Homology, Homotopy and Applications, vol. 20(1), 2018, pp.5–36

THE HOMOTOPY TYPES OF $U(n)$-GAUGE GROUPS OVER S^4 AND $\mathbb{C}P^2$

TYRONE CUTLER

(*communicated by Donald M. Davis*)

Abstract

The homotopy types of $U(n)$-gauge groups over the two most fundamental 4-manifolds S^4 and $\mathbb{C}P^2$ are studied. We give homotopy decompositions of the $U(n)$-gauge groups over S^4 in terms of certain $SU(n)$- and $PU(n)$-gauge groups and use these decompositions to enumerate the homotopy types of the $U(2)$-, $U(3)$- and $U(5)$-gauge groups. Over $\mathbb{C}P^2$ we provide bounding results on the number of homotopy types of $U(n)$-gauge groups, provide p-local decompositions and give homotopy decompositions of certain $U(n)$-gauge groups in terms of certain $SU(n)$-gauge groups. Applications are then given to count the number of homotopy types of $U(2)$-gauge groups over $\mathbb{C}P^2$.

1. Introduction

If G is a topological group and $P \xrightarrow{p} X$ is a principal G-bundle over a space X then a natural object to study is the gauge group $\mathcal{G}(P)$ of the bundle. This is the group, under composition, of G-equivariant maps $P \to P$ that cover the identity on X.

The study of the homotopy types of certain gauge groups has been a topic of much recent interest and perhaps the most interesting examples arise when G is a subgroup of one of the linear groups. In particular, the cases $G = U(n)$ with X a Riemann surface and $G = SU(n)$ with X a simply connected 4-manifold have applications to geometry [6, 7] and physics [4, 18]. Sutherland [19] and Theriault [20, 23] have contributed valuable work towards the understanding of the first problem, whilst Kono [15] and Theriault [21, 22] have provided limited solutions of the second for certain small values of n.

In each of the cases cited the number of isomorphism classes of principal G-bundles over X is countably infinite, yet in a key paper Crabb and Sutherland [5] have demonstrated that if G is a compact, connected Lie group and X is a finite complex, then the number of distinct homotopy types amongst all the gauge groups of principal G-bundles over X is finite. Consideration of the problem, on the other hand, shows that this number seems to be proportional to the topological and geometric complexity of the space X and the group G. As cells are added to X and the rank of G grows, the number of distinct homotopy types of these gauge groups may grow quickly.

Received January 27, 2017, revised April 14, 2017; published on December 19, 2017.
2010 Mathematics Subject Classification: 55P15, 54C35.
Key words and phrases: gauge group, homotopy type, homotopy decomposition, function space.
Article available at http://dx.doi.org/10.4310/HHA.2018.v20.n1.a2

The case when $X = S^4$ and $G = SU(n) \subseteq U(n)$ was first tackled by Kono [14] and is a natural problem to study for many reasons. In particular, any $U(n)$-bundle over S^4 has a reduction of structure to an $SU(n)$-bundle and it is logical to first study the gauge group associated to the second, simpler, object. The problem for $U(n)$-bundles over a simply connected 4-manifold X is different, however, due to the presence of extra low-dimensional topological information. Specifically there are now two independent obstructions to a $U(n)$-bundle being trivial, namely the first and second Chern classes $(c_1, c_2) \in H^2(X;\mathbb{Z}) \oplus H^4(X;\mathbb{Z})$ which may be used to index the isomorphism classes of $U(n)$-bundles over X.

This double-index makes the problem much more delicate than, say, the study of $SU(n)$-bundles over X. The first Chern class represents a twisting of the bundle over the 2-skeleton $X_2 \simeq \vee S^2$ whilst a non-vanishing second Chern class corresponds to non-triviality of the bundle over the closed 4-cell $e^4 \subseteq X$. In this context the problem has not previously been studied and it is the goal of this paper to begin its consideration by determining the homotopy types of $U(n)$-gauge groups over the 4-sphere S^4 and the complex projective plane $\mathbb{C}P^2$ – two fundamental examples of simply-connected 4-manifolds.

First we examine the homotopy types of the gauge groups belonging to $U(n)$-bundles over S^4. Since any such $U(n)$-bundle has a reduction of structure to an $SU(n)$-bundle one may hope that the homeomorphism $U(n) \cong S^1 \times SU(n)$ is somehow reflected in the topology of the gauge groups of these bundles and we show that this is indeed the case.

For $G = U(n), SU(n), PU(n)$ let $\mathcal{G}_k(S^4, G)$ denote the gauge group of the principal G-bundle over S^4 with second Chern class $k \in H^4(S^4;\mathbb{Z})$.

Theorem 1.1. *The following statements hold:*

1. *For $n \geqslant 3$ there is an isomorphism of principal $\mathcal{G}_k(S^4, SU(n))$-bundles over S^1*
$$\mathcal{G}_k(S^4, U(n)) \cong \mathcal{G}_k(S^4, SU(n)) \times S^1.$$
2. *For $n = 2$ and $k = 2l$ there is an isomorphism of principal $\mathcal{G}_{2l}(S^4, SU(2))$-bundles over S^1*
$$\mathcal{G}_{2l}(S^4, U(2)) \cong \mathcal{G}_{2l}(S^4, SU(2)) \times S^1.$$
3. *For $n = 2$ and $k = 2l+1$ there is an isomorphism of principal S^1-bundles over $\mathcal{G}_{2l+1}(S^4, PU(2))$*
$$\mathcal{G}_{2l+1}(S^4, U(2)) \cong S^1 \times \mathcal{G}_{2l+1}(S^4, PU(2)).$$

It was shown by Kono in [14] that $\mathcal{G}_k(S^4, SU(2)) \simeq \mathcal{G}_l(S^4, SU(2))$ if and only if $(12, k) = (12, l)$ and shown by Kamiyama, Kishimoto, Kono and Tsukuda in [13] that $\mathcal{G}_k(S^4, PU(2)) \simeq \mathcal{G}_l(S^4, PU(2))$ if and only if $(12, k) = (12, l)$. Likewise, it was shown by Hamanaka and Kono in [9] (see also [21]) that $\mathcal{G}_k(S^4, SU(3)) \simeq \mathcal{G}_l(S^4, SU(3))$ if and only if $(k, 24) = (l, 24)$ and shown by Theriault in [22] that $\mathcal{G}_k(S^4, SU(5))_{(p)} \simeq \mathcal{G}_l(S^4, SU(5))_{(p)}$ when rationalised or localised at any prime p if and only if $(k, 120) = (l, 120)$. Therefore Theorem 1.1 immediately gives the following corollary.

Corollary 1.2. *The following statements hold:*

1. *There is a homotopy equivalence $\mathcal{G}_k(S^4, U(2)) \simeq \mathcal{G}_l(S^4, U(2))$ if and only if $(12, k) = (12, l)$.*

2. *There is a homotopy equivalence* $\mathcal{G}_k(S^4, U(3)) \simeq \mathcal{G}_l(S^4, U(3))$ *if and only if* $(k, 24) = (l, 24)$.
3. *There is a local homotopy equivalence* $\mathcal{G}_k(S^4, U(5))_{(p)} \simeq \mathcal{G}_l(S^4, U(5))_{(p)}$ *when rationalised or localised at any prime* p *if and only if* $(k, 120) = (l, 120)$.

Turning now to the complex projective plane it has a cell structure $\mathbb{C}P^2 = S^2 \cup_\eta e^4$, where η is the Hopf map. In this case it is not automatic for a $U(n)$-bundle to have a reduction of structure to an $SU(n)$-bundle and the study of the homotopy types of $U(n)$-gauge groups over $\mathbb{C}P^2$ is a new and entirely unexplored area. If $c_1 = 0$ then the bundle does have a reduction of structure to an $SU(n)$-bundle and there is some hope that this simplification will be reflected in the topology of its gauge group. If $c_1 \neq 0$ then there is a new twisting to the bundle whose effects are previously unstudied. In fact, when c_1 is nontrivial, the problem is very intricate and a complete solution to the problem is beyond the reach of current techniques. In this case not even the homotopy types of the based gauge groups, consisting of those bundle automorphisms that restrict to the identity on the fibre over the basepoint, are well understood.

Let $\mathcal{G}^{(k,l)}(\mathbb{C}P^2, U(n))$ be the gauge group of the $U(n)$-bundle over $\mathbb{C}P^2$ with first and second Chern classes $(c_1, c_2) = (k, l) \in H^2(\mathbb{C}P^2) \oplus H^4(\mathbb{C}P^2)$ and let $\mathcal{G}_*^{(k,l)}(\mathbb{C}P^2, U(n))$ be the based gauge group associated to the same bundle. Also let $\mathcal{G}^{(k,l)}(\mathbb{C}P^2, PU(n))$ be the gauge group of the $PU(n)$-bundle with the indicated characteristic classes and let $\mathcal{G}^l(\mathbb{C}P^2, SU(n))$ be the gauge group of the $SU(n)$-bundle over $\mathbb{C}P^2$ with second Chern class $c_2 = l \in H^4(\mathbb{C}P^2)$.

Many results for general $U(n)$-gauge groups are stated and proved in section 4. These are then refined and applied to the case of $U(2)$ to produce our most complete results. We prove the following.

Theorem 1.3. *Let* k, l *be integers. Then the following hold:*

1. *There is a homotopy equivalence*
$$\mathcal{G}^{(k,l)}(\mathbb{C}P^2, U(2)) \simeq \begin{cases} \mathcal{G}^{(0,l')}(\mathbb{C}P^2, U(2)), & k \text{ even}, \\ \mathcal{G}^{(1,l')}(\mathbb{C}P^2, U(2)), & k \text{ odd} \end{cases}$$
for a suitable integer l'.
2. *There is a homotopy equivalence*
$$\mathcal{G}^{(k,l)}(\mathbb{C}P^2, U(2)) \simeq \mathcal{G}^{(k,l+12)}(\mathbb{C}P^2, U(2)).$$
3. *There is an isomorphism of principal* $\mathcal{G}^l(\mathbb{C}P^2, SU(n))$*-bundles over* S^1
$$\mathcal{G}^{(0,l)}(\mathbb{C}P^2, U(n)) \cong S^1 \times \mathcal{G}^l(\mathbb{C}P^2, SU(n)).$$
4. *When localised away from 2 there is a product splitting*
$$\mathcal{G}^{(1,l)}(\mathbb{C}P^2, U(2)) \simeq S^1 \times \mathcal{G}^{4l-1}(\mathbb{C}P^2, SU(2)).$$
5. *For any integer values of* l, l' *it holds that*
$$\mathcal{G}^{(0,l)}(\mathbb{C}P^2, U(2)) \not\simeq \mathcal{G}^{(1,l')}(\mathbb{C}P^2, U(2)).$$
6. *For any integer values of* l, l' *it holds for the based gauge groups that*
$$\mathcal{G}_*^{(0,l)}(\mathbb{C}P^2, U(2)) \not\simeq \mathcal{G}_*^{(1,l')}(\mathbb{C}P^2, U(2)).$$

The $SU(2)$-gauge groups $\mathcal{G}^l(\mathbb{C}P^2, SU(2))$ appear in 3 and 4 of Theorem 1.3. These gauge groups were studied by Kono and Tsukuda in [15] and their Theorem 1.2 may be applied to yield the following corollary.

Corollary 1.4. *There is a homotopy equivalence* $\mathcal{G}^{(0,l)}(\mathbb{C}P^2, U(2)) \simeq \mathcal{G}^{(0,l')}(\mathbb{C}P^2, U(2))$ *if and only if* $(6,l) = (6,l')$. *When localised away from 2 there is a homotopy equivalence* $\mathcal{G}^{(1,l)}(\mathbb{C}P^2, U(2)) \simeq \mathcal{G}^{(1,l')}(\mathbb{C}P^2, U(2))$ *if and only if* $(4l-1, 6) = (4l'-1, 6)$. *In particular, when localised at an odd prime* $p \geqslant 5$, *the gauge group* $\mathcal{G}^{(1,l)}$ *has the trivial homotopy type*

$$\mathcal{G}^{(1,l)}(\mathbb{C}P^2, U(2)) \simeq U(2) \times Map_*(\mathbb{C}P^2, U(2)).$$

A similar statement can be made for $U(3)$-gauge groups using the material in section 4 and the information on $SU(3)$-gauge groups contained in [21]. We leave its formulation to the interested reader.

What we show leaves us just one step short of a complete classification of the $U(2)$-gauge groups over $\mathbb{C}P^2$. We partition the homotopy types into the two non-intersecting sets $\{\mathcal{G}^{(0,l)}(\mathbb{C}P^2, U(2))\}$, $\{\mathcal{G}^{(1,l)}(\mathbb{C}P^2, U(2))\}$ and give a full classification of the homotopy types in the first set and a full classification of the odd-primary homotopy types in the second. The discrepancy that remains is in the second set. In particular, 2 of Theorem 1.3 shows that there are at most 4 distinct homotopy types amongst the 2-local gauge groups $\mathcal{G}^{(1,l)}(\mathbb{C}P^2, U(2))$. In section 5.2 we examine the low dimensional homotopy of their classifying spaces and show that there are at least 2 distinct homotopy types. It is undecided whether the actual number of distinct homotopy types of these gauge groups is 2, 3 or 4. A complete integral statement would be the most desirable resolution.

The layout of the paper is as follows. In section 2 we present preliminary information and set up notation for gauge groups. We examine the relationship between $U(n)$, $SU(n)$ and $PU(n)$ and give a classification of their principal bundles over S^4 and $\mathbb{C}P^2$. In section 3 we study the homotopy types of $U(n)$-gauge groups over S^4 and prove Theorem 1.1. Section 4 is dedicated to examining the homotopy types of $U(n)$-gauge groups over $\mathbb{C}P^2$. This section is broken into three subsections. In the first we study the connecting maps for certain evaluation fibrations, in the second we give p-local decompositions of the $U(n)$-gauge groups and in the third we examine the relationship between certain $U(n)$- and $SU(n)$-gauge groups. Finally, in section 5, we apply what we have collected to the case of $U(2)$-gauge groups over $\mathbb{C}P^2$. In 5.1 we study the homotopy types of the based $U(2)$-gauge groups and their classifying spaces and in 5.2 we study the homotopy types of the full $U(2)$-gauge groups and complete the proof of Theorem 1.3.

2. Preliminaries

Let G be a connected, compact Lie group and $P \xrightarrow{p} X$ a principal G-bundle over a connected, finite complex X. Let $\mathcal{G}(P)$ denote the gauge group of P and $\mathcal{G}_*(P)$ denote the based gauge group consisting of those gauge transformations of P that reduce to the identity on the fibre over the basepoint. Let BG be the classifying space for G and $EG \to BG$ the universal G-bundle with contractible total space. Then it is well known that the isomorphism classes of principal G-bundles over X are in one-to-one correspondence with the homotopy classes of maps $X \to BG$ via the prescription $f \mapsto f^*EG$ which forms the pullback bundle $f^*EG \to X$.

Choose a map $f\colon X \to BG$ that classifies the bundle $P \xrightarrow{p} X$. Then Gottlieb [8] has shown the existence of homotopy equivalences

$$B\mathcal{G}(P) \simeq Map^f(X, BG), \qquad B\mathcal{G}_*(P) \simeq Map^f_*(X, BG), \tag{2.1}$$

where $B\mathcal{G}(P)$ and $B\mathcal{G}_*(P)$ are the classifying spaces of $\mathcal{G}(P)$ and $\mathcal{G}_*(P)$ respectively, $Map^f(X, BG)$ denotes the path component of f in the space of free maps $X \to BG$ and $Map^f_*(X, BG)$ denotes the path component of f in the space of based maps $X \to BG$.

Using the representations (2.1) leads to the evaluation fibration for $\mathcal{G}(P)$

$$\cdots \to \mathcal{G}(P) \to G \xrightarrow{\delta} B\mathcal{G}_*(P) \to B\mathcal{G}(P) \xrightarrow{ev} BG, \tag{2.2}$$

where $ev\colon B\mathcal{G}(P) \simeq Map^f(X, BG) \to BG$ is given by evaluation at the basepoint of X. In (2.2) we have made explicit the homotopy equivalence $G \simeq \Omega BG$ and denoted the fibration connection map by $\delta\colon G \to B\mathcal{G}_*(P)$. This map shall play a prominent rôle in the following.

The situation with which we shall primarily be concerned is where $X = S^4$ or $X = \mathbb{C}P^2$ and $G = U(n)$, however, we shall also have need to consider $SU(n)$- and $PU(n)$-bundles over these spaces and their gauge groups. For this purpose we shall now briefly discuss the relationship between these groups and set up the notation required in later sections. In the following we shall always assume that $n \geqslant 2$. The special case of $U(1)$ will be treated separately.

