

THE MAYER-VIETORIS SEQUENCE FOR GRAPHS OF GROUPS, PROPERTY (T), AND THE FIRST ℓ^2 -BETTI NUMBER

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Abstract

We explore the Mayer-Vietoris sequence developed by Chiswell for the fundamental group of a graph of groups when vertex groups satisfy some vanishing assumption on the first cohomology (e.g. property (T), or vanishing of the first ℓ^2 -Betti number). We characterize the vanishing of first reduced cohomology of unitary representations when vertex stabilizers have property (T). We find necessary and sufficient conditions for the vanishing of the first ℓ^2 -Betti number. We also study the associated Haagerup cocycle and show that it vanishes in first reduced cohomology precisely when the action is elementary.

1. Introduction

1.1. 1-cohomology

The first cohomology of a locally compact group G with coefficients in a unitary representation $\pi: G \rightarrow \mathcal{U}(\mathcal{H})$ is an object whose study encompasses many interesting themes, such as property (T), the Haagerup property, and other relatives, as well as the first ℓ^2 -Betti number. For a discrete group G acting on a tree, Chiswell introduced a Mayer-Vietoris sequence for the cohomology of G , in terms of the cohomologies of the vertex groups and edge groups [Chi76]. The class of groups which admit such actions is quite large and includes limit groups and non-finitely generated countable groups, among others.

In this paper, we first extend Chiswell's sequence, in low degrees, to arbitrary topological groups acting on trees. We study these under the assumption that vertex stabilizers satisfy some condition on the vanishing of H^1 (property (T) or vanishing of the first ℓ^2 -Betti number), which leads to two major applications that we now discuss. One concerns the first ℓ^2 -Betti number $\beta^1(G)$; the other, the fact that the first cohomology of $\mathrm{PSL}_2\mathbb{Q}_p$ is non-vanishing precisely when the irreducible representation is special (which is a result of Nebbia [Neb12]).

Let G be a topological group acting without inversion on a tree \mathcal{T} , with quotient graph $X =: G \backslash \mathcal{T}$. Classical by now is the fact that G admits a graph of groups decomposition $\pi_1(\mathcal{G}, X, T)$, where $T \subset X$ is a maximal tree and \mathcal{G} represents the

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local groups (see Section 2 for more details on Bass-Serre theory). Throughout this paper, we will take such actions and decompositions as interchangeable. Also, in order to avoid cumbersome notation, we assume the graph of groups to be reduced (see Definition 5.5 in Section 5.3).

Theorem 1.1. *Let X be a graph, T a maximal tree in X , (\mathcal{G}, X) a reduced graph of discrete groups, and $G = \pi_1(\mathcal{G}, X, T)$. Assume that $\beta^1(G_v) = 0$ for every $v \in V$ and $\sum \frac{1}{|G_e|} < \infty$. Then $\beta^1(G) = 0$ if and only if G belongs to one of the following cases:*

1. *The graph X is a single vertex and $G = G_v$.*
2. *The graph X is a single loop and $G = \mathbb{Z} \ltimes G_v$.*
3. *The graph X is a single edge and there is an exact sequence*

$$1 \rightarrow G_e \rightarrow G \rightarrow \mathbb{Z}/2 * \mathbb{Z}/2 \rightarrow 1.$$

4. *Every edge group is infinite.*

For a locally compact group G , we denote the collection of irreducible unitary representations (up to unitary equivalence) of G by \widehat{G} . The second main result is an elementary proof of the following theorem, the hypotheses of which guarantee that there is a unique special representation of G (see Section 6 for more details).

Theorem 1.2 ([Neb12]). *Let \mathcal{T} be a locally finite, bi-regular tree. Let G be a closed subgroup of $\text{Aut}(\mathcal{T})$, acting transitively on $\partial\mathcal{T}$ and with two orbits on V . Let σ denote the special representation of G on the first ℓ^2 -cohomology on \mathcal{T} . If $\pi \in \widehat{G} \setminus \{\sigma\}$ then $H^1(G, \pi) = 0$; on the other hand $H^1(G, \sigma)$ is one-dimensional.*

1.2. Chiswell’s Mayer-Vietoris sequence

If G is a discrete group acting on a tree without inversion, and M is any G -module, the cohomology $H^*(G, M)$ can be computed by means of a Mayer-Vietoris sequence due to Chiswell [Chi76]:

$$0 \longrightarrow M^G \xrightarrow{\Delta} \prod_{v \in V} M^{G_v} \xrightarrow{\iota} \prod_{e \in A} M^{G_e} \xrightarrow{\partial} H^1(G, M) \xrightarrow{\Delta} \prod_{v \in V} H^1(G_v, M) \xrightarrow{\iota} \prod_{e \in A} H^1(G_e, M) \longrightarrow \dots, \tag{1}$$

where:

- V (resp. E) is the vertex set (resp. oriented edge set) of X , and A is an orientation, i.e. a choice of one edge in every pair of two edges with opposite orientation in E ;
- G_v (resp. G_e) is the vertex group (resp. edge group) attached with $v \in V$ (resp. $e \in E$);
- for every subgroup $H \subset G$, the sub-module of H -fixed points in M is M^H .

The maps Δ, ι, ∂ will be described later.

For a topological group G , a G -module M is *unitary* if M is a Hilbert space on which G acts through a strongly continuous unitary representation. In that case, we also consider the *reduced* cohomology $\overline{H}^*(G, M)$, i.e. the quotient of cocycles by

the closure of coboundaries, where the closure is taken in the topology of uniform convergence on compact subsets.

