If u^2 is spherical, then the $4n$-skeleton of $\Omega P^{2n+2}(2)$ is homotopy equivalent to $P^{2n+1}(2) \vee S^{4n}$. On the other hand, it is easy to see that the attaching map of the $4n$-cell in $\Omega P^{2n+2}(2)$ is given by the Whitehead square $[i_{2n}, i_{2n}]$ which is nontrivial as $n > 1$, whence $P^{2n+1} \cup_{[i_{2n}, i_{2n}]} e^{4n} \not\simeq P^{2n+1}(2) \vee S^{4n}$.

(a) is equivalent to (d): Since the Whitehead product $[i'_{2n-1}, j'_{2n}] \in \pi_{4n-2}(V_{2n+1,2}; \mathbb{Z}/2\mathbb{Z})$ is the obstruction to extending $i'_{2n-1} \vee j'_{2n}\colon S^{2n-1} \vee P^{2n}(2) \to V_{2n+1,2}$ to $S^{2n-1} \times P^{2n}(2)$, this follows immediately from Proposition 3.1.

As described in [**17**], applying the Hopf construction to a map $f\colon S^{2n-1} \times P^{2n}(2) \to V_{2n+1,2}$ as in (d) yields a map $H(f)\colon P^{4n}(2) \to \Sigma V_{2n+1,2}$ with Sq^{2n} acting nontrivially on $H^{2n}(C_{H(f)})$. Since $\Sigma^2 V_{2n+1,2} \simeq P^{2n+2}(2) \vee S^{4n+1}$, composing the suspension of the Hopf construction $H(f)$ with a retract $\Sigma^2 V_{2n+1,2} \to P^{2n+2}(2)$ defines a map $g\colon P^{4n+1}(2) \to P^{2n+2}(2)$ with Sq^{2n} acting nontrivially on $H^{2n+1}(C_g)$, so (d) implies (c).

By the proof of [**17**, Theorem 3.1], (c) implies (e), and (e) implies (f). The triviality of the Whitehead product $[i'_{2n-1}, j'_{2n}] \in \pi_{4n-2}(V_{2n+1,2}; \mathbb{Z}/2\mathbb{Z})$ when $n = 1, 2$ or 4 is implied by [**17**, Theorem 2.1], for example, and Proposition 3.1 implies $[i'_{15}, j'_{16}] \in \pi_{30}(V_{17,2}; \mathbb{Z}/2\mathbb{Z})$ is trivial as well since $[i_{16}, i_{16}] \in \pi_{31}(P^{17}(2))$ is divisible by 2 by [**10**, Lemma 3.10]. Thus (f) implies (d). $\square$

4. A loop space decomposition of $J_3(S^2)$

In this section, we consider some relations between the fibre bundle $S^{4n-1} \to V_{4n+1,2} \to S^{4n}$ defined by projection onto the first vector of an orthonormal 2-frame in $\mathbb{R}^{4n+1}$ (equivalently, the unit tangent bundle over S^{4n}) and the fibration $BW_n \to \Omega^2 S^{4n+1}\{2\} \to W_{2n}$ of Lemma 2.2. Letting $\partial\colon \Omega S^{4n} \to S^{4n-1}$ denote the connecting map of the first fibration, we will show that there is a morphism of homotopy fibrations

$$\begin{array}{ccccc} \Omega^2 S^{4n} & \xrightarrow{\Omega\partial} & \Omega S^{4n-1} & \longrightarrow & \Omega V_{4n+1,2} \\ \downarrow & & \downarrow & & \downarrow \\ \Omega W_{2n} & \longrightarrow & BW_n & \longrightarrow & \Omega^2 S^{4n+1}\{2\}, \end{array} \tag{2}$$

from which it will follow that for $n = 1, 2$ or 4, $\Omega\partial$ lifts through $\Omega\phi_n\colon \Omega^3 S^{4n+1} \to \Omega S^{4n-1}$. If this lift can be chosen to be $\Omega^2 E$, then it follows that there is a homotopy pullback diagram