The special unitary group $SU(n)$ is related to $U(n)$ by the fibration sequence

$$SU(n) \xrightarrow{j} U(n) \xrightarrow{det} S^1,$$

which is split by the inclusion $S^1 \cong U(1) \hookrightarrow U(n)$. There results a homeomorphism $U(n) \cong SU(n) \times S^1$ although this map does not respect the H-space structures.

The projective unitary group $PU(n)$ is the quotient of $U(n)$ by its centre, which is comprised of the diagonal matrices $\{\lambda I_n | \lambda \in S^1\} \cong S^1$. Equivalently it is the quotient of $SU(n)$ by its centre $\mathbb{Z}_n$ of nth roots of unity along the diagonal matrices. This is displayed in the following homotopy commutative diagram whose columns and rows are homotopy fibrations which we use to define the projection maps π, ρ, and the centre inclusion Δ

$$\begin{array}{ccccc}
\mathbb{Z}_n & \xrightarrow{\delta} & S^1 & \xrightarrow{n} & S^1 \\
\downarrow & & \downarrow{\scriptstyle \Delta} & & \| \\
SU(n) & \xrightarrow{j} & U(n) & \xrightarrow{det} & S^1 \\
\downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \pi} & & \downarrow \\
PU(n) & = & PU(n) & \longrightarrow & * \\
\downarrow & & \downarrow & & \downarrow \\
K(\mathbb{Z}_n, 1) & \xrightarrow{\delta} & BS^1 & \xrightarrow{n} & BS^1.
\end{array} \tag{2.3}$$

The maps labelled n denote the degree n self maps on the Eilenberg-Mac Lane spaces. Something to take note of at this point is that the groups $U(n)$, $SU(n)$ and $PU(n)$ all share the same higher dimensional homotopy and, in particular, there are homotopy equivalences

$$\Omega^k SU(n) \overset{\Omega^k j}{\simeq} \Omega^k U(n) \overset{\Omega^k \pi}{\simeq} \Omega^k PU(n), \qquad \text{for } k \geqslant 2. \tag{2.4}$$

The last thing to address is the classification of principal G-bundles over $X = S^4, \mathbb{C}P^2$ for $G = U(n)$, $SU(n)$ and $PU(n)$ that shall provide our scheme for labelling the components of the mapping spaces in (2.1). For this we need to enumerate the elements of the homotopy set $[X, BG]$ for each possible case. To begin note that S^4 and $\mathbb{C}P^2$ are 4-dimensional, so by connectivity the second Chern class induces a bijection of sets

$$c_2\colon [S^4, BSU(n)] \xrightarrow{\cong} H^4(S^4;\mathbb{Z}) \cong \mathbb{Z},$$
$$c_2\colon [\mathbb{C}P^2, BSU(n)] \xrightarrow{\cong} H^4(\mathbb{C}P^2;\mathbb{Z}) \cong \mathbb{Z}.$$

Now consider the homotopy fibrations $BSU(n) \xrightarrow{Bj} BU(n) \xrightarrow{c_1} K(\mathbb{Z}, 2)$ and $BSU(n) \xrightarrow{B\rho} BPU(n) \xrightarrow{\omega_2} K(\mathbb{Z}_n, 2)$. For $X = S^4$ the maps Bj, $B\rho$ induce bijections

$$c_2\colon [S^4, BU(n)] \cong [S^4, BPU(n)] \cong [S^4, BSU(n)] \xrightarrow{\cong} H^4(S^4) \cong \mathbb{Z}$$

so that isomorphism classes of principal $U(n)$-, $PU(n)$- and $SU(n)$-bundles over S^4 are all in correspondence according to the integer value of their second Chern class.

Similarly, for $X = \mathbb{C}P^2$ we get bijections

$$(c_1, c_2)\colon [\mathbb{C}P^2, BU(n)] \xrightarrow{\cong} H^2(\mathbb{C}P^2;\mathbb{Z}) \oplus H^4(\mathbb{C}P^2;\mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z},$$
$$(w_2, c_2)\colon [\mathbb{C}P^2, BPU(n)] \xrightarrow{\cong} H^2(\mathbb{C}P^2;\mathbb{Z}_n) \oplus H^4(\mathbb{C}P^2;\mathbb{Z}) \cong \mathbb{Z}_n \oplus \mathbb{Z}.$$

3. Homotopy types of $U(n)$-gauge groups over S^4

Let G be a compact, connected, simple Lie group and $p_k\colon P_k^G \to S^4$ the principal G-bundle classified by the degree k map $S^4 \to BG$. In this section we work only over the base space S^4 but shall be interested in comparing the gauge groups of the bundles P_k^G as G and k vary. Since no confusion can arise we introduce the notation $\mathcal{G}_k^G = \mathcal{G}(P_k^G) = \mathcal{G}_k(S^4, G)$ for the gauge group of the bundle P_k^G, with similar notation for the based gauge groups. In subsequent sections we shall revert to the notation introduced previously as we shall have need to differentiate between the gauge groups belonging to bundles over different spaces.

From equation (2.1) we get a model for the classifying space of the gauge group $\mathcal{G}_k^G$

$$B\mathcal{G}_k^G \simeq Map^k(S^4, BG) \tag{3.1}$$

and this leads to the evaluation fibration sequence of equation (2.2)

$$\cdots \to G \xrightarrow{\delta_k^G} B\mathcal{G}_{*k}^G \xrightarrow{i^G} B\mathcal{G}_k^G \xrightarrow{e^G} BG. \tag{3.2}$$

Then using (2.1) again we obtain a string of homotopy equivalences for the classifying space of the based gauge group $\mathcal{G}_{*k}^G$

$$B\mathcal{G}^G_{*k} \simeq Map^k_*(S^4, BG) = \Omega^4_k BG \simeq \Omega^3_k G \simeq \Omega^3_0 G \tag{3.3}$$

and with respect to these homotopy equivalences we have the following result due to Lang.

Theorem 3.1 (Lang [16])**.** *The triple adjoint of δ^G_k is the generalised Samelson product*

$$\langle k \cdot \epsilon_3, id_G \rangle \colon S^3 \wedge G \to G, \tag{3.4}$$

where $\epsilon_3 \in \pi_3(G) \cong \mathbb{Z}$ is a generator. □

3.1. Proof of Theorem 1.1

We begin by relating the evaluation fibration sequences (3.2) for $G = U(n)$ and $G = SU(n)$ using the map $Bj \colon BSU(n) \to BU(n)$. In cohomology $Bj^* \colon H^4(BU(n)) \to H^4(BSU(n))$ is epic, taking the second Chern class c_2 isomorphically between the two groups. Since this corresponds with our labelling of the components of the mapping spaces (3.1) there results a map $\widehat{Bj} \colon B\mathcal{G}^{SU(n)}_k \to B\mathcal{G}^{U(n)}_k$ which in turn, as $Bj \circ e^U(f) = Bj(f(*)) = e^{SU} \circ \widehat{Bj}(f)$ for $f \in B\mathcal{G}^{U(n)}_k$, gives a map between the evaluation fibrations. Given the homotopy equivalences (3.3), (2.4) it is clear that the induced map of fibres is the homotopy equivalence $\Omega^3 j$.

We remark at this stage that for $n = 1$ the problem is trivial: $U(1) \cong S^1$ is abelian and $BS^1 \simeq K(\mathbb{Z}, 2)$ is an Eilenberg-Mac Lane space. Over S^4 there is only the trivial $U(1)$-bundle and with regards to its gauge group $\mathcal{G}$ we have $B\mathcal{G} \simeq Map^0(S^4, K(\mathbb{Z}, 2)) \simeq K(\mathbb{Z}, 2)$ and $B\mathcal{G}_* \simeq Map^0_*(S^4, K(\mathbb{Z}, 2)) \simeq *$ so that $\mathcal{G} \cong Map(S^4, S^1) \simeq S^1$ and $\mathcal{G}_* \simeq *$. With this simple result out of the way we shall henceforth only consider the values $n \geqslant 2$, and we tacitly assume this is so in all of the following.

We assemble the previous information into a commutative diagram whose columns and rows are homotopy fibrations

$$\begin{array}{ccccc}
* & \longrightarrow & S^1 & = & S^1 \\
\downarrow & & \downarrow{\scriptstyle \epsilon} & & \downarrow{\scriptstyle *} \\
\Omega^3_0 SU(n) & \xrightarrow{i^{SU}} & B\mathcal{G}^{SU(n)}_k & \xrightarrow{e^{SU}} & BSU(n) \\
{\scriptstyle \simeq}\downarrow{\scriptstyle \Omega^3 j} & & \downarrow{\scriptstyle \widehat{Bj}} & (*) & \downarrow{\scriptstyle Bj} \\
\Omega^3_0 U(n) & \xrightarrow{i^U} & B\mathcal{G}^{U(n)}_k & \xrightarrow{e^U} & BU(n).
\end{array} \tag{3.5}$$

The homotopy equivalence $\Omega^3 j$ serves to identify the square labelled $(*)$ as a homotopy pullback. This in turn identifies the contractible space situated in the top left corner of the diagram and from this it follows that the fibre of $\widehat{Bj} \colon B\mathcal{G}^{SU(n)}_k \to B\mathcal{G}^{U(n)}_k$ has the homotopy type of S^1. The triviality of the map $S^1 \to BSU(n)$ on the right hand side of the diagram follows from the connectivity of $BSU(n)$.

Now extend the homotopy commutative diagram (3.5) upwards. Each column becomes a homotopy fibration sequence and, in particular, there is a sequence

$$\cdots \to \mathcal{G}^{SU(n)}_k \xrightarrow{\Omega\widehat{Bj}} \mathcal{G}^{U(n)}_k \to S^1 \xrightarrow{\epsilon} B\mathcal{G}^{SU(n)}_k \to \cdots \tag{3.6}$$

in which $\mathcal{G}^{U(n)}_k$ appears as the homotopy fibre of the map $\epsilon \in \pi_1(B\mathcal{G}^{SU(n)}_k)$.

Proof of Theorem 1.1.1. For $n \geqslant 3$ we have $\pi_1(BSU(n)) = 0$ and $\pi_1(\Omega^3 SU(n)) = \pi_4(SU(n)) = 0$. Therefore it follows from the middle row of (3.5) that $\pi_1(B\mathcal{G}_k^{SU(n)}) = 0$ and the map ϵ in (3.6) is null-homotopic. The result now follows from the general theory of principal bundles. □

We now focus on the case $n = 2$.

Lemma 3.2. *The following hold:*

$$\pi_1\left(B\mathcal{G}_k^{SU(2)}\right) \cong \mathbb{Z}_2, \tag{3.7}$$

$$\pi_1\left(B\mathcal{G}_k^{U(2)}\right) \cong \begin{cases} 0 & k \text{ odd}, \\ \mathbb{Z}_2 & k \text{ even}, \end{cases} \tag{3.8}$$

$$\pi_2\left(B\mathcal{G}_k^{SU(2)}\right) \cong \mathbb{Z}_2, \tag{3.9}$$

$$\pi_2\left(B\mathcal{G}_k^{U(2)}\right) \cong \mathbb{Z} \oplus \mathbb{Z}_2. \tag{3.10}$$

Proof. The statements (3.7), (3.9) follow easily from the homotopy exact sequence of the evaluation fibration (3.2) for $B\mathcal{G}_k^{SU(2)}$. For the other groups we use Lang's Theorem 3.4 to get that the map induced on homotopy groups by the connecting map $\delta_k^U : U(2) \to \Omega_0^3 U(2)$ is given by the Samelson product

$$\pi_r(U(2)) \to \pi_{r+3}(U(2)), \qquad \alpha \mapsto \langle k\epsilon_3, \alpha\rangle = k\langle \epsilon_3, \alpha\rangle,$$

where $\epsilon_3 \in \pi_3(U(2)) \cong \mathbb{Z}$ is a generator. Bott [2] has calculated the value of this product for $r = 1$ and shown that if $\epsilon_1 \in \pi_1(U(2)) \cong \mathbb{Z}$ is a generator, then $\langle \epsilon_3, \epsilon_1\rangle$ generates $\pi_4(U(2)) \cong \mathbb{Z}_2$.

In the fibre sequence (3.2) we have $\pi_1(BU(2)) = 0$ from which it follows that

$$\pi_1(B\mathcal{G}_k^{U(2)}) \cong \operatorname{coker}(\delta_k^U) \cong \operatorname{coker}\left(\pi_1(U(2)) \xrightarrow{k\langle \epsilon_3, -\rangle} \pi_4(U(2))\right) \cong \mathbb{Z}_{gcd(2,k)} \tag{3.11}$$

and we get (3.8).

Again in (3.2) we have $\pi_3(BU(2)) = 0$ so we obtain a short exact sequence

$$0 \to \pi_5(U(n)) \cong \mathbb{Z}_2 \to \pi_2(B\mathcal{G}_k^{U(2)}) \to \ker(\delta_k^U) \to 0.$$

By (3.11) $\ker \delta_k^U$ is either $\mathbb{Z}$ or $2\mathbb{Z}$ so this sequence must split to give (3.10). □

This lemma implies that there are exactly two isomorphism classes of principal $\mathcal{G}_k^{SU(2)}$-bundles over S^1, one of which is trivial and the other of which is represented by the generator of $\pi_1(B\mathcal{G}_k^{SU(2)}) \cong \mathbb{Z}_2$. Owing to the fibre sequence (3.6), the gauge group $\mathcal{G}_k^{U(2)}$ must belong to one of these, and to which is decided by the homotopy class of the map ϵ.

Proof of Theorem 1.1.2. Apply the functor π_1 to the homotopy commutative diagram (3.5). There are two cases to consider and the information from the previous lemma allows us to fill in the groups.

$$
\begin{array}{ccccccc}
 & & & \pi_2(B\mathcal{G}_k^{U(2)}) & & \\
 & & & \downarrow & & \\
 & 0 & & \pi_1(S^1)\cong\mathbb{Z} & & \\
 & \downarrow & & \downarrow{\scriptstyle \epsilon_*} & & \\
0 & \longrightarrow \mathbb{Z}_2 & \xrightarrow{\cong} & \pi_1(B\mathcal{G}_k^{SU(2)}) & \longrightarrow & 0 \\
 & \downarrow{\scriptstyle\cong} & & \downarrow & & \\
\mathbb{Z} & \xrightarrow{\delta_k^U} \mathbb{Z}_2 & \longrightarrow & 0 & & \\
 & \downarrow & & & & \\
 & 0 & & & &
\end{array}
\qquad
\begin{array}{ccccccc}
 & & & \pi_2(B\mathcal{G}_k^{U(2)}) & & \\
 & & & \downarrow & & \\
 & 0 & & \pi_1(S^1)\cong\mathbb{Z} & & \\
 & \downarrow & & \downarrow{\scriptstyle \epsilon_*} & & \\
0 & \longrightarrow \mathbb{Z}_2 & \xrightarrow{\cong} & \pi_1(B\mathcal{G}_k^{SU(2)}) & \longrightarrow & 0 \\
 & \downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle \widehat{Bj}_*} & & \\
0 & \longrightarrow \mathbb{Z}_2 & \xrightarrow{\cong} & \pi_1(B\mathcal{G}_k^{U(2)}) & \longrightarrow & 0 \\
 & \downarrow & & \downarrow & & \\
 & 0 & & 0 & &
\end{array}
$$

k odd $\qquad\qquad$ k even

For k odd the map ϵ_* is an epimorphism onto a non-trivial group so that ϵ itself must be essential. For k even it follows from the exactness of the diagram that $\widehat{Bj}_*$ is an isomorphism and the map $\epsilon = \epsilon_*(1) = 0$ is trivial. The statements of the theorem now follow by arguing as in the proof of Theorem 1.1.1. ☐

There is more to say about the case $n = 2$ and $k = 2l + 1$. For this we return to some generality and introduce the projective unitary group $PU(n)$. We have the homotopy equivalences of equation (2.4) and using these we follow the same steps that we did for the creation of diagram (3.5) to build the following homotopy commutative diagrams

$$
\begin{array}{ccccc}
* & \longrightarrow & K(\mathbb{Z},2) & = & K(\mathbb{Z},2) \\
\downarrow & & \downarrow & & \downarrow \\
\Omega_0^3 U(n) & \xrightarrow{i^U} & B\mathcal{G}_k^{U(n)} & \xrightarrow{e^U} & BU(n) \\
\simeq\downarrow{\scriptstyle \Omega^3\pi} & & \downarrow{\scriptstyle \widehat{B\pi}} & (*) & \downarrow{\scriptstyle B\pi} \\
\Omega_0^3 PU(n) & \xrightarrow{i^{PU}} & B\mathcal{G}_k^{PU(n)} & \xrightarrow{e^{PU}} & BPU(n) \\
\downarrow & & \downarrow{\scriptstyle \eta} & & \downarrow{\scriptstyle \chi} \\
* & \longrightarrow & K(\mathbb{Z},3) & = & K(\mathbb{Z},3)
\end{array}
\qquad
\begin{array}{ccccc}
* & \longrightarrow & K(\mathbb{Z}_n,1) & = & K(\mathbb{Z}_n,1) \\
\downarrow & & \downarrow & & \downarrow \\
\Omega_0^3 SU(n) & \xrightarrow{i^{SU}} & B\mathcal{G}_k^{SU(n)} & \xrightarrow{e^{SU}} & BSU(n) \\
\simeq\downarrow{\scriptstyle \Omega^3\rho} & & \downarrow{\scriptstyle \widehat{B\rho}} & (*) & \downarrow{\scriptstyle B\rho} \\
\Omega_0^3 PU(n) & \xrightarrow{i^{PU}} & B\mathcal{G}_k^{PU(n)} & \xrightarrow{e^{PU}} & BPU(n) \\
\downarrow & & \downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle \omega_2} \\
* & \longrightarrow & K(\mathbb{Z}_n,2) & = & K(\mathbb{Z}_n,2)
\end{array}
\tag{3.12}
$$

Each rown and column in these diagrams is a homotopy fibration sequence and the labelled squares are homotopy pullbacks. We use the fact that the fibration sequence down the right hand side of each diagram is principal to define the bottom rows, the map θ from the class $\omega_2 \in H^2(BPU_n;\mathbb{Z}_n)$ and the map η from the integral class $\chi = \delta\omega_2$.

