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ON THE BOREL TRANSGRESSION IN THE FIBRATION $G \to G/T$

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Abstract

Let G be a semisimple Lie group with a maximal torus T. We present an explicit formula for the Borel transgression $\tau\colon H^1(T) \to H^2(G/T)$ of the fibration $G \to G/T$. This formula corrects an error in the paper [**9**], and has been applied to construct the integral cohomology rings of compact Lie groups in the sequel works [**4, 6**].

1. Introduction

A Lie group is called *semisimple* if its center is finite, and is called *adjoint* if its center is trivial. In this paper the Lie groups G under consideration are compact, connected and semisimple. The homology and cohomology are over the ring of integers, unless otherwise stated.

For a Lie group G with a maximal torus T let $\pi\colon G \to G/T$ be the quotient fibration. Consider the diagram with top row the cohomology exact sequence of the pair (G, T)

$$\begin{array}{ccccccccc} 0 \longrightarrow H^1(G) & \xrightarrow{i^*} & H^1(T) & \xrightarrow{\delta} & H^2(G,T) & \xrightarrow{j^*} & H^2(G) \longrightarrow \cdots, \\ & & & \searrow^{\tau} & \cong \uparrow \pi^* & & \\ & & & & H^2(G/T) & & \end{array}$$

where, since the pair (G, T) is 1-connected, the induced map π^* is an isomorphism. The *Borel transgression* [**12**, p. 185] in the fibration π is the composition

$$\tau = (\pi^*)^{-1} \circ \delta\colon H^1(T) \to H^2(G/T).$$

It is irrelevant to the choice of a maximal torus on G since, if T' is another maximal torus then the relation $T' = gTg^{-1}$ holds on G for some $g \in G$.

The transgression τ is an essential ingredient of the Leray-Serre spectral sequence $\{E_r^{*,*}(G; \mathcal{R}), d_r \mid r \geqslant 2\}$ of the fibration π, where the coefficient ring $\mathcal{R}$ is either the ring $\mathbb{Z}$ of integers, the field $\mathbb{R}$ of reals, or the cyclic ring $\mathbb{Z}_p$ of finite order p. Precisely, as the quotient manifold G/T is always 1-connected, the Leray-Serre Theorem states that

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- $E_2^{*,*}(G;\mathcal{R}) = H^*(G/T) \otimes H^*(T;\mathcal{R})$;
- the differential $d_2\colon E_2^{*,*} \to E_2^{*,*}$ is determined by τ as
$$d_2(x \otimes t) = (x \cup \tau(t)) \otimes 1,$$
where $x \in H^*(G/T)$, $t \in H^1(T)$.

Furthermore, regarding both $E_3^{*,*}(G;\mathcal{R})$ and $H^*(G;\mathcal{R})$ as graded groups, Leray [**10**] and Reeder [**13**] have shown that
$$E_3^{*,*}(G;\mathbb{R}) = H^*(G;\mathbb{R});$$
Kač [**9**] claimed that, if p is a prime, then
$$E_3^{*,*}(G;\mathbb{Z}_p) = H^*(G;\mathbb{Z}_p).$$
Marlin [**11**] conjectured that, if the group G is 1-connected, then
$$E_3^{*,*}(G;\mathbb{Z}) = H^*(G).$$

Conceivably, an explicit formula for the transgression τ is requested by the spectral sequence approach to the cohomology theory of Lie groups.

Our main result is Theorem 2.5, where with respect to explicitly constructed bases on $H^2(G/T)$ and $H^1(T)$ a formula for τ is obtained. This formula corrects an error concerning the differential d_2 on $E_2^{*,*}$ occurring in [**9**]. It has also been applied in our sequel works [**4, 6**] to construct the integral cohomology rings of compact Lie groups, see Remark 3.4.

2. A formula for the transgression τ

For a semisimple Lie group G with a maximal torus T the tangent space $L(G)$ (resp. $L(T)$) to G at the group unit $e \in G$ (resp. to T at $e \in T$) is also known as the *Lie algebra* (resp. the *Cartan subalgebra*) of G. The exponential map exp of G at the unit e builds up the commutative diagram
$$\begin{array}{ccc} L(T) & \longrightarrow & L(G) \\ \big\downarrow{\scriptstyle \exp} & & \big\downarrow{\scriptstyle \exp} \\ T & \longrightarrow & G, \end{array}$$
where the horizontal maps are the obvious inclusions. Equip the real vector space $L(G)$ with an inner product $(\ ,\ )$ so that the adjoint representation acts as isometries on $L(G)$, and assume that $n = \dim T$.