The Delorme-Guichardet Theorem (see [Gui72], or Section 2.12 in [BdlHV08]) says that, for a locally compact, second-countable group G , Kazhdan’s property (T) is equivalent to $H^1(G, M) = 0$ for every unitary G -module M . Since then, Shalom [Sha00] proved that, for G compactly generated, property (T) is equivalent to the vanishing of $\overline{H}^1(G, V)$, for every unitary G -module M . (The relative analogue of Shalom’s theorem fails in general [FVM12].)

A locally compact group G has Serre’s property (FA) if every continuous, isometric action of G on a tree preserves a vertex or an edge. It is a result by Watatani [Wat82] that property (T) implies property (FA) (see also Lemma 3.2 below).

As a consequence, the fundamental group G of a graph of groups, provided it does not coincide with some vertex group, does *not* have property (T), as G acts without fixed point on the universal cover of the graph of groups (see [Ser77, Section 5.4]). The Mayer-Vietoris sequence (1) allows in principle to characterize the unitary G -modules M for which $H^1(G, M) \neq 0$. One question we address in this paper is to characterize the unitary G -modules M such that $\overline{H}^1(G, M) \neq 0$. It is possible to write down the analogue of (1) in reduced cohomology, but easy examples show that it will not in general be exact. However, when the vertex groups have property (T) (in particular, when they are compact, so that our result covers the case of locally compact groups acting properly on trees), we can characterize those unitary G -modules M with $\overline{H}^1(G, M) \neq 0$:

Theorem 1.3. *Let G be a locally compact group acting without inversion on a tree, with vertex stabilizers having property (T); let M be a unitary G -module.*

- i) $H^1(G, M) = 0$ if and only if the map $\iota: \prod_{v \in V} M^{G_v} \rightarrow \prod_{e \in A} M^{G_e}$ is onto;*
- ii) $\overline{H}^1(G, M) = 0$ if and only if the map $\iota: \prod_{v \in V} M^{G_v} \rightarrow \prod_{e \in A} M^{G_e}$ has dense image (where $\prod_{v \in V} M^{G_v}$ and $\prod_{e \in A} M^{G_e}$ are endowed with the product topology).*

Of course the first part of Theorem 1.3 is an immediate consequence of the Mayer-Vietoris sequence (1), we record it only to contrast it with the second part. As semi-direct products by \mathbb{Z} can be viewed as particular HNN-extensions, we may apply Theorem 1 to them. We will prove:

Corollary 1.4. *Let θ be an automorphism of a locally compact group Γ with property (T). Let M be a unitary G -module, with $G =: \Gamma \rtimes_{\theta} \mathbb{Z}$. Let t be the generator of \mathbb{Z} such that $t h t^{-1} = \theta(h)$ ($h \in \Gamma$).*

- i) $H^1(G, M) = 0$ if and only if 1 is not a spectral value of $t|_{M^{\Gamma}}$;*
- ii) $\overline{H}^1(G, M) = 0$ if and only if 1 is not an eigenvalue of $t|_{M^{\Gamma}}$.*

This paper is organized as follows. Section 2 gives a direct construction of Chiswell’s Mayer-Vietoris sequence, in degrees 0 and 1: this does not seem to appear explicitly in the literature and is needed for our computations. In Section 3, we study a specific 1-cocycle $b \in Z^1(G, \ell^2(\mathcal{E}))$, where \mathcal{E} is the set of oriented edges of \mathcal{T} : it is the cocycle such that $\|b(g)\|^2 = 2d(gx_0, x_0)$, used to prove that a group acting properly on a tree

has the Haagerup property; we call b the *Haagerup cocycle*. We re-prove the known fact that b is trivial in $H^1(G, \ell^2(\mathcal{E}))$ if and only if G has a fixed vertex; our proof provides a cohomological characterization of Serre’s property (FA). We also prove that b is trivial in $\overline{H}^1(G, \ell^2(\mathcal{E}))$ if and only if the G -action on \mathcal{T} is elementary, i.e. G has a finite orbit in $\mathcal{T} \cup \partial\mathcal{T}$. Section 4 is dedicated to the proof of Theorem 1.3 and Corollary 1.4. In Section 5, we study the consequences of the Mayer-Vietoris sequence on the vanishing of the first ℓ^2 -Betti number for discrete groups acting on trees; in particular we prove Theorem 1.1. Section 6 studies the connection with “large” closed groups of automorphisms of a locally finite tree, and it is there that one will find the proof of Nebbia’s Theorem 1.2.

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2. Preliminaries

The natural framework for our study is Bass-Serre Theory (Section 5 in [Ser77]), of which we first recall the relevant parts.

A *graph* is a pair $X = (V, E)$ where V is the set of vertices, E is the set of oriented edges; E is equipped with a fixed-point free involution $e \mapsto \bar{e}$ and with maps $E \rightarrow V : e \mapsto e_+$ and $E \rightarrow V : e \mapsto e_-$ (e_- is the initial vertex and e_+ the terminal vertex of the edge e), such that $\bar{e}_+ = e_-$ for every $e \in E$. An *orientation* of X is the choice of a fundamental domain for the involution on E .

A *graph of groups* (\mathcal{G}, X) is the data of a connected graph $X = (V, E)$ and, for every $v \in V$ a discrete group G_v , for every edge $e \in E$ a discrete group G_e such that $G_e = G_{\bar{e}}$ and a monomorphism $\sigma_e : G_e \rightarrow G_{e_+}$.