Now fix $n = 2$. Then $SU(2) \cong Spin(3)$ is the 2-connected cover of $PU(2) \cong SO(3)$. Kamiyama, Kishimoto, Kono and Tsukuda [13] have examined the gauge groups of $SO(3)$-bundles over S^4 and calculated enough Samelson products in $SO(3)$ so as to obtain the low-dimensional homotopy groups of $B\mathcal{G}_k^{PU(2)}$.

Lemma 3.3 (Kamiyama, Kishimoto, Kono, Tsukuda [13])**.**

$$\pi_1(B\mathcal{G}_k^{PU(2)}) \cong \begin{cases} 0 & k \text{ odd}, \\ \mathbb{Z}_2 & k \text{ even}, \end{cases}$$

$$\pi_2(B\mathcal{G}_k^{PU(2)}) \cong \begin{cases} \mathbb{Z}_2 & k \text{ odd}, \\ \mathbb{Z}_4 & k \equiv 2 \text{ mod } 4, \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 & k \equiv 0 \text{ mod } 4. \end{cases} \qquad \square$$

We now have enough information to complete the proof of 1.1.

Proof of Theorem 1.1.3. For $k = 2l + 1$, Lemma 3.3 and the Hurewicz Theorem give $H_1(B\mathcal{G}_{2l+1}^{PU(2)}) = 0$ and $H_2(B\mathcal{G}_{2l+1}^{PU(2)}) \cong \mathbb{Z}_2$. Furthermore, rational homotopy shows us that $H_3(B\mathcal{G}_{2l+1}^{PU(2)})$ is torsion. From the universal coefficient theorem it now follows that

$$H^1(B\mathcal{G}_{2l+1}^{PU(2)}) = 0, \qquad H^2(B\mathcal{G}_{2l+1}^{PU(2)}) = 0, \qquad H^3(B\mathcal{G}_{2l+1}^{PU(2)}) \cong \mathbb{Z}_2.$$

Consider then the Serre spectral sequence with integral coefficients for the evaluation fibration of $B\mathcal{G}_{2l+1}^{PU(2)}$. Using the cohomology groups above it follows from connectivity that there is an exact sequence

$$\begin{array}{ccccccccc} 0 \longrightarrow & H^2(\Omega_0^3 PU(2)) & \xrightarrow{\tau} & H^3(BPU(2)) & \xrightarrow{e^{PU*}} & H^3(B\mathcal{G}_{2l+1}^{PU(2)}) & \longrightarrow & H^3(\Omega_0^3 PU(2)) \\ & \| & & \| & & \| & & \\ & \mathbb{Z}_2 & & \mathbb{Z}_2 & & \mathbb{Z}_2 & & \end{array}$$

The transgression τ must be injective in degree 2 by exactness and so must be an isomorphism. The map $e^{PU*}\colon H^3(BPU) \to H^3(B\mathcal{G}_{2l+1}^{PU(2)})$ is therefore trivial. Returning to the diagrams (3.12) we see that the classifying map $\eta\colon B\mathcal{G}_{2l+1}^{PU(2)} \to K(\mathbb{Z}, 3)$ factors through the map $\chi\colon BPU(n) \to K(\mathbb{Z}, 3)$ as

$$\eta = \chi \circ e^{PU} = e^{PU*}(\chi).$$

Since e^{PU*} is trivial in degree 3 cohomology, so is the map $\eta = e^{PU*}(\chi)$. Thus the homotopy fibration $S^1 \to \mathcal{G}_{2l+1}^{U(2)} \to \mathcal{G}_{2l+1}^{PU(2)}$ is classified by the trivial map $\Omega\eta \simeq *$ and splits. $\square$

Remark 3.4. In the case that $k = 2l$ it is not hard to see that there is a nontrivial relationship between the gauge groups $\mathcal{G}_{2l}^{SU(2)}$ and $\mathcal{G}_{2l}^{PU(2)}$, and between $\mathcal{G}_{2l}^{U(2)}$ and $\mathcal{G}_{2l}^{PU(2)}$. The maps η, θ are essential in these cases and the bundle structure is twisted.

Remark 3.5. The proof of 3 of Theorem 1.1 actually allows for a slightly stronger conclusion to be drawn. It is shown that $\eta\colon B\mathcal{G}_k^{PU(n)} \to K(\mathbb{Z}, 3)$ is trivial for odd k.

Since this map classifies the principal fibration $\widehat{B\pi}\colon B\mathcal{G}_k^{U(n)} \to B\mathcal{G}_k^{PU(n)}$ it is possible to use the principal action to construct a homotopy equivalence $B\mathcal{G}_k^{U(n)} \simeq B\mathcal{G}_k^{PU(n)} \times K(\mathbb{Z},2)$. Thus the claimed splitting actually happens on the level of classifying spaces.

4. Homotopy types of $U(n)$-gauge groups over $\mathbb{C}P^2$

Let $E_{(k,l)} \xrightarrow{p_{(k,l)}} \mathbb{C}P^2$ be the principal $U(n)$-bundle over $\mathbb{C}P^2$ with Chern classes $(c_1, c_2) = (k,l) \in H^2(\mathbb{C}P^2) \oplus H^4(\mathbb{C}P^2)$ and let $\mathcal{G}^{(k,l)}(\mathbb{C}P^2, U(n))$ denote the gauge group of this bundle. We assume that $n \geqslant 2$. From (2.1) there are homotopy equivalences

$$B\mathcal{G}^{(k,l)}(\mathbb{C}P^2, U(n)) \simeq Map^{(k,l)}(\mathbb{C}P^2, BU(n)),$$
$$B\mathcal{G}_*^{(k,l)}(\mathbb{C}P^2, U(n)) \simeq Map_*^{(k,l)}(\mathbb{C}P^2, BU(n)),$$

and these spaces sit in the evaluation fibration

$$\mathcal{G}^{(k,l)}(\mathbb{C}P^2, U(n)) \to U(n) \xrightarrow{\lambda_{(k,l)}} B\mathcal{G}_*^{(k,l)}(\mathbb{C}P^2, U(n)) \to B\mathcal{G}^{(k,l)}(\mathbb{C}P^2, U(n)) \xrightarrow{e} BU(n) \tag{4.1}$$

where $\lambda_{(k,l)}\colon U(n) \to B\mathcal{G}_*^{(k,l)}(\mathbb{C}P^2, U(n))$ denotes the fibration connecting map.

Now the cellular structure of $\mathbb{C}P^2$ gives rise to a cofiber sequence

$$S^3 \xrightarrow{\eta} S^2 \xrightarrow{i} \mathbb{C}P^2 \xrightarrow{q} S^4 \to \cdots, \tag{4.2}$$

where η is the Hopf map. This sequence comes furnished with a coaction $c\colon \mathbb{C}P^2 \to \mathbb{C}P^2 \vee S^4$ produced by pinching a sphere out of the top cell. Application of the functor $Map_*(-, BU(n))$ to (4.2) yields a principal fibring

$$\Omega^4 BU(n) \xrightarrow{q^*} Map_*(\mathbb{C}P^2, BU(n)) \xrightarrow{i^*} \Omega^2 BU(n) \tag{4.3}$$

and the coaction induces a homotopy action

$$\mu\colon Map_*(\mathbb{C}P^2, BU(n)) \times \Omega^4 BU(n) \to Map_*(\mathbb{C}P^2, BU(n)), (f, \omega_l) \mapsto \nabla \circ (f \vee \omega_l) \circ c.$$

In (4.3) we have identified $Map_*(S^r, BU(n)) = \Omega^r BU(n)$ and used the homeomorphism $Map_*(\mathbb{C}P^2 \vee S^4, BU(n)) \cong Map_*(\mathbb{C}P^2, BU(n)) \times \Omega^4 BU(n)$.

In cohomology the coaction induces the map $c^*\colon H^*(\mathbb{C}P^2) \oplus H^*(S^4) \to H^*(\mathbb{C}P^2)$ which in degree 4 satisfies $c^*(l \cdot x^2, l' \cdot s_4) = l \cdot x^2 + l' \cdot q^* s_4 = (l + l') \cdot x^2$, where $x \in H^2(\mathbb{C}P^2)$, $s_4 \in H^4(S^4)$ are integral generators. From this it is clear that after taking components there results a map

$$\mu\colon Map_*^{(k,l)}(\mathbb{C}P^2, BU(n)) \times \Omega^4_{l'} BU(n) \to Map_*^{(k,l+l')}(\mathbb{C}P^2, BU(n)). \tag{4.4}$$

For each nonnegative integer r fix an element $\omega_r \in \Omega^4_r BU(n)$ and extend this to all integers by using the inverse to define $\omega_{-r} = -\omega_r$. Then with $B\mathcal{G}_*^{(k,l)}(\mathbb{C}P^2, BU(n)) \simeq Map_*^{(k,l)}(\mathbb{C}P^2, BU(n))$ there results, for each triple of integers k, l and r, a map $S_r\colon B\mathcal{G}_*^{(k,l)}(\mathbb{C}P^2, BU(n)) \to B\mathcal{G}_*^{(k,l+r)}(\mathbb{C}P^2, BU(n))$ defined by

$$S_r(f) = \mu(f, \omega_r)\colon \mathbb{C}P^2 \xrightarrow{c} \mathbb{C}P^2 \vee S^4 \xrightarrow{f \vee \omega_r} BU(n) \vee BU(n) \xrightarrow{\nabla} BU(n).$$

That this map is well defined follows from the properties of the coaction [1]. Moreover, it is straightforward to show using these properties that S_r is a homotopy equivalence with inverse S_{-r} [17]. Using these maps we get the following.

Proposition 4.1. *For each integer $k \in \mathbb{Z}$ and any $l, l' \in \mathbb{Z}$ there are homotopy equivalences*

$$B\mathcal{G}_*^{(k,l)}(\mathbb{C}P^2, BU(n)) \simeq B\mathcal{G}_*^{(k,l')}(\mathbb{C}P^2, BU(n)). \qquad \square$$

This proposition cannot be upgraded to a statement about the classifying space of the full gauge group due to its reliance on the coaction c. We still are able to get the following result, however.

Theorem 4.2. *With respect to the evaluation fibrations, for each pair of integers (k, l) and any integer r, there are fibre homotopy equivalences over $BU(n)$*

$$B\mathcal{G}^{(k,l)}(\mathbb{C}P^2, U(n)) \simeq B\mathcal{G}^{(k+rn, l+kr(n-1)+1/2n(n-1)r^2)}(\mathbb{C}P^2, U(n)).$$

These restrict to homotopy equivalences

$$B\mathcal{G}_*^{(k,l)}(\mathbb{C}P^2, U(n)) \simeq B\mathcal{G}_*^{(k+rn, l+kr(n-1)+1/2n(n-1)r^2)}(\mathbb{C}P^2, U(n)).$$

Proof. Let $E(\eta) \xrightarrow{\eta} \mathbb{C}P^2$ denote the canonical line bundle with $c_1(\eta) = x \in H^2(\mathbb{C}P^2)$ a generator. Now let E be a $U(n)$-bundle over $\mathbb{C}P^2$, $r \in \mathbb{Z}$ and form the tensor product $E \otimes \eta^r$, where

$$\eta^r = \begin{cases} \otimes^r \eta & r > 0, \\ \epsilon^1 & r = 0, \\ \otimes^{-r} \bar{\eta} & r < 0 \end{cases}$$

with ϵ^1 denoting the trivial line bundle and $\bar{\eta}$ denoting the conjugate bundle of η. Then $E \otimes \eta^r$ is again a $U(n)$-bundle and calculating with the Chern character shows that

$$\begin{aligned} c_1(E \otimes \eta^r) &= c_1(E) + (rn) \cdot x, \\ c_2(E \otimes \eta^r) &= c_2(E) + r(n-1) \cdot c_1(E) \cup x + \tfrac{1}{2} r^2 n(n-1) \cdot x^2. \end{aligned} \tag{4.5}$$

We use the tensor product with η^r as above to define a map

$$\begin{aligned} T_r = ((-) \otimes \eta^r) \colon & Map^{(k,l)}(\mathbb{C}P^2, BU(n)) \\ & \to Map^{(k+rn, l+kr(n-1)+1/2n(n-1)r^2)}(\mathbb{C}P^2, BU(n)) \end{aligned}$$

as follows. Let $f \colon \mathbb{C}P^2 \to BU(n)$ represent the $U(n)$-bundle E. Then $E \otimes \eta^r$ may be represented by the composite

$$T_r(f) = T_r(E) \colon \mathbb{C}P^2 \xrightarrow{\Delta} \mathbb{C}P^2 \times \mathbb{C}P^2 \xrightarrow{f \times \eta^r} BU(n) \times BU(1) \xrightarrow{m_n} BU(n), \tag{4.6}$$

where m_n is the H-action of $BU(1)$ on $BU(n)$ induced by the tensor product and η^r here is the composite $\mathbb{C}P^2 \xrightarrow{\Delta} \Pi^r \mathbb{C}P^2 \xrightarrow{\Pi^r \eta} \Pi^r BU(1) \xrightarrow{m_1 \circ (1 \times m_1) \circ \cdots \circ (1 \times \cdots \times 1 \times m_1)} BU(1)$ (with η replaced by $\bar{\eta}$ if $r < 0$, and $\eta^0 = *$ is defined to be the constant map). If $f \in Map^{(k,l)}(\mathbb{C}P^2, BU(n))$, then it is seen from (4.5) that $T_r(f)$ lies in the correct component and everything is well-defined. We take (4.6) as the definition of T_r.

We may now use elementary properties of the tensor product to determine some of the features of T_r. We abuse notation by blurring the distinction between a bundle and a representing map for it. It follows from the associativity of the tensor product that

$$T_{-r} \circ T_r(E) = (E \otimes \eta^r) \otimes \eta^{-r} \cong E \otimes (\eta^r \otimes \eta^{-r}) \cong E \otimes \epsilon^1 \cong E,$$

which demonstrates that T_r has a homotopy inverse and is a homotopy equivalence.

Since we may choose the map classifying η to be basepoint preserving and we may assume that the action m_n is strictly unital in the sense that $m_n(x, *) = x$ for all $x \in BU(n)$ and $m_n(*, z) = B\Delta(z)$ for all $z \in BU(1)$, we see the compatibility with the evaluation fibration,

$$e \circ T_r(f) = m_n \circ (f \times (r\eta)) \circ \Delta(*) = m_n(f(*), *) = f(*) = e(f).$$

Hence by [3] the map T_r is a fibre homotopy equivalence and thus restricts to gives a homotopy equivalence between the fibres. □

Before moving on we record one further lemma which will be of use in our applications. In the literature, for simply connected G, a standard technique is to take components in the fibration (4.3) and attempt to factor the connecting map δ through the fibre inclusion $q^*\colon \Omega_0^3 G \to Map_*(\mathbb{C}P^2, BG)$. In our case the problem is different. Once components are taken it is not even clear what the homotopy fibre of i^* actually is. What allows for progress to be made is the action μ; its presence makes the map i^* into what Zabrodsky [26] calls a weakly principal fibration.

Let $i^*\colon Map_*^{(k,l)}(\mathbb{C}P^2, BU(n)) \to \Omega_k^2 BU(n)$ be the map induced by $i\colon S^2 \hookrightarrow \mathbb{C}P^2$ and let $X(k, l; n)$ denote the homotopy fibre of this map. Then for any space X the map μ produces an operation of the homotopy set $[X, \Omega_0^3 U(n)]$ on $[X, Map_*^{(k,l)}(\mathbb{C}P^2, BU(n))]$ and we denote this action by the symbol $+$ in the following.