The exponential map $\exp\colon L(T) \to T$ defines a set $\mathcal{S}(G) = \{L_1, \dots, L_m\}$ of $m = \frac{1}{2}(\dim G - n)$ hyperplanes on $L(T)$, namely, the set of *singular hyperplanes* through the origin in $L(T)$ [**2**, p. 168]. The map exp carries the normal line l_k to L_k through the origin $0 \in L(T)$ onto a circle subgroup on T. Let $\pm\alpha_k \in l_k$ be the non-zero vectors with minimal length so that
$$\exp(\pm\alpha_k) = e, \quad 1 \leqslant k \leqslant m.$$
The subset $\Phi = \{\pm\alpha_k \in L(T) \mid 1 \leqslant k \leqslant m\}$ will be called *the root system of* G.

In addition, the planes in $\mathcal{S}(G)$ divide $L(T)$ into finitely many convex regions, called the *Weyl chambers* of G. Fix a regular point $x_0 \in L(T)$, and let $\mathcal{F}(x_0)$ be the closure of the Weyl chamber containing x_0. Assume that $L(x_0) = \{L_1, \ldots, L_n\} \subset \mathcal{S}(G)$ is the subset consisting of the walls of $\mathcal{F}(x_0)$, and let $\alpha_i \in \Phi$ be the root normal to L_i and pointing toward x_0.

Definition 2.1. The subset $\Delta = \{\alpha_1, \ldots, \alpha_n\} \subset L(T)$ will be called *the set of simple roots* of G relative to the Weyl chamber $\mathcal{F}(x_0)$ [**8**, p. 49].

The *Cartan matrix* of G is the $n \times n$ matrix $A = (b_{ij})_{n \times n}$ defined by $b_{ij} = \frac{2(a_i, \alpha_j)}{(\alpha_j, \alpha_j)}$.

Remark 2.2. Our introduction to the simple roots, as well as the Cartan matrix of G, is *dual* to those that are commonly used in literatures, e.g. [**1, 8**]. Indeed, with respect to the inner product $(\ , \ )$ on $L(T)$ the space $L(T)$ is naturally isomorphic to its dual space $L(T)^*$, and the roots are identified with "coroots".

The present approach to the root system of semisimple Lie groups allows us to perform subsequent construction and calculation coherently on the Euclidean space $L(T)$ without referring to its dual space $L(T)^*$.

On the Euclidean space $L(T)$ there are three distinguished lattices. Firstly, the set $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ of simple roots generates over the integers *the root lattice* Λ_r of G. Next, the pre-image of the exponential map $\exp\colon L(T) \to T$ at the group unit $e \in T$ gives rise to the *unit lattice* $\Lambda_e := \exp^{-1}(e)$ of G. Thirdly, using simple roots one defines the set $\Omega = \{\phi_1, \ldots, \phi_n\} \subset L(T)$ of *fundamental dominant weights* of G by the formula

$$2(\phi_i, \alpha_j)/(\alpha_j, \alpha_j) = \delta_{i,j}$$

that generates over the integers *the weight lattice* Λ_ω of G. It is known that (see [**5**, (3.4)]):

Lemma 2.3. *On the Euclidean space* $L(T)$

$$\Lambda_r \subseteq \Lambda_e \subseteq \Lambda_\omega.$$

In addition

i) *the group G is 1-connected if and only if $\Lambda_r = \Lambda_e$;*
ii) *the group G is adjoint if and only if $\Lambda_e = \Lambda_\omega$;*
iii) *the basis Δ on Λ_r can be expressed by the basis Ω on Λ_ω by*

$$\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} = A \begin{pmatrix} \phi_1 \\ \vdots \\ \phi_n \end{pmatrix},$$

where A is the Cartan matrix of G.