$$\begin{array}{ccccc} \Omega^2 V_{4n+1,2} & \longrightarrow & \Omega^2 S^{4n} & \xrightarrow{\Omega\partial} & \Omega S^{4n-1} \\ \downarrow & & \downarrow{\scriptstyle \Omega^2 E} & & \Vert \\ W_n & \xrightarrow{\Omega j} & \Omega^3 S^{4n+1} & \xrightarrow{\Omega\phi_n} & \Omega S^{4n-1} \\ \downarrow & & \downarrow{\scriptstyle \Omega^2 H} & & \\ \Omega^3 S^{8n+1} & = & \Omega^3 S^{8n+1}, & & \end{array} \tag{3}$$

which identifies $\Omega^2 V_{4n+1,2}$ with $\Omega M_3(n)$ where $\{M_k(n)\}_{k\geqslant 1}$ is the filtration of BW_n studied in [**7**] beginning with the familiar spaces $M_1(n) \simeq \Omega S^{4n-1}$ and $M_2(n) \simeq S^{4n-1}\{\underline{2}\}$. (Spaces are localized at an odd prime throughout [**7**] but the construction of the filtration works in the same way for $p = 2$.) We verify this (and deloop it) for $n = 1$ since it leads to an interesting loop space decomposition which gives isomorphisms $\pi_k(V_{5,2}) \cong \pi_k(J_3(S^2))$ for all $k \geqslant 3$.

In his factorization of the 4th-power map on $\Omega^2 S^{2n+1}$ through the double suspension, Theriault constructs in [**18**] a space A and a map $\overline{E}\colon A \to \Omega S^{2n+1}\{2\}$ with the following properties:

(a) $H_*(A) \cong \Lambda(x_{2n-1}, x_{2n})$ with Bockstein $\beta x_{2n} = x_{2n-1}$;

(b) $\overline{E}$ induces a monomorphism in homology;

(c) There is a homotopy fibration $S^{2n-1} \to A \to S^{2n}$ and a homotopy fibration diagram

$$\begin{array}{ccccc} S^{2n-1} & \longrightarrow & A & \longrightarrow & S^{2n} \\ \downarrow{\scriptstyle E^2} & & \downarrow{\scriptstyle \overline{E}} & & \downarrow{\scriptstyle E} \\ \Omega^2 S^{2n+1} & \longrightarrow & \Omega S^{2n+1}\{2\} & \longrightarrow & \Omega S^{2n+1}. \end{array}$$

Noting that the homology of A is isomorphic to the homology of the unit tangent bundle $\tau(S^{2n})$ as a coalgebra over the Steenrod algebra, Theriault raises the question of whether A is homotopy equivalent to $\tau(S^{2n}) = V_{2n+1,2}$. Our next proposition shows this is true for any space A with the properties above.

Proposition 4.1. *There is a homotopy equivalence $A \simeq V_{2n+1,2}$.*

Proof. First we show that A splits stably as $P^{2n} \vee S^{4n-1}$. As in [**18**], let Y denote the $(4n-1)$-skeleton of $\Omega S^{2n+1}\{2\}$. Consider the homotopy fibration

$$\Omega S^{2n+1}\{2\} \longrightarrow \Omega S^{2n+1} \xrightarrow{\ 2\ } \Omega S^{2n+1}$$

and recall that $H_*(\Omega S^{2n+1}\{2\}) \cong H_*(\Omega S^{2n+1}) \otimes H_*(\Omega^2 S^{2n+1})$. Restricting the fibre inclusion to Y and suspending once we obtain a homotopy commutative diagram

$$\begin{array}{ccccccc} & & & \overset{\ell}{\nearrow} & S^{2n+1} & \xrightarrow{\ \underline{2}\ } & S^{2n+1} \\ & & & & \downarrow & & \downarrow \\ \Sigma Y & \longrightarrow & \Sigma\Omega S^{2n+1}\{2\} & \longrightarrow & \Sigma\Omega S^{2n+1} & \xrightarrow{\ \Sigma 2\ } & \Sigma\Omega S^{2n+1}, \end{array}$$