Proposition 4.3. *There is a homotopy equivalence*

$$X(k, l; n) \simeq \Omega_0^4 BU(n) \simeq \Omega_0^3 SU(n).$$

Moreover, in the resulting homotopy fibration sequence

$$\Omega_0^4 BU(n) \xrightarrow{j} Map_*^{(k,l)}(\mathbb{C}P^2, BU(n)) \xrightarrow{i^*} \Omega_k^2 BU(n) \tag{4.7}$$

we may identify the fibre inclusion j as the map given by

$$j(\omega) = f_{(k,l)} + \omega,$$

where $f_{(k,l)} \in Map^{(k,l)}(\mathbb{C}P^2, BU(n))$ is a chosen basepoint preserving map.

Proof. Consider the following homotopy commutative diagram in which each row and column is a cofiber sequence, c is the coaction and C_c is its mapping cone:

$$\begin{array}{ccccc}
S^2 & \xrightarrow{i} & \mathbb{C}P^2 & \xrightarrow{q} & S^4 \\
\downarrow{\scriptstyle i} & \text{(hpo)} & \downarrow{\scriptstyle incl.} & & \downarrow{\scriptstyle \gamma} \\
\mathbb{C}P^2 & \xrightarrow{c} & \mathbb{C}P^2 \vee S^4 & \xrightarrow{r} & C_c \\
\downarrow{\scriptstyle q} & & \downarrow & & \downarrow \\
S^4 & = & S^4 & \longrightarrow & *.
\end{array}$$

The fact that the diagram homotopy commutes follows from the properties of the coaction and so the square labelled (hpo) is a homotopy pushout by construction. From this one infers that the induced map of cofibers $\gamma\colon S^4 \xrightarrow{\simeq} C_c$ is a homotopy equivalence. Moreover, this identification makes it possible to identify the map r as $r = (q, -1)\colon \mathbb{C}P^2 \vee S^4 \to S^4$.

Now with the identifications $C_c \simeq S^4$ and $r = (q, -1)$ in place, apply $Map_*(-, BU(n))$ to the diagram and use (4.4) to replace the map induced by the coaction c with the action μ. The result of this is the following diagram in which each column and row is a homotopy fibration sequence and the square labelled (hpb) is a homotopy pullback

$$\begin{array}{ccccc}
* & \longrightarrow & \Omega^4 BU(n) & = & \Omega^4 BU(n) \\
\downarrow & & \downarrow{\scriptstyle in_2} & & \downarrow{\scriptstyle q^*} \\
\Omega^4 BU(n) & \xrightarrow{(q^*,-1)} & Map_*(\mathbb{C}P^2, BU(n)) \times \Omega^4 BU(n) & \xrightarrow{\mu} & Map_*(\mathbb{C}P^2, BU(n)) \\
\| & & \downarrow{\scriptstyle pr_1} & \text{(hpb)} & \downarrow{\scriptstyle i^*} \\
\Omega^4 BU(n) & \xrightarrow{q^*} & Map_*(\mathbb{C}P^2, BU(n)) & \xrightarrow{i^*} & \Omega^2 BU(n).
\end{array}$$

Restricting to components of the mapping spaces in the bottom right-hand square of this diagram will now produce a new homotopy pullback square. Including homotopy fibres it appears as

$$\begin{array}{ccccc}
* & \longrightarrow & \Omega^4_l BU(n) & \xrightarrow[e]{\simeq} & X(k,l;n) \\
\downarrow & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle j'} \\
\Omega^4_l BU(n) & \xrightarrow{u} & Map_*^{(k,0)}(\mathbb{C}P^2, BU(n)) \times \Omega^4_l BU(n) & \xrightarrow{\mu} & Map_*^{(k,l)}(\mathbb{C}P^2, BU(n)) \\
\downarrow{\scriptstyle \simeq} & & \downarrow{\scriptstyle pr_1} & \text{(hpb)} & \downarrow{\scriptstyle i^*} \\
X(k,0;n) & \longrightarrow & Map_*^{(k,0)}(\mathbb{C}P^2, BU(n)) & \xrightarrow{i^*} & \Omega^2_k BU(n),
\end{array}$$

where $u\colon \Omega^4_l BU(n) \longrightarrow Map_*^{(k,0)}(\mathbb{C}P^2, BU(n)) \times \Omega^4_l BU(n)$ is defined by $u(\omega) = (f_{(k,l)} - \omega, \omega)$ for a chosen basepoint map $f_{(k,l)} \in Map_*^{(k,l)}(\mathbb{C}P^2, BU(n))$, $v\colon \Omega^4_l BU(n) \to Map_*^{(k,0)}(\mathbb{C}P^2, BU(n)) \times \Omega^4_l BU(n)$ is defined by $v(\omega) = (f_{(k,l)} - \omega_0, \omega)$ for a chosen basepoint loop $\omega_0 \in \Omega^4_l BU(n)$ and e is defined by the diagram.

Since the bottom right-hand square is a homotopy pullback, the map e is a homotopy equivalence which we take as an identification $e : \Omega^4_l BU(n) \simeq X(k,l;n)$ to get the homotopy type of $X(k,l;n)$ and a homotopy fibration sequence

$$\Omega^4_l BU(n) \xrightarrow{j''} Map_*^{(k,l)}(\mathbb{C}P^2, BU(n)) \xrightarrow{i^*} \Omega^2 BU(n)$$

with $j'' \simeq j' \circ e \simeq \mu \circ v$ given by

$$j''(\omega) = \mu \circ v(\omega) = \mu(f_{(k,l)} - \omega_0, \omega) = f_{(k,l)} + (\omega - \omega_0).$$

Finally, we choose the homotopy equivalence $\Omega^4_0 BU(n) \simeq \Omega^4_l BU(n)$ to be given by $\omega \mapsto \omega + \omega_0$ and define j in (4.7) to be the composite of j'' with this map. We get

$$j(\omega) = (f_{(k,l)} + \omega + \omega_0) - \omega_0 = f_{(k,l)} + \omega. \qquad \square$$

Before moving on we comment on the homotopy types of $U(1)$-gauge groups over $\mathbb{C}P^2$. Since $BU(1) \simeq K(\mathbb{Z}, 2)$ we have $[\mathbb{C}P^2, BU(1)] \cong H^2\mathbb{C}P^2 \cong \mathbb{Z}$ so isomorphism classes of principal $U(1)$-bundles are in correspondence with the integers. Using a theorem of Thom [24], however, we get from equation (2.1) that $B\mathcal{G}^k(\mathbb{C}P^2, U(1)) \simeq Map^k(\mathbb{C}P^2, BU(1)) \simeq BU(1)$ and $B\mathcal{G}^k_*(\mathbb{C}P^2, U(1)) \simeq Map^k_*(\mathbb{C}P^2, BU(1)) \simeq *$, so there is a unique homotopy type amongst the classifying spaces of these gauge groups. It follows that $\mathcal{G}^k(\mathbb{C}P^2, U(1)) \simeq S^1$ and $\mathcal{G}^k(\mathbb{C}P^2, U(1)) \simeq *$. For the remainder of the paper we will always assume that $n \geqslant 2$.

4.1. A decomposition of the connecting map $\lambda_{(k,l)}$

For each pair of integers (k, l) we have the evaluation fibration (4.1). There is also the evaluation fibration for the gauge group of a $U(n)$-bundle over S^4 with second Chern class $l \in H^4(S^4) \cong \mathbb{Z}$

$$\cdots \to \mathcal{G}^l(S^4, U(n)) \to U(n) \xrightarrow{\delta_l} \Omega^3_0 U(n) \to B\mathcal{G}^l(S^4, U(n)) \xrightarrow{e} BU(n).$$

The connecting map $\delta_l \colon U(n) \to \Omega^3_0 U(n)$ of this sequence is more amenable to study than $\lambda_{(k,l)}$ since by Theorem 3.1 it is adjoint to the Samelson product $\langle \epsilon_3, id_{U(n)} \rangle \colon S^3 \wedge U(n) \to U(n)$. The purpose of this section is to relate the two maps $\lambda_{(k,l)}$, δ_l in such a way as to allow more easily obtained information about δ_l to be passed to $\lambda_{(k,l)}$. For the following recall the action $\mu \colon Map^{(k,l)}_*(\mathbb{C}P^2, BU(n)) \times \Omega^4_{l'} BU(n) \to Map^{(k,l+l')}_*(\mathbb{C}P^2, BU(n))$ and the notation $+$ for the induced operation of the homotopy set $[X, \Omega^3_0 U(n)]$ on $[X, Map^{(k,l)}_*(\mathbb{C}P^2, BU(n))]$.

Theorem 4.4. *In the homotopy set $[U(n), Map^{(k,l)}_*(\mathbb{C}P^2, BU(n))]$ there is equality*

$$\lambda_{(k,l)} = \lambda_{(k,0)} + \delta_l.$$

Proof. Since $BU(n)$ is connected the components of the unbased mapping space $Map(\mathbb{C}P^2 \vee S^4, BU(n))$ are in one-to-one correspondence with those of the based mapping space $Map_*(\mathbb{C}P^2 \vee S^4, BU(n))$. Thus we may consider the map of evaluation fibrations induced by c in the diagram

$$\begin{array}{ccccccc}
U(n) & \xrightarrow{\Delta_{(k,0;l)}} & Map^{(k,0;l)}_*(\mathbb{C}P^2 \vee S^4, BU(n)) & \longrightarrow & Map^{(k,0;l)}(\mathbb{C}P^2 \vee S^4, BU(n)) & \xrightarrow{e} & BU(n) \\
\| & & \downarrow{\scriptstyle c^*} & & \downarrow{\scriptstyle c^*} & & \| \\
U(n) & \xrightarrow{\lambda_{(k,l)}} & Map^{(k,l)}_*(\mathbb{C}P^2, BU(n)) & \longrightarrow & Map^{(k,l)}(\mathbb{C}P^2, BU(n)) & \xrightarrow{e} & BU(n)
\end{array} \tag{4.8}$$

which we use to define the fibration connecting map $\Delta_{(k,0;l)}$ belonging to the top row. Here we label the components of the mapping space on the top row using the triple of integers $(k, 0; l)$ according to $Map^{(k,0;l)}_*(\mathbb{C}P^2 \vee S^4, BU(n)) \cong Map^{(k,0)}_*(\mathbb{C}P^2, BU(n)) \times \Omega^4_l BU(n)$.

We wish to identify the connecting map $\Delta_{(k,0;l)}$ under the homeomorphism

$$\theta = (in_0^*, in_1^*) \circ \Delta : Map_*(\mathbb{C}P^2 \vee S^4, BU(n)) \cong Map_*(\mathbb{C}P^2, BU(n)) \times \Omega^4 BU(n)$$

where $in_0 \colon \mathbb{C}P^2 \to \mathbb{C}P^2 \vee S^4$ and $in_1 \colon S^4 \to \mathbb{C}P^2 \vee S^4$ are the inclusions. These inclusions induce natural maps $in_0^* \colon Map^{(k,0;l)}(\mathbb{C}P^2 \vee S^4, BU(n)) \to Map^{(k,0)}(\mathbb{C}P^2,$

$BU(n))$ and $in_1^*\colon Map^{(k,0;l)}(\mathbb{C}P^2\vee S^4,BU(n))\to Map^l(S^4,BU(n))$ which are compatible with all three evaluation fibrations and produce factorisations

$$\begin{array}{ccc} U(n) & \xrightarrow{\Delta_{(k,0;l)}} & Map^{(k,0;l)}(\mathbb{C}P^2\vee S^4,BU(n)) \\ \| & & \downarrow{\scriptstyle in_0^*} \\ U(n) & \xrightarrow{\lambda_{(k,0)}} & Map^{(k,0)}(\mathbb{C}P^2,BU(n)) \end{array} \qquad \begin{array}{ccc} U(n) & \xrightarrow{\Delta_{(k,0;l)}} & Map^{(k,0;l)}(\mathbb{C}P^2\vee S^4,BU(n)) \\ \| & & \downarrow{\scriptstyle in_0^*} \\ U(n) & \xrightarrow{\delta_l} & \Omega^4_l BU(n). \end{array}$$

Since θ sends $f\in Map^{(k,0;l)}(\mathbb{C}P^2,\,BU(n))$ to the pair $(f\circ in_0,\,f\circ in_1)\in Map_*^{(k,0)}(\mathbb{C}P^2,BU(n))\times\Omega^4_l BU(n)$ the above factorisations show that for $x\in U(n)$ we have $\theta\left(\Delta_{(k,0;l)}(x)\right)=(\Delta_{(k,0;l)}(x)\circ in_0,\Delta_{(k,0;l)}(x)\circ in_1)=(\lambda_{(k,0)}(x),\delta_l(x))$.

Now let us return to the diagram (4.8) and use the homeomorphism θ to alter the left-most square, which will now appear as

$$\begin{array}{ccc} U(n) & \xrightarrow{(\lambda_{(k,0)},\delta_l)} & Map_*^{(k,0)}(\mathbb{C}P^2,BU(n))\times\Omega_l BU(n) \\ \| & & \downarrow{\scriptstyle \mu} \\ U(n) & \xrightarrow{\lambda_{(k,l)}} & Map_*^{(k,l)}(\mathbb{C}P^2,BU(n)). \end{array} \tag{4.9}$$

To identify the action μ appearing here we simply note that the inverse to θ is the map

$$\theta^{-1}=\nabla\circ(-\vee-):Map_*(\mathbb{C}P^2,BU(n))\times\Omega^4BU(n)\cong Map_*(\mathbb{C}P^2\vee S^4,BU(n)), \tag{4.10}$$

which sends the pair $(f,g)\in Map_*^{(k,0)}(\mathbb{C}P^2,BU(n))\times\Omega^4_lBU(n)$ to the composite $\nabla\circ(f\vee g)\colon\mathbb{C}P^2\vee S^4\to BU(n)$. Thus $c^*\left(\theta^{-1}(f,g)\right)=c^*(\nabla\circ(f\vee g))=\nabla\circ(f\vee g)\circ c=\mu(f,g)=f+g$. Combining (4.9) and (4.10) gives us the claimed equality of homotopy classes

$$\lambda_{(k,l)}=\mu(\lambda_{(k,0)},\delta_l)=\lambda_{(k,0)}+\delta_l. \qquad \square$$

Now we have noted that the connecting map $\delta_l\colon U(n)\to\Omega^3_0U(n)$ is adjoint to a Samelson product in $U(n)$. It follows from the linearity of the Samelson product that $\delta_l\simeq l\cdot\delta_1$. It is also true [12] that δ_1 is rationally trivial and has finite order in the group $[U(n),\Omega^3_0U(n)]$. Denote the order of δ_1 by $|\delta_1|$. Then for each fixed integer k, the following corollary gives an upper bound on the number of homotopy types amongst the gauge groups $\mathcal{G}^{(k,l)}(\mathbb{C}P^2,U(n))$.

Corollary 4.5. *For each pair of integers (k,l) there is a homotopy equivalence*

$$\mathcal{G}^{(k,l)}(\mathbb{C}P^2,U(n))\simeq\mathcal{G}^{(k,l+|\delta_1|)}(\mathbb{C}P^2,U(n)).$$

Proof. Recall that $\mathcal{G}^{(k,l)}(\mathbb{C}P^2,U(n))$ is the homotopy fibre of the connecting map $\lambda_{(k,l)}$, which, according to Theorem 4.4, has a decomposition $\lambda_{(k,l)}=\lambda_{(k,0)}+\delta_l$. Here we will further decompose this by focusing on the map $\delta_l\colon U(n)\to\Omega^3_0U(n)$.

