

USING TORSION THEORY TO COMPUTE THE ALGEBRAIC STRUCTURE OF HOCHSCHILD (CO)HOMOLOGY

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Abstract

The aim of this article is to provide explicit formulas for the cup product on the Hochschild cohomology of any nonnegatively graded connected algebra A and for the cap products on the Hochschild homology of A with coefficients in any graded bimodule M at the level of the complexes $\mathrm{Hom}_{A^e}(P_\bullet, A)$ and $M \otimes_{A^e} P_\bullet$, resp., where P_\bullet is a minimal projective resolution of the A -bimodule A , based on the A_∞ -algebra structure of $\mathcal{E}xt_A^\bullet(k, k)$. We remark that we do not (need to) construct any comparison map between P_\bullet and the Hochschild resolution of A , or any lift $\Delta: P \rightarrow P \otimes_A P$ of the identity of A . The main tools we use come from torsion theory of A_∞ -algebras and of their A_∞ -bimodules.

1. Introduction

The goal of this article is to show that, for a nonnegatively graded connected algebra A and a graded A -bimodule M , it is possible to compute the cup product of the Hochschild cohomology of A and the cap product on the Hochschild homology of A with coefficients in M at the level of the complexes $\mathrm{Hom}_{A^e}(P_\bullet, A)$ and $M \otimes_{A^e} P_\bullet$, resp., where P_\bullet is a minimal projective resolution of the A -bimodule A , **without** constructing any comparison map between P_\bullet and the Hochschild resolution of A , or any lift $\Delta: P \rightarrow P \otimes_A P$ of the identity of A , but by making use of the A_∞ -algebra structure of $\mathcal{E}xt_A^\bullet(k, k)$. More precisely, the main result of this work is the following (see Theorem 4.3).

Theorem. *Let A be a nonnegatively graded connected algebra over a field k and consider $C = \mathrm{Tor}_\bullet^A(k, k)$ provided with a coaugmented A_∞ -coalgebra structure whose dual is isomorphic to the augmented A_∞ -algebra $\mathcal{E}xt_A^\bullet(k, k)$. Then, there exists a twisted cochain τ in $\mathcal{H}om(C, A)$ such that:*

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1. there is a quasi-isomorphism of A_∞ -algebras from the complex $C^\bullet(A, A)$ computing the Hochschild cohomology of A to $\mathcal{H}om^\tau(C, A)$, which, in particular, induces an isomorphism of graded algebras from $HH^\bullet(A)$ to $H^\bullet(\mathcal{H}om^\tau(C, A))$;
2. given any graded A -bimodule M and using the previous quasi-isomorphism of A_∞ -algebras, there exists a quasi-isomorphism of A_∞ -bimodules over $C^\bullet(A, A)$ from $M \otimes_\tau C$ to the chain complex $C_\bullet(A, M)$ computing the Hochschild homology of A with coefficients in M , which gives thus an isomorphism of graded bimodules over $HH^\bullet(A)$ from $H_\bullet(A \otimes_\tau C)$ to $H_\bullet(A, M)$.

The proof is based on the torsion theory of A_∞ -algebras and of A_∞ -bimodules over them by elements satisfying the Maurer-Cartan equation. We also want to remark that there is a very simple condition to check if a given coaugmented A_∞ -coalgebra structure on $\mathrm{Tor}_\bullet^A(k, k)$ is dual to the augmented A_∞ -algebra $\mathcal{E}xt_A^\bullet(k, k)$ that also provides the twisting cochain τ (see Theorem 4.2). Moreover, we want to stress that our main result gives explicit expressions of the cup and cap products (see Addendum 4.4). We give, in particular, an application computing the cup and cap products on Hochschild cohomology and homology, resp., of a generalized Koszul algebra A (see Theorem 4.5). The part of this last result concerning Hochschild cohomology was already observed by Xu and Xiang in [15, Thm. 3.2]. Our proof is, however, completely different. The part concerning Hochschild homology does not seem to have been observed so far. Furthermore, our results can be applied to any nonnegatively graded connected algebras for which the A_∞ -algebra structure of the Yoneda algebra is known (e.g., the *multi-Koszul* algebras studied in [8]).

The article is structured as follows. In Section 2 we review the basic definitions and results about the main algebraic objects we shall use: coaugmented A_∞ -coalgebras and A_∞ -bimodules over augmented A_∞ -algebras. For the theory of A_∞ -algebras we refer the reader to the nice exposition [11], whose convention we shall follow, and to standard references like [10] and [13]. The section is concluded with two constructions that will be used in the sequel: an augmented A_∞ -algebra $\mathcal{H}om(C, A)$ from a coaugmented A_∞ -coalgebra C and an augmented dg algebra A , called the *convolution A_∞ -algebra*, and, given a dg A -bimodule M , an A_∞ -bimodule $M \otimes C$ over $\mathcal{H}om(C, A)$.

Section 3 deals with the twisting theory of topological augmented A_∞ -algebras and of A_∞ -bimodules over them satisfying a particular discreteness assumption. We surmise that the contents of Subsections 3.2 and 3.4, specially Lemmas 3.4, 3.5, 3.12 and 3.13, might be well-known among the experts, but we could not find any specific reference for them. Our main interest in these constructions is to show that, if C is an Adams connected coaugmented A_∞ -coalgebra and τ is a twisting cochain, one can twist the convolution A_∞ -algebra $\mathcal{H}om(C, A)$ and the A_∞ -bimodule $M \otimes C$ over the former. The new constructions are denoted by $\mathcal{H}om^\tau(C, A)$ and $M \otimes_\tau C$, respectively. The contents of Subsection 3.3 are standard with the possible exception of Lemma 3.11, which we believe is new even though not completely unexpected. The reason for providing most of the results of this subsection was in order to be able to prove the mentioned lemma, as well as to establish the notation we use in the sequel.

The last Section is divided in three parts. Subsection 4.1 recalls some easy facts about the twisted convolution algebra and the twisted tensor product, and presents a theorem announced by B. Keller at the X ICRA of Toronto, Canada, in 2002 (see

Theorem 4.2), which is of key importance in the proof of our main result. The latter is proved in Subsection 4.2 (see Theorem 4.3). Finally, Subsection 4.3 provides an application, computing the cup and cap products for the Hochschild (co)homology of a generalized Koszul algebra.

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2. Preliminaries on basic algebraic structures

We recall the following basic facts, which will also establish the notation. From now on, k will denote a field (which we also consider as a unitary graded ring concentrated in degree zero). We will denote by \mathbb{N} the set of (strictly) positive integers and by $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$. In this article A_∞ -algebras (and coalgebras) and their (co)modules will be graded with respect to $G = \mathbb{Z} \times \mathbb{Z}$, where the first grading is called *(co)homological* and the second one is called *Adams*. When applying the Koszul sign rule, we will only take the cohomological degree into account (for a nice exposition on basic homological algebra of dg modules over dg algebras and a detailed account on the sign rule we refer to [1]). We shall denote the cohomological degree of a homogeneous element x by $\deg x$. We recall that a family of linear maps $\{f_i: M \rightarrow M_i\}_{i \in \mathbb{N}}$, where M and M_i , for $i \in \mathbb{N}$, are vector spaces, is called *locally finite* if, for all $m \in M$, there exists a finite subset $S \subseteq \mathbb{N}$, which depends on m , such that $f_i(m)$ vanishes for all $i \in \mathbb{N} \setminus S$. Given $n_0 \in \mathbb{Z}$ and $V = \bigoplus_{(n,m) \in \mathbb{Z}^2} V^{(n,m)}$ any graded object (decorated perhaps with further adjectives), we shall denote by $V[n_0]$ the *shift* of V , defined as $V[n_0]^{(n,m)} = V^{(n+n_0,m)}$ for all $n, m \in \mathbb{Z}$, and by $s_{V,n_0}: V \rightarrow V[n_0]$ the morphism of cohomological degree $-n_0$, called the *suspension on V of degree n_0* , whose underlying set-theoretic map is the identity. In this work we shall never consider shifts of the Adams degree. If $n_0 = 1$ we will just write s_V and call it the *suspension on V* . We shall say that a graded vector space $V = \bigoplus_{(m,n) \in \mathbb{Z}^2} V^{(m,n)}$ over k is *Adams connected* if $V^{(0,0)} = k$ and $V^\wedge = \bigoplus_{(m,n) \in \mathbb{Z}^2 \setminus \{(0,0)\}} V^{(m,n)}$ is concentrated in either strictly positive or strictly negative Adams degrees, and each homogeneous component of V^\wedge of a fixed Adams degree d (but including all cohomological degrees) is locally finite dimensional (cf. [12, Def. 2.1]). By changing if necessary the Adams grading of V by its opposite, we may (and shall) assume without loss of generality that an Adams connected graded vector space V satisfies that V^\wedge is concentrated in Adams positive degrees.

2.1. Generalities on A_∞ -(co)algebras

For the general theory of A_∞ -algebras and A_∞ -coalgebras we refer to [10, Chs. 1–3], or [13, Ch. 3], even though we shall not follow the sign conventions of those works. The precise definition and sign conventions that we shall use of augmented A_∞ -algebras, morphisms among them, their bar construction and modules are given in the very nice exposition [11], to which we refer. We will present, however, the basic definitions for the reader's convenience.

Let A be a graded vector space provided with two maps $\eta_A: k \rightarrow A$ and $\epsilon_A: A \rightarrow k$ of degree zero, that we call the *unit* and the *augmentation* of A , respectively. Set $I_A = \text{Ker}(\epsilon_A)$ and $1_A = \eta(1_k)$. This defines an isomorphism of graded vector spaces

$A \simeq k \cdot 1_A \oplus I_A$. An *augmented A_∞ -algebra* is a coderivation B_A of cohomological degree 1 on the coaugmented graded coalgebra $T(I_A[1]) = \bigoplus_{i \in \mathbb{N}_0} I_A[1]^{\otimes i}$ with the coproduct given by deconcatenation, the obvious counit ϵ_T and coaugmentation η_T , such that $B_A \circ \eta_T = 0$ and $B_A \circ B_A = 0$. This coaugmented dg coalgebra is called the *bar construction* of A and is denoted by $B^+(A)$.

Taking into account that $B^+(A)$ is a coaugmented tensor graded coalgebra, it is cocomplete and cofree, so the coderivation B_A is uniquely determined by $\pi_1 \circ B_A$, where $\pi_1: B^+(A) \rightarrow I_A[1]$ is the canonical projection (see [10, Lemme 1.1.2.2]). Let us write this composition map by $b = \sum_{i \in \mathbb{N}} b_i$, where $b_i: I_A[1]^{\otimes i} \rightarrow I_A[1]$. For $i \in \mathbb{N}$, define $\bar{m}_i: I_A^{\otimes i} \rightarrow I_A$ by means of $b_i = -s_{I_A} \circ m_i \circ (s_{I_A}^{\otimes i})^{-1}$, and extend it to a map $m_i: A^{\otimes i} \rightarrow A$ given by the composition of $\bar{m}_i \circ p^{\otimes i}$ and the canonical inclusion $I_A \rightarrow A$ if $i \neq 2$, where $p: A \rightarrow I_A$ is the canonical projection, and by $m_2|_{I_A^{\otimes 2}} = \bar{m}_2$ and $m_2(1_A \otimes a) = a = m_2(a \otimes 1_A)$ for all $a \in A$. If the reader translates the previous definition of augmented A_∞ -algebra into a condition about the maps $\{m_i\}_{i \in \mathbb{N}}$, he will find precisely the classical notion appearing in [11].¹

A *morphism of augmented A_∞ -algebras* $f_\bullet: A \rightarrow B$ between two augmented A_∞ -algebras A and B is a morphism of coaugmented dg coalgebras $B^+(f): B^+(A) \rightarrow B^+(B)$. Since $B^+(B)$ is a coaugmented tensor graded coalgebra, the morphism $B^+(f)$ is uniquely determined by its composition with the canonical projection $B^+(B) \rightarrow I_B[1]$, that we write $F = \sum_{i \in \mathbb{N}} F_i$, where $F_i: I_A[1]^{\otimes i} \rightarrow I_B[1]$. Set $\bar{f}_i: I_A^{\otimes i} \rightarrow I_B$ by means of $F_i = s_{I_B} \circ \bar{f}_i \circ (s_{I_A}^{\otimes i})^{-1}$, for $i \in \mathbb{N}$, and define $f_i: A^{\otimes i} \rightarrow B$ as the composition of $\bar{f}_i \circ p^{\otimes i}$ and the canonical inclusion $I_B \rightarrow B$ if $i \neq 1$, where $p: A \rightarrow I_A$ is the canonical projection, and by $f_1(1_A) = 1_B$ and $f_1|_{I_A} = \bar{f}_1$. The translation of the definition of morphism of augmented A_∞ -algebras into a condition about the maps $\{f_i\}_{i \in \mathbb{N}}$ gives precisely the classical notion that appears in [11].