Let T be a maximal tree in X . Let $F(E) = \langle t_e (e \in E) \mid \emptyset \rangle$ be the free group on E . The *fundamental group* $G =: \pi_1(\mathcal{G}, X, T)$ is the quotient of the free product $(\ast_{v \in V} G_v) \ast F(E)$ by the following set of relations:

$$\begin{cases} t_e \sigma_e(g_e) t_e^{-1} = \sigma_{\bar{e}}(g_e) & (e \in E, g_e \in G_e); \\ t_e t_{\bar{e}} = 1 & (e \in E); \\ t_e = 1 & (e \in E(T)), \end{cases}$$

where $E(T)$ is the edge set of T . Assume that some orientation A has been chosen. For $e \in A$, we shall identify G_e with $\sigma_e(G_e)$, and we shall denote by θ_e (instead of $\sigma_{\bar{e}}$) the monomorphism $G_e \rightarrow G_{e_-}$. Then G can also be described (see [Chi76, p. 67]) as the quotient of the free product $(\ast_{v \in V} G_v) \ast F(A)$ by the relations:

$$\begin{cases} t_e g_e t_e^{-1} = \theta_e(g_e) & (e \in A, g_e \in G_e); \\ t_e = 1 & (e \in E(T) \cap A). \end{cases}$$

Recall from [Ser77] that the universal cover \mathcal{T} of X has vertex set $\tilde{V} = \sqcup_{v \in V} G/G_v$ and oriented edge set $\tilde{A} = \sqcup_{e \in A} G/G_e$. Furthermore, if $\pi : \mathcal{T} \rightarrow X$ denotes the canonical

projection, then there is a lifting $X \rightarrow \mathcal{T}$ denoted by $x \mapsto \tilde{x}$ such that $\pi(\tilde{x}) = x$, and \tilde{T} , the lift of the maximal tree T is a subtree of \mathcal{T} . With this notation in place, we have that,¹ for $e \in A$:

$$(g\tilde{e})_+ = g\tilde{e}_+ \quad \text{and} \quad (g\tilde{e})_- = gt_e^{-1}\tilde{e}_-. \tag{2}$$

This gives rise to the following short exact sequences of G -modules

$$0 \longrightarrow \mathbb{Z}(\tilde{A}) \xrightarrow{\delta} \mathbb{Z}(\tilde{V}) \xrightarrow{q} \mathbb{Z} \longrightarrow 0. \tag{3}$$

As G -modules, these decompose as $\mathbb{Z}(\tilde{A}) = \bigoplus_{e \in A} \mathbb{Z}(G.\tilde{e})$ and $\mathbb{Z}(\tilde{V}) = \bigoplus_{v \in V} \mathbb{Z}(G.\tilde{v})$, so that $\delta(g.\tilde{e}) = g\tilde{e}_+ - gt_e^{-1}\tilde{e}_-$ is just the boundary operator on the tree \mathcal{T} , and q is the augmentation defined by $q(g.\tilde{v}) = 1$.

Now, for two vertices $v, w \in V$, we denote by $[v, w]$ (respectively $[g\tilde{v}, g'\tilde{w}]$), the unique *oriented* edge path from v to w (respectively from $g\tilde{v}$ to $g'\tilde{w}$) in the maximal tree T , (respectively in \mathcal{T}).

For $e \in E$ or $e \in \tilde{E}$, we define

$$\varepsilon_{vw}(e) = \begin{cases} 0 & \text{if } e \notin [v, w]; \\ +1 & \text{if } e \in [v, w] \text{ and } e \text{ points away from } v; \\ -1 & \text{if } e \in [v, w] \text{ and } e \text{ points towards } v. \end{cases}$$

Note that if $e \in E \setminus E(T)$ then $\varepsilon_{vw}(e) = 0$ for every $v, w \in V$. Furthermore, since T lifts to \tilde{T} we have that $[v, w] = [\tilde{v}, \tilde{w}]$ and so $\pi[\tilde{v}, \tilde{w}] = \pi[v, w] = [v, w]$, in particular, $\varepsilon_{\tilde{v}\tilde{w}}(\tilde{e}) = \varepsilon_{vw}(e)$ for every $v, w \in V$ and $e \in E$.

Remark 2.1. Observe that the following hold for all vertices $u, v, w \in V$, or $u, v, w \in \tilde{V}$:

$$\varepsilon_{vw} = -\varepsilon_{wv};$$

$$\varepsilon_{uv} + \varepsilon_{vw} = \varepsilon_{uw}.$$

This allows us to define $\int_v^w [v, w] = \sum \varepsilon_{vw}(e)e$, where the sum is taken over $e \in A$, or $e \in \tilde{A}$, according to whether $v, w \in V$ or \tilde{V} .

Using this, we fix a base vertex $v_0 \in V$ and define $s: \mathbb{Z}(\tilde{V}) \rightarrow \mathbb{Z}(\tilde{A})$ by

$$s(g\tilde{v}) = \int_{\tilde{v}_0}^{g\tilde{v}} [\tilde{v}_0, g\tilde{v}].$$

It is then straightforward to verify that s is a left-inverse to δ :

$$s \circ \delta(g\tilde{e}) = \int_{\tilde{v}_0}^{(g\tilde{e})_+} [\tilde{v}_0, (g\tilde{e})_+] - \int_{\tilde{v}_0}^{(g\tilde{e})_-} [\tilde{v}_0, (g\tilde{e})_-] = g\tilde{e}.$$

Now, for abelian groups C, D denote by $\text{hom}(C, D)$ the abelian group of all homomorphisms $C \rightarrow D$. If furthermore, C and D are G -modules, then $\text{hom}(C, D)$ is also

¹We note that our convention here does not agree with Serre's, who identifies edge groups G_e with their image in G_{e_+} .

a G -module by taking $g.f := g \circ f \circ g^{-1}$ and so $\text{hom}(C, D)^G$ is the collection of G -equivariant homomorphisms (i.e. intertwiners) from C to D . For $f: C \rightarrow D$ a homomorphism, and M an abelian group, the *transposed* homomorphism

$$f^t: \text{hom}(D, M) \rightarrow \text{hom}(C, M)$$

is defined by $f^t(g) = g \circ f$.

