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ON A BASE CHANGE CONJECTURE FOR HIGHER ZERO-CYCLES

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(*communicated by Charles A. Weibel*)

Abstract

We show the surjectivity of a restriction map for higher $(0,1)$-cycles for a smooth projective scheme over an excellent henselian discrete valuation ring. This gives evidence for a conjecture by Kerz, Esnault and Wittenberg saying that base change holds for such schemes in general for motivic cohomology in degrees (i,d) for fixed d being the relative dimension over the base. Furthermore, the restriction map we study is related to a finiteness conjecture for the n-torsion of $\mathrm{CH}_0(X)$, where X is a variety over a p-adic field.

1. Introduction

Let $\mathcal{O}_K$ be an excellent henselian discrete valuation ring with quotient field K and residue field $k = \mathcal{O}_K/\pi\mathcal{O}_K$ and always assume that $1/n \in k$. Let X be a regular scheme, flat and projective over $\mathrm{Spec}\mathcal{O}_K$ of fibre dimension d. Let X_K denote the generic fibre and X_0 the reduced special fibre. Let $\Lambda = \mathbb{Z}/n\mathbb{Z}$.

In [**SS**, Cor. 9.5] and [**EWB**, App.] it is shown that for $X \to \mathrm{Spec}\,\mathcal{O}_K$ smooth and projective and k finite or algebraically closed, the restriction map

$$\mathrm{CH}_1(X)_\Lambda \xrightarrow{\simeq} \mathrm{CH}_0(X_0)_\Lambda$$

is an isomorphism of Chow groups with coefficients in Λ. This result is reproven in [**KEW**] for more general residue fields and generalised to the case that X_0 is a simple normal crossings divisor. In that case one needs to replace $\mathrm{CH}_0(X_0)$ by $H^{2d}_{\mathrm{cdh}}(X_0, \mathbb{Z}/n\mathbb{Z}(d))$, i.e. the hypercohomology of the motivic complex $\mathbb{Z}/n\mathbb{Z}(d)$ in the cdh-topology, which is isomorphic to $\mathrm{CH}_0(X_0)$ for X_0/k smooth. The result then says that if k is finite, or algebraically closed, or $(d-1)!$ prime to m, or A is of equal characteristic, or $X/\mathcal{O}_K$ is smooth with perfect residue field k, then there is an isomorphism

$$\mathrm{CH}_1(X)_\Lambda \xrightarrow{\simeq} H^{2d}_{\mathrm{cdh}}(X_0, \mathbb{Z}/n\mathbb{Z}(d))$$

which is induced by restricting a one-cycle in general position to a zero-cycle on X_0^{sm}. Generalising this result, the following conjecture is stated in Section 10 of [**KEW**]:

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Conjecture 1.1. *The restriction homomorphism*

$$res\colon H^{i,d}(X,\mathbb{Z}/n\mathbb{Z}) \to H^{i,d}_{\mathrm{cdh}}(X_0,\mathbb{Z}/n\mathbb{Z})$$

is an isomorphism for all $i \geqslant 0$.

Here $H^{i,d}(X,\mathbb{Z}/m\mathbb{Z}) = H^i(X,\mathbb{Z}/m\mathbb{Z}(d))$ are the motivic cohomology groups for schemes over Dedekind rings defined in [**Sp**]. In this article we consider the corresponding restriction map on higher Chow groups of zero-cycles with Λ-coefficients

$$res^{\mathrm{CH}}\colon \mathrm{CH}^d(X,2d-i)_\Lambda \to \mathrm{CH}^d(X_0,2d-i)_\Lambda$$

for $X/\mathcal{O}_K$ smooth which we define to be induced by the following composition:

$$res^{\mathrm{CH}}\colon \mathrm{CH}^n(X,m) \to \mathrm{CH}^n(X_K,m) \xrightarrow{\cdot(-\pi)} \mathrm{CH}^{n+1}(X_K,m+1) \xrightarrow{\partial} \mathrm{CH}^n(X_0,m).$$

Here $\cdot(-\pi)$ is the product with $-\pi \in \mathrm{CH}^1(K,1) = K^\times$ defined in [**Bl**, Sec. 5], π is a local parameter for the discrete valuation on K and ∂ is the boundary map coming from the localization sequence for higher Chow groups (see [**Le1**]). We call the composition

$$sp_\pi^{\mathrm{CH}}\colon \mathrm{CH}^n(X_K,m) \xrightarrow{\cdot(-\pi)} \mathrm{CH}^{n+1}(X_K,m+1) \xrightarrow{\partial} \mathrm{CH}^n(X_0,m)$$

a specialisation map for higher Chow groups. We note that res^{CH} does not depend on the choice of π whereas sp_π^{CH} does. For a detailed discussion of the specialisation map see also [**ADIKMP**, Sec. 3].