Consider the degree l map $l\colon S^4\to S^4$ and the map of evaluation fibrations that it induces

$$\begin{array}{ccccccc} U(n) & \xrightarrow{\delta_1} & \Omega^3_1 U(n) & \longrightarrow & Map^1(S^4, BU(n)) & \xrightarrow{e} & BU(n) \\ \| & & \downarrow{\scriptstyle l^*} & & \downarrow{\scriptstyle l^*} & & \| \\ U(n) & \xrightarrow{\delta_l} & \Omega^3_l U(n) & \longrightarrow & Map^l(S^4, BU(n)) & \xrightarrow{e} & BU(n). \end{array}$$

The adjoint of the left-most square of this ladder is the diagram

$$\begin{array}{ccc} S^3 \wedge U(n) & \xrightarrow{\widetilde{\delta_1}} & U(n) \\ {\scriptstyle l\wedge 1}\uparrow & & \| \\ S^3 \wedge U(n) & \xrightarrow{\widetilde{\delta_l}} & U(n) \end{array}$$

which displays a factorisation of the adjoint map $\widetilde{\delta_l} \simeq \widetilde{\delta_1} \circ (l \wedge 1) \simeq l \cdot \widetilde{\delta_1}$. Adjointing back we get

$$\delta_l = l \cdot \delta_1 \in [U(n), \Omega^3_0 U(n)].$$

Now the coaction $c\colon \mathbb{C}P^2 \to \mathbb{C}P^2 \vee S^4$ satisfies $(c \vee 1) \circ c = (1 \vee \nu) \circ c$, where $\nu\colon S^4 \to S^4 \vee S^4$ is the coproduct. This means that for any space X and maps $f\colon \mathbb{C}P^2 \to X$, $\alpha, \beta\colon S^4 \to X$ we have in our notation $f + (\alpha + \beta) = (f + \alpha) + \beta$. Applying this to the case at hand gives us the decomposition

$$\lambda_{(k,l)} = \lambda_{(k,0)} + \delta_l = \lambda_{(k,0)} + l \cdot \delta_1.$$

Thus if $|\delta_1|$ is the order of δ_1 then

$$\lambda_{(k,l+|\delta_1|)} = \lambda_{(k,0)} + (l + |\delta_1|)\delta_1 = (\lambda_{(k,0)} + l \cdot \delta_1) + |\delta_1| \cdot \delta_1 = \lambda_{(k,0)} + l \cdot \delta_1 = \lambda_{(k,l)}.$$

Thus there is a homotopy $\lambda_{(k,l)} \simeq \lambda_{(k,l+|\delta_1|)}$ and since the gauge groups $\mathcal{G}^{(k,l)}(\mathbb{C}P^2, U(2))$ and $\mathcal{G}^{(k,l+|\delta_1|)}(\mathbb{C}P^2, U(2))$ are the homotopy fibres of $\lambda_{(k,l)}$ and $\lambda_{(k,l+|\delta_1|)}$, respectively, they share a common homotopy type. □

4.2. p-Local decompositions of $\mathcal{G}^{(k,l)}(\mathbb{C}P^2, U(n))$

Given the close relationship between the groups $U(n)$, $SU(n)$ and $PU(n)$ displayed in the fibration diagram (2.3), it is not surprising that the group $U(n)$ has particularly nice properties when localised at certain primes. The purpose of this section is to formalise this statement and examine how much of this behaviour is transferred to the $U(n)$-gauge groups over $\mathbb{C}P^2$.

Lemma 4.6. *For each pair of integers $k, l \in \mathbb{Z}$ there is a homotopy equivalence*

$$Map^{(k,l)}_*(\mathbb{C}P^2, BU(n)) \simeq Map^{(k,l)}_*(\mathbb{C}P^2, BPU(n)).$$

Proof. Apply the functor $Map_*(\mathbb{C}P^2, -)$ to the homotopy fibration $BU(n) \xrightarrow{B\pi} BPU(n) \xrightarrow{\chi} K(\mathbb{Z}, 3)$ to get the following homotopy fibration sequence

$$Map_*(\mathbb{C}P^2, BU(n)) \xrightarrow{B\pi_*} Map_*(\mathbb{C}P^2, BPU(n)) \xrightarrow{\chi_*} Map_*(\mathbb{C}P^2, K(\mathbb{Z}, 3)).$$

Observe from diagram (2.3) that $\Omega^2\chi\colon \Omega PU(n) \to S^1$ factors through a map $\mathbb{Z}_n \to S^1$ and is thus null-homotopic. Using the fact that Thom's theorem [24] gives a homotopy equivalence $Map_*(\mathbb{C}P^2, K(\mathbb{Z},3)) \simeq Map_*(S^2, K(\mathbb{Z},3)) \simeq S^1$ we find that the induced map χ_* factors through $\Omega^2\chi$ and is thus also null-homotopic. It follows that $B\pi_*$ admits a section and since the homotopy fibre of $B\pi_*$ is $Map_*(\mathbb{C}P^2, BS^1) \simeq \mathbb{Z}$ it is easy to see that this section becomes a homotopy equivalence after restricting to components. □

An easy consequence of this lemma is the following very useful homotopy pullback.

Lemma 4.7. *The following homotopy commutative diagram is a homotopy pullback:*

$$\begin{array}{ccc} B\mathcal{G}^{(k,l)}(\mathbb{C}P^2, U(n)) & \xrightarrow{e^U} & BU(n) \\ \downarrow{\scriptstyle \widehat{B\pi}} & (hpb) & \downarrow{\scriptstyle B\pi} \\ B\mathcal{G}^{(k,l)}(\mathbb{C}P^2, PU(n)) & \xrightarrow{e^{PU}} & BPU(n). \end{array} \tag{4.11}$$

Proof. The map $B\pi$ induces the map $\widehat{B\pi}\colon B\mathcal{G}^{(k,l)}(\mathbb{C}P^2, U(n)) \simeq Map^{(k,l)}(\mathbb{C}P^2, BU(n)) \to Map^{(k,l)}(\mathbb{C}P^2, BPU(n)) \simeq B\mathcal{G}^{(k,l)}(\mathbb{C}P^2, PU(n))$. The restriction of $\widehat{B\pi}$ to the homotopy fibre of e^U is the homotopy equivalence $B\pi_* : Map_*^{(k,l)}(\mathbb{C}P^2, BU(n)) \simeq Map_*^{(k,l)}(\mathbb{C}P^2, BPU(n))$ that was demonstrated in Lemma 4.6 and from this it follows that the square (4.11) is a homotopy pullback. □

The main application of this lemma is the following theorem, which is the main result of this section. It is of interest in its own right, however, and the reader should bear in mind the accidental isomorphism $PU(2) \cong SO(3)$ when we study $U(2)$-gauge groups in a subsequent section. In this case the lemma says the classifying spaces of the based $U(2)$- and $SO(3)$-gauge groups are of the same homotopy type.

Proposition 4.8. *Let p be a prime not dividing n. Then after localisation at p the following diagram becomes homotopy commutative*

$$\begin{array}{ccccccc} S^1 & \longrightarrow & * & \longrightarrow & BS^1 & = & BS^1 \\ \downarrow{\scriptstyle \Delta} & & \downarrow & & \downarrow & & \downarrow{\scriptstyle B\Delta} \\ U(n) & \xrightarrow{\lambda_{(k,l)}} & B\mathcal{G}_*^{(k,l)}(\mathbb{C}P^2, U(n)) & \longrightarrow & B\mathcal{G}^{(k,l)}(\mathbb{C}P^2, U(n)) & \longrightarrow & BU(n) \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle \simeq} & & \downarrow{\scriptstyle \widehat{B\pi}} & & \downarrow{\scriptstyle B\pi} \\ SU(n) & \xrightarrow{\delta_{l'}} & B\mathcal{G}_*^{l'}(\mathbb{C}P^2, SU(n)) & \longrightarrow & B\mathcal{G}^{l'}(\mathbb{C}P^2, SU(n)) & \longrightarrow & BSU(n), \end{array} \tag{4.12}$$

where

$$l' = \begin{cases} 2nl - (n-1)k^2, & n \text{ even}, \\ 2nl - \frac{(n-1)}{2}k^2, & n \text{ odd}. \end{cases}$$

Proof. Let p be a prime not dividing n and localise at p. Then the canonical projection $\rho\colon SU(n) \to PU(n)$ becomes a homotopy equivalence which we shall use to identify these groups and their classifying spaces. From Lemma 4.6 there is an integral homotopy equivalence $B\mathcal{G}_*^{(k,l)}(\mathbb{C}P^2, U(n)) \simeq B\mathcal{G}_*^{(k,l)}(\mathbb{C}P^2, PU(n))$ and we use the maps induced by $B\rho$ to identify $B\mathcal{G}^{(k,l)}(\mathbb{C}P^2, PU(n)) \simeq Map^{(k,l)}(\mathbb{C}P^2, BPU(n)) \simeq$

$B\mathcal{G}^{l'}(\mathbb{C}P^2, SU(n)) \simeq Map^{l'}(\mathbb{C}P^2, BSU(n))$ and $B\mathcal{G}_*^{(k,l)}(\mathbb{C}P^2, SU(n)) \simeq B\mathcal{G}_*^{l'}(\mathbb{C}P^2, SU(n))$. Note that we have used different integer labels for the components on each side of the equivalence. This a point to which we shall shortly return.

Now insert these identifications in the square (4.11) appearing in Lemma 4.7 and extend to get the following homotopy commutative diagram

$$\begin{array}{ccccccc}
S^1 & \longrightarrow & * & \longrightarrow & BS^1 & = & BS^1 \\
\downarrow{\scriptstyle \Delta} & & \downarrow & & \downarrow & & \downarrow{\scriptstyle B\Delta} \\
U(n) & \longrightarrow & B\mathcal{G}_*^{(k,l)}(\mathbb{C}P^2, U(n)) & \longrightarrow & B\mathcal{G}^{(k,l)}(\mathbb{C}P^2, U(n)) & \longrightarrow & BU(n) \\
\downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \simeq} & & \downarrow{\scriptstyle \widehat{B\pi}} & & \downarrow{\scriptstyle B\pi} \\
SU(n) & \xrightarrow{\delta_{l'}} & B\mathcal{G}_*^{l'}(\mathbb{C}P^2, SU(n)) & \longrightarrow & B\mathcal{G}^{l'}(\mathbb{C}P^2, SU(n)) & \longrightarrow & BSU(n).
\end{array} \tag{4.13}$$

At this stage we have left the notation of (4.11) in place: the maps in the diagram are not canonical projections and must be carefully calculated. Note that the bottom row of the diagram now appears as the evaluation fibration of the gauge group of an $SU(n)$-bundle over $\mathbb{C}P^2$.

We next identify the integer l'. The action of $B\Delta$ on the first two integral Chern classes was calculated by Woodward in [25] where it was shown that

$$\begin{aligned} B\Delta^* c_1 &= nx, \\ B\Delta^* c_2 &= \frac{n(n-1)}{2} x^2, \end{aligned}$$

where $x \in H^2(BS^1)$ is a generator. Using this he proves that the image of $B\pi^*\colon H^4(BPU(n)) \to H^4(BU(n))$ is generated by

$$\begin{cases} -(n-1)c_1^2 + 2nc_2, & n \text{ even}, \\ -\frac{(n-1)}{2}c_1^2 + 2nc_2, & n \text{ odd} \end{cases}$$

and so we obtain the action of $B\pi^*$ on the Pontryagin class $p_1 \in H^4(BPU(n))$

$$B\pi^* p_1 = \begin{cases} -(n-1)c_1^2 + 2nc_2, & n \text{ even}, \\ -\frac{(n-1)}{2}c_1^2 + 2nc_2, & n \text{ odd}. \end{cases}$$

With this the value of l' is now clear. If $f\colon \mathbb{C}P^2 \to BU(2)$ satisfies $f^*c_1 = k \cdot x$, $f^*c_2 = l \cdot x^2$ then

$$\begin{aligned} \widehat{B\pi}(f)^* p_1 &= (B\pi \circ f)^* p_1 \\ &= f^* \left(\begin{cases} -(n-1)c_1^2 + 2nc_2, & n \text{ even}, \\ -\frac{(n-1)}{2}c_1^2 + 2nc_2, & n \text{ odd} \end{cases} \right) \\ &= \begin{cases} (2nl - (n-1)k^2) \cdot x^2, & n \text{ even}, \\ \left(2nl - \frac{(n-1)}{2}k^2\right) \cdot x^2, & n \text{ odd} \end{cases} \end{aligned}$$

and we have

$$l' = \begin{cases} 2nl - (n-1)k^2, & n \text{ even}, \\ 2nl - \frac{(n-1)}{2}k^2, & n \text{ odd}. \end{cases}$$

With this in place it only remains to study the map π in (4.13). Consider the diagram

$$\begin{array}{ccc} S^1 \times SU(n) & \xrightarrow{n \cdot i + j} & U(n) \\ \downarrow{\scriptstyle pr_2} & & \downarrow{\scriptstyle \pi} \\ SU(n) & \xrightarrow{\rho} & PU(n), \end{array} \tag{4.14}$$

where $i\colon S^1 = U(1) \to U(n)$ is the inclusion. We claim that this diagram homotopy commutes. Indeed, the map π is a homomorphism so

$$\pi \circ (n \cdot i + j) = n \cdot (\pi \circ i) + \pi \circ j \simeq * + \rho \circ pr_2 \simeq \rho \circ pr_2.$$

Note that since $(n,p) = 1$, the map $n \cdot i + j$ on the top row is a p-local homotopy equivalence with inverse $(1/n \cdot det) \times q\colon U(n) \to S^1 \times SU(n)$ where q, i appear in the fibration sequence $S^1 \xrightarrow{i} U(n) \xrightarrow{q} U(n)/S^1 \cong SU(n)$.

The map we seek to identify is $\rho^{-1} \circ \pi\colon U(n) \to SU(n)$ and using (4.14) we may write this as

$$\rho^{-1} \circ \pi \simeq pr_2 \circ (n \cdot i + j)^{-1} \simeq pr_2 \circ (1/n \cdot det \times q) = q.$$

In short, up to equivalence, we may simply replace the map π in (4.13) with the projection q. This information is presented, with a slight abuse of notation, in the diagram (4.12). □

Corollary 4.9. *Let p be a prime not dividing n. Then after localisation at p there is a homotopy equivalence*

$$\mathcal{G}^{(k,l)}(\mathbb{C}P^2, U(n)) \simeq \mathcal{G}^{l'}(\mathbb{C}P^2, SU(n)) \times S^1,$$

where

$$l' = \begin{cases} 2nl - (n-1)k^2, & n \text{ even}, \\ 2nl - \frac{(n-1)}{2}k^2, & n \text{ odd}. \end{cases}$$

Proof. Loop diagram (4.12) of Proposition 4.8 to obtain

$$\begin{array}{ccccc} \mathcal{G}_*^{(k,l)}(\mathbb{C}P^2, U(n)) & \longrightarrow & \mathcal{G}^{(k,l)}(\mathbb{C}P^2, U(n)) & \longrightarrow & U(n) \\ \downarrow{\scriptstyle \simeq} & & \downarrow{\scriptstyle \Omega\widehat{B\pi}} & & \downarrow{\scriptstyle \pi} \\ \mathcal{G}_*^{l'}(\mathbb{C}P^2, SU(n)) & \longrightarrow & \mathcal{G}^{l'}(\mathbb{C}P^2, SU(n)) & \longrightarrow & SU(n) \\ \downarrow & & \downarrow{\scriptstyle \epsilon} & & \downarrow{\scriptstyle *} \\ * & \longrightarrow & BS^1 & = & BS^1. \end{array}$$

Since $SU(n)$ is 2-connected the connecting map $SU(n) \to BS^1$ of the right-hand vertical fibration is trivial. The connecting map $\epsilon\colon \mathcal{G}^{l'}(\mathbb{C}P^2, SU(n)) \to BS^1$ factors through this trivial map so it too is trivial. This observation induces the product decomposition $\mathcal{G}^{(k,l)}(\mathbb{C}P^2, U(n)) \simeq \mathcal{G}^{l'}(\mathbb{C}P^2, SU(n)) \times S^1$. □

Remark 4.10. It is interesting to observe that the splitting of Corollary 4.9 actually happens on the level of classifying spaces. Indeed, localised at a prime p not dividing n the map $\chi\colon BPU(n) \to K(\mathbb{Z},3)$ becomes trivial since it factors through the contractible space $K(\mathbb{Z}_n,2)_{(p)} \simeq K((\mathbb{Z}_n)_{(p)},2) \simeq *$. Using the principal action we get a local homotopy equivalence $BS^1 \times BSU(n) \simeq BS^1 \times BPU(n) \simeq BU(n)$ (see [11]) which induces a local homotopy equivalence $Map(\mathbb{C}P^2, BS^1) \times Map(\mathbb{C}P^2, BSU(n)) \simeq Map(\mathbb{C}P^2, SU(n))$. The inverse is not easily identified, however, and it is difficult to correctly match components.

4.3. The relationship between $B\mathcal{G}^{(0,l)}(\mathbb{C}P^2, U(n))$ and $B\mathcal{G}^{l}(\mathbb{C}P^2, SU(n))$

Consider the bundles $E_{(0,l)} \to \mathbb{C}P^2$ with trivial first Chern class. These bundles restrict to trivial bundles over S^2 so therefore $\mathcal{G}(E_{(0,l)}|_{S^2}) \simeq \mathcal{G}(S^2 \times U(n)) \simeq Map(S^2, U(n))$. This observation hints that the gauge groups $\mathcal{G}(E_{(0,l)})$ of the unrestricted bundles may have a particularly simple structure and this is indeed the case. In this section we examine the topologies of these gauge groups and uncover their relationship with certain $SU(n)$-gauge groups. Our first result relates to the classifying spaces of the based groups.