Let M be a G -module. Transposing the exact sequence (3), we get a new exact sequence:

$$0 \rightarrow M = \text{hom}(\mathbb{Z}, M) \xrightarrow{q^t} \text{hom}(\mathbb{Z}(\tilde{V}), M) \xrightarrow{\delta^t} \text{hom}(\mathbb{Z}(\tilde{A}), M) \rightarrow 0, \quad (4)$$

where $(q^t(m))(g\tilde{v}) \equiv m$, and $(\delta^t(f))(g\tilde{e}) = f(\delta(g\tilde{e})) = f((g\tilde{e})_+) - f((g\tilde{e})_-)$. Furthermore, we have for $\omega \in \text{hom}(\mathbb{Z}(\tilde{A}), M)$

$$(s^t(\omega))(g\tilde{v}) = \int_{\tilde{v}_0}^{g\tilde{v}} \omega := \omega\left(\int_{\tilde{v}_0}^{g\tilde{v}} [\tilde{v}_0, g\tilde{v}]\right).$$

We note that for $\omega = (\omega_e)_{e \in A} \in \prod_{e \in A} M^{G_e}$, and any three vertices $u, v, w \in V$, we then have that

$$\begin{aligned} \int_v^w \omega &= \int_{\tilde{v}}^{\tilde{w}} \omega, \\ \int_v^w \omega &= - \int_w^v \omega, \\ \int_u^v \omega + \int_v^w \omega &= \int_u^w \omega. \end{aligned}$$

Furthermore, if $e \in E(T) \cap A$ then

$$\int_{e_-}^{e_+} \omega = \omega_e. \quad (5)$$

Let $C^k(G, M)$ be the space of k -cochains on G with coefficients in M , i.e. the set of maps $G^k \rightarrow M$. Applying the functor $C^*(G, \cdot)$ to the short exact sequence (4), we get a commutative diagram:

$$\begin{array}{ccccccc} & & \uparrow & & \uparrow & & \uparrow \\ 0 & \longrightarrow & C^1(G, M) & \longrightarrow & C^1(G, \text{hom}(\mathbb{Z}(\tilde{V}), M)) & \longrightarrow & C^1(G, \text{hom}(\mathbb{Z}(\tilde{A}), M)) \longrightarrow 0 \\ & & \uparrow & & \partial_0 \uparrow & & \uparrow \\ 0 & \longrightarrow & M & \longrightarrow & \text{hom}(\mathbb{Z}(\tilde{V}), M) & \longrightarrow & \text{hom}(\mathbb{Z}(\tilde{A}), M) \longrightarrow 0, \end{array}$$

where $\partial_0: \text{hom}(\mathbb{Z}(\tilde{V}), M) \rightarrow C^1(G, \text{hom}(\mathbb{Z}(\tilde{V}), M))$ is the map given by $(\partial_0\phi)(g) = (g - 1)\phi$. This in turn yields the long exact sequence [Chi76, Theorem 2]:

$$0 \rightarrow M^G \xrightarrow{q_*^t} \text{hom}(\mathbb{Z}(\tilde{V}), M)^G \xrightarrow{\delta_*^t} \text{hom}(\mathbb{Z}(\tilde{A}), M)^G \xrightarrow{\partial} H^1(G, M) \rightarrow \dots,$$

where the connecting map is given by the Snake Lemma:

$$\tilde{\partial}\omega = ((q_*^t)^{-1} \circ \partial_0 \circ (\delta_*^t)^{-1})(\omega).$$

The decomposition of \tilde{V} and \tilde{A} into G orbits yields that $H^k(\mathbb{Z}(\tilde{V}), M) =$

$\prod_{v \in V} H^k(\mathbb{Z}(G/G_v), M)$ and $H^k(\mathbb{Z}(\tilde{A}), M) = \prod_{e \in A} H^k(\mathbb{Z}(G/G_e), M)$. By Shapiro's Lemma [Bro94, Proposition 6.2] we have that for any $x \in V \sqcup A$

$$H^k(G_x, M) \cong H^k(G, \text{hom}(\mathbb{Z}(G/G_x), M)).$$

Hence, recalling that $M^{G_x} \cong \text{hom}(\mathbb{Z}(G/G_x), M)^G$ and setting $\Delta = q_*^t$ and $\iota = \delta_*^t$ we get Chiswell's Mayer-Vietoris sequence:

$$0 \longrightarrow M^G \xrightarrow{\Delta} \prod_{v \in V} M^{G_v} \xrightarrow{\iota} \prod_{e \in A} M^{G_e} \xrightarrow{\partial} H^1(G, M) \longrightarrow \dots$$

We proceed to record the maps Δ and ι appearing in the Mayer-Vietoris sequence (1) and observe that these are well defined for G a topological group.