Our main theorem is the following:

Theorem 1.2. *Let* $X/\mathcal{O}_K$ *be smooth. Then the restriction map*

$$res^{\mathrm{CH}}\colon \mathrm{CH}^d(X,1)_\Lambda \twoheadrightarrow \mathrm{CH}^d(X_0,1)_\Lambda$$

is surjective. This implies in particular the surjectivity part of Conjecture 1.1 for the pair $(2d-1,d)$.

This implies the following corollary:

Corollary 1.3. *Let* $X/\mathcal{O}_K$ *be smooth. Then the specialisation map*

$$sp_\pi^{\mathrm{CH}}\colon \mathrm{CH}^d(X_K,1)_\Lambda \twoheadrightarrow \mathrm{CH}^d(X_0,1)_\Lambda$$

is surjective.

The restriction map in the degree of Theorem 1.2 is of particular interest since it is related to a conjecture on the finiteness of $\mathrm{CH}^d(X_K)[n]$ for K a p-adic field. This is shown in Section 3 as well as the injectivity for $d=2$. Furthermore, Theorem 1.2 together with the main result of [**KEW**] may be considered as a generalization to perfect residue fields of the vanishing of the Kato homology group $KH_3(X,\mathbb{Q}_\ell/\mathbb{Z}_\ell)$ defined in [**SS**] where it was proven for k finite or separably closed.

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2. Main result

Let $\mathcal{O}_K$ be an excellent henselian discrete valuation ring with quotient field K and residue field $k = \mathcal{O}_K/\pi\mathcal{O}_K$ and always assume that $1/n \in k$. From now on let X be a smooth and projective scheme over $\mathrm{Spec}\mathcal{O}_K$ of fibre dimension d in which case we also say that X is of relative dimension d over $\mathcal{O}_K$. Let X_K denote the generic fibre and X_0 the reduced special fibre. By $X_{(p)}$ we denote the set of points $x \in X$ such that $\dim(\overline{\{x\}}) = p$, where $\overline{\{x\}}$ denotes the closure of x in X.

We are going to use the following notation for Rost's Chow groups with coefficients in Milnor K-theory (see [**Ro**, Sec. 5]):

$$\begin{aligned} C_p(X,m) &= \bigoplus_{x\in X_{(p)}} (K^M_{m+p}k(x)) \otimes \mathbb{Z}/n\mathbb{Z}, \\ Z_p(X,m) &= \ker[\partial\colon C_p(X,m) \to C_{p-1}(X,m)], \\ A_p(X,m) &= H_p(C_*(X,m)). \end{aligned}$$

We write $Z_k(X)$ for the group of k-cycles on X, i.e. the free abelian group generated by k-dimensional closed subschemes of X.

Let π be some fixed a local parameter of $\mathcal{O}_K$. We define the restriction map

$$res_\pi\colon C_p(X,m) \to C_{p-1}(X_0, m+1)$$

to be the composition

$$res_\pi\colon C_p(X,m) \to C_{p-1}(X_K, m+1) \xrightarrow{\cdot\{-\pi\}} C_{p-1}(X_K, m+2) \xrightarrow{\partial} C_{p-1}(X_0, m+1).$$

In the above composition the map $C_p(X,m) \to C_{p-1}(X_K,m)$ is defined to be the identity on all elements supported on $X_{(p)} \setminus X_{0(p)}$ and zero on $X_{0(p)}$. The map ∂ is defined to be the boundary map induced by the tame symbol on Milnor K-theory for discrete valuation rings. More precisely, ∂ is defined as follows: Let $\overline{\{x\}}$ be the subscheme corresponding to $x \in X_{(p)}$. Let us assume for simplicity that $\overline{\{x\}}$ is normal. Otherwise we take the normalisation and use the norm map. Now if $y \in \overline{\{x\}}_{(p-1)}$, then y defines a discrete valuation on $k(x)$. Let π' be a local parameter of $k(x)$. Let $\partial^x_y\colon K^M_{n+1}k(x) \to K^M_n k(y)$ be the tame symbol defined by sending $\{\pi', u_1, \ldots, u_n\}$ to $\{\bar{u_1}, \ldots, \bar{u_n}\}$, where the u_i are units in the discrete valuation ring of $k(x)$ and the $\bar{u_i}$ their images in $k(y)$. ∂ is defined to be the sum of all ∂^x_y taken over all $x \in X_{(p)}$ and all $y \in \overline{\{x\}}_{(p-1)}$. Note that the restriction map res_π has to be distinguished from the specialisation map

$$sp^x_{y,\pi'} = \partial^x_y \circ \{-\pi'\}\colon K^M_n k(x) \to K^M_n k(y).$$

$sp^x_{y,\pi'}$ sends $\{\pi'^{i_1}u_1, \ldots, \pi'^{i_n}u_n\}$ to $\{\bar{u_1}, \ldots, \bar{u_n}\}$, where again the u_i are units in the discrete valuation ring of $k(x)$ and the $\bar{u_i}$ their images in $k(y)$.