We shall briefly recall the dual notions, because they are no so common in the literature. Let C be a graded vector space provided two maps $\epsilon_C: C \rightarrow k$ and $\eta_C: k \rightarrow C$ of degree zero, called the *counit* and the *coaugmentation* of C , respectively. Define $1_C = \eta_C(1_k)$ and $J_C = \text{Coker}(\eta_C)$, and consider the isomorphism of graded vector spaces $C \simeq k \cdot 1_C \oplus J_C$ via the identification $J_C \simeq \text{Ker}(\epsilon_C)$.

A *coaugmented A_∞ -coalgebra* is a derivation D_C of cohomological degree 1 on the augmented graded tensor algebra $T(J_C[-1])$ provided with the concatenation, and the obvious unit η_T and augmentation ϵ_T , such that $\epsilon_T \circ D_C = 0$ and $D_C \circ D_C = 0$. The previous augmented dg algebra is called the *cobar construction* of C and is typically denoted by $\Omega^+(C)$.

Since $\Omega^+(C)$ is a free graded algebra, D_C is uniquely determined by its restriction to $J_C[-1]$, which we denote by $d = \sum_{i \in \mathbb{N}} d_i$ for $d_i: J_C[-1] \rightarrow J_C[-1]^{\otimes i}$. Set $\bar{\Delta}_i: J_C \rightarrow J_C^{\otimes i}$ by means of $d_i = (-1)^i (s_{J_C[-1]}^{\otimes i})^{-1} \circ \bar{\Delta}_i \circ s_{J_C[-1]}$, and define $\Delta_i: C \rightarrow C^{\otimes i}$ as the composition of the canonical projection $C \rightarrow J_C$, $\bar{\Delta}_i$ and the inclusion $J_C^{\otimes i} \rightarrow C^{\otimes i}$ if $i \neq 2$, and as $\Delta_2(1_C) = 1_C \otimes 1_C$ and $\Delta_2(\bar{c}) = \bar{\Delta}_2(\bar{c}) + 1_C \otimes \bar{c} + \bar{c} \otimes 1_C$ for $\bar{c} \in J_C$. Then the collection of maps $\Delta_i: C \rightarrow C^{\otimes i}$ for $i \in \mathbb{N}$ is locally finite, each

¹Unfortunately, the choice of sign appearing in the bar construction of Section 9 of [11] is mistaken, for it is not compatible with their Definition 2.1 if they want their Lemma 9.1 to hold. Using the notation of that article, a coherent choice of sign is given by taking $\bar{m}_n = -m_n$, and by replacing $(|a_{j+t}| + 1)$ by $|a_{j+t}|$ in their definition of w at p. 22.

of homological degree $i - 2$ and Adams degree zero, satisfying the following identities

$$\sum_{(r,s,t) \in \mathcal{I}_n} (-1)^{rs+t} (\text{id}_C^{\otimes r} \otimes \Delta_s \otimes \text{id}_C^{\otimes t}) \circ \Delta_{r+1+t} = 0,$$

for $n \in \mathbb{N}$, where $\mathcal{I}_n = \{(r, s, t) \in \mathbb{N}_0 \times \mathbb{N} \times \mathbb{N}_0 : r + s + t = n\}$, $(\text{id}_C^{\otimes r} \otimes \epsilon_C \otimes \text{id}_C^{\otimes t}) \circ \Delta_i$ vanishes for all $i \neq 2$ and $r, t \geq 0$ such that $r + 1 + t = i$, $(\text{id}_C \otimes \epsilon_C) \circ \Delta_2 = \text{id}_C = (\epsilon_C \otimes \text{id}_C) \circ \Delta_2$, $\epsilon_C \circ \eta_C = \text{id}_k$, $\Delta_2 \circ \eta_C(1_k) = \eta_C(1_k)^{\otimes 2}$, and $\Delta_i \circ \eta_C(1_k) = 0$ for all $i \in \mathbb{N} \setminus \{2\}$. Reciprocally, starting from a locally finite collection of maps $\Delta_i: C \rightarrow C^{\otimes i}$ fulfilling the previous properties we obtain a coaugmented A_∞ -coalgebra structure. A coaugmented A_∞ -coalgebra C is called *minimal* if $\Delta_1 = 0$.

Note that a coaugmented A_∞ -coalgebra C is also a complex of vector spaces for the differential Δ_1 . Moreover, a coaugmented dg coalgebra structure on C is equivalent to a coaugmented A_∞ -coalgebra structure with vanishing *higher comultiplications* Δ_n for $n \geq 3$, where the differential is Δ_1 and the coproduct is Δ_2 . Note that our definition of the cobar construction for the special case of a coaugmented dg coalgebra coincides with the one given in [11, Section 8].

Given two coaugmented A_∞ -coalgebras C and C' , a *morphism* $f_\bullet: C \rightarrow C'$ is a morphism of augmented dg algebras $\Omega^+(f_\bullet): \Omega^+(C) \rightarrow \Omega^+(C')$ of the cobar constructions. Since $\Omega^+(C)$ is a free graded algebra, such a morphism is completely determined by its restriction to $J_C[-1]$, which we denote by $F = \sum_{i \in \mathbb{N}} F_i$, where $F_i: J_C[-1] \rightarrow J_{C'}[-1]^{\otimes i}$. Define $\bar{f}_i: J_C \rightarrow J_{C'}^{\otimes i}$ by $F_i = (-1)^{i+1} (s_{J_{C'}[-1]}^{\otimes i})^{-1} \circ \bar{f}_i \circ s_{J_C[-1]}$, for $i \in \mathbb{N}$, and set $f_i: C \rightarrow (C')^{\otimes i}$ as the composition of the canonical projection $C \rightarrow J_C$, \bar{f}_i and the inclusion $J_{C'}^{\otimes i} \rightarrow (C')^{\otimes i}$ if $i \neq 1$, and $f_1(1_C) = 1_{C'}$ and $f_1(\bar{c}) = \bar{f}_1(\bar{c})$ for $\bar{c} \in J_C$. Then $f_i: C \rightarrow (C')^{\otimes i}$ is a locally finite collection of maps, each of homological degree $i - 1$ and Adams degree zero for $i \in \mathbb{N}$, such that

$$\sum_{(r,s,t) \in \mathcal{I}_n} (-1)^{rs+t} (\text{id}_D^{\otimes r} \otimes \Delta_s^D \otimes \text{id}_D^{\otimes t}) \circ f_{r+1+t} = \sum_{q \in \mathbb{N}} \sum_{\bar{i} \in \mathbb{N}^{q,n}} (-1)^{w'} (f_{i_1} \otimes \cdots \otimes f_{i_q}) \circ \Delta_q^C,$$

where $w' = \sum_{j=1}^q (j-1)(i_j+1)$ and $\mathbb{N}^{q,n}$ is the subset of elements \bar{i} of \mathbb{N}^q satisfying that $|\bar{i}| = i_1 + \cdots + i_q = n$, $\epsilon_{C'} \circ f_1 = \epsilon_C$, $(\text{id}_{C'}^{\otimes (j-1)} \otimes \epsilon_{C'} \otimes \text{id}_{C'}^{\otimes (i-j)}) \circ f_i$ vanishes for all $i \geq 2$ and $j \in \{1, \dots, i\}$, $f_1 \circ \eta_C = \eta_{C'}$ and $f_i \circ \eta_C$ vanishes for $i \geq 2$. Reciprocally, starting from a locally finite collection of maps $f_i: C \rightarrow (C')^{\otimes i}$ fulfilling the previous properties we obtain a morphism of coaugmented A_∞ -coalgebras.

Notice that f_1 is also a morphism of dg vector spaces for the underlying structures on C and C' . A morphism of coaugmented A_∞ -coalgebras is *strict* if f_i vanishes for $i \geq 2$ and it is called a *quasi-isomorphism* if the map f_1 is so. We remark that our definition of $\Omega^+(f_\bullet)$ agrees with the corresponding one given in [11, Section 8], for coaugmented dg coalgebras in the case the morphism f_\bullet is further assumed to be strict.

If $f_\bullet: C \rightarrow C'$ and $g_\bullet: C' \rightarrow D$ are morphisms of coaugmented A_∞ -coalgebras, we can consider their composition $\Omega^+(g_\bullet) \circ \Omega^+(f_\bullet)$. Using the comments in the previous paragraphs we see that $\Omega^+(g_\bullet) \circ \Omega^+(f_\bullet) = \Omega^+(h_\bullet)$ where $\{h_n: C \rightarrow D^{\otimes n}\}_{n \in \mathbb{N}}$ is of the form

$$h_n = \sum_{q \in \mathbb{N}} \sum_{\bar{i} \in \mathbb{N}^{q,n}} (-1)^{w'} (g_{i_1} \otimes \cdots \otimes g_{i_q}) \circ f_q,$$

and $w' = \sum_{j=1}^q (j-1)(i_j+1)$.

2.2. Generalities on A_∞ -bimodules

For the following definitions we refer to [10, Ch. 2, Section 5], even though our sign conventions are distinct. Given an augmented A_∞ -algebra A , an A_∞ -bimodule over A is a graded vector space M and a bicoderivation B_M on the graded counitary bicomodule $B^+(A) \otimes M[1] \otimes B^+(A)$ over $B^+(A)$ such that $B_M \circ B_M = 0$. We shall denote the previous bicomodule by $B^+(A, M, A)$. Since $B^+(A) \otimes M[1] \otimes B^+(A)$ is a cofree graded bicomodule, a bicoderivation is uniquely determined by its composition with $\epsilon_{B^+(A)} \otimes \text{id}_{M[1]} \otimes \epsilon_{B^+(A)}$, which is a sum of mappings of the form $b_{p,q}: I_A[1]^{\otimes p} \otimes M[1] \otimes I_A[1]^{\otimes q} \rightarrow M[1]$, for $p, q \in \mathbb{N}_0$. Define $m_{p,q}^M: A^{\otimes p} \otimes M \otimes A^{\otimes q} \rightarrow M$ as the composition of the canonical projection $A^{\otimes p} \otimes M \otimes A^{\otimes q} \rightarrow I_A^{\otimes p} \otimes M \otimes I_A^{\otimes q}$ and $-s_M^{-1} \circ b_{p,q} \circ (s_{I_A}^{\otimes p} \otimes s_M \otimes s_{I_A}^{\otimes q})$ if $p + q \neq 1$, and by $m_{0,1} \circ (\text{id}_M \otimes \eta_A) = \text{id}_M = m_{1,0} \circ (\eta_A \otimes \text{id}_M)$ as well as the equalities $m_{0,1} \circ (\text{id}_M \otimes i_A) = -s_M^{-1} \circ b_{0,1} \circ (s_M \otimes s_{I_A})$ and $m_{1,0} \circ (i_A \otimes \text{id}_M) = -s_M^{-1} \circ b_{1,0} \circ (s_{I_A} \otimes s_M)$, where $i_A: I_A \rightarrow A$ is the canonical inclusion.

They satisfy the identity

$$\sum_{(r,s,t) \in \mathcal{I}_{n'+n''+1}} (-1)^{r+st} \tilde{m}_{r,t}^M \circ (\text{id}^{\otimes r} \otimes \tilde{m}_s \otimes \text{id}^{\otimes t}) = 0,$$

for all $n', n'' \in \mathbb{N}_0$, where we recall that $\mathcal{I}_n = \{(r, s, t) \in \mathbb{N}_0 \times \mathbb{N} \times \mathbb{N}_0 : r + s + t = n\}$, and where \tilde{m}_s is interpreted as the corresponding multiplication map m_s of A if either $r + s \leq n'$ or $s + t \leq n''$, and it is understood as $m_{n'-r, n''-t}^M$ else. In the first case, $\tilde{m}_{r,t}^M$ is $m_{n'-s+1, n''}^M$ if $r + s \leq n'$ or $m_{n', n''-s+1}^M$ if $s + t \leq n''$, and it is $m_{r,t}^M$ else. We also have that $m_{p,q}^M \circ (\text{id}^{\otimes r} \otimes \eta_A \otimes \text{id}^{\otimes t})$ vanishes for $r \neq p$ and $(p, q) \notin \{(0, 1), (1, 0)\}$, and that $m_{1,0}^M \circ (\eta_A \otimes \text{id}_M) = \text{id}_M = m_{0,1}^M \circ (\text{id}_M \otimes \eta_A)$. Reciprocally, given any collection of maps $m_{p,q}^M: A^{\otimes p} \otimes M \otimes A^{\otimes q} \rightarrow M$ fulfilling the previous properties, it defines an A_∞ -bimodule structure on M over A . Note that an augmented A_∞ -algebra is also an A_∞ -bimodule for the structure maps $m_{p,q} = m_{p+q+1}$, where $p, q \in \mathbb{N}_0$.

