K-THEORETIC TORSORS FOR INFINITE DIMENSIONAL VECTOR BUNDLES OF LOCALLY COMPACT TYPE

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Abstract

Drinfeld observed that there were apparently two notions of K-theory torsor one might expect to associate to a Tate R-module, and that these should be equivalent. The purpose of the present note is to explain this equivalence as a direct consequence of the author's delooping theorem and Drinfeld's theorem that the first negative K-group vanishes Nisnevich locally.

1. Introduction

Drinfeld observed in [**3**] that there were apparently two notions of K-theory torsor one might expect to associate to a topological module called a Tate R-module, and that these should be equivalent. The purpose of the present note is to explain this equivalence as a direct consequence of the author's delooping theorem of [**8**] and Drinfeld's theorem of [**3**] that the first negative K-group vanishes Nisnevich locally.

Let S be a quasi-compact, quasi-separated noetherian scheme of finite Krull dimension with an ample family of line bundles. We denote by $\mathrm{Shv}_{(\mathrm{Spaces})}(NS_{\mathrm{Nis}})$ the ∞-topos of sheaves of spaces on the small Nisnevich site S_{Nis} of S; by $\mathcal{K}$ the sheaf of K-theory spaces $S' \mapsto \Omega^\infty K(S')$ on S_{Nis} (here $K(-)$ is the connective K-theory spectrum) regarded as a group object in $\mathrm{Shv}_{(\mathrm{Spaces})}(NS_{\mathrm{Nis}})$; and by $B\mathcal{K}$ the classifying space for $\mathcal{K}$-torsors. (See section 2 below for more details.)

Recall the construction $\underleftrightarrow{\lim}$ introduced in [**1**, A.3], that associates to an exact category $\mathcal{A}$ a new one $\underleftrightarrow{\lim}\,\mathcal{A} \subset \mathrm{Ind}\,\mathrm{Pro}(\mathcal{A})$. We write $\mathrm{Vect}_{\mathrm{Tate}}(S)$ for the idempotent completion of $\underleftrightarrow{\lim}\,\mathrm{Vect}_{\mathrm{finite}}(S)$, where $\mathrm{Vect}_{\mathrm{finite}}(S)$ is the exact category of finite-rank vector bundles. If $S = \mathrm{Spec}\,R$ is affine, $\mathrm{Vect}_{\mathrm{Tate}}(S)$ embeds into the category of Tate R-modules in the sense of [**3**], being equivalent to the full subcategory of Tate R-modules of countable type; see [**2**, Theorem 5.30] for details. Denote by $\mathcal{K}_{\mathrm{Tate}}$ the sheaf of spaces $S' \mapsto \Omega^\infty \mathbb{K}(\mathrm{Vect}_{\mathrm{Tate}}(S'))$ on S_{Nis}, where $\mathbb{K}(-)$ denotes the non-connective K-theory spectrum. (See section 2 below for more details.)

In this short note, we prove the following results.

Theorem 1.1. *There is an equivalence* $B\mathcal{K} \xrightarrow{\sim} \mathcal{K}_{\mathrm{Tate}}$ *in the* ∞*-topos* $\mathrm{Shv}_{(\mathrm{Spaces})}(NS_{\mathrm{Nis}})$.

Let $\mathcal{E}$ be an object of $\mathrm{Vect}_{\mathrm{Tate}}(S)$ and $[\mathcal{E}] \in \mathcal{K}_{\mathrm{Tate}}(S) = \Omega^\infty \mathbb{K}(\mathrm{Vect}_{\mathrm{Tate}}(S))$ the associated point in the K-theory space. The map $S \xrightarrow{[\mathcal{E}]} \mathcal{K}_{\mathrm{Tate}}$ corresponds by the above theorem to a $\mathcal{K}$-torsor $\mathfrak{D}_{\mathcal{E}}$ on S. Write $\mathrm{Aut}(\mathcal{E})$ for the sheaf of groups on S_{Nis} given by $S' \mapsto \mathrm{Aut}_{\mathrm{Vect}_{\mathrm{Tate}}(S')} \mathcal{E} \times_S S'$. This is a group object of the 1-topos $\mathrm{Shv}_{(\mathrm{Sets})}(S_{\mathrm{Nis}})$, and hence of the ∞-topos $\mathrm{Shv}_{(\mathrm{Spaces})}(NS_{\mathrm{Nis}}) \supset \mathrm{Shv}_{(\mathrm{Sets})}(S_{\mathrm{Nis}})$.

Theorem 1.2. *The automorphism group* $\mathrm{Aut}\,\mathcal{E}$ *acts on the $\mathcal{K}$-torsor* $\mathfrak{D}_{\mathcal{E}}$.