The map res_π depends on the choice of π but the induced map on homology

$$res\colon A_p(X,m) \to A_{p-1}(X_0, m+1)$$

is independent of the choice. This can be seen as follows: Let $u \in \mathcal{O}_K^\times$ and $\alpha \in C_p(X,m)$. Then $res_{u\pi}(\alpha) = \partial(\{-\pi u\}\cdot\alpha) = \partial(\{-\pi\}\cdot\alpha) + \partial(\{u\}\cdot\alpha)$ is equal to the sum $res_\pi(\alpha) + \partial(\{u\}\cdot\alpha)$. Now if $\alpha \in A_p(X,m)$, then $\partial(\{u\}\cdot\alpha) = 0$ and $res_{u\pi}(\alpha) = res_\pi(\alpha)$. In the following we will write res for res_π, fixing a local parameter $\pi \in O_K$.

We now turn to our principle interest of study, the restriction map

$$res\colon C_2(X,-1) \to C_1(X_0,0).$$

We start with the following lemma:

Lemma 2.1. *The map $res\colon C_2(X,-1) \to C_1(X_0,0)$, after having fixed π, is surjective.*

Proof. Let $\bar{u} \in K_1^M k(x)$ for some $x \in X_0^{(d-1)}$. As in the proof of [**SS**, Lem. 7.2] we can find a relative surface $Z \subset X$ containing x, being regular at x and such that $Z \cap X_0$ contains $\overline{\{x\}}$ with multiplicity 1. Let $Z_0 = \cup_{i\in I} Z_0^{(i)} \cup \overline{\{x\}}$ be the union of the pairwise different irreducible components of the special fiber of Z with those irreducible components different from $\overline{\{x\}}$ indexed by I. Since all maximal ideals, m_i corresponding to $Z_0^{(i)}$ and m_x corresponding to $\overline{\{x\}}$, in the semi-local ring $\mathcal{O}_{Z,Z_0}$ are coprime, the map $\mathcal{O}_{Z,Z_0} \to \prod_{i\in I} \mathcal{O}_{Z,Z_0}/m_i \times \mathcal{O}_{Z,Z_0}/m_x$ is surjective. Therefore, we can find a lift $u \in K_1^M k(z)$, z being the generic point of Z, of $\bar{u}$ which specialises to $\bar{u}$ in $K(\overline{\{x\}})^\times$ and to 1 in $K(Z_0^{(i)})^\times$ for all $i \in I$. □

The main result we are going to prove is the following:

Proposition 2.2. *The restriction map $res\colon A_2(X,-1) \to A_1(X_0,0)$ is surjective.*

It will be implied by the following key lemma:

Key lemma 2.3. *Let $\xi \in \ker[Z_1(X)/n \overset{res}{\to} Z_0(X_0)/n]$, then there is a*

$$\xi' \in \ker[C_2(X,-1) \overset{res}{\to} C_1(X_0,0)]$$

with $\partial(\xi') = \xi$.

Proof of Proposition 2.2. Let $\xi_0 \in \ker[C_1(X_0,0) \overset{\partial}{\to} C_0(X_0,0)]$. By Lemma 2.1 there is a $\xi \in C_2(X,-1)$ with $res(\xi) = \xi_0$. As $res(\partial(\xi)) = \partial(res(\xi)) = 0$, Key lemma 2.3 tells us that there is a $\xi' \in \ker(C_2(X,-1) \to C_1(X_0,0))$ with $\partial\xi' = \partial\xi$. As res is a homomorphism, it follows that $\xi_0 = res(\xi - \xi')$ and $\partial(\xi - \xi') = 0$. Hence $res\colon Z_2(X,-1) \to Z_1(X_0,0)$ is surjective and the commutativity of ∂ and res implies that $res\colon A_2(X,-1) \to A_1(X_0,0)$ is surjective. □