If M and N are two A_∞ -bimodules, a *morphism $f_{\bullet, \bullet}$ of A_∞ -bimodules from M to N* is a morphism of counitary dg bicomodules $B^+(f_{\bullet, \bullet}): B^+(A, M, A) \rightarrow B^+(A, N, A)$. Since $B^+(A) \otimes M[1] \otimes B^+(A)$ is a cofree graded bicomodule, $B^+(f_{\bullet, \bullet})$ is uniquely determined by its composition with $\epsilon_{B^+(A)} \otimes \text{id}_{N[1]} \otimes \epsilon_{B^+(A)}$, which is a sum of maps $F_{p,q}: I_A^{\otimes p} \otimes M[1] \otimes I_A[1]^{\otimes q} \rightarrow N[1]$. Define $f_{p,q}^M: A^{\otimes p} \otimes M \otimes A^{\otimes q} \rightarrow N$ as the composition of the canonical projection $A^{\otimes p} \otimes M \otimes A^{\otimes q} \rightarrow I_A^{\otimes p} \otimes M \otimes I_A^{\otimes q}$ and $s_N^{-1} \circ f_{p,q} \circ (s_{I_A}^{\otimes p} \otimes s_M \otimes s_{I_A}^{\otimes q})$. The collection of morphisms $f_{p,q}: A^{\otimes p} \otimes M \otimes A^{\otimes q} \rightarrow N$ satisfy

$$\begin{aligned} \sum_{(r,s,t) \in \mathcal{I}_{n'+n''+1}} (-1)^{r+st} f_{r',t'} \circ (\text{id}^{\otimes r} \otimes \tilde{m}_s \otimes \text{id}^{\otimes t}) \\ = \sum_{(a,k,l,b) \in \mathbb{N}_{0,n',n''}} (-1)^{b(-k-l)} m_{a,b}^N \circ (\text{id}_A^{\otimes a} \otimes f_{k,l} \otimes \text{id}_A^{\otimes b}), \end{aligned}$$

for all $n', n'' \in \mathbb{N}_0$, where $\mathbb{N}_{0,n',n''}^4$ is the subset of \mathbb{N}_0^4 of elements (a, k, l, b) such that $a + k = n'$ and $l + b = n''$, and where we should understand \tilde{m}_s as m_s^A if either $r + s \leq n'$ or $s + t \leq n''$, or as $m_{n'-r, n''-t}^M$ else. The indices (r', t') are completely determined from the previous cases. We also have that $f_{p,q} \circ (\text{id}^{\otimes r} \otimes \eta_A \otimes \text{id}^{\otimes t})$ vanishes for $r \neq p$

and $(p, q) \notin \{(0, 0)\}$. Reciprocally, given any collection of maps $f_{p,q}^M: A^{\otimes p} \otimes M \otimes A^{\otimes q} \rightarrow N$ fulfilling the previous properties, it defines a morphism of A_∞ -bimodules from M to N over A . We say that it is *strict* if $f_{p,q}$ vanishes for all $(p, q) \neq (0, 0)$.

The *composition* of two morphisms $f_{\bullet,\bullet}: M \rightarrow N$ and $g_{\bullet,\bullet}: N \rightarrow P$ is defined by the usual formula $B^+(g_{\bullet,\bullet}) \circ B^+(f_{\bullet,\bullet})$. It is easy to see that the composition is given by $B^+(h_{\bullet,\bullet})$, where

$$h_{p,q} = \sum_{(a,k,l,b) \in \mathbb{N}_{0,p,q}} (-1)^{b(-k-l)} g_{a,b} \circ (\text{id}_A^{\otimes a} \otimes f_{k,l} \otimes \text{id}_A^{\otimes b}).$$

If $f_\bullet: A' \rightarrow A$ is a morphism of augmented A_∞ -algebras and N is an A_∞ -bimodule over A whose bar construction $B^+(A, N, A)$ has coderivation B_N , then N can be regarded as an A_∞ -bimodule over A' with the unique coderivation B'_N determined by $(\epsilon_{B^+(A)} \otimes \text{id}_{N[1]} \otimes \epsilon_{B^+(A)}) \circ B_N \circ (B^+(f_\bullet) \otimes \text{id}_{N[1]} \otimes B^+(f_\bullet))$. Note that

$$B_N \circ (B^+(f_\bullet) \otimes \text{id}_{N[1]} \otimes B^+(f_\bullet)) = (B^+(f_\bullet) \otimes \text{id}_{N[1]} \otimes B^+(f_\bullet)) \circ B'_N, \quad (1)$$

for the compositions of both members with $(\epsilon_{B^+(A)} \otimes \text{id}_{N[1]} \otimes \epsilon_{B^+(A)})$ coincide. Equivalently, if $m_{\bullet,\bullet}$ are the multiplications of N over A , then the structure of an A_∞ -bimodule over A' is given by

$$m'_{p,q} = \sum_{r,s \in \mathbb{N}_0} \sum_{(\bar{i}, \bar{j}) \in \mathbb{N}^{r,p} \times \mathbb{N}^{s,q}} (-1)^\varepsilon m_{r,s} \circ (f_{i_1} \otimes \cdots \otimes f_{i_r} \otimes \text{id}_N \otimes f_{j_1} \otimes \cdots \otimes f_{j_s}), \quad (2)$$

where we recall that $\mathbb{N}^{m,n}$ is the subset of \mathbb{N}^m of elements $\bar{i} = (i_1, \dots, i_m)$ such that $|\bar{i}| = i_1 + \cdots + i_m = n$, and $\varepsilon = \sum_{u=1}^r (r+s+1-u)(i_u-1) + \sum_{u=1}^s (s-u)(j_u-1)$. We shall denote the A_∞ -bimodule over A' with structure maps $m'_{\bullet,\bullet}$ by $f_\bullet N$.

2.3. Two constructions

If C is a coaugmented A_∞ -coalgebra and A is an augmented dg algebra, the dg vector space $\mathcal{H} = \mathcal{H}om(C, A)$ has the structure of an augmented A_∞ -algebra, which we call the *convolution A_∞ -algebra*, where $m_1^{\mathcal{H}}$ is given by the usual differential $d_A \circ \phi - (-1)^{\deg \phi} \phi \circ \Delta_1^C$,

$$m_n^{\mathcal{H}}(\phi_1 \otimes \cdots \otimes \phi_n) = (-1)^{n(\deg \phi_1 + \cdots + \deg \phi_n + 1)} \mu_A^{(n)} \circ (\phi_1 \otimes \cdots \otimes \phi_n) \circ \Delta_n^C, \quad (3)$$

for $n \geq 2$, $1_{\mathcal{H}om(C,A)} = \eta_A \circ \epsilon_C$ and $\epsilon_{\mathcal{H}om(C,A)}(\phi) = \epsilon_A \circ \phi \circ \eta_C(1_k)$.

If $f_\bullet: C \rightarrow D$ is a morphism of coaugmented A_∞ -coalgebras, then the collection of morphisms

$$(f_n)_*: \mathcal{H}om(D, A)^{\otimes n} \rightarrow \mathcal{H}om(C, A), \quad (4)$$

for $n \in \mathbb{N}$, of graded vector spaces of cohomological degree $1 - n$ given by $(f_1)_*(\phi) = \phi \circ f_1$, and by

$$(f_n)_*(\phi_1 \otimes \cdots \otimes \phi_n) = (-1)^{(n-1)(\deg \phi_1 + \cdots + \deg \phi_n)} \mu_A^{(n)} \circ (\phi_1 \otimes \cdots \otimes \phi_n) \circ f_n,$$

for $n \geq 2$, is a morphism of augmented A_∞ -algebras. We remark that the previous construction defines an augmented A_∞ -algebra structure on the graded dual $C^\#$ of C .

If M is a dg A -bimodule over an augmented dg algebra A and C is a coaugmented

A_∞ -coalgebra, then $M \otimes C$ is an A_∞ -bimodule over $\mathcal{H}om(C, A)$ with the structure morphisms given by $m_{0,0}^{M \otimes C} = d_M \otimes \text{id}_C + \text{id}_M \otimes \Delta_1^C$, and, for $p + q \geq 1$,

$$\begin{aligned} m_{p,q}^{M \otimes C}(\phi_1 \otimes \cdots \otimes \phi_p \otimes (m \otimes c) \otimes \psi_1 \otimes \cdots \otimes \psi_q) \\ = (-1)^{\epsilon'}(\phi_1(c_{(q+2)}) \cdots \phi_p(c_{(q+p+1)})) \cdot m.(\psi_1(c_{(1)}) \cdots \psi_q(c_{(q)})) \otimes c_{(q+1)}, \end{aligned} \quad (5)$$

where $\Delta_{p+q+1}^C(c) = c_{(1)} \otimes \cdots \otimes c_{(p+q+1)}$, and

$$\begin{aligned} \epsilon' = pq + \deg c \deg m + (p + q + 1) \sum_{i=1}^p \deg \phi_i + \sum_{\substack{1 \leq i \leq p \\ q+2 \leq i' \leq q+i}} \deg c_{(i')} \deg \phi_i \\ + \sum_{\substack{1 \leq j \leq q \\ 1 \leq j' < j}} \deg c_{(j')} \deg \psi_j + (\deg m + \sum_{i=1}^p \deg c_{(q+1+i)} + \sum_{j=1}^q \deg \psi_j) \left(\sum_{j=1}^{q+1} \deg c_{(j)} \right). \end{aligned} \quad (6)$$

It is rather long but straightforward to prove that it is indeed an A_∞ -bimodule over $\mathcal{H}om(C, A)$. If M is only a left (resp., right) dg module over A , we can (and shall) regard it as a dg A -bimodule by means of the augmentation ϵ_A , *i.e.* $a.m.a' = \epsilon_A(a')a.m$ (resp., $a.m.a' = \epsilon_A(a)m.a'$), so we may apply the previous construction. It will be denoted by M_{ϵ_A} (resp., ${}_{\epsilon_A}M$).

If $g: M \rightarrow N$ is a morphism of dg A -bimodules over an augmented dg algebra A and C is a coaugmented A_∞ -coalgebra, then the map $g \otimes \text{id}_C: M \otimes C \rightarrow N \otimes C$ is a strict morphism of A_∞ -bimodules over $\mathcal{H}om(C, A)$.

Let $f_\bullet: C \rightarrow D$ be a morphism of coaugmented A_∞ -coalgebras and let A be an augmented dg algebra. It induces a morphism of augmented A_∞ -algebras $(f_\bullet)_*: \mathcal{H}om(D, A) \rightarrow \mathcal{H}om(C, A)$, as seen in (4). In particular, given any dg A -bimodule M , this allows to consider $M \otimes C$ as an A_∞ -bimodule over $\mathcal{H}om(D, A)$ by means of (2). Then, the collection of morphisms

$$F_{p,q}: \mathcal{H}om(D, A)^{\otimes p} \otimes (M \otimes C) \otimes \mathcal{H}om(D, A)^{\otimes q} \rightarrow M \otimes D \quad (7)$$

given by

$$\begin{aligned} F_{p,q}(\phi_1 \otimes \cdots \otimes \phi_p \otimes (m \otimes c) \otimes \psi_1 \otimes \cdots \otimes \psi_q) \\ = (-1)^{\epsilon'}(\phi_1(d_{(q+2)}) \cdots \phi_p(d_{(q+p+1)})) \cdot m.(\psi_1(d_{(1)}) \cdots \psi_q(d_{(q)})) \otimes d_{(q+1)}, \end{aligned}$$

where $f_{p+q+1}(c) = d_{(1)} \otimes \cdots \otimes d_{(p+q+1)}$, and ϵ' is defined by the same expression as ϵ' in (6) replacing all occurrences of $c_{(i)}$ by $d_{(i)}$, defines a morphism of A_∞ -bimodules over $\mathcal{H}om(D, A)$.

3. Twists of A_∞ -algebras

We will now recall the twisting procedure of A_∞ -algebras by a Maurer-Cartan element, which was introduced in [6, Chs. 3 and 4] (see also [5], and [10, Ch. 6]).