Granted with the notion introduced above we turn to the construction of a canonical basis of the second cohomology $H^2(G/T)$ of the base manifold G/T. For a simple root $\alpha \in \Delta$ let $K(\alpha) \subset G$ be the subgroup with Lie algebra $l_\alpha \oplus L_\alpha$ [**2**, p. 238, Exercise 6], where $l_\alpha \subset L(T)$ is the 1-dimensional subspace spanned by α, and $L_\alpha \subset L(G)$ is the root space (viewed as an oriented real 2-plane) belonging to the root α [**8**, p. 35].

Then the circle subgroup $S^1 = \exp(l_\alpha)$ is a maximal torus on $K(\alpha)$, while the quotient manifold K_α/S^1 is diffeomorphic to the 2-dimensional sphere S^2. Moreover, the inclusion $(K_\alpha, S^1) \subset (G, T)$ of subgroups induces an embedding

$$s_\alpha \colon S^2 = K_\alpha/S^1 \to G/T,$$

whose image is known as the *Schubert variety* associated to the root α [**7**]. By the basis theorem of Chevalley [**3**] the maps s_α with $\alpha \in \Delta$ represent a basis of the second homology $H_2(G/T)$. As a result, if one lets $\omega_i \in H^2(G/T)$ be the Kronecker dual of the homology class represented by the map s_{α_i}, $1 \leqslant i \leqslant n$, then one has that

Lemma 2.4. *The set $\{\omega_1, \dots, \omega_n\}$ is a basis of the cohomology group $H^2(G/T)$.*

On the other hand let $\Theta = \{\theta_1, \dots, \theta_n\}$ be an ordered basis of the unit lattice Λ_e. It defines n oriented circle subgroups on the maximal torus

$$\widetilde{\theta}_i \colon S^1 = \mathbb{R}/\mathbb{Z} \to T \text{ by } \widetilde{\theta}_i(t) := \exp(t\theta_i),\ 1 \leqslant i \leqslant n,$$

that represent also an ordered basis of the first homology $H_1(T)$. As result if we let $t_i \in H^1(T)$ be the class Kronecker dual to the map $\widetilde{\theta}_i$, then

$$H^*(T) = \Lambda(t_1, \dots, t_n) \text{ (i.e. the exterior ring generated by } t_1, \dots, t_n).$$

In view of the relation $\Lambda_r \subseteq \Lambda_e$ by Lemma 2.3 there exists a unique integer matrix $C(\Theta) = (c_{i,j})_{n\times n}$ expressing the ordered basis Δ of Λ_r by the ordered basis Θ of Λ_e. That is, the relation

$$\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} = C(\Theta) \begin{pmatrix} \theta_1 \\ \vdots \\ \theta_n \end{pmatrix}$$

holds on $L(T)$. Our main result is

Theorem 2.5. *With respect to the basis $\{t_1, \dots, t_n\}$ on $H^1(T)$ and the basis $\{\omega_1, \dots, \omega_n\}$ on $H^2(G/T)$, the transgression τ is given by the formula*

$$\begin{pmatrix} \tau(t_1) \\ \vdots \\ \tau(t_n) \end{pmatrix} = C(\Theta)^\tau \begin{pmatrix} \omega_1 \\ \vdots \\ \omega_n \end{pmatrix}, \tag{1}$$

where $C(\Theta)^\tau$ is the transpose of the matrix $C(\Theta)$.

Proof. We begin with the simple case where the group G is 1-connected. Then a basis Θ of the unit lattice $\Lambda_e = \Lambda_r$ can be taken to be $\Delta = \{\alpha_1, \dots, \alpha_n\}$ by i) of Lemma 2.3. Since $C(\Theta)$ is now the identity matrix we are bound to show that

$$\tau(t_i) = \omega_i,\ 1 \leqslant i \leqslant n.$$

For a simple root $\alpha_i \in \Delta$ the inclusion $(K(\alpha_i), S^1) \subset (G, T)$ of subgroups induces the following bundle map over s_{α_i}:

$$\begin{array}{ccc} S^1 & \xrightarrow{\widetilde{\alpha}_i} & T \\ \downarrow & & \downarrow \\ K(\alpha_i) = S^3 & \longrightarrow & G \\ \pi_i \downarrow & & \downarrow \pi \\ K(\alpha_i)/S^1 = S^2 & \xrightarrow{s_{\alpha_i}} & G/T, \end{array}$$