- The map $\Delta: H^k(G, M) \rightarrow \prod_{v \in V} H^k(G_v, M)$ arises by restricting the action from G to the G_v 's; in particular, $\Delta: M^G \rightarrow \prod_{v \in V} M^{G_v}$ is given by $(\Delta m)_v = m$ (where $m \in M^G, v \in V$).
- The map $\iota: \prod_{v \in V} H^k(G_v, M) \rightarrow \prod_{e \in A} H^k(G_e, M)$ is given at the level of k -cocycles by

$$(\iota \omega)_e = \omega_{e_+}|_{G_e} - t_e^{-1}(\omega_{e_-} \circ \theta_e),$$

where $e \in A, \omega \in \prod_{v \in V} Z^k(G_v, M)$. In particular, $\iota: \prod_{v \in V} M^{G_v} \rightarrow \prod_{e \in A} M^{G_e}$ is given by $(\iota f)_e = f_{e_+} - t_e^{-1}f_{e_-}$, for $f \in \prod_{v \in V} M^{G_v}$.

Example 2.2. Let $G = \mathbb{F}_2 = \langle t_1, t_2 \mid \emptyset \rangle$ be the free group on two generators, viewed as the fundamental group of the graph of groups with one vertex and two edges, and all groups trivial. For M a G -module, the map $\iota: M \rightarrow M \oplus M$ is given by $f \mapsto ((1 - t_1^{-1})f, (1 - t_2^{-1})f)$. If M is a unitary G -module, it was shown by Guichardet [Gui72] that ι is *not* onto, so that $H^1(\mathbb{F}_2, M) \neq 0$ for every unitary G -module M . Note that examples of unitary, irreducible G -modules M with $\overline{H}^1(G, M) = 0$ were constructed in [MV10].

Next we consider the connecting map $\partial: \prod_{e \in A} M^{G_e} \rightarrow H^1(G, M)$: it is not described explicitly in Chiswell's paper [Chi76], nor are we aware of any published description of this map,² although it might of course be known to experts. It is important for our main results to have this map explicitly, where we will also not assume that groups are discrete; we therefore will spend some time developing it.

Let $\omega \in \text{hom}(\mathbb{Z}(\tilde{A}), M)^G$ so that

$$\begin{aligned} ((\partial \circ s^t)(\omega)(g_1))(g\tilde{v}) &= ((g_1 - 1)s^t(\omega))(g\tilde{v}) = g_1 \left(\int_{\tilde{v}_0}^{g_1^{-1}g\tilde{v}} \omega \right) - \int_{\tilde{v}_0}^{g\tilde{v}} \omega \\ &= \int_{g_1\tilde{v}_0}^{g\tilde{v}} \omega + \int_{g\tilde{v}}^{\tilde{v}_0} \omega = - \int_{\tilde{v}_0}^{g\tilde{v}_0} \omega, \end{aligned}$$

that is,

$$\tilde{\partial}\omega(g) := ((q_*^t)^{-1} \circ \partial \circ ((\delta_*^t)^{-1})(\omega))(g) = - \int_{\tilde{v}_0}^{g\tilde{v}_0} \omega. \tag{6}$$

²Chiswell's paper was preceded by papers of Swan [Swa69] where (1) is established for amalgamated products, and Bieri [Bie75] where (1) is established for HNN extensions. But these papers do not contain any explicit description of ∂ either.

However, we would like to have a description of $\tilde{\partial}$ purely in terms of the graph of groups data.

Proposition 2.3. *Let $\omega \in \prod_{e \in A} M^{G_e}$.*

1. *For $v \in V$ and $g_v \in G_v$, we have: $(\partial\omega)(g_v) = (g_v - 1) \int_{v_0}^v \omega$.*
2. *For $e \in A \setminus E(T)$, we have: $(\partial\omega)(t_e) = t_e \int_{v_0}^{e+} \omega - t_e \omega_e - \int_{v_0}^{e-} \omega$.*

Proof. Fix $\omega \in \prod_{e \in A} M^{G_e}$; the canonical isomorphism $\prod_{e \in A} M^{G_e} \rightarrow \text{hom}(\mathbb{Z}(\tilde{A}), M)^G$ maps ω to $\tilde{\omega}$ where $\tilde{\omega}_{g\tilde{e}} = g\omega_e$ for $e \in A$ and $g \in G$.

1. If $g_v \in G_v$ then using the fact that $g_v \cdot \tilde{v} = \tilde{v}$ we have by (6):

$$\begin{aligned} (\partial\omega)(g_v) &= - \int_{\tilde{v}_0}^{g_v \tilde{v}_0} \tilde{\omega} \\ &= \int_{g_v \tilde{v}_0}^{\tilde{v}} \omega + \int_{\tilde{v}}^{\tilde{v}_0} \tilde{\omega} \\ &= (g_v - 1) \int_{\tilde{v}_0}^{\tilde{v}} \tilde{\omega} = (g_v - 1) \int_{v_0}^v \omega, \end{aligned}$$

where we have used $\pi[\widetilde{v}, w] = [v, w]$, as observed earlier.

2. For $e \in A \setminus E(T)$, we have by (6):

$$\begin{aligned} (\tilde{\partial}\omega)(t_e) &= \int_{t_e \tilde{v}_0}^{\tilde{v}_0} \tilde{\omega} \\ &= \int_{t_e \tilde{v}_0}^{t_e \tilde{e}_+} \tilde{\omega} + \int_{t_e \tilde{e}_+}^{\tilde{e}_-} \tilde{\omega} + \int_{\tilde{e}_-}^{\tilde{v}_0} \tilde{\omega} \\ &= t_e \int_{\tilde{v}_0}^{\tilde{e}_+} \tilde{\omega} + t_e \int_{\tilde{e}_+}^{t_e^{-1} \tilde{e}_-} \tilde{\omega} - \int_{\tilde{v}_0}^{\tilde{e}_-} \tilde{\omega}. \end{aligned}$$