where $\underline{2}$ is the degree 2 map, the vertical maps are inclusions of the bottom cell of $\Sigma\Omega S^{2n+1}$ and a lift ℓ inducing an isomorphism in $H_{2n+1}(\ )$ exists since ΣY is a $4n$-dimensional complex and $\mathrm{sk}_{4n}(\Sigma\Omega S^{2n+1}) = S^{2n+1}$. It follows from the James splitting $\Sigma\Omega S^{2n+1} \simeq \bigvee_{i=1}^{\infty} S^{2ni+1}$ and the commutativity of the diagram that $\underline{2} \circ \ell$ is nullhomotopic, so, in particular, $\Sigma\ell$ lifts to the fibre $S^{2n+2}\{\underline{2}\}$ of the degree 2 map on S^{2n+2}. Since $H_*(S^{2n+2}\{\underline{2}\}) \cong \mathbb{Z}/2\mathbb{Z}[u_{2n+1}] \otimes \Lambda(v_{2n+2})$ with $\beta v_{2n+2} = u_{2n+1}$, this implies $\Sigma\ell$ factors through a map $r\colon \Sigma^2 Y \to P^{2n+2}(2)$ which is an epimorphism in homology by naturality of the Bockstein, and hence $P^{2n+2}(2)$ is a retract of $\Sigma^2 Y$. (Alternatively, r can be obtained by suspending a lift $\Sigma Y \to S^{2n+1}\{\underline{2}\}$ of ℓ and using the well-known fact that $\Sigma S^{2n+1}\{\underline{2}\}$ splits as a wedge of Moore spaces.) Now since $\overline{E}\colon A \to \Omega S^{2n+1}\{2\}$ factors through Y and induces a monomorphism in homology, composing $\Sigma^2 A \to \Sigma^2 Y$ with the retraction r shows that $\Sigma^2 A \simeq \Sigma^2(P^{2n}(2) \vee S^{4n-1})$.

Next, let $E^\infty\colon A \to QA$ denote the stabilization map and let F denote the homotopy fibre of a map $g\colon QP^{2n}(2) \to K(\mathbb{Z}/2\mathbb{Z}, 4n-2)$ representing the mod-2 cohomology class $u_{2n-1}^2 \in H^{4n-2}(QP^{2n}(2))$. A homology calculation shows that the $(4n-1)$-skeleton of F is a three-cell complex with homology isomorphic to $\Lambda(x_{2n-1}, x_{2n})$ as a coalgebra. The splitting $\Sigma^2 A \simeq \Sigma^2(P^{2n}(2) \vee S^{4n-1})$ gives rise to a map $\pi_1\colon QA \simeq QP^{2n}(2) \times QS^{4n-1} \to QP^{2n}(2)$ inducing isomorphisms in $H_{2n-1}(\ )$ and $H_{2n}(\ )$, and since the composite $g \circ \pi_1 \circ E^\infty\colon A \to K(\mathbb{Z}/2\mathbb{Z}, 4n-2)$ is nullhomotopic, there is a lift $A \to F$ inducing isomorphisms in $H_{2n-1}(\ )$ and $H_{2n}(\ )$. The coalgebra structure of $H_*(A)$ then implies this lift is a $(4n-1)$-equivalence and the result follows as $V_{2n+1,2}$ can similarly be seen to be homotopy equivalent to the $(4n-1)$-skeleton of F. □