Remark 1.3. Considering the composite map $S \xrightarrow{[\mathcal{E}]} \mathcal{K}_{\mathrm{Tate}} = B\mathcal{K} \to B\tau_{\leqslant 1}\mathcal{K}$ instead of $[\mathcal{E}]$ itself (where $B\mathcal{K} \to B\tau_{\leqslant 1}\mathcal{K}$ is induced from the truncation map), we get a $\tau_{\leqslant 1}\mathcal{K}$-torsor $D_{\mathcal{E}}$. The proof of Theorem 1.2 shows $\mathrm{Aut}\,\mathcal{E}$ acts on $D_{\mathcal{E}}$ as well, defining an enriched $\mathbb{G}_m$-central extension of $\mathrm{Aut}\,\mathcal{E}$. Cf. [**3**, section 5.5].

2. ∞-topos formalism

In this preliminary section, we give a detailed account on the notions and objects involved in Theorems 1.1 and 1.2, in order to ensure that the statements make sense to the reader even if he or she is unfamiliar with the language of ∞-topoi.

Write Set_Δ for the category of simplicial sets, which is a combinatorial, simplicial model category with the Kan model structure. Recall that S_{Nis} denotes the small Nisnevich site of the scheme S. We write $\mathrm{Set}_\Delta^{S_{\mathrm{Nis}}^{\mathrm{op}}}$ for the combinatorial, simplicial model category of simplicial presheaves on the underlying category of S_{Nis} with the injective model structure, and $(\mathrm{Set}_\Delta^{S_{\mathrm{Nis}}^{\mathrm{op}}})^\circ$ for its fibrant-cofibrant objects. By Proposition 4.2.4.4 of [**5**], there is an equivalence of ∞-categories

$$\theta \colon N(\mathrm{Set}_\Delta^{S_{\mathrm{Nis}}^{\mathrm{op}}})^\circ \xrightarrow{\sim} \mathrm{Fun}(NS_{\mathrm{Nis}}^{\mathrm{op}}, (\mathrm{Spaces})) = \mathrm{Preshv}_{(\mathrm{Spaces})}(NS_{\mathrm{Nis}}).$$

Here N denotes the simplicial nerve, $\mathrm{Fun}(-, -)$ the ∞-category of functors, (Spaces) the ∞-category of spaces, which is by definition the simplicial nerve of the simplicial category of Kan complexes, and $\mathrm{Preshv}_{(\mathrm{Spaces})}(NS_{\mathrm{Nis}})$ the ∞-category of presheaves of spaces on NS_{Nis}. Let $\mathrm{Set}_{\Delta,\mathrm{loc}}^{S_{\mathrm{Nis}}^{\mathrm{op}}}$ denote the combinatorial, simplicial model category of simplicial presheaves on the site S_{Nis} with respect to Jardine's local model structure [**4**], and $(\mathrm{Set}_{\Delta,\mathrm{loc}}^{S_{\mathrm{Nis}}^{\mathrm{op}}})^\circ$ its fibrant-cofibrant objects. Then Proposition 6.5.2.14 of [**5**] shows that the above equivalence θ restricts to an equivalence

$$\theta \colon N(\mathrm{Set}_{\Delta,\mathrm{loc}}^{S_{\mathrm{Nis}}^{\mathrm{op}}})^\circ \xrightarrow{\sim} \mathrm{Shv}_{(\mathrm{Spaces})}(NS_{\mathrm{Nis}})^\wedge \xrightarrow{\sim} \mathrm{Shv}_{(\mathrm{Spaces})}(NS_{\mathrm{Nis}}),$$

where $\mathrm{Shv}_{(\mathrm{Spaces})}(NS_{\mathrm{Nis}}) \subset \mathrm{Preshv}_{(\mathrm{Spaces})}(NS_{\mathrm{Nis}})$ is the ∞-topos of sheaves of spaces on NS_{Nis} [**5**, 6.1.0.4 and 6.2.2], and $\mathrm{Shv}_{(\mathrm{Spaces})}(NS_{\mathrm{Nis}})^\wedge$ denotes its hypercompletion [**5**, 6.5.2], with the inclusion functor $\mathrm{Shv}_{(\mathrm{Spaces})}(NS_{\mathrm{Nis}})^\wedge \xrightarrow{\sim} \mathrm{Shv}_{(\mathrm{Spaces})}(NS_{\mathrm{Nis}})$ being an equivalence by [**6**, 2.25], since S is noetherian and of finite Krull dimension.