Proof of Key lemma 2.3. We start with the case of relative dimension $d = 1$, i.e. X is a smooth fibered surface over $\mathcal{O}_K$, and consider the following diagram:

$$\begin{array}{ccc} C_2(X,-1) = K(X)^* \otimes \mathbb{Z}/n\mathbb{Z} & \xrightarrow{res} & C_1(X_0,0) = K(X_0)^* \otimes \mathbb{Z}/n\mathbb{Z} \\ \downarrow \partial & & \downarrow \partial \\ Z_1(X)/n & \xrightarrow{res} & Z_0(X_0)/n \end{array}$$

where we write $Z_i(X)/n$ for $C_i(X,-i)$ which are just the cycles of dimension i modulo n. The restriction map in the lowest degree $res\colon Z_1(X)/n \to Z_0(X_0)/n$ agrees with the specialisation map on cycles defined by Fulton in [**Fu**, Rem. 2.3] since X_0 is a principle Cartier divisor and $\partial_y^x(\{-\pi\}) = \mathrm{ord}_{\mathcal{O}_{\overline{\{x\}},y}}(\pi)$. Modifying $\xi \in \ker[Z_1(X)/n \overset{res}{\to} Z_0(X_0)/n]$ by elements equivalent to zero in $Z_1(X)/n$, we may represent it by an element $x \in \ker[Z_1(X) \to Z_0(X_0)]$.

We consider the following short exact sequence of sheaves:

$$0 \to \mathcal{O}^*_{X;X_0} \to \mathcal{M}^*_{X;X_0} \to \mathcal{D}iv(X,X_0) \to 0, \tag{1}$$

where $\mathcal{M}^*_{X;X_0}$(resp. $\mathcal{O}^*_{X;X_0}$) denotes the sheaf of invertible meromorphic functions (resp. invertible regular functions) relative to $\operatorname{Spec}\mathcal{O}_K$ and congruent to 1 in the generic point of X_0, i.e. in $\mathcal{O}_{X,\mu}$, where μ is the generic point of X_0, and $\mathcal{D}iv(X,X_0)$ is the sheaf associated to $\mathcal{M}^*_{X;X_0}/\mathcal{O}^*_{X;X_0}$. In other words, $\mathcal{D}iv(X,X_0)(U)$ is the set of relative Cartier divisors on $U\subset X$ which specialise to zero in X_0. For the concept of relative meromorphic functions and divisors see [**EGA4**, Sec. 20, 21.15].

We want to show that $(\mathcal{D}iv(X,X_0)(X)/\mathcal{M}^*_{X;X_0}(X))/n=0$.

Claim 2.4. $\operatorname{Pic}(X,X_0)\cong\mathcal{D}iv(X,X_0)(X)/\mathcal{M}^*_{X;X_0}(X)$.

Short exact sequence (1) induces the following exact sequence:

$$\mathcal{O}^*_{X;X_0}(X)\to\mathcal{M}^*_{X;X_0}(X)\to\mathcal{D}iv(X,X_0)(X)\to\operatorname{Pic}(X,X_0)\to H^1(X,\mathcal{M}^*_{X;X_0}).$$

Now $\operatorname{Pic}(X,X_0)=H^1(X,\mathcal{O}^*_{X;X_0})$ can also be described as the group of isomorphism classes of pairs $(\mathcal{L},\psi)$ of an invertible sheaf $\mathcal{L}$ with a trivialisation $\psi:\mathcal{L}|_{X_0}\cong\mathcal{O}_{X_0}$ (see e.g. [**SV**, Lem. 2.1]).

The following argument shows that the map $\mathcal{D}iv(X,X_0)(X)\to\operatorname{Pic}(X,X_0)$ is surjective: Let $(\mathcal{L},\psi)\in\operatorname{Pic}(X,X_0)$. The trivialisation ψ gives an isomorphism $\psi\colon\mathcal{L}\otimes_{\mathcal{O}_X}\mathcal{O}_{X_0}\xrightarrow{\cong}\mathcal{O}_{X_0}$ and by localising an isomorphism $\psi_\mu\colon\mathcal{L}_\mu\otimes_{\mathcal{O}_{X,\mu}}\mathcal{O}_{X_0,\mu}\xrightarrow{\cong}\mathcal{O}_{X_0,\mu}$, where μ again denotes the generic point of X_0. Let s denote a lift of $\psi_\mu^{-1}(1)$ under the surjective map $\mathcal{L}_\mu\twoheadrightarrow\mathcal{L}_\mu\otimes_{\mathcal{O}_{X,\mu}}\mathcal{O}_{X_0,\mu}$. Then s is a meromorphic section of $\mathcal{L}$ and the divisor $\operatorname{div}(s)\in\mathcal{D}iv(X,X_0)(X)$ maps to $(\mathcal{L},\psi)$.