3.1. Topological definitions

The following definitions are standard. A *topology* on a dg vector space M is a decreasing sequence $\{F^i M\}_{i \in \mathbb{N}_0}$ of dg vector subspaces of M , where $F^0 M = M$ (cf. [4, Ch. III, §2, n° 5]). We understand each dg vector subspace $F^i M$ as a *neighborhood* of the zero element of M , and we say that M is a *topological dg vector space*. Note that $F^i M$ is closed for all $i \in \mathbb{N}_0$. The topology is *Hausdorff* if $\bigcap_{i \in \mathbb{N}_0} F^i M = 0$. We regard k with the discrete topology, *i.e.* the one given by the filtration $F^i k = \{0\}$ for $i \in \mathbb{N}$. We define a *homogeneous morphism of topological dg vector spaces* $f: M \rightarrow N$ to be a homogeneous morphism that is continuous. It is said to be *contracting* if $f(F^i M) \subseteq F^i N$, for all $i \in \mathbb{N}_0$. Given two topological dg vector spaces M and N , the tensor product $M \otimes N$ has the topology

$$F^i(M \otimes N) = \sum_{i_1+i_2 \geq i} F^{i_1} M \otimes F^{i_2} N, \tag{8}$$

for $i \in \mathbb{N}_0$. We remark that this topology is equivalent to the one considered in [14, 1.5], since

$$F^i(M \otimes N) \subseteq F^{\lfloor i/2 \rfloor} M \otimes N + M \otimes F^{\lfloor i/2 \rfloor} N$$

and

$$F^i M \otimes N + M \otimes F^j N \subseteq F^{\min(i,j)}(M \otimes N),$$

where $\lfloor x \rfloor \in \mathbb{Z}$ denotes the floor function of $x \in \mathbb{R}$. The definition of completion of a topological dg vector space M is the standard one via inverse limits

$$\hat{M} = \lim_{\leftarrow \mathbb{N}_0} M/F^j M, \text{ where } F^i \hat{M} = \lim_{\leftarrow \mathbb{N}_0} F^i M/F^j M \tag{9}$$

gives the topology (cf. [4, Ch. III, §2, n° 6]). Moreover, the *completed tensor product* $M \hat{\otimes} N$ is the completion of $M \otimes N$ for the previous topology (8).

We remark that there is an obvious notion of *complete dg coalgebra*, which is just a coalgebra in the monoidal category of complete topological dg vector spaces and continuous morphisms of dg vector spaces (of degree zero) provided with completed tensor products (see [10, Subsection 6.2.1]). The same comments apply to their (complete) dg (bi)comodules.

All the previous definitions can also be made for the category of graded vector spaces, and they are obtained from the previous ones by imposing the differentials to be zero.

3.2. Topological A_∞ -algebras and their bimodules

A *topological augmented A_∞ -algebra* is an augmented A_∞ -algebra such that the underlying graded vector space has a Hausdorff topology satisfying that all morphisms m_i , for $i \in \mathbb{N}$, and ϵ_A are contracting morphisms of topological graded vector spaces. We note that the unit morphism η_A is always contracting. A *morphism of topological augmented A_∞ -algebras* $f_\bullet: A \rightarrow B$ is a morphism of augmented A_∞ -algebras such that f_i is a contracting morphism of topological graded vector spaces for all $i \in \mathbb{N}$.

The next statement is easy to prove.

Proposition 3.1. *Let V be a topological graded vector space with $F^1 V = V$, and let $C(V)$ be the complete coaugmented graded coalgebra $\prod_{n \in \mathbb{N}_0} V^{\otimes n}$ with coproduct*

given by deconcatenation, the obvious counit and coaugmentation, and the filtration $\{\prod_{m \in \mathbb{N}_0} F^n(V^{\hat{\otimes} m})\}_{n \in \mathbb{N}_0}$. Then

- (i) the topology stated before coincides with the usual product topology, and, as a consequence, if $\iota: T^c(V) \rightarrow C(V)$ denotes the canonical inclusion of the graded tensor coalgebra $T^c(V)$, its image is dense in $C(V)$;
- (ii) $\Delta_{C(V)} \circ \iota = (\iota \otimes \iota) \circ \Delta_{T^c(V)}$ and ι commutes with the corresponding coaugmentations and counits.

In particular, if V is complete, $C(V)$ is the cofree counitary (but not necessarily coaugmented) topological coalgebra over V in the monoidal category of complete topological vector spaces provided with continuous morphisms, i.e. given C any complete topological counitary graded coalgebra C and a continuous morphism $f: C \rightarrow V$, there exists a unique morphism of topological counitary graded coalgebras $F: C \rightarrow C(V)$ such that $\pi_1 \circ F = f$, where $\pi_1: C(V) \rightarrow V$ is the canonical projection. It is explicitly given by

$$F(c) = \epsilon_C(c) + f(c) + \sum_{n \in \mathbb{N}_{\geq 2}} f^{\hat{\otimes} n} \circ \Delta_C^{(n)}.$$

Proof. Note that $F^1V = V$ yields that $F^n(V^{\hat{\otimes} m}) = V^{\hat{\otimes} m}$ if $m \geq n$. Indeed, using identities (8) and (9), $F^n(V^{\hat{\otimes} m})$ is given as a sum, one of whose terms is precisely $(F^1V)^{\hat{\otimes} n} \hat{\otimes} (F^0V)^{\hat{\otimes} (m-n)} = V^{\hat{\otimes} m}$. As a consequence,

$$\prod_{m \in \mathbb{N}_0} F^n(V^{\hat{\otimes} m}) = \prod_{m=0}^{n-1} F^n(V^{\hat{\otimes} m}) \times \prod_{m \geq n} V^{\hat{\otimes} m}. \quad (10)$$

This is clearly an open neighborhood of the origin of $C(V)$ for the usual product topology, and conversely, any open neighborhood of the origin of $C(V)$ for the usual product topology includes a subspace of the form (10) for n large enough. This implies the first part of item (i). The second part of it follows from [14, Prop. 1.6 and Eq. (1.9)]. Item (ii) is clear. For the last part, see [14, Thm. 3.1]. \square

The following statement is the topological analogue of a well-known result (cf. for instance [11, Lemma 9.1]), and can be regarded as an extension of [6, Lemma 3.2.17].

Lemma 3.2. *Let A be a complete Hausdorff topological graded vector space provided with contracting maps $\epsilon_A: A \rightarrow k$ and $\eta_A: k \rightarrow A$ of degree zero satisfying that $\epsilon_A \circ \eta_A = \text{id}_k$. Let $I_A = \text{Ker}(\epsilon_A)$ and $\mathcal{B}(A) = C(I_A[1])$. Then the set of structures of topological augmented A_∞ -algebra structures on A with unit η_A and augmentation ϵ_A is in bijection with the set of contracting coderivations of $\mathcal{B}(A)$ that vanish on the image of the coaugmentation of $\mathcal{B}(A)$ and turn $\mathcal{B}(A)$ into a complete coaugmented dg coalgebra.*

Proof. Let A be a topological augmented A_∞ -algebra, and consider the topology of I_A induced by that of A , i.e. $F^p I_A = F^p A \cap I_A$. By Proposition 3.1, the inclusion $\iota: B^+(A) \rightarrow \mathcal{B}(A)$ has dense image. Since the maps m_\bullet are contracting, the same is true for $\pi_1 \circ B$, where $\pi_1: B^+(A) \rightarrow I_A[1]$ is the canonical projection. Hence $\pi_1 \circ B$ is continuous for the induced topology of $\mathcal{B}(A)$, so it can be thus uniquely extended to a continuous linear map \hat{b} from $\mathcal{B}(A)$ to $I_A[1]$. As the filtration defining the topology of $I_A[1]^{\hat{\otimes} n}$ is the completion of that of $I_A[1]^{\otimes n}$, and $\pi_1 \circ B$ is contracting,

the restriction of \hat{b} to $I_A[1]^{\hat{\otimes} n}$ is also contracting. Define \hat{B} from $\mathcal{B}(A)$ to itself as the unique continuous coderivation satisfying that $(\hat{\pi}_1 \circ \hat{B})$ is \hat{b} , where $\hat{\pi}_1: \hat{B}^+(A) \rightarrow I_A[1]$ is the canonical projection. Note that $\hat{B} \circ \eta_{\mathcal{B}(A)} = 0$ and $\iota: B^+(A) \rightarrow \mathcal{B}(A)$ commutes with the differentials. The last property follows from $\hat{\pi}_1 \circ \hat{B} \circ \iota = \pi_1 \circ B$ and Proposition 3.1. Hence, $\hat{B} \circ \hat{B} \circ \iota = \iota \circ B \circ B = 0$. The continuity of $\hat{B} \circ \hat{B}$ and the density of ι imply thus that \hat{B} is a differential. This tells us that $\mathcal{B}(A)$ is a complete coaugmented dg coalgebra whose differential vanishes on the image of its coaugmentation and is contracting.

Conversely, if \hat{B} is a contracting coderivation of $\mathcal{B}(A)$ that is a differential and it vanishes on the image of $\eta_{\mathcal{B}(A)}$, we define $b = \hat{\pi}_1 \circ \hat{B} \circ \iota$, where $\iota: B^+(A) \rightarrow \mathcal{B}(A)$ is the canonical inclusion, and extend b to a coderivation B of $B^+(A)$. Since ι commutes with the coaugmentations, $B \circ \eta_{B^+(A)}$ vanishes. The fact that \hat{B} is a differential implies that B is also, for ι commutes with the differentials and it is injective. Moreover, since \hat{B} is contracting, the multiplications m_\bullet determined by B are also contracting, and the statement follows. \square

We shall denote the coaugmented dg coalgebra of the lemma by $\hat{B}^+(A)$, and call it the *complete bar construction* of A (cf. [6, Def. 3.2.16]).

The proof of the next result is similar to the one given for the previous lemma (cf. [11, Lemma 9.3]).

Lemma 3.3. *Let A and A' be two complete topological augmented A_∞ -algebras, and let $B^+(A)$ and $B^+(A')$ be the corresponding complete bar constructions. Then the set of morphisms $\{f_n: A^{\otimes n} \rightarrow A'\}_{n \in \mathbb{N}}$ of topological augmented A_∞ -algebras is in bijection with the set of contracting morphisms of complete coaugmented dg coalgebras from $\hat{B}^+(A)$ to $\hat{B}^+(A')$.*

We shall denote the last morphism of the lemma by $\hat{B}^+(f_\bullet)$.

Let M be an A_∞ -bimodule over a complete topological augmented A_∞ -algebra A . We say that M is *almost discrete* if there exists an increasing filtration $\{F_i M\}_{i \in \mathbb{N}_0}$ of A_∞ -subbimodules of M (i.e. the inclusion $F_i M$ in M is a strict morphism of A_∞ -bimodules over A) that is exhaustive (i.e. $\cup_{i \in \mathbb{N}_0} F_i M = M$) and such that, for all $i \in \mathbb{N}_0$ there exists $\ell \in \mathbb{N}$ satisfying that $m_{p,q}(\omega \otimes m \otimes \omega')$ vanishes for all $m \in F_i M$, $\omega \in I_A[1]^{\otimes p}$ and $\omega' \in I_A[1]^{\otimes q}$ such that $\omega \otimes \omega' \in F^\ell(I_A[1]^{\otimes(p+q)})$. The next result shows that this property is essentially some kind of discreteness assumption on M .

Lemma 3.4. *Let (A, m_\bullet) be a complete topological augmented A_∞ -algebra, and let M be a graded vector space provided with an exhaustive increasing filtration $\{F_i M\}_{i \in \mathbb{N}_0}$.*

If M is an A_∞ -bimodule over A with structure maps $m_{\bullet,\bullet}$ such that each $F_i M$ is a strict A_∞ -subbimodule of M and it is almost discrete with respect to the given filtration, then the bicoderivation B_M of the bar construction $B^+(A, M, A)$ can be extended to a unique continuous bicoderivation \hat{B}_M of the counitary graded bicomodule over $\hat{B}^+(A)$ given by $\hat{B}^+(A, M, A) = \cup_{i \in \mathbb{N}_0} \hat{B}^+(A) \hat{\otimes} F_i M[1] \hat{\otimes} \hat{B}^+(A)$ that respects the filtration and is a differential, where we assume in the latter space that each $F_i M$ has the discrete topology and the union has the direct limit topology.

Conversely, assume that there exists a continuous bicoderivation B' of the counitary graded bicomodule $\hat{B}^+(A, M, A) = \cup_{i \in \mathbb{N}_0} \hat{B}^+(A) \hat{\otimes} F_i M[1] \hat{\otimes} \hat{B}^+(A)$ over $\hat{B}^+(A)$ that preserves the filtration and is a differential, where each $F_i M$ has the discrete

topology and the union has the direct limit topology. Then M is an almost discrete A_∞ -bimodule over A with respect to the given filtration and B' coincides with the bicoderivation \hat{B}_M mentioned previously.