The homotopy commutative diagram (2) is now obtained by noting that the composite $\Omega S^{4n-1} \to \Omega V_{4n+1,2} \xrightarrow{\Omega\overline{E}} \Omega^2 S^{4n+1}\{2\}$ is homotopic to $\Omega S^{4n-1} \xrightarrow{\Omega E^2} \Omega^3 S^{4n+1} \longrightarrow \Omega^2 S^{4n+1}\{2\}$, which in turn is homotopic to a composite $\Omega S^{4n-1} \to BW_n \to \Omega^2 S^{4n+1}\{2\}$ since by Theorem 2.1 there is a homotopy fibration diagram

$$\begin{array}{ccccccc}
\Omega S^{4n-1} & \longrightarrow & BW_n & \xrightarrow{j} & \Omega^2 S^{4n+1} & \xrightarrow{\phi_n} & S^{4n-1} \\
\downarrow{\scriptstyle \Omega E^2} & & \downarrow & & \| & & \downarrow{\scriptstyle E^2} \\
\Omega^3 S^{4n+1} & \longrightarrow & \Omega^2 S^{4n+1}\{2\} & \longrightarrow & \Omega^2 S^{4n+1} & \xrightarrow{2} & \Omega^2 S^{4n+1}.
\end{array}$$

Specializing to the case $n = 1$, the proof of Proposition 4.3 will show that $\Omega V_{5,2}$ fits in a delooping of diagram (3) and hence that $\Omega V_{5,2} \simeq M_3(1)$. We will need the following cohomological characterization of $V_{5,2}$.

Lemma 4.2. *Let E be the total space of a fibration $S^3 \to E \to S^4$. If E has integral cohomology group $H^4(E;\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$ and mod-2 cohomology ring $H^*(E)$ an exterior algebra $\Lambda(u, v)$ with $|u| = 3$ and $|v| = 4$, then E is homotopy equivalent to the Stiefel manifold $V_{5,2}$.*

Proof. As shown in [**22**, Theorem 5.8], the top row of the homotopy pullback diagram

$$\begin{array}{ccccc}
X^4 & \longrightarrow & P^4(2) & \longrightarrow & BS^3 \\
\downarrow & & \downarrow{\scriptstyle q} & & \| \\
S^7 & \xrightarrow{\nu} & S^4 & \longrightarrow & BS^3
\end{array}$$

induces a split short exact sequence

$$0 \longrightarrow \mathbb{Z}/4\mathbb{Z} \longrightarrow \pi_6(P^4(2)) \longrightarrow \pi_5(S^3) \longrightarrow 0,$$

from which it follows that $\pi_6(P^4(2)) = \mathbb{Z}/4\mathbb{Z}\{\lambda\} \oplus \mathbb{Z}/2\mathbb{Z}\{\widetilde{\eta}_3^2\}$ where λ is the attaching map of the top cell of $V_{5,2}$ and $\widetilde{\eta}_3^2$ maps to the generator η_3^2 of $\pi_5(S^3)$. It follows from the cohomological assumptions that $E \simeq P^4(2) \cup_f e^7$, where $f = a\lambda + b\widetilde{\eta}_3^2$ for some $a \in \mathbb{Z}/4\mathbb{Z}$, $b \in \mathbb{Z}/2\mathbb{Z}$, and that $H_*(\Omega E)$ is isomorphic to a polynomial algebra $\mathbb{Z}/2\mathbb{Z}[u_2, v_3]$. Since the looped inclusion $\Omega P^4(2) \to \Omega E$ induces the abelianization map $T(u_2, v_3) \to \mathbb{Z}/2\mathbb{Z}[u_2, v_3]$ in homology, it is easy to see that the adjoint $f' \colon S^5 \to \Omega P^4(2)$ of f has Hurewicz image $[u_2, v_3] = u_2 \otimes v_3 + v_3 \otimes u_2$ and hence f is not divisible by 2. Moreover, since E is an S^3-fibration over S^4, the pinch map $q \colon P^4(2) \to S^4$ must extend over E. This implies the composite $S^6 \xrightarrow{f} P^4(2) \xrightarrow{q} S^4$ is nullhomotopic and therefore $b = 0$ by the commutativity of the diagram above. It now follows that $f = \pm\lambda$ which implies $E \simeq V_{5,2}$. □

Proposition 4.3. *There is a homotopy fibration*

$$V_{5,2} \longrightarrow J_3(S^2) \longrightarrow K(\mathbb{Z}, 2),$$

which is split after looping.