It follows that $\operatorname{Pic}(X,X_0)\cong\mathcal{D}iv(X,X_0)(X)/\mathcal{M}^*_{X;X_0}(X)$. ☐

Claim 2.5. $\operatorname{Pic}(X,X_0)$ *is uniquely* n*-divisible.*

Since

$$\operatorname{Pic}(X,X_0)\cong\varprojlim_m\operatorname{Pic}(X_m,X_0)\cong\varprojlim_m H^1(X_0,1+\pi\mathcal{O}_{X_m}),$$

where the first isomorphism follows from [**EGA3**, Thm. 5.1.4], it suffices to show that $H^1(X_0,1+\pi\mathcal{O}_{X_m})$ is uniquely n-divisible. This can be seen as follows:

$$1+\pi\mathcal{O}_{X_m}\supset1+\pi^2\mathcal{O}_{X_m}\supset\cdots\supset1$$

defines a finite filtration on the sheaf $1+\pi\mathcal{O}_{X_m}$ with the following graded pieces: $gr^n=(\pi)^n/(\pi)^{n+1}\cong\mathcal{O}_{X_0}\otimes(\pi)^n$. We use this filtration to define a filtration on $H^1(X_0,1+\pi\mathcal{O}_{X_m})$ by

$$F^n:=\operatorname{Im}(H^1(X_0,1+\pi^n\mathcal{O}_{X_m})\to H^1(X_0,1+\pi\mathcal{O}_{X_m})).$$

The unique divisibility of $H^1(X_0,1+\pi\mathcal{O}_{X_m})$ follows now by descending induction from the exact sequence

$$0\to1+\pi^{n+1}\mathcal{O}_{X_m}\to1+\pi^n\mathcal{O}_{X_m}\to gr^n\to0,$$

the unique divisibility of $H^i(X_0,\mathcal{O}_{X_0}\otimes\pi^n)$ as a finitely generated k-module and the five-lemma. ☐

It follows that $\operatorname{Pic}(X,X_0)/n\cong(\mathcal{D}iv(X,X_0)(X)/\mathcal{M}^*_{X;X_0}(X))/n=0$ and therefore that the class of x in $Z_1(X)/n$, i.e. ξ, is in the image of $\ker[C_2(X,-1)\xrightarrow{res}C_1(X_0,0)]$ under ∂.

We now do the induction step for X of arbitrary relative dimension $d > 1$ over Spec$\mathcal{O}_K$, assuming that the key lemma holds for relative dimension $d-1$, using an idea of Bloch put forward in [**EWB**, App.]. By a standard norm argument we may from now on assume that k is infinite.

As above we may represent ξ by an element of $\ker[Z_1(X) \to Z_0(X_0)]$ and as in the proof of [**KEW**, Prop. 4.1] we may assume that ξ is represented by a cycle of the form $[x] - r[y] \in \ker[Z_1(X) \to Z_0(X_0)]$ with x and y integral and such that y is regular and has intersection number 1 with X_0. Let us recall the argument: First note that one can lift a reduced closed point of X_0 to an integral horizontal one-cycle having intersection number 1 with X_0. Now if $\xi = \sum_{i=1}^{s} n_i[x_i] \in \ker[Z_1(X) \to Z_0(X_0)]$, then we lift $(x_i \cap X_0)_{\mathrm{red}}$ to a one-cycle y_i of the aforementioned type. Furthermore, we choose the same y_i for all the x_i intersecting X_0 in the same closed point. Let r_i be the intersection multiplicity of x_i with X_0. Then also $\sum_{i=1}^{s} n_i r_i[y_i] \in \ker[Z_1(X) \to Z_0(X_0)]$ and it suffices to show the statement for each $x_i - r_i y_i$ separately, i.e. the claim follows.

Let $\tilde{x}$ be the normalisation of x. Since $\mathcal{O}_K$ is excellent, $\tilde{x}$ is finite over x. This implies that there is an imbedding $\tilde{x} \hookrightarrow X' := X \times_{\operatorname{Spec}\mathcal{O}_K} \mathbb{P}^N$ such that the following diagram commutes:

$$\begin{array}{ccc} \tilde{x} & \longrightarrow & X' = X \times_{\operatorname{Spec}\mathcal{O}_K} \mathbb{P}^N \\ \downarrow & & \downarrow {\scriptstyle pr_X} \\ x & \longrightarrow & X \\ \downarrow & & \downarrow \\ \operatorname{Spec}\mathcal{O}_K & \xrightarrow{=} & \operatorname{Spec}\mathcal{O}_K \end{array}$$

Let $[\tilde{x} \cap X'_0] = r'[\bar{z}]$ for $\bar{z}$ an integral zero-dimensional subscheme of X'_0. We take a regular lift z of $\bar{z}$ in $y \times \mathbb{P}^N \subset X'$ which has intersection number 1 with X'_0 and get that $[\tilde{x}] - r'[z] \in \ker[Z_1(X') \to Z_0(X'_0)]$ and $pr_{X*}([\tilde{x}] - r'[z]) = [x] - r[y] = \xi$.