But $\tilde{e}_+ = \tilde{e}_+$ and $t_e^{-1} \tilde{e}_- = \tilde{e}_-$ by (2), so $(\tilde{\partial}\omega)(t_e) = t_e \int_{v_0}^{e+} \omega - t_e \int_{e-}^{e+} \omega - \int_{e-}^{v_0} \omega$. Finally, $\int_{e-}^{e+} \omega = \omega_e$ by (5), which concludes the proof. \square

Lemma 2.4. *The class of $\tilde{\partial}\omega$ is independent of choice of the base-vertex v_0 .*

Proof. Consider $\tilde{\partial}'\omega$ defined similarly to $\tilde{\partial}\omega$ but with respect to a base vertex v_1 . It is a straightforward computation, which uses Remark 2.1, that for each $g \in (\sqcup_{v \in V} G_v) \sqcup \{t_e : e \in A\}$ the difference is a co-boundary. Namely,

$$\tilde{\partial}\omega(g) - \tilde{\partial}'\omega(g) = (g - 1) \int_{v_0}^{v_1} \omega. \quad \square$$

Then, $\partial: \prod_{e \in A} M^{G_e} \rightarrow H^1(G, M)$ as the composition of $\tilde{\partial}$ with the canonical map $Z^1(G, M) \rightarrow H^1(G, M)$.

3. On the Haagerup cocycle

Let G be a topological group acting without inversion on the tree \mathcal{T} . Denote by \mathcal{V} and \mathcal{E} the set of vertices and oriented edges, respectively, of the tree \mathcal{T} . We then have a natural action of G on $\ell^2(\mathcal{E})$ which we will now study.

Recall that the removal of an (open) edge disconnects the tree into two connected components, called *half-trees*. To each $h \in \mathcal{E}$, we will associate the connected component which contains h_+ . This way, we identify \mathcal{E} with the set of half-trees: as a half-tree, the edge h corresponds to $\{x \in \mathcal{V} : d(x, h_+) < d(x, h_-)\}$. This allows us to write: $x \in h$.

Let $x \in \mathcal{V}$ and consider the characteristic function $\mathbf{1}_x = \{h \in \mathcal{E} : x \in h\}$ of the set of edges pointing towards x . With this notation, we fix an initial vertex $x_0 \in \mathcal{V}$ and consider

$$b(g) = (g - 1) \cdot \mathbf{1}_{x_0} = \mathbf{1}_{gx_0} - \mathbf{1}_{x_0}.$$

Since b is a formal cocycle, the observation that $\|b(g)\|_2^2 = 2d(gx_0, x_0)$ shows immediately that $[b] \in H^1(G, \ell^2(\mathcal{E}))$. Furthermore, b is bounded if and only if G has a fixed vertex, and similarly, G acts properly on \mathcal{T} if and only if b is proper on G . We note that the class of b is independent of the base vertex x_0 . We will call b the *Haagerup cocycle with respect to the base vertex x_0* . This cocycle is a witness to the fact that groups that admit a proper action on a tree have the Haagerup property. Note that the class $[b]$ is clearly independent of the choice of x_0 ; this will allow us to choose x_0 in an appropriate way, when studying cohomological properties of b .

We wish to now understand when this cocycle is trivial in the context of Chiswell’s Mayer-Vietoris sequence. Let $X = (V, E)$ be the quotient graph $G \backslash \mathcal{T}$. Fix a base vertex $v_0 \in V$, a maximal tree T of X , and an orientation A of E . As before, denote by $\pi: \mathcal{T} \rightarrow X$ the quotient map, with section $x \mapsto \tilde{x}$ for $x \in V \cup T$. This lifting has the property that $\tilde{e}_+ = \tilde{e}_-$. Note that $\tilde{e} = \tilde{e}$.

For $e \in A \cap E(T)$, let us denote by $\vartheta_e: G_e \rightarrow G_{e_-}$ the natural inclusion. For $e \in A \setminus E(T)$, recall that $\tilde{e}_- = t_e^{-1} \tilde{e}$, and we set $\vartheta_e(g_e) = t_e g_e t_e^{-1}$ for $g_e \in G_e$.

We now observe that b is clearly bounded, hence cohomologically trivial on each vertex group G_v ($v \in V$). This means that $[b]$ is in the kernel of $\Delta: H^1(G, \ell^2(\mathcal{E})) \rightarrow \prod_{v \in V} H^1(G_v, \ell^2(\mathcal{E}))$, so by the Mayer-Vietoris sequence (1) it is in the image of ∂ . We explicitly describe an element of $\prod_{e \in A} \ell^2(\mathcal{E})^{G_e}$ that maps to $[b]$.

Let us define $\omega_e = \delta_{\tilde{e}} - \delta_{\tilde{e}}$ for $e \in A$; observe that $\omega \in \prod_{e \in A} \ell^2(\mathcal{E})^{G_e}$.

Lemma 3.1. *Let $b \in Z^1(G, \ell^2(\mathcal{E}))$ be the Haagerup cocycle with respect to the base vertex \tilde{v}_0 . Then $\tilde{\partial}(\omega) = b$, with ω as above.*

Proof. Observe first that for $e \in A$:

$$\omega_e = \mathbf{1}_{\tilde{e}_-} - \mathbf{1}_{\tilde{e}_+}.$$

Formally $\omega|_T = df$, where $f_v = -\mathbf{1}_{\tilde{v}}$ ($v \in V$). So:

$$\int_{v_0}^v \omega = -\mathbf{1}_{\tilde{v}} + \mathbf{1}_{\tilde{v}_0}.$$

Now, for $g_v \in G_v$ we have:

$$\begin{aligned} \tilde{\partial}\omega(g_v) &= (g_v - 1) \int_{v_0}^v \omega \\ &= (g_v - 1)(-\mathbf{1}_{\tilde{v}} + \mathbf{1}_{\tilde{v}_0}) \\ &= (g_v - 1)\mathbf{1}_{\tilde{v}_0} \\ &= b(g_v). \end{aligned}$$