where, since the group G is 1-connected, the subgroup $K(\alpha_i)$ is isomorphic to the 3-sphere S^3 (i.e. the group of unit quaternions), while the map π_i is the Hopf fibration over S^2. This implies that, in the homotopy exact sequence of the fibration π

$$\cdots \longrightarrow \pi_2(G) \xrightarrow{\pi_*} \pi_2(G/T) \xrightarrow{\partial} \pi_1(T) \longrightarrow \cdots,$$

the connecting homomorphism ∂ satisfies the relation

$$\partial\,[s_{\alpha_i}] = [\widetilde{\alpha}_i]\,,\ 1 \leqslant i \leqslant n. \tag{2}$$

Since both of the Hurewicz homomorphisms

$$\pi_1(T) \to H_1(T) \quad \text{and} \quad \pi_2(G/T) \to H_2(G/T)$$

are isomorphisms, and since the transgression τ is Kronecker dual to ∂ in the sense that

$$\tau = Hom(\partial, 1)\colon Hom(\pi_1(T), \mathbb{Z}) \to Hom(\pi_2(G/T), \mathbb{Z})$$

one obtains from (2) that $\tau(t_i) = \omega_i$, $1 \leqslant i \leqslant n$.

Turning to a general situation assume that the group G is semisimple. Let $d\colon (G_0, T_0) \to (G, T)$ be the universal cover of G with T_0 the maximal torus on G_0 that corresponds to T under d. Then, with respect to the canonical identifications (induced by the tangent map of d at the group unit)

$$L(G_0) = L(G) \quad \text{and} \quad L(T_0) = L(T),$$

the exponential map of G admits the decomposition

$$\exp = d \circ \exp_0\colon (L(G_0), L(T_0)) \to (G_0, T_0) \to (G, T),$$

where $\exp_0$ is the exponential map of G_0. It follows that, if we let

$$p(\Lambda_r, \Lambda_e)\colon T_0 = L(T_0)/\Lambda_r \to T = L(T_0)/\Lambda_e$$

be the covering map induced by the inclusion $\Lambda_r \subseteq \Lambda_e$ of the lattices, then

$$d \mid T_0 = p(\Lambda_r, \Lambda_e)\colon T_0 \to T. \tag{3}$$

Note that the induced map $p(\Lambda_r, \Lambda_e)_*$ on $\pi_1(T_0)$ is determined by the matrix $C(\Theta) = (c_{ij})_{n\times n}$ as

$$p(\Lambda_r, \Lambda_e)_*[\widetilde{\alpha}_i] = c_{i,1}[\widetilde{\theta}_1] + \cdots + c_{i,n}[\widetilde{\theta}_n]. \tag{4}$$

On the other hand, by the naturality of homotopy exact sequence of fibrations, the restriction $d \mid T_0$ fits into the commutative diagram

$$\begin{array}{ccc} \pi_2(G_0/T_0) & \xrightarrow[\cong]{\partial_0} & \pi_1(T_0) \\ \| & & \downarrow (d|T_0)_* \\ \pi_2(G/T) & \xrightarrow{\partial} & \pi_1(T), \end{array} \tag{5}$$

where the vertical identification on the left comes from the fact that the covering $d\colon (G_0, T_0) \to (G, T)$ induces a diffeomorphism $G_0/T_0 \cong G/T$, and where ∂_0, ∂ are the connecting homomorphisms in the homotopy exact sequences of the bundles $G_0 \to G_0/T_0$, $G \to G/T$, respectively. It follows that, for a simple root $\alpha_i \in \Delta$, one has

$$\begin{aligned} \partial\,[s_{\alpha_i}] &= (d \mid T_0)_* \circ \partial_0\,[s_{\alpha_i}] && \text{(by the diagram (5))} \\ &= (d \mid T_0)_*\,[\widetilde{\alpha}_i] && \text{(by the proof of the previous case)} \\ &= p(\Lambda_r, \Lambda_e)_*([\widetilde{\alpha}_i]) && \text{(by (3)).} \end{aligned}$$

The proof is now completed by (4), together with the fact (again) that the map τ is Kronecker dual to ∂. □

3. Applications

In a concrete situation formula (1) is ready to apply to evaluate the transgression τ (henceforth, the differential d_2 on $E_2^{*,*}(G)$) associated to the fibration $G \to G/T$. We present below three examples.