Lemma 4.11. *For each integer l there is a homotopy equivalence*

$$B\mathcal{G}^{(0,l)}_*(\mathbb{C}P^2, U(n)) \simeq B\mathcal{G}^{l}_*(\mathbb{C}P^2, SU(n)).$$

Proof. Start with the fibration sequence $SU(n) \xrightarrow{j} U(n) \xrightarrow{det} S^1$. Note that this sequence is split by the inclusion $S^1 \cong U(1) \hookrightarrow U(n)$ which, being itself a group homomorphism, induces a splitting of the fibring of classifying spaces

$$BSU(n) \xrightarrow{Bj} BU(n) \xrightarrow{Bdet} BS^1. \tag{4.15}$$

Now consider the fibre sequence of functors $\Omega^4(-) \xrightarrow{q^*} Map_*(\mathbb{C}P^2, -) \xrightarrow{i^*} \Omega^2(-)$ and take its product with the fibration (4.15). The result of this is a homotopy commutative diagram in which each row and column is a fibre sequence

$$\begin{array}{ccccc}
\Omega^4 BSU(n) & \xrightarrow{q^*} & Map_*(\mathbb{C}P^2, BSU(n)) & \xrightarrow{i^*} & \Omega^2 BSU(n) \\
\simeq \downarrow \Omega^4 Bj & & \downarrow Bj_* & (*) & \downarrow \Omega^2 Bj \\
\Omega^4 BU(n) & \xrightarrow{q^*} & Map_*(\mathbb{C}P^2, BU(n)) & \xrightarrow{i^*} & \Omega^2 BU(n) \simeq \mathbb{Z} \times \Omega^2 BSU(n) \\
\downarrow & & \downarrow Bdet_* & & \downarrow \Omega^2 Bdet \\
* & \longrightarrow & Map_*(\mathbb{C}P^2, BS^1) & \underset{\simeq}{\xrightarrow{i^*}} & \Omega^2 BS^1 \simeq \mathbb{Z}.
\end{array}$$

By a result of Thom [24] $Map_*(\mathbb{C}P^2, BS^1) \simeq Map_*(\mathbb{C}P^2, K(\mathbb{Z}, 2)) \simeq K(H^2(\mathbb{C}P^2), 0) \simeq K(H^2(S^2), 0)$ so the map i^* along the bottom of the diagram is a homotopy equivalence. Since also $\Omega^4 Bj$ on the left-hand side of the diagram is a homotopy equivalence it follows that the square $(*)$ is a homotopy pullback.

Restricting the downwards arrow on the right-hand side of $(*)$ to the basepoint component converts it into a homotopy equivalence $\Omega^2 Bj : \Omega^2 BSU(n) \simeq \Omega^2_0 BU(n)$ and i^* takes the component $Map^{(0,l)}_*(\mathbb{C}P^2, BU(n))$ into $\Omega^2_0 BU(n)$. Since homotopy classes $\mathbb{C}P^2 \to BSU(n)$ are classified by their action on the second Chern class, the map Bj_* sends the component $Map^l_*(\mathbb{C}P^2, BSU(n))$ into $Map^{(0,l)}_*(\mathbb{C}P^2, BU(n))$.

From these observations we obtain, through restriction, another homotopy pullback, here labelled (hpb)

$$\begin{array}{ccccc}
\Omega^4_l BSU(n) & \xrightarrow{q^*} & Map^l_*(\mathbb{C}P^2, BSU(n)) & \xrightarrow{i^*} & \Omega^2 BSU(n) \\
\simeq \big\downarrow \Omega^4_l Bj & & \simeq \big\downarrow Bj_*| \quad \text{(hpb)} & & \simeq \big\downarrow \Omega^2_0 Bj \\
\Omega^4_l BU(n) & \xrightarrow{q^*} & Map^{(0,l)}_*(\mathbb{C}P^2, BU(n)) & \xrightarrow{i^*} & \Omega^2_0 BU(n),
\end{array}$$

which clearly displays the restriction $Bj_*|$ as a homotopy equivalence. □

The previous lemma leads immediately to the next, and the pair is subsequently used to relate the classifying spaces of the full $U(n)$- and $SU(n)$-gauge groups.

Lemma 4.12. *The following homotopy commutative square is a homotopy pullback*

$$\begin{array}{ccc}
B\mathcal{G}^l(\mathbb{C}P^2, SU(n)) & \xrightarrow{e^{SU}} & BSU(n) \\
\big\downarrow \widehat{Bj} & (hpb) & \big\downarrow Bj \\
B\mathcal{G}^{(0,l)}(\mathbb{C}P^2, U(n)) & \xrightarrow{e^U} & BU(n).
\end{array} \tag{4.16}$$

Proof. Introduce the evaluation fibrations for each of the gauge groups $\mathcal{G}^{(0,l)}(\mathbb{C}P^2, U(n))$ and $\mathcal{G}^l(\mathbb{C}P^2, SU(n))$ and use the map $Bj\colon BSU(n) \to BU(n)$ to induce the map $\widehat{Bj}\colon B\mathcal{G}^l(\mathbb{C}P^2, SU(n)) \to B\mathcal{G}^{(0,l)}(\mathbb{C}P^2, U(n))$. Using Lemma 4.11 we see that the induced map of fibres is the homotopy equivalence $Bj_*\colon B\mathcal{G}^l_*(\mathbb{C}P^2, SU(n)) \xrightarrow{\simeq} B\mathcal{G}^{(0,l)}_*(\mathbb{C}P^2, U(n))$ and we use this to identify (4.16) as a homotopy pullback. □

Theorem 4.13. *There is an isomorphism of principal $\mathcal{G}^l(\mathbb{C}P^2, SU(n))$-bundles over S^1:*

$$\mathcal{G}^{(0,l)}(\mathbb{C}P^2, U(n)) \cong S^1 \times \mathcal{G}^l(\mathbb{C}P^2, SU(n)).$$

Proof. Consider diagram (4.16). Since it is a homotopy pullback the homotopy fibre of $\widehat{Bj}$ and Bj share a common homotopy type, namely S^1, and it follows that there is a homotopy fibration sequence

$$\cdots \to \mathcal{G}^l(\mathbb{C}P^2, SU(n)) \xrightarrow{\Omega\widehat{Bj}} \mathcal{G}^{(0,l)}(\mathbb{C}P^2, U(n)) \xrightarrow{\xi} S^1 \xrightarrow{\nu} B\mathcal{G}^l(\mathbb{C}P^2, SU(n)) \to \cdots$$

for some maps ξ, ν. We shall show that the map $\nu\colon S^1 \to B\mathcal{G}^l(\mathbb{C}P^2, SU(n))$ must be null-homotopic by demonstrating that $\pi_1(B\mathcal{G}^l(\mathbb{C}P^2, SU(n))) = 0$ and from this we will be able to conclude the theorem.

To this end we note that $\pi_1(BSU(n)) = 0$ so the homotopy exact sequence of the top line in (4.16) gives an epimorphism $\pi_1(B\mathcal{G}^l_*(\mathbb{C}P^2, SU(n))) \to \pi_1(B\mathcal{G}^l(\mathbb{C}P^2, SU(n)))$ and it will therefore be sufficient to show that $\pi_1(B\mathcal{G}^l_*(\mathbb{C}P^2, SU(n))) = 0$. Using the action of $\Omega^4 BSU(n)$ we obtain a homotopy equivalence $B\mathcal{G}^l_*(\mathbb{C}P^2, SU(n)) \simeq Map^l(\mathbb{C}P^2, BSU(n)) \simeq Map^0_*(\mathbb{C}P^2, BSU(n))$ and we find this latter space in the homotopy fibration sequence $\Omega^4_0 BSU(n) \xrightarrow{q^*} Map^0_*(\mathbb{C}P^2, BSU(n)) \xrightarrow{i^*} \Omega^2 BSU(n)$

which is induced by the cofibring $S^3 \xrightarrow{\eta} S^2 \xrightarrow{i} \mathbb{C}P^2 \xrightarrow{q} S^4$. The long exact homotopy sequence of this fibration then gives us the following exact sequence

$$\pi_4(BSU(n)) \cong \mathbb{Z} \xrightarrow{\eta^*} \pi_5(BSU(n)) \xrightarrow{q^*} \pi_1(Map_*^0(\mathbb{C}P^2, BSU(n))) \xrightarrow{i^*} 0 = \pi_3(BSU(n)).$$

For $n \geqslant 3$ we have $\pi_5(BSU(n)) = 0$ and so in this case it must be that $\pi_1(Map_*^0(\mathbb{C}P^2, BSU(n))) = 0$. When $n = 2$ we have $BSU(2) \simeq BS^3$ with $\pi_4(BS^3) \cong \mathbb{Z}$ generated by the inclusion of the bottom cell $i\colon S^4 \hookrightarrow BS^3$, and $\pi_5(BS^3) \cong \mathbb{Z}_2$ generated by $i \circ \eta$. Thus $\eta^*\colon \pi_4(BS^3) \to \pi_5(BS^3)$ is onto in this case and once again we have $\pi_1(Map_*^0(\mathbb{C}P^2, BSU(2))) = 0$. From the proceeding comments we are now able to conclude the stated result. □

5. Homotopy types of $U(2)$-gauge groups over $\mathbb{C}P^2$

In this section we give application for the various theorems we have collected by studying the homotopy types of $U(2)$-gauge groups over $\mathbb{C}P^2$. We first examine the homotopy types of the based $U(2)$-gauge groups over $\mathbb{C}P^2$ and their classifying spaces. The results collected here are then used to complete the proof of Theorem 1.3 in section 5.2.

5.1. The based gauge groups

We focus here on the classifying spaces of the based gauge groups. Using Theorem 4.2 and Proposition 4.1 we see that there are at most two distinct types, with representatives $B\mathcal{G}_*^{(0,0)}$ and $B\mathcal{G}_*^{(1,0)}$. The main result of this section is that these two spaces do, in fact, represent distinct homotopy types. The proof proceeds by an investigation of the low-dimensional homotopy groups of each of the types.

This result is in strong contrast to previous cases that appear in the literature. For instance when studying the gauge groups associated to any simply connected, compact Lie group over a simply connected 4-manifold, the analogue of Proposition 4.1 is enough to ensure that there is a unique homotopy type amongst the classifying spaces of the based gauge groups.

Lemma 5.1.

$$\pi_1(B\mathcal{G}_*^{(0,0)}(\mathbb{C}P^2, U(2))) = 0, \tag{5.1}$$

$$\pi_1(B\mathcal{G}_*^{(1,0)}(\mathbb{C}P^2, U(2))) \cong \mathbb{Z}_2. \tag{5.2}$$

Proof. We first show (5.1). We identify $B\mathcal{G}_*^{(0,0)}(\mathbb{C}P^2, U(2)) \simeq Map_*^{(0,0)}(\mathbb{C}P^2, BU(2))$. Then we have a homotopy fibration sequence

$$\cdots \to \Omega^2 S^3 \xrightarrow{\eta^*} \Omega_0^3 S^3 \xrightarrow{q^*} Map_*^{(0,0)}(\mathbb{C}P^2, BU_2) \xrightarrow{i^*} \Omega S^3,$$

which is the restriction of the sequence (4.3) to the basepoint components. Examining the homotopy exact sequence of this fibration in low dimensions leads to the exact sequence

$$\pi_2(B\mathcal{G}_*^{(0,0)}(\mathbb{C}P^2, U(2))) \xrightarrow{i^*} \pi_3(S^3) \cong \mathbb{Z} \xrightarrow{\eta^*} \pi_4(S^3)$$
$$\cong \mathbb{Z}_2 \xrightarrow{q^*} \pi_1(B\mathcal{G}_*^{(0,0)}(\mathbb{C}P^2, U(2))) \to 0$$

with q^* onto since $\pi_1(\Omega S^3) \cong \pi_2(S^3) = 0$. As $\pi_3(S^3) \cong \mathbb{Z}$ is generated by the identity

id_{S^3} and $\pi_4(S^3) \cong \mathbb{Z}_2$ is generated by the Hopf map η, the middle map of this sequence is, in fact, onto and it must be that $\pi_1(B\mathcal{G}_*^{(0,0)}) = 0$, proving (5.1).

Turning now to (5.2) we first recall that $B\mathcal{G}^{(1,0)}(\mathbb{C}P^2, U(2)) \simeq B\mathcal{G}^{(1,1)}(\mathbb{C}P^2, U(2))$ by Proposition 4.1 so to prove the statement it will be sufficient to calculate $\pi_1(B\mathcal{G}^{(1,1)}(\mathbb{C}P^2, U(2)))$. We have a fibring

$$\Omega_0^3 S^3 \xrightarrow{j} Map_*^{(1,1)}(\mathbb{C}P^2, BU(2)) \xrightarrow{i^*} \Omega S^3$$

granted by Proposition 4.3 but in this case there is no easy identification of the fibration connecting map. In any case the homotopy exact sequence shows us that $\pi_1(Map_*^{(1,1)}(\mathbb{C}P^2, BU(2)))$ is a quotient of $\pi_1(\Omega_0^3 S^3) \cong \mathbb{Z}_2$ and so it is either trivial or $\mathbb{Z}_2$. We will show it is the latter.

Consider the $U(2)$-bundle $U(2) \hookrightarrow SU(3) \to \mathbb{C}P^2$ obtained by letting $SU(3)$ act on $\mathbb{C}P^2$ in the standard way and let $f\colon \mathbb{C}P^2 \to BU(2)$ be a classifying map for this bundle. Then a simple calculation with the Serre spectral sequence shows that this bundle has Chern classes $(f^*c_1, f^*c_2) = (x, x^2)$ so that f is an element of $Map^{(1,1)}(\mathbb{C}P^2, BU(2)) \simeq B\mathcal{G}^{(1,1)}(\mathbb{C}P^2, U(2))$.

Now the homotopy fibration sequence

$$SU(3) \to \mathbb{C}P^2 \xrightarrow{f} BU(2)$$

gives rise to a short exact sequence of homotopy groups

$$0 \to \pi_5(SU(3)) \cong \mathbb{Z} \to \pi_5(\mathbb{C}P^2) \cong \mathbb{Z} \xrightarrow{f_*} \pi_5(BU(2)) \cong \mathbb{Z}_2 \to 0. \tag{5.3}$$

Here $\pi_6(BU(2)) \cong \pi_5(U(2)) \cong \mathbb{Z}_2$ so the left-hand map is monic and $\pi_4(SU(3)) = 0$ so f_* is epic.

Now $\pi_5(\mathbb{C}P^2)$ is generated by the Hopf map $\gamma\colon S^5 \to \mathbb{C}P^2$, so (5.3) shows that $f_*(\gamma) = f \circ \gamma = \gamma^*(f)$ is a generator, $\tilde{\epsilon}_5$, of $\pi_5(BU(2))$. It follows that γ induces a map of evaluation fibrations as in the diagram

$$\begin{array}{ccccccccc} \cdots \longrightarrow & U(2) & \xrightarrow{\lambda_{(1,1)}} & B\mathcal{G}_*^{(1,1)}(\mathbb{C}P^2, U(2)) & \longrightarrow & B\mathcal{G}^{(1,1)}(\mathbb{C}P^2, U(2)) & \xrightarrow{e} & BU(2) \\ & \| & & \downarrow \gamma^* & & \downarrow \gamma^* & & \| \\ \cdots \longrightarrow & U(2) & \xrightarrow{\delta'} & \Omega_0^4 U(2) & \longrightarrow & Map^{\tilde{\epsilon}_5}(S^5, BU(2)) & \longrightarrow & BU(2). \end{array} \tag{5.4}$$

Here we see a factorisation of the connecting map δ' of the bottom sequence through $\lambda_{(1,1)}$.

By the work of Lang [16] the adjoint of δ' is the Samelson product $\langle \epsilon_4, id_{U(2)} \rangle\colon S^4 \wedge U(2) \to U(2)$, where ϵ_4 generates $\pi_4(U(2)) \cong \mathbb{Z}_2$ and is adjoint to $\tilde{\epsilon}_5$. On the fundamental group we then have $\delta'_*\colon \pi_1(U(2)) \to \pi_5(U(2))$ taking a generator $\epsilon_1 \in \pi_1(U(2))$ to the Samelson product $\langle \epsilon_4, \epsilon_1 \rangle$.

We claim that this Samelson product is non-trivial and is, in fact, a generator of the group. Indeed, if $\epsilon_3 \in \pi_3(U(2)) \cong \mathbb{Z}$ is a generator, then $\langle \epsilon_3, \epsilon_1 \rangle = \epsilon_4$ generates $\pi_4(U(2)) \cong \mathbb{Z}_2$ [2]. Now using that the inclusion $j\colon S^3 \cong SU(2) \hookrightarrow U(2)$ gives an isomorphism on homotopy groups in dimension greater than 1 we have an identification of the generators as $\epsilon_3 = j$ and $\epsilon_4 = \langle \epsilon_3, \epsilon_1 \rangle = j_*\eta$. From this we get that

$$\langle \epsilon_4, \epsilon_1 \rangle = \langle j_*\eta, \epsilon_1 \rangle = \langle \epsilon_3, \epsilon_1 \rangle \circ (\eta \wedge 1) = j \circ \eta \circ \Sigma\eta$$

is the generator of $\pi_5(U(2)) \cong \mathbb{Z}_2$ and the Samelson product is non-trivial.