We now use a Bertini theorem by Altman and Kleiman to prove Key lemma 2.3 by an induction on the relative dimension of X over $\mathcal{O}_K$.

Lemma 2.6. *There exist smooth closed subschemes $Z, Z' \subset X'$ with the following properties:*

1. *Z has fiber dimension one, Z' has fiber dimension $d-1$.*
2. *Z contains $\tilde{x}$, Z' contains z.*
3. *The intersection $Z \cap Z' \cap X'_0$ consist of reduced points.*

Proof. First note that for a sheaf of ideals $\mathcal{J} \subset \mathcal{O}_{X'}$ we have the following short exact sequence:

$$0 \to \mathcal{J} \otimes_{\mathcal{O}_{X'}} \mathcal{O}_{X'}(-[X'_0])(M) \to \mathcal{J} \otimes_{\mathcal{O}_{X'}} \mathcal{O}_{X'}(M) \to \mathcal{J} \otimes_{\mathcal{O}_{X'}} i_*\mathcal{O}_{X'_0}(M) \to 0$$

for $i\colon X'_0 \hookrightarrow X'$ and $M \in \mathbb{Z}$. For $M \gg 0$ Serre vanishing implies that $H^1(X', \mathcal{F}(M))$ is 0 for $\mathcal{F}$ coherent and therefore that the map

$$\Gamma(\mathcal{J} \otimes_{\mathcal{O}_{X'}} \mathcal{O}_{X'}(M)) \twoheadrightarrow \Gamma(\mathcal{J} \otimes_{\mathcal{O}_{X'}} \mathcal{O}_{X'_0}(M))$$

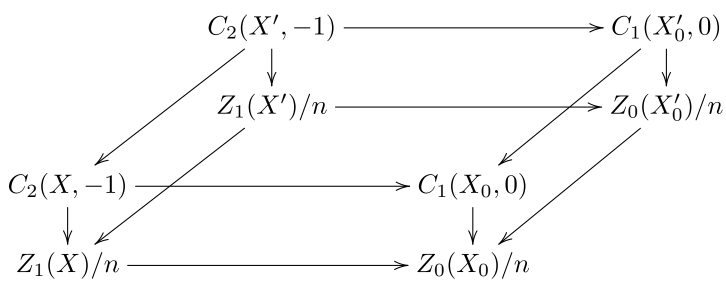
is surjective. This allows us to lift the sections on the right defining subvarieties of X'_0 to sections of a twisted sheaf of ideals on X'.

Let $\mathcal{J}_{\tilde{x}}$ be the sheaf of ideals defining $\tilde{x}$ and $\mathcal{J}_z$ be the sheaf of ideals defining z. Let $p \in \tilde{x} \cap X_0'$ ($q \in z \cap X_0'$). Then $\dim_{X_0}(p) = d \geqslant 2$ and since $\tilde{x}$ (resp. z) is regular, we have that $e_{\tilde{x}\cap X_0'}(p) \leqslant e_{\tilde{x}}(p) = \dim_{k(p)}(\Omega^1_{\tilde{x}}(p)) = 1 < 2$, where $e_{\tilde{x}}(p)$ is the embedding dimension of $\tilde{x}$ at p and analogously for q. Therefore by [**AK**, Thm. 7], we can find sections in $\bar{\sigma}_1, \dots, \bar{\sigma}_{d+N-1} \in \mathcal{J}_{\tilde{x}}|_{X_0'}(M)$ (resp. $\bar{\sigma}' \in \mathcal{J}_{\tilde{x}}|_{X_0'}(M)$) defining smooth subschemes containing p (resp. q) that intersect transversally. Let $\sigma_1, \dots, \sigma_{d+N-1}$ (resp. σ') be liftings under the surjections $\Gamma(\mathcal{J}_{\tilde{x}} \otimes_{\mathcal{O}_{X'}} \mathcal{O}_{X'}(M)) \twoheadrightarrow \Gamma(\mathcal{J}_{\tilde{x}} \otimes_{\mathcal{O}_{X'}} \mathcal{O}_{X_0'}(M))$ and $\Gamma(\mathcal{J}_z \otimes_{\mathcal{O}_{X'}} \mathcal{O}_{X'}(M)) \twoheadrightarrow \Gamma(\mathcal{J}_z \otimes_{\mathcal{O}_{X'}} \mathcal{O}_{X_0'}(M))$. Then the complete intersections $Z := V(\sigma_1, \dots, \sigma_{d+N-1})$ and $Z' := V(\sigma')$ have the desired properties. □