Assume firstly that the group G is 1-connected. By i) of Lemma 2.3 one can take the set $\Delta = \{\alpha_1, \dots, \alpha_n\}$ of simple roots as a preferable basis of the unit lattice Λ_e. The transition matrix $C(\Theta)$ is then the identity matrix. Theorem 2.5 implies that

Corollary 3.1. *If the group G is 1-connected, there exists a basis $\{t_1, \dots, t_n\}$ on $H^1(T)$ so that $\tau(t_i) = \omega_i$, $1 \leqslant i \leqslant n$.*

Suppose next that the group G is adjoint. According to ii) of Lemma 2.3 the set $\Omega = \{\phi_1, \dots, \phi_n\}$ of fundamental dominant weights is a basis of Λ_e, and the transition matrix $C(\Theta)$ from Λ_e to Λ_r is the Cartan matrix A of G by iii) of Lemma 2.3. We get from Theorem 2.5 that

Corollary 3.2. *If the group G is adjoint, there exists a basis $\{t_1, \dots, t_n\}$ on $H^1(T)$ so that*

$$\begin{pmatrix} \tau(t_1) \\ \vdots \\ \tau(t_n) \end{pmatrix} = A^\tau \begin{pmatrix} \omega_1 \\ \vdots \\ \omega_n \end{pmatrix},$$

where A is the Cartan matrix of the group G.

Since both of the groups $H^1(T)$ and $H^2(G/T)$ are torsion free, formula (1) is also applicable to evaluate the transgression

$$\tau\colon H^1(T; \mathbb{Z}_p) \to H^2(G/T; \mathbb{Z}_p),$$

for the cohomologies with coefficients in the cyclic ring $\mathbb{Z}_p$ of order $p > 1$. Given a Lie group G denote by $PG := G/\mathcal{Z}(G)$ the associated Lie group of the adjoint type, where $\mathcal{Z}(G)$ is the center of G. In what follows we shall assume that G is one of the 1-connected Lie groups $Sp(n)$, E_6 or E_7, and shall adopt the following conventions:

- a set $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ of simple roots of G is given and ordered as the vertex of the Dykin diagram of G pictured on [**8**, p. 58];
- the preferable basis of the unit lattice Λ_e of the group PG is taken to be Ω.

Granted with the Cartan matrix of G presented on [**8**, p. 59] Corollary 3.2 implies that

Corollary 3.3. *For each of the pairs $(G,p) = (Sp(n),2)$, $(E_6,3)$ and $(E_7,2)$ the kernel and cokernel of the transgression*

$$\tau\colon H^1(T;\mathbb{Z}_p) \to H^2(PG/T;\mathbb{Z}_p)$$

are both isomorphic to $\mathbb{Z}_p$, whose generators are specified in the following table:

(G,p)	Generator of $\ker\tau = \mathbb{Z}_p$	Generator of coker $\tau = \mathbb{Z}_p$
$(Sp(n),2)$	t_n	ω_1
$(E_6,3)$	$t_1 - t_3 + t_5 - t_6$	ω_1
$(E_7,2)$	$t_2 + t_5 + t_7$	ω_2

Remark 3.4. Let G be a compact semisimple Lie group and let p be prime. In [**9**, formula (4)] Kač stated a formula for the differential d_2 on $E_2^{*,*}(G;\mathbb{Z}_p)$ which implies that the transgression τ in the characteristic p is always an isomorphism. This contradicts to Corollary 3.3.

On the other hand, in the context of Schubert calculus a unified presentation of the cohomology ring of the quotient manifolds G/T has been obtained in [**7**, Theorem 1.2]. Combining this result with Theorem 2.5 of the present paper, method to construct the integral cohomology rings of compact connected Lie groups using the spectral sequence $\{E_r^{*,*}(G), d_r\}$ has been developed in the works [**4, 6**].

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