Proof. Suppose that M is almost discrete and set $\pi_1^M: B^+(A, M, A) \rightarrow M[1]$ the projection given by $\epsilon_{B^+(A)} \otimes \text{id}_{M[1]} \otimes \epsilon_{B^+(A)}$. Let $\iota_M: B^+(A, M, A) \rightarrow \hat{B}^+(A, M, A)$ denote the canonical inclusion. We will show that $\pi_1^M \circ B_M$ can be extended to a unique continuous linear map \hat{b}_M from $\hat{B}^+(A, M, A)$ to $M[1]$ as follows. Since $F_i M$ is a strict A_∞ -subbimodule of M , $\pi_1^M \circ B_M$ sends $B^+(A) \otimes F_i M[1] \otimes B^+(A)$ to $F_i M$. The assumption that M is almost discrete tells us that the previous map vanishes on $F^\ell(B^+(A) \otimes F_i M[1] \otimes B^+(A))$ for some ℓ , so it is continuous for the topology induced from $\hat{B}^+(A) \hat{\otimes} F_i M[1] \hat{\otimes} \hat{B}^+(A)$. Since a continuous morphism of graded vector spaces is uniformly continuous, it can be uniquely extended to the completion, and we obtain thus a continuous morphism \hat{b}_M^i from $\hat{B}^+(A) \hat{\otimes} F_i M[1] \hat{\otimes} \hat{B}^+(A)$ to $F_i M[1]$. As a continuous bicoderivation of the latter space over the complete dg coalgebra $\hat{B}^+(A)$ is determined by the continuous morphism from $\hat{B}^+(A) \hat{\otimes} F_i M[1] \hat{\otimes} \hat{B}^+(A)$ to $F_i M[1]$ given by composition with the canonical projection (cf. [10, Lemme 2.1.2.1]), \hat{b}_M^i determines a unique continuous bicoderivation \hat{B}_M^i on $\hat{B}^+(A) \hat{\otimes} F_i M[1] \hat{\otimes} \hat{B}^+(A)$. Finally, $\hat{B}_M^i \circ \hat{B}_M^i$ vanishes. Indeed, this follows from the verification of a well-known condition on the map $\hat{\pi}_1^M \circ \hat{B}_M^i$ that is analogous to the one in [10, Lemme 2.1.2.1], where $\hat{\pi}_1^M$ denotes $\epsilon_{\hat{B}^+(A)} \hat{\otimes} \text{id}_{M[1]} \hat{\otimes} \epsilon_{\hat{B}^+(A)}$. The mentioned identity for \hat{b}_M^i is just the continuous extension of the analogous condition for the map $\pi_1^M \circ B_M$ restricted to $F_i M$, which holds for B_M is a differential. Now, it is trivial to see that all the maps \hat{b}_M^i (or \hat{B}_M^i) are compatible for all $i \in \mathbb{N}_0$, so they define a unique continuous bicoderivation of $\hat{B}^+(A, M, A)$ over $\hat{B}^+(A)$ which is also a differential. It is clear that the inclusion ι_M of $B^+(A, M, A)$ into $\hat{B}^+(A, M, A)$ commutes with the differentials.

Conversely, if there is a continuous bicoderivation B' of the counitary graded bicomodule $\hat{B}^+(A, M, A) = \cup_{i \in \mathbb{N}_0} \hat{B}^+(A) \hat{\otimes} F_i M[1] \hat{\otimes} \hat{B}^+(A)$ over $\hat{B}^+(A)$ that respects the filtration and is a differential, where each $F_i M$ has the discrete topology and the union has the direct limit topology, $\hat{\pi}_1^M \circ B'$ induces for each $i \in \mathbb{N}_0$ a continuous map from $\hat{B}^+(A) \hat{\otimes} F_i M[1] \hat{\otimes} \hat{B}^+(A)$ to $F_i M[1]$. The fact that $F_i M$ is discrete implies that there exists $\ell \in \mathbb{N}$ such that $\hat{\pi}_1^M \circ B'$ vanishes on $F^\ell(\hat{B}^+(A) \hat{\otimes} F_i M[1] \hat{\otimes} \hat{B}^+(A))$. Define b^i as the composition of the inclusion of $B^+(A) \otimes F_i M[1] \otimes B^+(A)$ inside $\hat{B}^+(A) \hat{\otimes} F_i M[1] \hat{\otimes} \hat{B}^+(A)$ together with $\hat{\pi}_1^M$, and take B^i as the unique bicoderivation of $B^+(A) \otimes F_i M[1] \otimes B^+(A)$ over $B^+(A)$ induced by b^i . Then $F_i M$ is an A_∞ -bimodule for all $i \in \mathbb{N}_0$ such that the inclusion $F_i M \rightarrow F_{i+1} M$ is a strict morphism of A_∞ -bimodules. This induces the structure of an A_∞ -bimodule on their union M , which is almost discrete with respect to the filtration $\{F_i M\}_{i \in \mathbb{N}_0}$. The rest of the proof follows from the comments in the previous paragraph. \square

Let M and N be two almost discrete A_∞ -bimodules over a complete topological augmented A_∞ -algebra A with respect to the increasing filtrations $\{F_i M\}_{i \in \mathbb{N}_0}$ and $\{F_i N\}_{i \in \mathbb{N}_0}$, respectively, and let $f_{\bullet, \bullet}: M \rightarrow N$ be a morphism of A_∞ -bimodules over A . We say that $f_{\bullet, \bullet}$ is *almost discrete (with respect to the given filtrations)* if for each $i \in \mathbb{N}_0$, $f_{p,q}(I_A[1]^p \otimes F_i M \otimes I_A[1]^q) \subseteq F_i N$ for all $p, q \in \mathbb{N}_0$, and there is $\ell \in \mathbb{N}$ such that $f_{p,q}(\omega \otimes m \otimes \omega')$ vanishes for all $m \in F_i M$, $\omega \in I_A[1]^{\otimes p}$ and $\omega' \in I_A[1]^{\otimes q}$ such that $\omega \otimes \omega' \in F^\ell(I_A[1]^{\otimes(p+q)})$.

The proof of the following result is analogous to the one of the previous lemma.

Lemma 3.5. *Let (A, m_\bullet) be a complete topological augmented A_∞ -algebra, and let $(M, m_{\bullet,\bullet}^M)$ and $(N, m_{\bullet,\bullet}^N)$ be two almost discrete A_∞ -bimodules over it, with fixed increasing filtrations.*

If $f_{\bullet,\bullet}: (M, m_{\bullet,\bullet}^M) \rightarrow (N, m_{\bullet,\bullet}^N)$ is almost discrete, then the morphism of counitary dg bicomodules $B^+(f_{\bullet,\bullet})$ can be extended to a unique continuous morphism of counitary dg bicomodules $\hat{B}^+(f_{\bullet,\bullet})$ from $\hat{B}^+(A, M, A)$ to $\hat{B}^+(A, N, A)$, that respects the filtration stated in Lemma 3.4 and where we consider the topologies indicated in that result.

Conversely, given a continuous morphism of counitary dg bicomodules F' from $\hat{B}^+(A, M, A)$ to $\hat{B}^+(A, N, A)$ that respects the filtrations mentioned previously, where we consider the topologies indicated there, it induces a unique almost discrete morphism $f_{\bullet,\bullet}: (M, m_{\bullet,\bullet}^M) \rightarrow (N, m_{\bullet,\bullet}^N)$ of A_∞ -bimodules such that F' coincides with the morphism $\hat{B}^+(f_{\bullet,\bullet})$.

3.3. Twists of topological A_∞ -algebras and A_∞ -bimodules

Let A be a topological augmented A_∞ -algebra, which we assume to be A complete.

We recall that k is a complete counitary dg coalgebra with the discrete topology and zero differential. Fix an element a in the neighborhood F^1A of cohomological degree 1 and zero Adams degree, and define a morphism of topological graded counitary (not necessarily coaugmented) coalgebras $\xi_a: k \rightarrow \hat{B}^+(A)$ as the unique one satisfying that $\hat{\pi}_1 \circ \xi_a$ sends 1_k to $-[a]$, where $\hat{\pi}_1: \hat{B}^+(A) \rightarrow I_A[1]$ is the canonical projection. It exists by the universal property of $\hat{B}^+(A)$ stated in Proposition 3.1. We say that a satisfies the (topological) *Maurer-Cartan equation* if ξ_a commutes with the differentials, i.e. $\hat{B} \circ \xi_a = 0$, and we shall denote the set of such elements by $\text{MC}(A)$. Equivalently, a satisfies the Maurer-Cartan equation if the sum

$$\sum_{n \in \mathbb{N}} (-1)^{\frac{n(n+1)}{2}} m_n(a^{\otimes n}) \tag{11}$$

converges to zero. Note that the previous sum always converges because A is complete.

Let $a \in \text{MC}(A)$. Let $t_a: \hat{B}^+(A) \rightarrow I_A[1]$ be the sum of $\hat{\pi}_1$ and the composition of the canonical projection $\epsilon_{\hat{B}^+(A)}: \hat{B}^+(A) \rightarrow k$ together with the map sending 1_k to $-[a]$. Then, there exists a unique morphism of topological counitary graded coalgebras $T_a: \hat{B}^+(A) \rightarrow \hat{B}^+(A)$ such that $\hat{\pi}_1 \circ T_a = t_a$. Note that T_a is contracting, $T_a \circ \eta_{\hat{B}^+(A)} = \xi_a$ and that $\hat{\pi}_1 \circ T_a \circ T_{-a} = \hat{\pi}_1 \circ T_{-a} \circ T_a = \hat{\pi}_1$. The last chain of identities and the universal property of $\hat{B}^+(A)$ imply that T_a is an isomorphism with inverse T_{-a} .

Given $l, n \in \mathbb{N}_0$, define

$$p_{l,n}^a: A^{\otimes n} \rightarrow A^{\otimes(l+n)}$$

as follows. If $n = 0$, we set $p_{l,0}^a(1_k) = a^{\otimes l}$, where $a^{\otimes 0} = 1_k \in k$. If $n \geq 1$,

$$p_{l,n}^a(x_1 \otimes \cdots \otimes x_n) = \sum_{\bar{l} \in \mathbb{N}_0^{n+1, l}} (-1)^{w''} a^{\otimes l_1} \otimes x_1 \otimes a^{\otimes l_2} \otimes x_2 \otimes \cdots \otimes a^{\otimes l_n} \otimes x_n \otimes a^{\otimes l_{n+1}},$$

where $w'' = \sum_{j=2}^{n+1} l_j (\deg x_1 + \cdots + \deg x_{j-1} + j - 1)$. Note that $p_{l,n}^a$ is a contracting

morphism of topological graded vector spaces of cohomological degree l and Adams degree zero.

Proposition 3.6. *Let (A, m_\bullet) be a complete topological augmented A_∞ -algebra with strict unit 1_A and augmentation ϵ_A , and let $a \in \text{MC}(A)$. Define*

$$m_n^a = \sum_{l \in \mathbb{N}_0} (-1)^{\frac{l(l+1)}{2} + ln} m_{n+l} \circ p_{l,n}^a, \quad (12)$$

for $n \in \mathbb{N}$. It converges by the completeness hypothesis on A . Then (A, m_\bullet^a) is a topological augmented A_∞ -algebra with strict unit 1_A and augmentation ϵ_A .

Proof. It is easy to verify that m_n^a has cohomological degree $2 - n$ and zero Adams degree, and it is contracting, for all $n \in \mathbb{N}$. Consider the differentials B and \hat{B} of the bar constructions $B^+(A)$ and $\hat{B}^+(A)$ of (A, m_\bullet) . Define now the new differential $\hat{B}^a = T_a^{-1} \circ \hat{B} \circ T_a$ for $\mathcal{B}(A)$. The fact that $a \in \text{MC}(A)$ implies that $\hat{B}^a \circ \eta_{\hat{B}^+(A)} = 0$, so \hat{B}^a is a continuous coderivation of the coaugmented graded coalgebra structure of $\mathcal{B}(A)$. Furthermore, it is contracting, because \hat{B} , T_a and its inverse are so. The multiplications of the topological A_∞ -algebra structure it determines by means of Lemma 3.2 are exactly the ones given in (12). The statement thus follows from Lemma 3.2. \square

We shall denote the topological augmented A_∞ -algebra described in the previous proposition by (A, m_\bullet^a) . It will be called the *twisted topological augmented A_∞ -algebra of (A, m_\bullet) by a* . Note that these definitions coincide with the corresponding ones for augmented dg algebras, in the case that m_i vanishes for $i \geq 3$.