Using these subschemes, we can now do the induction step and finish the proof of the key lemma. Since $Z \cap Z' \cap X_0'$ consists of reduced points, the component z' of $Z \cap Z'$ that contains $z \cap X_0'$ has intersection number 1 with X_0' and is a regular curve as it is regular over the closed point of $\mathrm{Spec}\mathcal{O}_K$. Now since Z' is of relative dimension $d-1$ and z and z' both lie in Z' and satisfy $res([z'] - [z]) = 0$, we get by the induction assumption that there is a ξ with support on Z' restricting to 1 and with $\partial(\xi) = [z'] - [z]$.

By the relative dimension one case proved in the beginning we get that for $\tilde{x}, z' \subset Z$ and $[\tilde{x}] - r'[z']$, which also restricts to 0, there is a ξ' with support on Z such that $res(\xi') = 0$ and $\partial(\xi') = [\tilde{x}] - r'[z']$. It follows that $res(\xi' + r\xi) = 1$ and $\partial(\xi' + r\xi) = [\tilde{x}] - r'[z]$.

By the commutativity of the following diagram we get the result.



The commutativity of the diagram follows from [**Ro**, Sec. 4] since all the maps in question are defined in terms of the 'four basic maps' which are compatible. □

Corollary 2.7. *The restriction map*

$$res^{\mathrm{CH}} \colon \mathrm{CH}^d(X, 1)_\Lambda \to \mathrm{CH}^d(X_0, 1)_\Lambda$$

defined in the introduction is surjective.

Proof. We first show that the homology of the sequence

$$\oplus_{x \in X_0^{(d-2)}} K_2^M k(x) \to \oplus_{x \in X_0^{(d-1)}} K_1^M k(x) \to \oplus_{x \in X_0^{(d)}} K_0^M k(x)$$

is isomorphic to $\mathrm{CH}^d(X_0, 1)$ which implies that $A_1(X_0, 0) \cong \mathrm{CH}^d(X_0, 1)_\Lambda$. This follows from the spectral sequence

$$E_1^{p,q} = \oplus_{x \in X_0^{(p)}} \mathrm{CH}^{r-p}(\mathrm{Spec}\, k(x), -p-q) \Rightarrow \mathrm{CH}^r(X_0, -p-q) \qquad (2)$$

(see [**Bl**, Sec. 10]) for $r = d = \dim X_0$, the fact that $\mathrm{CH}^r(k(x), r) \cong K_r^M(k(x))$ and the vanishing of $\mathrm{CH}^r(\mathrm{Spec}\, k(x), j)$ for $r > j$ as well as the vanishing of $\mathrm{CH}^0(k(x), 1)$.

Using a limit argument and the localization sequence for schemes over a regular noetherian base B of dimension one constructed in [**Le1**], we also get the existence of spectral sequence (2) for $X/\mathcal{O}_K$. Now for the same reasons as above this spectral sequence implies that the homology of

$$\oplus_{x\in X^{(d-2)}} K_2^M k(x) \to \oplus_{x\in X^{(d-1)}} K_1^M k(x) \to \oplus_{x\in X^{(d)}} K_0^M k(x)$$

is isomorphic to $\mathrm{CH}^d(X,1)$ which implies that $A_2(X,-1) \cong \mathrm{CH}^d(X,1)_\Lambda$.

The result now follows from Proposition 2.2 and the compatibility of res and res^{CH}. □

Remark 2.8. The isomorphism $A_1(X_0,0) \cong \mathrm{CH}^d(X_0,1)_\Lambda$ also follows from the isomorphism $\mathrm{CH}^p(X_0,1) \cong H^{p-1}(X_0,\mathcal{K}_p)$ for $p \geqslant 0$ and $\mathcal{K}_p$ the K-theory sheaf (see e.g. [**M**, Cor. 5.3]).

3. Remarks on the injectivity of res

In this section we prove the injectivity of the restriction map for $d = 2$ in our case and remark on implications of the conjectured injectivity.

Conjecture 3.1. *The map* $res\colon A_2(X,-1) \to A_1(X_0,0)$ *is injective.*

Proposition 3.2. *Conjecture 3.1 holds for* $X/\mathcal{O}_K$ *of relative dimension* 2.

Proof. Let $\Lambda := \mathbb{Z}/n$ and $\Lambda(q) := \mu_n^{\otimes q}$. We use the coniveau spectral sequence

$$E_1^{p,q}(X,\Lambda(c)) = \coprod_{x\in X^p} H_x^{p+q}(X,\Lambda(c)) \Rightarrow H_{\text{ét}}^{p+q}(X,\Lambda(c)),$$

where H_x^* is étale cohomology with support in x.