Consider the simplicial presheaf of K-theory spaces $S' \mapsto \Omega^\infty K(S')$ on the Nisnevich site S_{Nis}, where $K(S')$ is the connective K-theory spectrum. Since the connective and non-connective K-theory spectra of a scheme have the same non-negative homotopy groups, the canonical map $K(-) \to \mathbb{K}(-)$ induces a weak equivalence $\Omega^\infty K(-) \xrightarrow{\sim} \Omega^\infty \mathbb{K}(-)$ of simplicial presheaves in $\mathrm{Set}_\Delta^{S_{\mathrm{Nis}}^{\mathrm{op}}}$, where $\mathbb{K}$ denotes the non-connective K-theory spectrum. By Thomason's theorem, non-connective K-theory

satisfies Nisnevich descent. It follows that the simplicial presheaf $\Omega^\infty K(-)$ is a fibrant object of $\mathrm{Set}_{\Delta,\mathrm{loc}}^{S_{\mathrm{Nis}}^{\mathrm{op}}}$, so that it defines an object of the ∞-topos $\mathrm{Shv}_{(\mathrm{Spaces})}(NS_{\mathrm{Nis}})$ by the above equivalence θ. We write $\mathcal{K}$ for this object $\theta(\Omega^\infty K(-))$ and call it the *sheaf of K-theory spaces* on S.

Since the K-theory space is the infinite loop space associated to a connective spectrum, $\mathcal{K}$ has a structure of a group object in $\mathrm{Shv}_{(\mathrm{Spaces})}(NS_{\mathrm{Nis}})$. (More precisely, by [**5**, 7.2.2.11-(1)], to give a group structure on an object G of an ∞-topos $\mathfrak{X}$ is equivalent to give a locally connected delooping BG of G. Consider the simplicial presheaf $S' \mapsto \Omega^\infty \Sigma K(S')$ on S_{Nis} and its image $\theta(\Omega^\infty \Sigma K(-))$ in the ∞-category $\mathrm{Preshv}_{(\mathrm{Spaces})}(NS_{\mathrm{Nis}})$ under the equivalence θ. The sheafification $a(\theta(\Omega^\infty \Sigma K(-))) \in \mathrm{ob}\,\mathrm{Shv}_{(\mathrm{Spaces})}(NS_{\mathrm{Nis}})$ of $\theta(\Omega^\infty \Sigma K(-))$ serves as a locally connected delooping of $\mathcal{K}$.) In particular, there is a notion of $\mathcal{K}$-*torsor* in $\mathrm{Shv}_{(\mathrm{Spaces})}(NS_{\mathrm{Nis}})$, as a particular case of the notion of G-torsor for a general group object G defined in [**7**, Definition 3.4], and we have the *classifying space $B\mathcal{K}$* that classifies $\mathcal{K}$-torsors in the sense that for every object $\mathcal{F}$ of $\mathrm{Shv}_{(\mathrm{Spaces})}(NS_{\mathrm{Nis}})$ the ∞-groupoid of $\mathcal{K}$-torsors on $\mathcal{F}$ is equivalent to the mapping space $\mathrm{Map}_{\mathrm{Shv}_{(\mathrm{Spaces})}(NS_{\mathrm{Nis}})}(\mathcal{F}, B\mathcal{K})$.

Recall that $\mathrm{Vect}_{\mathrm{Tate}}(S)$ denotes the idempotent completion of $\varprojlim_{\longleftrightarrow} \mathrm{Vect}_{\mathrm{finite}}(S)$, where $\mathrm{Vect}_{\mathrm{finite}}(S)$ is the exact category of finite-rank vector bundles on S and $\varprojlim_{\longleftrightarrow}$ is the construction on exact categories introduced in [**1**, A.3]. By Theorem 1.2 of [**8**], there is an equivalence $\Sigma \mathbb{K}(S) \xrightarrow{\sim} \mathbb{K}(\mathrm{Vect}_{\mathrm{Tate}}(S))$. The presheaf of spectra on S_{Nis} given by $S' \mapsto \Sigma \mathbb{K}(S') = \mathbb{K}(\mathrm{Vect}_{\mathrm{Tate}}(S'))$ satisfies Nisnevich descent by Thomason's theorem, and hence the simplicial presheaf on S_{Nis} given by $S' \mapsto \Omega^\infty \mathbb{K}(\mathrm{Vect}_{\mathrm{Tate}}(S'))$ is a fibrant object of $\mathrm{Set}_{\Delta,\mathrm{loc}}^{S_{\mathrm{Nis}}^{\mathrm{op}}}$, and thus defines via the equivalence θ an object of the ∞-topos $\mathrm{Shv}_{(\mathrm{Spaces})}(NS_{\mathrm{Nis}})$. We write $\mathcal{K}_{\mathrm{Tate}}$ for this object $\theta(\Omega^\infty \mathbb{K}(\mathrm{Vect}_{\mathrm{Tate}}(-)))$.