Let $a \in \text{MC}(A)$, and let $f_\bullet: A \rightarrow B$ be a morphism of topological augmented A_∞ -algebras. Define b as the unique element of B satisfying that $\hat{B}^+(f_\bullet) \circ \xi_a = \xi_b$. We call it the *image of a under f_\bullet* . Equivalently, b is defined as the convergent sum

$$\sum_{n \in \mathbb{N}} (-1)^{\frac{n(n+1)}{2} + 1} f_n(a^{\otimes n}).$$

Proposition 3.7. *Let $f_\bullet: A \rightarrow B$ be a morphism of complete topological augmented A_∞ -algebras, and let $a \in \text{MC}(A)$. Then, the image b of a under f_\bullet satisfies the Maurer-Cartan equation.*

Proof. This follows directly from

$$\hat{B}' \circ \xi_b = \hat{B}' \circ \hat{B}^+(f_\bullet) \circ \xi_a = \hat{B}^+(f_\bullet) \circ \hat{B} \circ \xi_a = 0,$$

where \hat{B}' denotes the differential of $B^+(B)$ and \hat{B} that of $B^+(A)$. \square

Proposition 3.8. *Let $f_\bullet: A \rightarrow B$ be a morphism of complete topological augmented A_∞ -algebras, and let $a \in \text{MC}(A)$. Take b as the image of a under f_\bullet and define*

$$f_n^a = \sum_{l \in \mathbb{N}_0} (-1)^{\frac{l(l+1)}{2} + ln} f_{n+l} \circ p_{l,n}^a,$$

for $n \in \mathbb{N}$, which converges by the completeness hypothesis on B . Then f_\bullet^a defines a morphism of topological augmented A_∞ -algebras from $(A, m_\bullet^{A,a})$ to $(B, m_\bullet^{B,b})$.

Proof. The statement follows from the fact that f_{\bullet}^a is the morphism of complete topological augmented A_{∞} -algebras satisfying that $\hat{B}^+(f_{\bullet}^a) = T_b^{-1} \circ \hat{B}^+(f_{\bullet}) \circ T_a$. \square

Given $l', l'' \in \mathbb{N}_0$ and $n', n'' \in \mathbb{N}_0$, let us define

$$p_{l', n', l'', n''}^a: A^{\otimes n'} \otimes M \otimes A^{\otimes n''} \rightarrow A^{\otimes (l'+n')} \otimes M \otimes A^{\otimes (l''+n'')}$$

as $p_{l', n'}^a \otimes \text{id}_M \otimes p_{l'', n''}^a$, where the first and last tensor factors morphisms were introduced at the beginning of this subsection.

The proof of the next result is analogous to the one given for Proposition 3.6.

Proposition 3.9. *Let (A, m_{\bullet}) be a complete topological augmented A_{∞} -algebra, $a \in \text{MC}(A)$ and $(M, m_{\bullet, \bullet})$ an almost discrete A_{∞} -bimodule over it. Define*

$$m_{n', n''}^a = \sum_{l', l'' \in \mathbb{N}_0} (-1)^{\frac{(l'+l'')(l'+l''+1)}{2} + l'(n'+n''+1) + l''n''} m_{n'+l', n''+l''} \circ p_{l', n', l'', n''}^a, \quad (13)$$

for $n', n'' \in \mathbb{N}_0$. Then $(M, m_{\bullet, \bullet}^a)$ is an A_{∞} -bimodule over (A, m_{\bullet}^a) .

Proof. Note that (13) is well-defined, since the almost discrete assumption of M implies that $\{m_{n'+l', n''+l''} \circ p_{l', n', l'', n''}^a\}_{l', l'' \in \mathbb{N}_0}$ is locally finite. Define

$$\hat{B}_M^a = (T_a^{-1} \otimes \text{id}_{M[1]} \otimes T_a^{-1}) \circ \hat{B}_M \circ (T_a \otimes \text{id}_{M[1]} \otimes T_a).$$

The statement follows, since $m_{\bullet, \bullet}^a$ are the higher multiplications induced by the unique bicoderivation of $B^+(A, M, A)$ over $B^+(A, m_{\bullet}^a)$ given in Lemma 3.4 associated to \hat{B}_M^a . \square

The new A_{∞} -bimodule will be denoted by $(M, m_{\bullet, \bullet}^a)$, and it is called the *twisted A_{∞} -bimodule of $(M, m_{\bullet, \bullet})$ by a* . Notice that these definitions coincide with the corresponding ones for bimodules over augmented dg algebras, in the case that $m_{i', i''}$ vanishes for $i' + i'' \in \mathbb{N}$.

The proof of the next result is analogous to one given in Proposition 3.9 and it is based on Lemma 3.5.

Proposition 3.10. *Let (A, m_{\bullet}) be a complete topological augmented A_{∞} -algebra, $a \in \text{MC}(A)$, $(M, m_{\bullet, \bullet}^M)$ and $(N, m_{\bullet, \bullet}^N)$ be two almost discrete A_{∞} -bimodules over it, and $f_{\bullet, \bullet}: M \rightarrow N$ be a morphism of A_{∞} -bimodules over A . We assume that $f_{\bullet, \bullet}$ is almost discrete. Define*

$$f_{n', n''}^a = \sum_{l', l'' \in \mathbb{N}_0} (-1)^{\frac{(l'+l'')(l'+l''+1)}{2} + l'(n'+n''+1) + l''n''} f_{n'+l', n''+l''} \circ p_{l', n', l'', n''}^a, \quad (14)$$

for $n', n'' \in \mathbb{N}_0$. Then $(f_{\bullet, \bullet}^a)$ is a morphism of A_{∞} -bimodules from $(M, m_{\bullet, \bullet}^{M, a})$ to $(N, m_{\bullet, \bullet}^{N, a})$ over (A, m_{\bullet}^a) .

Proof. The expression (14) is well-defined, since the almost discrete assumption of $f_{\bullet, \bullet}$ implies that the family of maps $\{f_{n'+l', n''+l''} \circ p_{l', n', l'', n''}^a\}_{l', l'' \in \mathbb{N}_0}$ is locally finite. Define

$$\hat{B}^+(f_{\bullet, \bullet}^a) = (T_b^{-1} \otimes \text{id}_{N[1]} \otimes T_b^{-1}) \circ \hat{B}^+(f_{\bullet, \bullet}) \circ (T_a \otimes \text{id}_{M[1]} \otimes T_a).$$

The statement holds, since $f_{\bullet, \bullet}^a$ is the unique morphism of counitary dg bicomodules over $B^+(A, m_{\bullet}^a)$ given in Lemma 3.5 associated to $\hat{B}^+(f_{\bullet, \bullet}^a)$. \square

The following result will be useful in the sequel.

Lemma 3.11. *Let $f_\bullet: A \rightarrow B$ be a morphism of topological augmented A_∞ -algebras and $a \in \text{MC}(A)$. Take $b \in B$ the image of a under f_\bullet . If $(N, m_{\bullet, \bullet})$ is an almost discrete A_∞ -bimodule over B , then the identity of N gives a strict isomorphism of A_∞ -bimodules over (A, m_\bullet^a) between the almost discrete A_∞ -bimodules $(f_\bullet N, (m'_{\bullet, \bullet})^a)$ and $f_\bullet^a(N, m_{\bullet, \bullet}^b)$, where $m'_{\bullet, \bullet}$ is given by (2).*

Proof. Using Propositions 3.9 and 3.10, the statement is a direct consequence of the following commuting diagram

$$\begin{array}{ccccc}
 & & \hat{B}^+(A, N, A) & \xrightarrow{\hat{B}^+(f_\bullet) \otimes \text{id}_{N[1]} \otimes \hat{B}^+(f_\bullet)} & \hat{B}^+(B, N, B) \\
 & \nearrow T_a \otimes \text{id}_{N[1]} \otimes T_a & \downarrow & & \nearrow T_b \otimes \text{id}_{N[1]} \otimes T_b \\
 \hat{B}^+(A, N, A) & \xrightarrow{\hat{B}^+(f_\bullet^a) \otimes \text{id}_{N[1]} \otimes \hat{B}^+(f_\bullet^a)} & \hat{B}^+(B, N, B) & & \downarrow \hat{B}_N \\
 \downarrow (\hat{B}'_N)^a & & \downarrow \hat{B}'_N & & \\
 & \nearrow T_a \otimes \text{id}_{N[1]} \otimes T_a & \hat{B}^+(A, N, A) & \xrightarrow{\hat{B}^+(f_\bullet) \otimes \text{id}_{N[1]} \otimes \hat{B}^+(f_\bullet)} & \hat{B}^+(B, N, B) \\
 & & \downarrow \hat{B}^b_N & & \downarrow \\
 \hat{B}^+(A, N, A) & \xrightarrow{\hat{B}^+(f_\bullet^a) \otimes \text{id}_{N[1]} \otimes \hat{B}^+(f_\bullet^a)} & \hat{B}^+(B, N, B) & & \nearrow T_b \otimes \text{id}_{N[1]} \otimes T_b
 \end{array}$$

together with Lemmas 3.4 and 3.5. We remark that the commutativity of the back square face is just the completed version of equation (1) (which is a consequence of Lemmas 3.3 and 3.4). \square

3.4. The twisted convolution A_∞ -algebra and the twisted tensor product

Lemma 3.12. *Let A be an augmented dg algebra and C an Adams connected coaugmented A_∞ -coalgebra. Consider the convolution A_∞ -algebra $\mathcal{H} = \text{Hom}(C, A)$ introduced in Subsection 2.3, and define a topology on it by setting $F^i \mathcal{H}$ as the subset of \mathcal{H} given by maps which vanish on the subspace $F_{i-1} C$ of C formed by the sums of the homogeneous elements of Adams degree less than or equal to $i-1$. Then \mathcal{H} is a complete topological augmented A_∞ -algebra.*

Proof. It is easy to see that $\{F^i \mathcal{H}\}_{i \in \mathbb{N}_0}$ defines a Hausdorff decreasing filtration of graded vector subspaces of \mathcal{H} . We shall now prove that the topology is complete. For all $i \in \mathbb{N}_0$, we have the obvious isomorphism of graded vector spaces $\mathcal{H}/F^i \mathcal{H} \simeq \text{Hom}(F_{i-1} C, A)$ induced by $f \mapsto f|_{F_{i-1} C}$, for $f \in \mathcal{H}$. This implies that

$$\lim_{\leftarrow \mathbb{N}_0} \mathcal{H}/F^i \mathcal{H} \simeq \lim_{\leftarrow \mathbb{N}_0} \text{Hom}(F_{i-1} C, A) \simeq \text{Hom}(\lim_{\rightarrow \mathbb{N}_0} F_{i-1} C, A) = \text{Hom}(C, A).$$

Hence, the topological graded vector space \mathcal{H} is complete.

On the other hand, the augmentation of \mathcal{H} is clearly contracting. It thus suffices to prove that the multiplications $m_n^{\mathcal{H}}$ are morphisms of topological graded vector spaces for all $n \in \mathbb{N}$. Let $\phi \in F^i \mathcal{H}$. Then $m_1^{\mathcal{H}}(\phi) = d_A \circ \phi - (-1)^{\text{deg } \phi} \phi \circ \Delta_1$ vanishes on $F_{i-1} C$, because Δ_1 preserves the Adams degree. This implies that $m_1^{\mathcal{H}}(\phi) \in F^i \mathcal{H}$.