It follows that $\delta'_* \colon \pi_1(U(2)) \to \pi_5(U(2))$ is onto and the factorisation in the left-hand square of (5.4) shows that $\pi_1(B\mathcal{G}^{(1,1)}_*(\mathbb{C}P^2, U(2)))$ cannot be the trivial group. We now know that $\pi_1(B\mathcal{G}^{(1,1)}_*(\mathbb{C}P^2, U(2)))$ is a non-trivial quotient of $\mathbb{Z}_2$. It must therefore be $\mathbb{Z}_2$. □

We see from the lemma that there are, in fact, two distinct homotopy types amongst the classifying spaces of the based $U(2)$-gauge groups. Since the homotopical information presented in Lemma 5.1 is retained after looping, the distinction is passed onto the based gauge groups they classify as well. We record this in the following statement.

Theorem 5.2. *There are exactly two different homotopy types amongst the classifying spaces of based gauge groups of $U(2)$-bundles over $\mathbb{C}P^2$. A Similar statement holds for the based gauge groups they classify. In particular*

$$B\mathcal{G}^{(0,0)}_*(\mathbb{C}P^2, U(2)) \not\simeq B\mathcal{G}^{(1,0)}_*(\mathbb{C}P^2, U(2))$$

and

$$\mathcal{G}^{(0,0)}_*(\mathbb{C}P^2, U(2)) \not\simeq \mathcal{G}^{(1,0)}_*(\mathbb{C}P^2, U(2)).$$ □

It turns out the distinction between these classifying spaces is solely due to 2-local phenomena. The odd primary homotopy types of these based gauge groups are much more understandable.

Proposition 5.3. *Localised away from 2 there is a single homotopy type amongst the classifying spaces of based gauge groups of $U(2)$-bundles over $\mathbb{C}P^2$. A Similar statement holds for the based gauge groups they classify. In particular, there are local homotopy equivalences $B\mathcal{G}^{(0,0)}_*(\mathbb{C}P^2, U(2)) \simeq B\mathcal{G}^{(1,0)}_*(\mathbb{C}P^2, U(2))$ and $\mathcal{G}^{(1,0)}_*(\mathbb{C}P^2, U(2)) \simeq \mathcal{G}^{(0,0)}_*(\mathbb{C}P^2, U(2)) \simeq Map_*(\mathbb{C}P^2, U(2))$.*

Proof. Once 2 is inverted we have two chains of homotopy equivalences,

$$\begin{aligned} B\mathcal{G}^{(0,0)}_*(\mathbb{C}P^2, U(2)) \simeq Map^{(0,0)}_*(\mathbb{C}P^2, BU(2)) &\simeq Map^{(0,0)}_*(\mathbb{C}P^2, BPU(2)) \\ &\simeq Map^{0}_*(\mathbb{C}P^2, BS^3), \\ B\mathcal{G}^{(1,0)}_*(\mathbb{C}P^2, U(2)) \simeq Map^{(1,0)}_*(\mathbb{C}P^2, BU(2)) &\simeq Map^{(1,0)}_*(\mathbb{C}P^2, BPU(2)) \\ &\simeq Map^{0}_*(\mathbb{C}P^2, BS^3), \end{aligned}$$

which follow from Lemma 4.6 and Proposition 4.8. Combining these two chains yields the first statement.

With this established it is now immediate that

$$\begin{aligned} \mathcal{G}^{(1,0)}_*(\mathbb{C}P^2, U(2)) &\simeq \Omega B\mathcal{G}^{(1,0)}_*(\mathbb{C}P^2, U(2)) \\ &\simeq \Omega B\mathcal{G}^{(0,0)}_*(\mathbb{C}P^2, U(2)) \simeq \mathcal{G}^{(0,0)}_*(\mathbb{C}P^2, U(2)). \end{aligned}$$

Moreover, since $Map^{(0,0)}_*(\mathbb{C}P^2, BU(2))$ is the component containing the constant map it holds that

$$\begin{aligned}\mathcal{G}_*^{(0,0)}(\mathbb{C}P^2, U(2)) \simeq \Omega B\mathcal{G}_*^{(0,0)}(\mathbb{C}P^2, U(2)) &\simeq \Omega Map_*^{(0,0)}(\mathbb{C}P^2, BU(2)) \\ &\simeq Map_*^{(0,0)}(\Sigma\mathbb{C}P^2, BU(2)) \simeq Map_*^{(0,0)}(\mathbb{C}P^2, \Omega BU(2)) \\ &\simeq Map_*^{(0,0)}(\mathbb{C}P^2, U(2)).\end{aligned}$$

Thus

$$\mathcal{G}_*^{(1,0)}(\mathbb{C}P^2, U(2)) \simeq \mathcal{G}_*^{(0,0)}(\mathbb{C}P^2, U(2)) \simeq Map_*^{(0,0)}(\mathbb{C}P^2, U(2)). \qquad \square$$

For later use we record the following.

Lemma 5.4.

$$\pi_2(B\mathcal{G}_*^{(0,0)}) \cong \mathbb{Z}, \tag{5.5}$$

$$\pi_2(B\mathcal{G}_*^{(1,0)}) \cong \mathbb{Z} \oplus \mathbb{Z}_2. \tag{5.6}$$

Proof. We first show (5.5) using the homotopy exact sequence of the fibring $\Omega_0^3 S^3 \xrightarrow{q^*} Map_*^{(0,0)}(\mathbb{C}P^2, BU_2) \xrightarrow{i^*} \Omega S^3$, which, after making the appropriate identifications, displays as

$$\pi_4 S^3 \xrightarrow{\eta^*} \pi_5 S^3 \xrightarrow{q^*} \pi_2 Map_*^{(0,0)}(\mathbb{C}P^2, BU_2) \xrightarrow{i^*} \pi_3 S^3 \xrightarrow{\eta^*} \pi_4 S^4.$$

Now $\pi_4(S^3) \cong \mathbb{Z}_2$ is generated by η and $\pi_5(S^3) \cong \mathbb{Z}_2$ is generated by η^2 so the left hand η^* is an isomorphism and we get $\pi_2(Map_*^{(0,0)}(\mathbb{C}P^2, BU_2)) \cong \ker(\eta^* \colon \pi_3(S^3) \to \pi_4(S^3))$. From Lemma 5.1 we know that $\ker(\eta^* \colon \pi_3(S^3) \to \pi_4(S^3)) \cong 2\mathbb{Z}$ and so we are able to conclude (5.5).

Now we shall show (5.6). We begin by using the map induced by $i \colon S^2 \hookrightarrow \mathbb{C}P^2$ to compare the evaluation fibration sequence of the gauge group $\mathcal{G}^{(1,0)}(\mathbb{C}P^2, U(2))$ with that of the gauge group $\mathcal{G}^1(S^2, U(2))$ belonging to the restricted bundle. The result of this is the following homotopy commutative diagram

$$\begin{array}{ccccccc}
* & \longrightarrow & \Omega_0^3 S^3 & = & \Omega_0^3 S^3 & \longrightarrow & * \\
\downarrow & & \downarrow j & & \downarrow & & \downarrow \\
U(2) & \xrightarrow{\lambda_{(1,0)}} & Map_*^{(1,0)}(\mathbb{C}P^2, BU(2)) & \longrightarrow & B\mathcal{G}^{(1,0)}(\mathbb{C}P^2, U(2)) & \xrightarrow{e} & BU(2) \\
\| & & \downarrow i^* & \text{(hpb)} & \downarrow i^* & & \| \\
U(2) & \xrightarrow{\delta_1} & \Omega S^3 & \longrightarrow & B\mathcal{G}^1(S^2, U(2)) & \longrightarrow & BU(2).
\end{array} \tag{5.7}$$

By construction the square labelled (hpb) is a homotopy pullback. In this diagram we have used the homotopy fibration sequence

$$\Omega_0^3 S^3 \xrightarrow{j} Map_*^{(1,0)}(\mathbb{C}P^2, BU(2)) \xrightarrow{i^*} \Omega S^3 \tag{5.8}$$

supplied by Proposition 4.3 to identify the spaces appearing in the top line of the diagram. The point of which to take note is the factorisation of the connecting map δ_1 of the bottom evaluation sequence through both i^* and $\lambda_{(1,0)}$.

Now it was shown in Lemma 5.1 that $\pi_1(Map_*^{(1,0)}(\mathbb{C}P^2, BU(2))) \cong \mathbb{Z}_2$, and argued there also that the map $j_*\colon \pi_1(\Omega_0^3 S^3) \xrightarrow{\cong} \pi_1(Map_*^{(1,0)}(\mathbb{C}P^2, BU(2)))$ is an isomorphism. Shortly we shall show that the map $j_*\colon \pi_2(\Omega_0^3 S^3) \to \pi_2(Map_*^{(1,0)}(\mathbb{C}P^2, BU(2)))$ is injective. If we assume this for now and examine the homotopy exact sequence of the fibration (5.8) we find a short exact sequence

$$0 \to \pi_2(\Omega_0^3 S^3) \xrightarrow{j_*} \pi_2(Map_*^{(1,0)}(\mathbb{C}P^2, BU(2))) \xrightarrow{i^*} \pi_2(\Omega S^3) \to 0,$$

which must split for algebraic reasons since $\pi_2(\Omega S^3) \cong \pi_3(S^3) \cong \mathbb{Z}$ is free abelian. This then allows us to get (5.6) by concluding that

$$\pi_2(Map_*^{(1,0)}(\mathbb{C}P^2, BU(2))) \cong \pi_2(\Omega S^3) \oplus \pi_2(\Omega_0^3 S^3) \cong \mathbb{Z} \oplus \mathbb{Z}_2.$$

We shall now provide the details of the argument for $j_*\colon \pi_2(\Omega_0^3 S^3) \to \pi_2(Map_*^{(1,0)}(\mathbb{C}P^2, BU(2)))$ being injective. We have seen that the Samelson product $\langle \epsilon_1, \epsilon_3 \rangle$ is non-trivial and that the map $\delta_{1*}\colon \pi_3(U(2)) \to \pi_3(\Omega S^3) \cong \pi_4(U(2))$ is given the assignment $\alpha \mapsto \langle \epsilon_1, \alpha \rangle$. It must therefore be that δ_{1*} is a non-trivial surjection $\pi_3(U(2)) \cong \mathbb{Z} \to \pi_4(U(2)) \cong \mathbb{Z}_2$. In (5.7) we saw that δ_1 factors through $i^*\colon Map_*^{(1,0)}(\mathbb{C}P^2, BU(2)) \to \Omega S^3$ so it must be that i^* induces a non-trivial surjection in this degree. Consequently, the connecting map of the sequence $\Delta\colon \pi_3(\Omega S^3) \to \pi_2(\Omega^3 S^3)$ must be trivial and the following map of the sequence, $j_*\colon \pi_2(\Omega_0^3 S^3) \to \pi_2(Map_*^{(1,0)}(\mathbb{C}P^2, BU(2)))$, must be injective. □

5.2. The full gauge groups

Moving on now to the full gauge groups we first use Theorem 4.2 to reduce the enumeration problem to a more manageable size. Applying its statement in the case of $n = 2$ yields the following.

Proposition 5.5. *For all integers, k, l, r there are homotopy equivalences*

$$B\mathcal{G}^{(k,l)}(\mathbb{C}P^2, U(2)) \simeq B\mathcal{G}^{(k+2r, l+kr+r^2)}(\mathbb{C}P^2, U(2))$$

and, consequently, also homotopy equivalences

$$\mathcal{G}^{(k,l)}(\mathbb{C}P^2, U(2)) \simeq \mathcal{G}^{(k+2r, l+kr+r^2)}(\mathbb{C}P^2, U(2)).$$ □

Now assume given a particular gauge group $\mathcal{G}^{(k,l)}(\mathbb{C}P^2, U(2))$. Write $k = 2k' + \epsilon$ with $\epsilon \in \{0, 1\}$ and take $r = -k'$ in the formula of Proposition 5.5 to obtain a homotopy equivalence

$$B\mathcal{G}^{(k,l)}(\mathbb{C}P^2, U(2)) \simeq B\mathcal{G}^{(\epsilon, l-k'^2-\epsilon k')}(\mathbb{C}P^2, U(2))$$

with $\epsilon = 0$ or 1. We see that in studying the homotopy type of any given gauge group $\mathcal{G}^{(k,l)}(\mathbb{C}P^2, U(2))$ we can always find a homotopy equivalent object with $k = 0$ or 1. This observation reduces the problem of studying the homotopy types of all possible $U(2)$-gauge groups over $\mathbb{C}P^2$ to just the study of the homotopy types of the gauge groups

$$\mathcal{G}^{(0,l)}(\mathbb{C}P^2, U(2)) \qquad \mathcal{G}^{(1,l)}(\mathbb{C}P^2, U(2))$$

and their classifying spaces as l ranges over the integers. In the following we shall always therefore assume that k is an integer mod 2.

We next obtain an upper bound on the number of homotopy types in each class. We apply Corollary 4.5 and feed in the information from Theorem 1.1, that $|\delta_1| = 12$ [14], to get the following.

Proposition 5.6. *For each pair of integers k, l, there is a homotopy equivalence*

$$\mathcal{G}^{(k,l)}(\mathbb{C}P^2, U(2)) \simeq \mathcal{G}^{(k,l+12)}(\mathbb{C}P^2, U(2)). \qquad \square$$

This proposition gives an upper bound that is not necessarily met. Indeed there are fewer distinct homotopy types amongst the gauge groups in the first class. For these objects we have already solved the problem completely.

Proposition 5.7. *There is a homotopy equivalence $\mathcal{G}^{(0,l)}(\mathbb{C}P^2, U(2)) \simeq \mathcal{G}^{(0,l')}(\mathbb{C}P^2, U(2))$ if and only if $(6,l) = (6,l')$.*

Proof. From Theorem 4.13 we obtain a decomposition

$$\mathcal{G}^{(0,l)}(\mathbb{C}P^2, U(2)) \simeq S^1 \times \mathcal{G}^{l}(\mathbb{C}P^2, SU(2)). \tag{5.9}$$

Now the gauge groups of principal $SU(2)$-bundles over $\mathbb{C}P^2$ have been studied by Kono and Tsukuda in [15]. In fact, they give results relating to the homotopy types of $SU(2)$-bundles over any simply connected 4-manifold. To apply their results we simply note that the signature of the intersection form on $\mathbb{C}P^2$ is $+1$, so their Theorem 1.2 applies to give $\mathcal{G}^{l}(\mathbb{C}P^2, SU(2)) \simeq \mathcal{G}^{l'}(\mathbb{C}P^2, SU(2))$ if and only if $(6,l) = (6,l')$. Combining this information with the splitting in equation (5.9) completes the proof. $\square$

Turning now towards the study of the second class of gauge groups, namely the gauge groups $\mathcal{G}^{(1,l)}(\mathbb{C}P^2, U(2))$, we encounter a much more delicate problem. Integral results are particularly tricky and for the most part we make do with p-local statements. We already have the upper bound $\mathcal{G}^{(1,l)}(\mathbb{C}P^2, U(2)) \simeq \mathcal{G}^{(1,l+12)}(\mathbb{C}P^2, U(2))$ of Proposition 5.6. The following provides a lower bound of at least 2 distinct homotopy types. For its proof recall the notation of section 4.1; $\lambda_{(k,l)} \colon U(2) \to Map_*^{(k,l)}(\mathbb{C}P^2, BU(2))$ is the connecting map for the evaluation fibration of the gauge group $\mathcal{G}^{(k,l)}(\mathbb{C}P^2, U(2))$ and $\delta_l \colon U(2) \to \Omega_0^3 U(2)$ is the connecting map of the evaluation fibration sequence of the gauge group $\mathcal{G}^{l}(S^4, U(2))$.

Proposition 5.8.

$$\pi_1(B\mathcal{G}^{(1,l)}(\mathbb{C}P^2, U(2))) \cong \begin{cases} 0, & l \equiv 1 \pmod 2, \\ \mathbb{Z}_2, & l \equiv 0 \pmod 2. \end{cases}$$

Proof. The first step of the proof is to calculate the homomorphisms $\lambda_{(1,0)*} \colon \pi_1(U(2)) \to \pi_1(Map_*^{(1,0)}(\mathbb{C}P^2, BU(2)))$ and $\delta_l \colon \pi_1(U(2)) \to \pi_4(U(2))$. Following this we use the decomposition of Theorem 4.4 to obtain the action of $\lambda_{(1,l)*}$ for each l, and thus calculate $\pi_1 B\mathcal{G}^{(1,l)}(\mathbb{C}P^2, U(2)) \cong \operatorname{coker}(\lambda_{(1,l)*})$.