Cohomological purity (resp. absolute purity) gives isomorphisms $H_x^{p+q}(X,\Lambda(c)) \cong H^{q-p}(k(x),\Lambda(c-p))$ which lets us write the above spectral sequence in the following form:

$$E_1^{p,q}(X,\Lambda(c)) = \coprod_{x\in X^p} H^{q-p}(k(x),\Lambda(c-p)) \Rightarrow H_{\text{ét}}^{p+q}(X,\Lambda(c)).$$

For more details see, for example, [**CHK**]. Writing out this spectral sequence for X and X_0 respectively and using the norm residue isomorphism $K_n^M(k)/m \cong H^n(k,\mu_m^{\otimes n})$ for $n \leqslant 2$ (see [**MS**]), we get injective edge morphisms $A_2(X,-1) \hookrightarrow H_{\text{ét}}^3(X,\Lambda(2))$ and $A_1(X_0,-1) \hookrightarrow H_{\text{ét}}^3(X_0,\Lambda(2))$ for dimensional reasons. The restriction map induces a map between these spectral sequences and therefore a commutative diagram

$$\begin{array}{ccc} A_2(X,-1) & \longrightarrow & A_1(X_0,0) \\ \big\downarrow & & \big\downarrow \\ H_{\text{ét}}^3(X,\Lambda(2)) & \xrightarrow{\cong} & H_{\text{ét}}^3(X_0,\Lambda(2)) \end{array}$$

whose lower horizontal morphism is an isomorphism by proper base change. It follows that $A_2(X,-1) \to A_1(X_0,0)$ is injective. □

Remark 3.3. The injectivity of res would have implications for a finiteness conjecture on the n-torsion of $\mathrm{CH}_0(X_K)$ for X_K a smooth projective scheme over a p-adic field with good reduction (see, for example, [**Co**]). More precisely, using the coniveau spectral sequence, we can see that the group $A_1(X_K, 0)$ is isomorphic to $H^{2d-1}_{Zar}(X_K, \mathbb{Z}/n(d))$ and therefore surjects onto $\mathrm{CH}_0(X_K)[n]$. Furthermore it fits into the exact sequence (see [**Ro**, Sec. 5])

$$A_2(X, -1) \to A_1(X_K, 0) \to A_1(X_0, -1) \cong \mathrm{CH}_1(X_0)/n.$$

Now Conjecture 3.1 implies that there is a sequence of injections

$$A_2(X, -1) \hookrightarrow A_1(X_0, 0) \hookrightarrow H^{2d-1}_{\text{ét}}(X_0, \mathbb{Z}/n(d))$$

into the finite group $H^{2d-1}_{\text{ét}}(X_0, \mathbb{Z}/n(d))$. Note that the second injection follows from the Kato conjectures. More precisely, there is an exact sequence

$$KH_3(X_0, \mathbb{Z}/n\mathbb{Z}) \to A_1(X_0, 0) \cong \mathrm{CH}^d(X_0, 1)_\Lambda \to H^{2d-1}_{\text{ét}}(X_0, \mathbb{Z}/n(d))$$

(see [**JS**, Lem. 6.2]) and the Kato homology group $KH_3(X_0, \mathbb{Z}/n\mathbb{Z})$ is zero due to the Kato conjectures (see [**KS**]). Therefore the finiteness of $\mathrm{CH}_0(X_K)[n]$ would depend on the finiteness of $\mathrm{CH}_1(X_0)/n$.

In the case of relative dimension 2 the finiteness of $\mathrm{CH}_1(X_0)/n \cong \mathrm{Pic}(X_0)/n$ can be shown using the injection $\mathrm{Pic}(X_0)/n \hookrightarrow H^2_{\text{ét}}(X_0, \mu_n)$ and the finiteness of $H^2_{\text{ét}}(X_0, \mu_n)$ (see e.g. [**Mi**, VI.2.8]). Therefore Proposition 3.2 implies in particular the finiteness of $\mathrm{CH}_0(X_K)[n]$ for X_K a smooth projective surface over a p-adic field with good reduction. The finiteness of $\mathrm{CH}_0(X_K)[n]$ is known more generally for any smooth surface X_K over a p-adic field K by [**CSS1**] (see also [**CSS2**]).

Remark 3.4. In the light of Remark 3.3 and the base change conjecture for higher zero-cycles stated in the introduction one might ask if

$$\mathrm{CH}^d(X_K, i)[n]$$

is finite for all $i \geqslant 0$ for smooth schemes over p-adic fields.

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