Now, let $e \in A$; using $\tilde{e}_+ = \tilde{e}_+$ and $t_e(\tilde{e}_-) = \tilde{e}_-$:

$$\begin{aligned} (\tilde{\partial}\omega)(t_e) &= t_e \int_{v_0}^{e_+} \omega - \int_{v_0}^{e_-} \omega - t_e \omega_e \\ &= t_e(-\mathbf{1}_{\tilde{e}_+} + \mathbf{1}_{\tilde{v}_0}) + \mathbf{1}_{\tilde{e}_-} - \mathbf{1}_{\tilde{v}_0} - t_e(\mathbf{1}_{\tilde{e}_-} - \mathbf{1}_{\tilde{e}_+}) \\ &= (t_e - 1)\mathbf{1}_{\tilde{v}_0} \\ &= b(t_e). \end{aligned}$$

Since $\tilde{\partial}\omega$ agrees with b on the generators $(\sqcup_{v \in V} G_v) \sqcup \{t_e : e \in A\}$ by Lemma 2.4 we have that $\tilde{\partial}\omega = b$. □

Lemma 3.2. *Let G be a topological group acting on a tree \mathcal{T} without inversion. The following are equivalent:*

- a) $[b] = 0$ in $H^1(G, \ell^2(\mathcal{E}))$;
- b) G has a fixed vertex;
- c) $\Delta: H^1(G, \ell^2(\mathcal{E})) \rightarrow \prod_{v \in V} H^1(G_v, \ell^2(\mathcal{E}))$ is injective.

Proof. (a) \Leftrightarrow (b) Using the fact that

$$\|b(g)\|^2 = 2d(gx_0, x_0),$$

we see that $[b] = 0$ if and only if b is bounded, if and only if G admits a bounded orbit on vertices. By a standard argument (see, for example, [BH99, Proposition 2.7, Chapter II.2]), this is equivalent to G fixing a vertex.

(c) \Rightarrow (a) Follows immediately from the already observed fact that $[b] \in \ker \Delta$.

(b) \Rightarrow (c) G admits a globally fixed vertex x_0 , so that $G = G_{v_0}$, where $v_0 = \pi(x_0)$.

Since G is equal to a vertex stabilizer, the restriction of Δ to the corresponding factor of $\prod_v H^1(G_v, \ell^2(\mathcal{E}))$ is the identity, and Δ is injective. □

Remark 3.3. Assume that $[b] = 0$. Then, by the exact sequence (1), the vector $\omega \in \prod_{e \in A} \ell^2(\mathcal{E})^{G_e}$ from Lemma 3.1 is in the image of $\iota: \prod_{v \in V} \ell^2(\mathcal{E})^{G_v} \rightarrow \prod_{e \in A} \ell^2(\mathcal{E})^{G_e}$. This can be seen explicitly as follows. Suppose G fixes the vertex x_0 in \mathcal{T} , set $v_0 = p(x_0)$. Then $X = G \setminus \mathcal{T}$ is a tree. Set then $f_v = \int_{v_0}^v \omega$; then $f_v \in \ell^2(\mathcal{E})^{G_v}$, since G_v pointwise fixes the geodesic $[x_0, \tilde{v}]$; and clearly $\iota(f) = df = \omega$ (which is essentially Poincaré’s lemma).

From Lemma 3.2 and the Delorme-Guichardet Theorem, we immediately deduce the following cohomological characterization of Serre’s property (FA):

Corollary 3.4. *A topological group G has Serre’s property (FA) if and only if, for any action of G without inversion on a tree \mathcal{T} , the map*

$$\Delta: H^1(G, \ell^2(\mathcal{E})) \rightarrow \prod_{v \in V} H^1(G_v, \ell^2(\mathcal{E}))$$

is injective. In particular, if G is locally compact with Kazhdan's property (T), then G has Serre's property (FA).³ \square

We now explore the triviality of the Haagerup cocycle in reduced cohomology.

Lemma 3.5. *Suppose that $\mathbb{F}_2 \leq \text{Aut}(\mathcal{T})$ acts freely. Then $\ell^2(\mathcal{E})$ does not have \mathbb{F}_2 -almost invariant vectors.*

Proof. Let C denote a choice of one representative in each \mathbb{F}_2 -orbit in \mathcal{E} . Since the \mathbb{F}_2 -action is free, this choice identifies the $\ell^2(\mathcal{E}) \cong \bigoplus_{c \in C} \ell^2(\mathbb{F}_2)$, as an \mathbb{F}_2 -module, where the direct sum is endowed with the diagonal left regular representation of \mathbb{F}_2 . So the result follows from the observation that $\ell^2(\mathbb{F}_2)$ does not have \mathbb{F}_2 -almost invariant vectors, which is guaranteed by the non-amenability of \mathbb{F}_2 . \square

Definition 3.6. A group G acting on a tree \mathcal{T} is said to be *elementary* if it has a finite orbit in \mathcal{T} or $\partial\mathcal{T}$.

If one has a finite orbit in $\partial\mathcal{T}$ but not in \mathcal{T} , then the orbit must have size at most 2. This follows, for example, from Propositions 1 and 2 of [PV91] along with the classification of isometries.