Let $n \geq 2$ and take $\phi_1 \in F^{i_1}\mathcal{H}, \dots, \phi_n \in F^{i_n}\mathcal{H}$. By the definition of $m_n^{\mathcal{H}}$ given in (3) and the fact that Δ_n preserves the Adams degree, we see that $m_n^{\mathcal{H}}(\phi_1, \dots, \phi_n)$ vanishes on any element of $F_{i_1+\dots+i_n-1}C$, implying that $m_n^{\mathcal{H}}$ is contracting. The lemma is thus proved. \square

Lemma 3.13. *Let A be an augmented dg algebra, C an Adams connected coaugmented A_∞ -coalgebra, and M a dg bimodule over A . Then the A_∞ -bimodule $M \otimes C$ defined in Subsection 2.3 is almost discrete for the increasing filtration given by $F_i(M \otimes C) = M \otimes F_iC$, where F_iC is the subspace of C formed by the sums of homogeneous elements of Adams degree less than or equal to i . Moreover, if $f_\bullet : C \rightarrow D$ is a morphism of Adams connected coaugmented A_∞ -coalgebras, then the induced map described in (7) is almost discrete.*

Proof. Let $n', n'' \in \mathbb{N}_0$. Take $m \in M$ and $c \in F_iC$ homogeneous, $\omega = \phi_1 \otimes \dots \otimes \phi_p$ in $F^a\mathcal{H}om(C, A)^{\otimes p}$, and $\omega' = \psi_1 \otimes \dots \otimes \psi_q$ in $F^b\mathcal{H}om(C, A)^{\otimes q}$. Then, using Adams grading considerations and (5) we see that $m_{p,q}(\omega \otimes (m \otimes c) \otimes \omega')$ vanishes for $a + b \geq i + 1$, which proves the first part of the lemma. The second part is proved analogously. \square

A (generalized or homotopical) twisting cochain from C to A is an element $\tau \in \mathcal{H} = \mathcal{H}om(C, A)$ of cohomological degree 1 and zero Adams degree such that $\epsilon_A \circ \tau$ and $\tau \circ \eta_C$ vanish and that it satisfies the Maurer-Cartan equation on \mathcal{H} . Note that $\tau \circ \eta_C = 0$ means precisely that $\tau \in F^1\mathcal{H}$. Equation (11) translates explicitly into

$$d_A \circ \tau + \sum_{i \in \mathbb{N}} (-1)^{i(i+1)/2+1} \mu_A^{(i)} \circ \tau^{\otimes i} \circ \Delta_i = 0,$$

where $\mu_A^{(i)} : A^{\otimes i} \rightarrow A$ is the iterative application of the product of A . The twist of the convolution A_∞ -algebra $\mathcal{H}om(C, A)$ by a twisting cochain τ will be called the *twisted convolution algebra* and it will be denoted by $\mathcal{H}om^\tau(C, A)$. The twist of the A_∞ -bimodule $M \otimes C$ by a twisting cochain τ will be called the *twisted tensor product* (of M and C by τ) and it will be denoted by $M \otimes_\tau C$.

Let A be an augmented dg algebra and C an Adams connected coaugmented A_∞ -coalgebra. If $\text{Tw}(C, A)$ denotes the set of twisting cochains from C to A , we have a canonical map

$$\text{Hom}_{\text{aug-dg-alg}}(\Omega^+(C), A) \rightarrow \text{Tw}(C, A) \quad (15)$$

given by $g \mapsto g \circ \tau^C$, where $\tau^C : C \rightarrow \Omega^+(C)$ is the composition of the canonical projection $C \rightarrow C/J_C$, $s_{J_C[-1]}^{-1}$ and the canonical inclusion of $J_C[-1]$ inside $\Omega^+(C)$, where J_C is the cokernel of the coaugmentation η_C of C . It is clear that the map (15) is a bijection (see [13, Lemme 3.17]). Furthermore, by means of the previous morphism we can define the *composition twisting cochain* of a morphism of Adams connected coaugmented A_∞ -coalgebras $f_\bullet : C' \rightarrow C$ with a twisting cochain τ from C to A . Indeed, if $F_\tau \in \text{Hom}_{\text{aug-dg-alg}}(\Omega^+(C), A)$ is the morphism such that $F_\tau \circ \tau^C = \tau$, and $\Omega^+(f_\bullet)$ is the morphism of augmented dg algebras from $\Omega^+(C')$ to $\Omega^+(C)$, the composition twisting cochain $\tau \circ f_\bullet$ is defined as $F_\tau \circ \Omega^+(f_\bullet) \circ \tau^{C'}$. It is easy to see that the composition twisting cochain coincides exactly with the image of τ under the morphism $(f_\bullet)_* : \mathcal{H}om(C, A) \rightarrow \mathcal{H}om(C', A)$ introduced in (4).

4. Hochschild (co)homology of nonnegatively graded connected algebras

4.1. Some preparatory results

Let C be a coaugmented A_∞ -coalgebra and A be a nonnegatively (Adams) graded connected algebra, which we regard in zero cohomological degree. Denote by A^e the *enveloping algebra* of A defined as $A^{\text{op}} \otimes A$. It has two commuting A -bimodule actions: the *outer* structure given by $a.(a' \otimes b').b = a.a' \otimes b'.b$, and the *inner* one of the form $a.(a' \otimes b').b = a'.b \otimes a.b'$, for $a, a', b, b' \in A$ and where the products of the right members of the two definitions are in A .

The next result is clear.

Proposition 4.1. *Let C be a coaugmented A_∞ -coalgebra, let A be a nonnegatively (Adams) graded connected algebra, which we regard in zero cohomological degree, and let $\tau: C \rightarrow A$ be a twisting cochain. Then $A^e \otimes_\tau C$ is a complex of free graded A -bimodules provided with a map of complexes of A -bimodules from it to A of the form $(a \otimes a') \otimes c \mapsto a'.a.\epsilon_C(c)$. The previous twisted tensor product is formed by making use of the outer A -bimodule structure of A^e , whereas the remaining inner structure turns $A^e \otimes_\tau C$ into an A -bimodule. Moreover,*

- (i) *if we apply the functor $\text{Hom}_{A^e}(-, A)$ to $A^e \otimes_\tau C$, the complex obtained is naturally identified with the underlying complex of the twisted convolution A_∞ -algebra $\text{Hom}^\tau(C, A)$ given by twisting the topological A_∞ -algebra of Lemma 3.12;*
- (ii) *if M is a (non Adams) graded A -bimodule, which we regard in zero cohomological degree, the complex obtained by applying functor $M \otimes_{A^e} (-)$ to $A^e \otimes_\tau C$ is naturally identified with the underlying complex of the twisted tensor product A_∞ -bimodule $M \otimes_\tau C$ over $\text{Hom}^\tau(C, A)$, given by twisting the almost discrete A_∞ -bimodule of Lemma 3.13.*

We recall the following theorem, which must be well-known by the experts. It was announced by B. Keller at the X ICRA of Toronto, Canada, in 2002.

Theorem 4.2. *Let C be a minimal coaugmented A_∞ -coalgebra and A be a nonnegatively (Adams) graded connected algebra, which we regard in zero cohomological degree. Then, the following are equivalent:*

- (i) *There is a quasi-isomorphism of augmented minimal A_∞ -algebras*

$$\text{Ext}_A^\bullet(k, k) \rightarrow C^\#.$$

- (ii) *There is a twisting cochain $\tau: C \rightarrow A$ such that the twisted tensor product ${}_{\epsilon_A}A \otimes_\tau C$ is a minimal projective resolution of the trivial left A -module k , where the twisted tensor product is formed by using the A -bimodule structure ${}_{\epsilon_A}A$ with the action induced by the augmentation ϵ_A of A on the left and with the standard action on the right, whereas the left A -module structure of ${}_{\epsilon_A}A \otimes_\tau C$ comes from the remaining left action of A .*
- (iii) *There is a twisting cochain $\tau: C \rightarrow A$ such that the twisted tensor product $A^e \otimes_\tau C$ is a minimal projective resolution of the regular A -bimodule A , where the twisted tensor product is formed by using the outer A -bimodule structure of A^e , whereas the A -bimodule structure of $A^e \otimes_\tau C$ comes from the inner action of the A -bimodule A^e .*

Proof. A short proof of the equivalence between (i) and (ii) was given in [8, Thm. 4.7]. Furthermore, the condition (iii) implies (ii) by a standard argument. Indeed, since $A^e \otimes_\tau C$ is a minimal projective resolution of A , the cone of the quasi-isomorphism $A^e \otimes_\tau C \rightarrow A$ is exact. Since an exact complex of projective left A -modules is homotopically trivial, there exists a contracting homotopy s for it, which is A -linear. Then $\text{id}_k \otimes_A s$ is a contracting homotopy of the cone of $k \otimes_A (A^e \otimes_\tau C) \simeq {}_{\epsilon_A} A \otimes_\tau C \rightarrow k \otimes_A A \simeq k$. To prove that condition (ii) implies (iii), we only have to show that $A^e \otimes_\tau C$ is exact in positive homological degrees. This follows from [3, Prop. 4.1]. \square

Note that, according to our conventions, the differential of ${}_{\epsilon_A} A \otimes_\tau C$ is the map

$$\begin{aligned} m_{0,0}^{{}_{\epsilon_A} A \otimes_\tau C} &= \text{id}_A \otimes \Delta_1 \\ &+ \sum_{i \in \mathbb{N}} (-1)^{\frac{i(i+1)}{2}} (\mu_A^{(i+1)} \otimes \text{id}_C) \circ (\text{id}_A \otimes \tau^{\otimes i} \otimes \text{id}_C) \circ (\text{id}_A \otimes \Delta_{i+1}). \end{aligned}$$

4.2. Main result

We present now the main result of this article.

Theorem 4.3. *Let A be a nonnegatively (Adams) graded connected algebra, that we regard in zero (co)homological degree, let C be a minimal coaugmented A_∞ -coalgebra such that there is a quasi-isomorphism of augmented minimal A_∞ -algebras*

$$\mathcal{E}xt_A^\bullet(k, k) \rightarrow C^\#,$$

and let τ be the associated twisted cochain of Theorem 4.2.

Then, there exists a quasi-isomorphism of A_∞ -algebras from the cochain complex $C^\bullet(A, A)$ computing the Hochschild cohomology of A to $\mathcal{H}om^\tau(C, A)$, which, in particular, induces an isomorphism of graded algebras from $HH^\bullet(A)$ to $H^\bullet(\mathcal{H}om^\tau(C, A))$.

Given any graded A -bimodule M and using the previous quasi-isomorphism of A_∞ -algebras, there exists a quasi-isomorphism of A_∞ -bimodules over $C^\bullet(A, A)$ from $M \otimes_\tau C$ to the chain complex $C_\bullet(A, M)$ computing the Hochschild homology of A with coefficients in M , which, in particular, induces an isomorphism of graded bimodules from $H_\bullet(A \otimes_\tau C)$ to $H_\bullet(A, M)$ over $HH^\bullet(A)$.

Proof. By dualizing the quasi-isomorphism of augmented A_∞ -algebras

$$B^+(A)^\# \rightarrow \mathcal{E}xt_A^\bullet(k, k) \rightarrow C^\#,$$

we obtain a quasi-isomorphism of Adams connected coaugmented A_∞ -coalgebras

$$f_\bullet : C \rightarrow B^+(A).$$

Note that $B^+(A)$ is also Adams connected, for A is nonnegatively graded and connected (see [12, Lemma 2.2]). By the comments in Subsection 2.3, we obtain a morphism $(f_\bullet)_* : \mathcal{H}om(B^+(A), A) \rightarrow \mathcal{H}om(C, A)$ of augmented A_∞ -algebras, which are complete topological A_∞ -algebras by Lemma 3.12. Since f_\bullet respects the Adams degree, $(f_\bullet)_*$ is contracting, so a morphism of topological augmented A_∞ -algebras. By taking a look at the proof of Theorem 4.2 we see that the image under $(f_\bullet)_*$ of τ_A is τ , which yields a morphism of topological augmented A_∞ -algebras $(f_\bullet)_*^{\tau_A} : \mathcal{H}om^{\tau_A}(B^+(A), A) \rightarrow \mathcal{H}om^\tau(C, A)$. On the other hand, by Lemma 3.13

f_\bullet induces an almost discrete morphism of A_∞ -bimodules over $\mathcal{H}om(B^+(A), A)$ of the form

$$\bar{F}_{\bullet, \bullet}: (f_\bullet)_*(A^e \otimes C) \rightarrow A^e \otimes B^+(A),$$

such that $\bar{F}_{0,0} = \text{id}_{A^e} \otimes f_1$. We also have an almost discrete morphism of A_∞ -bimodules over $\mathcal{H}om(B^+(A), A)$

$$F_{\bullet, \bullet}: (f_\bullet)_*(M \otimes C) \rightarrow M \otimes B^+(A), \tag{16}$$

such that $F_{0,0} = \text{id}_M \otimes f_1$. By Proposition 3.10, the twist of $\bar{F}_{\bullet, \bullet}$ by τ_A yields a morphism of A_∞ -bimodules over $\mathcal{H}om^{\tau_A}(B^+(A), A)$ of the form

$$\bar{F}_{\bullet, \bullet}^{\tau_A}: (f_\bullet)_*^{\tau_A}(A^e \otimes_\tau C) \rightarrow A^e \otimes_{\tau_A} B^+(A),$$

where we are using Lemma 3.11 to identify $(f_\bullet)_*^{\tau_A}(A^e \otimes_\tau C)$ and the twist by τ_A of $(f_\bullet)_*(A^e \otimes C)$. It is clearly A^e -linear for the A -bimodule structures coming from the corresponding inner actions of A^e . We know that the codomain of this map is the usual Hochschild resolution of the A -bimodule A (see [9, Fact 2.1]), whereas the domain is the minimal projective resolution of the A -bimodule A by the previous theorem. The map $\bar{F}_{0,0}^{\tau_A}$ is a morphism between different free resolutions of A -bimodules of A , so it is a quasi-isomorphism. This implies that $\bar{F}_{\bullet, \bullet}^{\tau_A}$ is also a quasi-isomorphism.