3. Proofs

Recall:

Theorem 3.1 ([**8**], Theorem 1.2). *There is a natural equivalence of spectra $\Sigma \mathbb{K}(S') \xrightarrow{\sim} \mathbb{K}(\mathrm{Vect}_{\mathrm{Tate}}(S'))$ for every object S' in S_{Nis}.*

Theorem 3.2 ([**3**], Theorem 3.7). *Every element of the first negative K-group vanishes Nisnevich locally. I.e., the sheafification of the presheaf $S' \mapsto K_{-1}(S')$ on S_{Nis} is trivial.*

Theorem 3.3 ([**7**], Theorem 3.17). *Let $\mathfrak{X}$ be a general ∞-topos and G a group object. Then the classifying space BG for G-torsors is equivalent to the locally connected delooping of G.*

Proof of Theorem 1.1. By Theorem 3.1, the object $\mathcal{K}_{\mathrm{Tate}}$ is a delooping of $\mathcal{K}$, and is locally connected by Theorem 3.2. Theorem 3.3 now tells that $\mathcal{K}_{\mathrm{Tate}}$ is equivalent to the classifying space $B\mathcal{K}$, proving Theorem 1.1. $\qquad\square$

Recall:

Proposition 3.4 (See [**7**], section 3). *Let $\mathfrak{X}$ be a general ∞-topos and G a group object. Then, giving a G-action on an object P of $\mathfrak{X}$ is equivalent to giving a fiber sequence $P \to X \to BG$ for some object X.*

Proof of Theorem 1.2. By Proposition 3.4, constructing the desired $\mathrm{Aut}(\mathcal{E})$-action on the $\mathcal{K}$-torsor $\mathfrak{D}_{\mathcal{E}}$ amounts to describing $\mathfrak{D}_{\mathcal{E}}$ as a pullback $\mathfrak{D}_{\mathcal{E}} = \varprojlim(X \to B\,\mathrm{Aut}(\mathcal{E}) \leftarrow S)$ for some X and some map $X \to B\,\mathrm{Aut}(\mathcal{E})$. It turns out that it suffices to have a map $[[\mathcal{E}]] \colon B\,\mathrm{Aut}(\mathcal{E}) \to \mathcal{K}_{\mathrm{Tate}}$ whose precomposition with the base-point map $S \to B\,\mathrm{Aut}(\mathcal{E})$ is equivalent to the map $[\mathcal{E}] \colon S \to \mathcal{K}_{\mathrm{Tate}} = B\mathcal{K}$ classifying the $\mathcal{K}$-torsor $\mathfrak{D}_{\mathcal{E}}$.

We first note that, in general, for an exact category $\mathcal{A}$ and an object A of $\mathcal{A}$ the connected delooping $B\,\mathrm{Aut}_{\mathcal{A}} A$ of the group $\mathrm{Aut}_{\mathcal{A}} A$ admits a canonical map to the K-theory space $\Omega^{\infty} K(\mathcal{A})$ given as the composition $B\,\mathrm{Aut}_{\mathcal{A}} A \to Ni\mathcal{A} \to \Omega iS_{\bullet}\mathcal{A} = \Omega^{\infty} K(\mathcal{A})$, where $Ni\mathcal{A}$ denotes the nerve of the maximal groupoid of $\mathcal{A}$ and the latter arrow is the first structure map of the spectrum $K(\mathcal{A})$. We also remark that if the exact category $\mathcal{A}$ is idempotent complete then $\Omega^{\infty} K(\mathcal{A}) \xrightarrow{\sim} \Omega^{\infty}\mathbb{K}(\mathcal{A})$, and we thus get a map $B\,\mathrm{Aut}_{\mathcal{A}} A \to \Omega^{\infty}\mathbb{K}(\mathcal{A})$. In view of this we see that there is a map of simplicial presheaves $B\,\mathrm{Aut}_{\mathrm{Vect}_{\mathrm{Tate}}(-)} \mathcal{E} \times_S (-) \to \Omega^{\infty}\mathbb{K}(\mathrm{Vect}_{\mathrm{Tate}}(-))$ in $\mathrm{Set}_{\Delta}^{S_{\mathrm{Nis}}^{\mathrm{op}}}$. It induces a map $[[\mathcal{E}]] \colon a(\theta(B\,\mathrm{Aut}_{\mathrm{Vect}_{\mathrm{Tate}}(-)} \mathcal{E} \times_S (-))) = B\,\mathrm{Aut}\,\mathcal{E} \to \mathcal{K}_{\mathrm{Tate}}$ in $\mathrm{Shv}_{(\mathrm{Spaces})}(NS_{\mathrm{Nis}})$, where $a(-)$ denotes the sheafification. By construction, the precomposition of $[[\mathcal{E}]]$ with the base-point map $S \to B\,\mathrm{Aut}\,\mathcal{E} = a(\theta(B\,\mathrm{Aut}_{\mathrm{Vect}_{\mathrm{Tate}}(-)} \mathcal{E} \times_S (-)))$ corresponds to the point $[\mathcal{E}] \in \mathcal{K}_{\mathrm{Tate}}(S)$, and the proof of Theorem 1.2 is complete. $\square$

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