Proof. Let h denote the composite $\Omega S^3\langle 3\rangle \to \Omega S^3 \xrightarrow{H} \Omega S^5$ and consider the pullback

$$\begin{array}{ccc}
P & \longrightarrow & S^4 \\
\downarrow & & \downarrow{\scriptstyle E} \\
\Omega S^3\langle 3\rangle & \xrightarrow{h} & \Omega S^5.
\end{array}$$

Since h has homotopy fibre S^3, so does the map $P \to S^4$. Next, observe that P is the homotopy fibre of the composite $\Omega S^3\langle 3\rangle \xrightarrow{h} \Omega S^5 \xrightarrow{H} \Omega S^9$ and since ΩS^9 is 7-connected, the inclusion of the 7-skeleton of $\Omega S^3\langle 3\rangle$ lifts to a map $\mathrm{sk}_7(\Omega S^3\langle 3\rangle) \to P$. Recalling that $H^4(\Omega S^3\langle 3\rangle;\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$ and $H_*(\Omega S^3\langle 3\rangle) \cong \Lambda(u_3) \otimes \mathbb{Z}/2\mathbb{Z}[v_4]$ with generators in degrees $|u_3| = 3$ and $|v_4| = 4$, it follows that this lift must be a homology isomorphism and hence a homotopy equivalence. So P is homotopy equivalent to the total space of a fibration satisfying the hypotheses of Lemma 4.2 and there is a homotopy equivalence $P \simeq V_{5,2}$.

It is well known that the iterated composite of the pth James-Hopf invariant $H^{\circ k}\colon \Omega S^{2n+1} \to \Omega S^{2np^k+1}$ has homotopy fibre $J_{p^k-1}(S^{2n})$, the (p^k-1)st stage of the James construction on S^{2n}. The argument above identifies $V_{5,2}$ with the homotopy fibre of the composite

$$\Omega S^3\langle 3\rangle \longrightarrow \Omega S^3 \xrightarrow{H} \Omega S^5 \xrightarrow{H} \Omega S^9,$$

so there is a homotopy pullback diagram

$$\begin{array}{ccccc}
V_{5,2} & \longrightarrow & J_3(S^2) & \longrightarrow & K(\mathbb{Z},2) \\
\downarrow & & \downarrow & & \| \\
\Omega S^3\langle 3\rangle & \longrightarrow & \Omega S^3 & \longrightarrow & K(\mathbb{Z},2) \\
\downarrow{\scriptstyle H\circ h} & & \downarrow{\scriptstyle H^{\circ 2}} & & \\
\Omega S^9 & = & \Omega S^9, & &
\end{array}$$

where the maps into $K(\mathbb{Z},2)$ represent generators of $H^2(J_3(S^2);\mathbb{Z}) \cong \mathbb{Z}$ and $H^2(\Omega S^3;\mathbb{Z}) \cong \mathbb{Z}$. To see that the homotopy fibration along the top row splits after looping, note that the connecting map $\Omega K(\mathbb{Z},2) = S^1 \to V_{5,2}$ is nullhomotopic since $V_{5,2}$ is simply-connected. Therefore the looped projection map $\Omega J_3(S^2) \to S^1$ has a right homotopy inverse producing a splitting $\Omega J_3(S^2) \simeq S^1 \times \Omega V_{5,2}$. □

Corollary 4.4. *$\pi_k(J_3(S^2)) \cong \pi_k(V_{5,2})$ for all $k \geqslant 3$.*

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Steven Amelotte `steven.amelotte@mail.utoronto.ca`

Department of Mathematics, University of Toronto, 40 St. George Street, Toronto, Ontario, M5S 2E4, Canada