By item (i) of Proposition 4.1, $\mathcal{H}om_{A^e}(\bar{F}_{0,0}^{\tau_A}, A)$ is the first component of the morphism of augmented A_∞ -algebras $(f_\bullet)_*^{\tau_A}: \mathcal{H}om^{\tau_A}(B^+(A), A) \rightarrow \mathcal{H}om^\tau(C, A)$, which is thus a quasi-isomorphism. The first part of the theorem thus follows. In particular, there is an isomorphism of graded algebras from $HH^\bullet(A)$ to $H^\bullet(\mathcal{H}om^\tau(C, A))$.

Finally, item (ii) of Proposition 4.1 tells us that $\text{id}_M \otimes_{A^e} \bar{F}_{0,0}^{\tau_A}$ is the first component of the morphism of A_∞ -bimodules

$$F_{\bullet, \bullet}^{\tau_A}: (f_\bullet)_*^{\tau_A}(M \otimes_\tau C) \rightarrow M \otimes_{\tau_A} B^+(A)$$

over $\mathcal{H}om^{\tau_A}(B^+(A), A)$ given by twisting (16) by τ_A . Hence, $F_{\bullet, \bullet}^{\tau_A}$ is a quasi-isomorphism. This induces an isomorphism of graded bimodules over $HH^\bullet(A)$ from $H_\bullet(M \otimes_\tau C)$ to $H_\bullet(A, M)$. The theorem is thus proved. \square

Addendum 4.4. *By (3) and (12), we see that the product of $\mathcal{H} = \mathcal{H}om^\tau(C, A)$ is*

$$m_2^{\mathcal{H}, \tau}(\phi_1, \phi_2)(c) = \sum_{\substack{l \in \mathbb{N}_0 \\ \bar{l} \in \mathbb{N}_0^{3, l}}} (-1)^\sigma \tau(c_{(1)}) \cdots \tau(c_{(l_1)}) \phi_1(c_{(l_1+1)}) \tau(c_{(l_1+2)}) \cdots \\ \cdots \tau(c_{(L_2+1)}) \phi_2(c_{(L_2+2)}) \tau(c_{(L_2+3)}) \cdots \tau(c_{(l+2)}), \tag{17}$$

where $\sigma = l + l_2 + \text{deg } \phi_1 \text{ deg } \phi_2 + (l_2 + l_3) \text{ deg } \phi_1 + l_3 \text{ deg } \phi_2$, $L_2 = l_1 + l_2$, $\phi_1, \phi_2 \in \mathcal{H}$ and $c \in C$ are homogeneous, and where the sum in the expression of the comultiplication map $\Delta_{l+2}(c) = c_{(1)} \otimes \cdots \otimes c_{(l+2)}$ is omitted for simplicity.

Analogously, the left action of a homogeneous element $\phi \in \mathcal{H}$ on $m \otimes c \in M \otimes_\tau C$, that is also assumed to be homogeneous, is obtained by using (5) and (13) for $n' = 1$ and $n'' = 0$, and it gives

$$\sum_{\substack{l', l'' \in \mathbb{N}_0 \\ \bar{l}' \in \mathbb{N}_0^{2, l'}}} (-1)^{\sigma'} \tau(c_{(l''+2)}) \cdots \tau(c_{(L'_1+1)}) \phi(c_{(L'_1+2)}) \tau(c_{(L'_1+3)}) \cdots \\ \cdots \tau(c_{(l+2)}) m \tau(c_{(1)}) \cdots \tau(c_{(l'')}) \otimes c_{(l'+1)}, \tag{18}$$

where $\sigma' = l'' + l'_2 + \deg \phi(l'_1 + 1) + \deg c(\deg \phi + l')$, $L'_1 = l'' + l_1$, $l = l' + l''$, and where the sum in the expression of the comultiplication map $\Delta_{l+2}(c) = c_{(1)} \otimes \cdots \otimes c_{(l+2)}$ is omitted for simplicity.

4.3. An application

We shall now provide an application of the previous theorem, showing that the first part gives a different proof of the main result of [15, Thm. 3.2], that computes the cup product for the Hochschild cohomology of any s -Koszul algebra. From our point of view, this computation is just a direct consequence of Theorem 4.3 taking into account that the A_∞ -coalgebra C is just the dual to the A_∞ -algebra considered in [7, Thm. 6.5], and the twisting cochain is given by minus the composition of the canonical projection $C \rightarrow V$ and the canonical inclusion $V \rightarrow A$. However, our main result may also be applied to compute the algebraic structure of the Hochschild (co)homology of any nonnegatively graded connected algebra, e.g. if A is multi-Koszul in the sense of [8], by applying Thm. 4.8 of that article (taking into account, however, that the definition of Maurer-Cartan equation there has a different sign convention).

Let V be a finite dimensional vector space over k and $A = TV/(R)$ be an s -homogeneous algebra for $s \in \mathbb{N}_{\geq 2}$, i.e. $R \subseteq V^{\otimes s}$. We say that A is *generalized Koszul* (or *s -Koszul*, if we want to emphasize the degree of R) if the minimal projective resolution P_\bullet of the trivial left A -module k satisfies that P_n is (a graded free module) generated in degree $\xi_s(n)$, for all $n \in \mathbb{N}_0$, where $\xi_s(2m) = sm$ and $\xi_s(2m + 1) = sm + 1$, for all $m \in \mathbb{N}_0$.

The Koszul property of A implies that $C = \text{Tor}_\bullet^A(k, k)$ satisfies that

$$C_p = \text{Tor}_p^A(k, k) = \bigcap_{i=0}^{\xi_s(p-2)} V^{\otimes i} \otimes R \otimes V^{\otimes (\xi_s(p-2)-i)},$$

for $p \geq 2$, together with $C_0 = \text{Tor}_0^A(k, k) = k$ and $C_1 = \text{Tor}_1^A(k, k) = V$ (see [2, Eq. (2.5)]). From now on we shall only consider $s > 2$, for the case $s = 2$ has already been treated in [9, Prop. 2.2]. We give to C the following A_∞ -coalgebra structure. There are only two nonvanishing comultiplications, Δ_2 and Δ_s , which satisfy that

- (i) $(\mathfrak{p}_{p_1} \otimes \mathfrak{p}_{p_2}) \circ \Delta_2|_{C_p}$ is the canonical inclusion if $p_1 + p_2 = p$ and $\xi_s(p) = \xi_s(p_1) + \xi_s(p_2)$ for $p_1, p_2, p \in \mathbb{N}_0$, and zero else;
- (ii) $(\mathfrak{p}_{p_1} \otimes \cdots \otimes \mathfrak{p}_{p_s}) \circ \Delta_s|_{C_p}$ is the canonical inclusion if $p_1 + \cdots + p_s = p + s - 2$ and $\xi_s(p) = \xi_s(p_1) + \cdots + \xi_s(p_s)$ for $p_1, \dots, p_s, p \in \mathbb{N}$, and zero otherwise;

where $\mathfrak{p}_m : C \rightarrow C_m$ denotes the canonical projection. Note that the nonvanishing statement of item (i) implies that either p_1 or p_2 is even, whereas in the case of item (ii) it yields that p_1, \dots, p_s are odd (and p even). It is now trivial to verify ${}_{\epsilon_A} A \otimes_\tau C$ is the minimal projective (Koszul) resolution of ${}_A k$, where $\tau : C \rightarrow A$ is the twisting cochain given by the composition of the canonical projection $C \rightarrow V$ together with minus the canonical inclusion. By Thm. 4.2, the graded dual of C is quasi-isomorphic to $\text{Ext}_A^\bullet(k, k)$, and it is precisely the one appearing in [7, Thm. 6.5].

If $\bar{n} = (n_1, \dots, n_s) \in \mathbb{N}_0^s$ is a partition of $n \in \mathbb{N}_0$, consider the canonical map $a_{\bar{n}} : V^{\otimes n} \rightarrow V^{\otimes n_1} \otimes \cdots \otimes V^{\otimes n_s}$. Given $n \in \mathbb{N}_0$ and $p_1, p_2 \in \mathbb{N}_0$ such that $n \geq \xi_s(p_1) +$

$\xi_s(p_2)$, define the set of its decompositions of type (p_1, p_2) of an element $\omega \in V^{\otimes n}$ as

$$\{a_{\bar{n}}(\omega) : \bar{n} \in \mathbb{N}_0^5 \text{ is a partition of } n \text{ such that } n_{2i} = \xi_s(p_i) \text{ for } i = 1 \text{ and } 2\}.$$

In what follows we shall denote $a_{\bar{n}}(\omega)$ by $\omega_1\omega'_1\omega_2\omega'_2\omega_3$.

The next result is a more general statement than the one appearing in [15, Thm. 3.2], which only covers the case of Hochschild cohomology. Our proof is completely different.

Theorem 4.5. *Let A be an s -Koszul algebra, with $s \in \mathbb{N}_{>2}$. By Theorem 4.3, the graded algebras $HH^\bullet(A)$ and $H^\bullet(\mathcal{H}om^\tau(C, A))$ are isomorphic, and the product $\phi_1 \cdot \phi_2$ of two cocycles $\phi_1 \in \mathcal{H}^{p_1} = \mathcal{H}om(C_{p_1}, A)$ and $\phi_2 \in \mathcal{H}^{p_2} = \mathcal{H}om(C_{p_2}, A)$ in the latter algebra is described in general by (17). In this particular case, we obtain:*

1. if either p_1 or p_2 is even, then $\phi_1 \cdot \phi_2 \in \mathcal{H}^{p_1+p_2}$ is

$$(\phi_1 \cdot \phi_2)(c) = \phi_1(c_{(1)}) \cdot \phi_2(c_{(2)}),$$

where $c = c_{(1)} \otimes c_{(2)}$, $c_{(1)} \in C_{p_1}$ and $c_{(2)} \in C_{p_2}$, and we have omitted the sum in the expression of c ;

2. if both p_1 and p_2 are odd, then $\phi_1 \cdot \phi_2 \in \mathcal{H}^{p_1+p_2}$ is

$$(\phi_1 \cdot \phi_2)(c) = -(-1)^s \sum \omega_1\phi_1(\omega'_1)\omega_2\phi_2(\omega'_2)\omega_3,^2$$

where the sum is indexed over the set of decompositions $\omega_1\omega'_1\omega_2\omega'_2\omega_3$ of type (p_1, p_2) of the fixed element $c \in C_{p_1+p_2}$, for which the sum and tensor symbols are omitted in order to reduce the notation, and $\omega_i \in V^{\otimes n_{2i}-1}$.

Given any graded A -bimodule, Theorem 4.3 tells us that there is an isomorphism of graded bimodules between $H_\bullet(A \otimes_\tau C)$ and $H_\bullet(A, M)$ over $HH^\bullet(A)$. The left action of a cocycle $\phi \in \mathcal{H}^p = \mathcal{H}om(C_p, A)$ on $m \otimes c \in M \otimes C_q$ is described in general by (18). In this case, we obtain:

1. if q is odd, or both p and q are even, then $\phi \cdot (m \otimes c) \in M \otimes C_{q-p}$ is

$$\phi(c_{(2)}) \cdot m \otimes c_{(1)},$$

where $c = c_{(1)} \otimes c_{(2)}$, $c_{(2)} \in C_p$, and we have omitted the sum in the expression of c ;

2. if p is odd and q are even, then $\phi \cdot (m \otimes c) \in M \otimes C_{q-p}$ is

$$- \sum (-1)^{n_1+n_3+n_5} (\omega_2\phi(\omega'_2)\omega_3m\omega_1) \otimes \omega'_1,$$

where the sum is indexed over the set of decompositions $\omega_1\omega'_1\omega_2\omega'_2\omega_3$ of type $(q-p, p)$ of $c \in C_q$ for which the sum and the tensor signs are omitted in order to reduce the notation, and $\omega_i \in V^{\otimes n_{2i}-1}$.

Proof. This is a direct consequence of Theorem 4.3, Addendum 4.4 and the description of the A_∞ -coalgebra structure of $\text{Tor}_\bullet^A(k, k)$ given in the previous paragraph. \square

²The sign difference with [15] comes from the fact that our definition of cup product respects the Koszul sign rule, whereas the corresponding one in the mentioned article does not.

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