To proceed we use the evaluation fibration for the gauge group $\mathcal{G}^{(1,0)}(\mathbb{C}P^2, U(2))$ to get an exact sequence

$$\begin{aligned} \pi_1(\mathcal{G}^{(1,0)}(\mathbb{C}P^2, U(2))) \xrightarrow{e_*} \pi_1(U(2)) \xrightarrow{\lambda_{(1,0)*}} \pi_1(&Map_*^{(1,0)}(\mathbb{C}P^2, BU(2))) \\ &\to \pi_1(B\mathcal{G}^{(1,0)}(\mathbb{C}P^2, U(2))) \to 0 \end{aligned} \tag{5.10}$$

since $BU(2)$ is simply connected. The map $e\colon \mathcal{G}^{(1,0)}(\mathbb{C}P^2, U(2)) \to U(2)$ is the evaluation. We claim that e_* is surjective and therefore $\lambda_{(1,0)}$ is trivial. This is seen as follows.

Since its second Chern class vanishes, the bundle $E_{(1,0)} \to \mathbb{C}P^2$ has a reduction of structure to a $U(1)$-bundle $\tilde{E}_1 \to \mathbb{C}P^2$ and there results an isomorphism of principal bundles $E_{(1,0)} \cong \tilde{E}_1 \times_{U(1)} U(2)$ over $\mathbb{C}P^2$, where $U(1)$ acts on $U(2)$ from the left via its inclusion $i_2\colon U(1) \hookrightarrow U(2)$ in the bottom right-hand corner. Define $\alpha\colon U(1) \to \mathcal{G}(\widetilde{E}_1 \times_{U(1)} U(2)) \cong \mathcal{G}^{(1,0)}(\mathbb{C}P^2, U(2))$ by

$$\alpha(\lambda)(x, A) = (x, i_1(\lambda) \cdot A), \qquad \lambda \in U(1),\ x \in \tilde{E}_1,\ A \in U(2),$$

where $i_1\colon U(1) \to U(2)$ is the inclusion in the top left-hand corner. It is not difficult to see that this is a well defined homomorphism and satisfies

$$e \circ \alpha = i_1. \tag{5.11}$$

Now recall that i_1 generates $\pi_1(U(n))$. It follows from (5.11) that e_* is surjective and $\lambda_{(1,0)*} = 0$.

On the other hand, $\delta_l\colon U(2) \to \Omega^3_0 U(2)$ is adjoint to the Samelson product $\langle l \cdot \epsilon_3, 1_{U(2)}\rangle\colon S^3 \wedge U(2) \to U(2)$ and on the fundamental group the induced homomorphism is given by $\delta_{l*}\colon \pi_1(U(2)) \cong \mathbb{Z} \to \pi_4(U(2)) \cong \mathbb{Z}_2,\ \epsilon_1 \mapsto \langle l \cdot \epsilon_3, \epsilon_1\rangle = l \cdot \epsilon_4$ where ϵ_i generates $\pi_i U(2)$. It is therefore trivial when l is even and a non-trivial surjection when l is odd.

Now we calculate the action of the combined map $\lambda_{(1,l)} = \lambda_{(1,0)} + \delta_l$. Consider the composite $\lambda_{(1,l)*}\epsilon_1 = \lambda_{(1,l)} \circ \epsilon_1$ as a class in $\pi_1(Map_*^{(1,l)}(\mathbb{C}P^2, BU(2)))$. With respect to the decomposition of Theorem 4.4 it is represented by the following map

$$\begin{aligned} &S^1 \to Map_*^{(1,l)}(\mathbb{C}P^2, BU(2)) \\ &z \mapsto \left[\mathbb{C}P^2 \xrightarrow{c} \mathbb{C}P^2 \vee S^4 \xrightarrow{(\lambda_{(1,0)*}\epsilon_1(z)) \vee (\delta_{l*}\epsilon_1(z))} BU(2) \vee BU(2) \xrightarrow{\nabla} BU(2)\right]. \end{aligned}$$

Since $\lambda_{(1,0)*}\epsilon_1 = 0$, it is homotopic to the constant loop at a chosen basepoint map $f_{(1,0)} \in Map_*^{(1,0)}(\mathbb{C}P^2, BU(2))$. Similarly, $\delta_{l*}\epsilon_1$ is either homotopic to a chosen basepoint loop $\omega_0 \in \Omega^4_l BU(2)$ when l is even, or, when l is odd, is a generator $\widehat{\epsilon}_1 \in \pi_1(\Omega^4_l BU(2)) \cong \mathbb{Z}_2$. Thus we have

$$\begin{aligned} \lambda_{(1,l)*}\epsilon_1(z) &= \lambda_{(1,0)*}\epsilon_1(z) + \delta_{l*}\epsilon_1(z) \\ &= f_{(1,0)} + \delta_{l*}\epsilon_1(z) \\ &= \begin{cases} f_{(1,0)} + \omega_0, & l \text{ even} \\ f_{(1,0)} + \widehat{\epsilon}_1(z), & l \text{ odd} \end{cases} \\ &= \begin{cases} f_{(1,l)}, & l \text{ even} \\ f_{(1,l)} + (-\omega_0 + \widehat{\epsilon}_1(z)), & l \text{ odd} \end{cases} \\ &= \begin{cases} j(\omega_0), & l \text{ even} \\ j(\widehat{\epsilon}_1(z)), & l \text{ odd,} \end{cases} \end{aligned} \tag{5.12}$$

where the last line follows from Proposition 4.3 where the fibre inclusion $j\colon \Omega^4_0 BU(2) \to Map_*^{(1,l)}(\mathbb{C}P^2, BU(2))$ was identified as $j(\omega) = f_{(1,l)} + \omega$.

Now equation (5.12) shows us that if l is even, then $\lambda_{(1,l)*}\epsilon_1$ is the constant loop at the basepoint map and is thus trivial. On the other hand, if l is odd, then $\lambda_{(1,l)*}\epsilon_1 = j_*\widehat{\epsilon}_1$. Since $j_*\colon \pi_1(\Omega^4_l BU(2)) \to \pi_1(Map^{(1,l)}_*(\mathbb{C}P^2, BU(2)))$ is an isomorphism by Lemma 5.1, this element is non-trivial and we conclude that $\lambda_{(1,l)_*}$ is surjective in this case.

Now (5.10) displays the fact that $\pi_1(B\mathcal{G}^{(1,l)}(\mathbb{C}P^2, U(2))) \cong \operatorname{coker}\left(\lambda_{(1,l)_*}\right)$. We have just seen that $\lambda_{(1,l)_*}$ is zero when l is even and surjective when l is odd. We know $\pi_1(Map^{(1,l)}_*(\mathbb{C}P^2, BU(2))) = \mathbb{Z}_2$ so therefore

$$\pi_1(B\mathcal{G}^{(1,l)}(\mathbb{C}P^2, U(2))) \cong \begin{cases} 0, & l \equiv 1 \pmod 2, \\ \mathbb{Z}_2, & l \equiv 0 \pmod 2 \end{cases}$$

and we are complete. □

The major obstacle to obtaining further integral information on the homotopy types of the gauge groups $\mathcal{G}^{(1,l)}(\mathbb{C}P^2, U(2))$ is actually a lack of 2-local information. Localised at an odd prime p, the homotopy types of these objects are much simpler.

Theorem 5.9. *When localised away from 2 there is a product splitting*

$$\mathcal{G}^{(1,l)}(\mathbb{C}P^2, U(2)) \simeq S^1 \times \mathcal{G}^{4l-1}(\mathbb{C}P^2, SU(2)).$$

It follows that when localised away from 2 there is a homotopy equivalence $\mathcal{G}^{(1,l)}(\mathbb{C}P^2, U(2)) \simeq \mathcal{G}^{(1,l')}(\mathbb{C}P^2, U(2))$ *if and only if* $(4l-1, 6) = (4l'-1, 6)$. *In particular, when localised at an odd prime* $p \geqslant 5$, *the gauge group* $\mathcal{G}^{(1,l)}(\mathbb{C}P^2, U(2))$ *has the trivial homotopy type*

$$\mathcal{G}^{(1,l)}(\mathbb{C}P^2, U(2)) \simeq U(2) \times Map^*(\mathbb{C}P^2, U(2)).$$

Proof. The decomposition statement follows from Corollary 4.9. After this is established it is a simple matter to apply the results of Kono and Tsukuda [15] on the homotopy types of the $\mathcal{G}^{4l-1}(\mathbb{C}P^2, SU(2))$ gauge groups to verify the second statement. The final statement is then an immediate consequence of this. □

Before closing we shall return to the integral world to answer one remaining question.

Theorem 5.10. *For any integer values of* l, l' *it holds that*

$$B\mathcal{G}^{(0,l)}(\mathbb{C}P^2, U(2)) \not\simeq B\mathcal{G}^{(1,l')}(\mathbb{C}P^2, U(2)),$$
$$\mathcal{G}^{(0,l)}(\mathbb{C}P^2, U(2)) \not\simeq \mathcal{G}^{(1,l')}(\mathbb{C}P^2, U(2)).$$

To demonstrate the statement we shall calculate π_2 of each classifying space. We shall find the result independent of the integers l, l' – up to isomorphism – but different for $B\mathcal{G}^{(0,l)}(\mathbb{C}P^2, U(2))$ and $B\mathcal{G}^{(1,l')}(\mathbb{C}P^2, U(2))$. From this we shall conclude that there are no values of l, l' for which the classifying spaces are homotopy equivalent. Since the information in π_2 is retained after looping we shall be able to conclude the statement for the gauge groups.

Lemma 5.11. *For any integer values of* l, l' *the following hold.*

$$\pi_2(B\mathcal{G}^{(0,l)}(\mathbb{C}P^2, U(2))) \cong \mathbb{Z} \oplus \mathbb{Z},$$
$$\pi_2(B\mathcal{G}^{(1,l')}(\mathbb{C}P^2, U(2))) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_2.$$

Proof. The first step comes from an examination of the homotopy exact sequences of the evaluation fibration sequences

$$\cdots U(2) \xrightarrow{\lambda_{(k,l)}} B\mathcal{G}_*^{(k,l)}(\mathbb{C}P^2, U(2)) \to B\mathcal{G}^{(k,l)}(\mathbb{C}P^2, U(2)) \xrightarrow{e} BU(2)$$

for $k = 0$, 1. Since $\pi_3(BU(2)) = 0$ we have in both cases an exact sequence

$$0 \to \pi_2(B\mathcal{G}_*^{(k,l)}(\mathbb{C}P^2, U(2))) \to \pi_2(B\mathcal{G}^{(k,l)}(\mathbb{C}P^2, U(2))) \xrightarrow{e_*} \pi_2(BU(2)) \xrightarrow{\lambda_{(k,l)*}} \pi_1(B\mathcal{G}_*^{(k,l)}(\mathbb{C}P^2, U(2)))$$

and we conclude from this that the following is short exact

$$0 \to \pi_2(B\mathcal{G}_*^{(k,l)}(\mathbb{C}P^2, U(2))) \to \pi_2(B\mathcal{G}^{(k,l)}(\mathbb{C}P^2, U(2))) \to \ker(\lambda_{(k,l)*}) \to 0. \quad (5.13)$$

Now Lemma 5.1 and Proposition 5.8 give

$$\ker(\lambda_{(k,l)*}) \cong \begin{cases} 2\mathbb{Z} & k = 1,\ l \text{ odd}, \\ \mathbb{Z} & \text{otherwise} \end{cases}$$

so in either case this means that the last group in (5.13) is free abelian and the sequence must split to give

$$\pi_2(B\mathcal{G}^{(k,l)}(\mathbb{C}P^2, U(2))) \cong \pi_2(Map_*^{(k,l)}(\mathbb{C}P^2, BU(2))) \oplus \mathbb{Z}.$$

Now the calculation of $\pi_2(Map_*^{(k,l)}(\mathbb{C}P^2, BU(2)))$ was completed in Lemma 5.4 and using this information completes the proof. □

Summarising the details presented in this section we now briefly prove Theorem 1.3.

Proof of Theorem 1.3. The first item 1 follows from Proposition 5.5 and the comments following it. Item 2 is Proposition 5.6. Item 3 follows from Theorem 4.13 and was included in the proof of Proposition 5.7. Item 4 is the statement of Theorem 5.9. Item 5 was shown in proposition 5.10 and item 6 was given in Theorem 5.2. □

References

[1] M. Arkowitz, *Introduction to Homotopy Theory*, Springer, New York, (2010).

[2] R. Bott, *A Note on the Samelson Product in the Classical Groups*, Comment. Math. Helv. **34**, (1960), 245–256.

[3] A, Dold, *Partitions of Unity in the Theory of Fibrations*, Ann. of Math. **78**, (1962), 223–255.

[4] T. Cheng, L. Li, *Gauge Theory of Elementary Particle Physics*, Clarendon Press, Oxford, (1982).

[5] M. Crabb, W. Sutherland, *Counting Homotopy Types of Gauge Groups*, Proc. Lond. Math. Soc. **83**, (2000), 747–768.

[6] S. Donaldson, *An Application of Gauge Theory to Four Dimensional Topology*, J. Differential Geom. **18**, (1983), 279–315.

[7] S. Donaldson, P. Kronheimer, *The Geometry of Four-Manifolds*, Oxford University Press, (1990).

[**8**] D. Gottlieb, *On Fibre Spaces and the Evaluation Map*, Ann. of Math. **87**, (1968), 42–55.

[**9**] H. Hamanaka, A. Kono, *Unstable K^1-Group $\left[\Sigma^{2n-2}\mathbb{C}P^2, U(n)\right]$ and Homotopy Type of Certain Gauge Groups*, Proc. Roy. Soc. Edinburgh Sect. A **136**, (2006), 149–155.

[**10**] A. Hatcher, *Algebraic Topology*, Cambridge University Press, (2002).

[**11**] K. Ishiguro, *A p-Local Splitting of BU(n)*, Proc. Amer. Math. Soc. **2**, (1985), 307–311.

[**12**] I. James, *On Lie Groups and Their Homotopy Groups*, Proc. Cambridge Philos. Soc. **55** (1959), 244–247.

[**13**] Y. Kamiyama, D. Kishimoto, A. Kono, S. Tsukuda, *Samelson Products of SO(3) and Applications*, Glasg. Math. J. **49**, (2007), 405–409.

[**14**] A. Kono, *A Note on the Homotopy Type of Certain Gauge Groups*, Proc. Roy. Soc. Edinburgh Sect. A **117**, (1991), 295–297.

[**15**] A. Kono, S. Tsukuda, *Remark on the Homotopy Type of Certain Gauge Groups*, J. Math. Kyoto Univ. **36**, (1996), 115–121.

[**16**] G. Lang, *The Evaluation Map and EHP Sequences*, Pacific J. Math. **44**, (1973), 201–210.

[**17**] G. Lupton, S. Smith, *Criteria for Components of a Function Space to be Homotopy Equivalent*, Math. Proc. Cambridge Philos. Soc. **145**, (2008), 95–106.

[**18**] I. Singer, *Some Remarks on the Gribov Ambiguity*, Comm. Math. Phys. **60**, (1978), 7–12.

[**19**] W. Sutherland, *Function Spaces Related to Gauge Groups*, Proc. Roy. Soc. Edinburgh Sect. A **121**, (1992), 185–190.

[**20**] S. Theriault, *Homotopy Decompositions of Gauge Groups Over Riemann Surfaces and Applications to Moduli Spaces*, Int. J. Math. **22**, (2011), 1711–1719.

[**21**] S. Theriault, *The Homotopy Types of SU(3)-gauge Groups over Simply Connected 4-manifolds*, Publ. Res. Inst. Math. Sci. **48**, (2012), 543–563.

[**22**] S. Theriault, *The Homotopy Types of SU(5)-Gauge Groups*, Osaka J. Math. **52**, (2015), 15–31.

[**23**] S. Theriault, *The Homotopy Types of Gauge Groups of Nonorientable Surfaces and Applications to Moduli Spaces*, Illinois J. Math. **57**, (2013), 59–85.

[**24**] R. Thom, *L'homologie des Espaces Fonctionnels*, Colloq. Topol. Algébrique, Louvain, (1956), 29–39.

[**25**] L. Woodward, *The Classification of Principal PU_n-Bundles Over a 4-Complex*, J. Lond. Math. Soc. **25**, (1982), 513–524.

[**26**] A. Zabrodsky, *Hopf Spaces*, North-Holland Publishing Company, (1976).

Tyrone Cutler tcutler@math.uni-bielefeld.de

Fakultät für Mathematik, Universität Bielefeld, Postfach 100131, 33501 Bielefeld, Germany