We now recall a little about the structure of the stabilizers $\text{Aut}(\mathcal{T})_{\xi_0}$ and $\text{Aut}(\mathcal{T})_{\{\xi_0, \xi_1\}}$ where $\xi_0, \xi_1 \in \partial\mathcal{T}$. The group $\text{Aut}(\mathcal{T})_{\xi_0}$ contains the collection of its elliptic elements as a normal subgroup $R_{\xi_0} = \cup_{v \in \mathcal{T}} \text{stab}[v, \xi_0)$. The map that associates to each element in $\text{Aut}(\mathcal{T})_{\xi_0}$ its signed translation length is a homomorphism to \mathbb{Z} with kernel R_{ξ_0} , see [PV91, Lemme 4]; we call it the *Busemann homomorphism*. Choosing $a \in \text{Aut}(\mathcal{T})_{\xi_0}$ a hyperbolic element with minimal translation length (or setting $a = 1$ otherwise) describes an isomorphism

$$\text{Aut}(\mathcal{T})_{\xi_0} \cong \langle a \rangle \rtimes R_{\xi_0}.$$

This provides $\text{Aut}(\mathcal{T})_{\xi_0}$ with a normal form, i.e. for each $g \in \text{Aut}(\mathcal{T})_{\xi_0}$ there is a unique $n \in \mathbb{Z}$ and $r \in R_{\xi_0}$ such that $g = a^n r$.

Next, consider $G = \text{Aut}(\mathcal{T})_{\{\xi_0, \xi_1\}}$; observe that it contains, as a subgroup of index at most two, $G_0 = \text{Aut}(\mathcal{T})_{\xi_0} \cap \text{Aut}(\mathcal{T})_{\xi_1}$ and $G_0 \cong \langle a \rangle \rtimes (R_{\xi_0} \cap R_{\xi_1})$.

Finally, we observe that these descriptions and canonical forms hold by restriction to any subgroup or $\text{Aut}(\mathcal{T})_{\xi_0}$ or $\text{Aut}(\mathcal{T})_{\{\xi_0, \xi_1\}}$.

To simplify notation which will quickly become cumbersome, let

$$\mathfrak{D}_{x,y} = \mathbb{1}_{[x,y]} - \mathbb{1}_{[y,x]}.$$

The fact that $\|\mathfrak{D}_{x,y}\|^2 = 2d(x,y)$ should give the reader an idea of why the notation was chosen this way. Observe that if b is the Haagerup cocycle with respect to base point x_0 then

$$b(g) = \mathfrak{D}_{x_0, gx_0}.$$

Theorem 3.7. *Let G be a discrete group acting on \mathcal{T} without inversion. Let $b \in Z^1(G, \ell^2(\mathcal{E}))$ denote the Haagerup cocycle. Then $[b]$ is trivial in $\overline{H}^1(G, \ell^2(\mathcal{E}))$ if and only if the G -action is elementary.*

³The latter statement was first proved by Watatani [Wat82].

Proof. Assume the G -action is non-elementary. Then, by [PV91] there exists a freely acting $F_2 \leq G$. By Lemma 3.5, $\ell^2(\mathcal{E})$ has no F_2 -almost invariant vectors, hence no G -almost invariant vectors. By Guichardet’s result: $\overline{H}^1(G, \ell^2(\mathcal{E})) = H^1(G, \ell^2(\mathcal{E}))$; so it is enough to show that $[b] \neq 0$ in $H^1(G, \ell^2(\mathcal{E}))$, i.e. that b is unbounded on G . But b is already unbounded on F_2 , as it acts freely.

Conversely, suppose that the action is elementary. If there is a finite orbit in \mathcal{T} , then by Lemma 3.2, $[b]$ is trivial in $H^1(G, \ell^2(\mathcal{E}))$ and hence in $\overline{H}^1(G, \ell^2(\mathcal{E}))$.

Therefore, assume that G does not have a finite orbit in \mathcal{T} . Then, either G has a fixed point in $\partial\mathcal{T}$ or there is a G -invariant set $\{\xi_0, \xi_1\} \subset \partial\mathcal{T}$ such that $G_0 := G \cap \text{Aut}(\mathcal{T})_{\xi_0} \cap \text{Aut}(\mathcal{T})_{\xi_1}$ has index 2 in G .

Case 1: G fixes ξ_0 .

Choosing a hyperbolic isometry $a \in G$ of minimal translation length $\ell(a)$, every element of G may be described uniquely as $a^N r$ for $N \in \mathbb{Z}$ and $r \in R_{\xi_0} \cap G$. Replacing a by a^{-1} if necessary, we may assume that ξ_0 is a contracting fixed point for a .

Let $F \subset G$ be a finite set. Then $F \subset \{a^N r : M' \leq N \leq M, r \in F_0\}$ where F_0 is a finite subset of R_{ξ_0} , with $1 \in F_0$. We begin by considering the case where $F = \{a^N r : 0 \leq N \leq M, r \in F_0\}$.

Let A be the axis of a . The elements of the finite set $F_0 \subset R_{\xi_0}$ must have a common fixed point t which allows us to choose $x_0 \in [t, \xi_0] \cap A$, that we take as base-point for the Haagerup cocycle. To simplify notation, let $x_n = a^n x_0$ for $n \in \mathbb{Z}$ and observe that if $\ell(a)$ is the translation length of a then $\|\mathfrak{2}_{x_k, x_{k+1}}\|^2 = 2\ell(a)$. With this, we have that, for $N \in \mathbb{Z}$ and $r \in F_0$

$$b(a^N r) = \mathfrak{2}_{x_0, x_N}.$$

Let $v_n = -\sum_{k=0}^n (1 - \frac{k}{n}) \mathfrak{2}_{x_k, x_{k+1}}$ and $b_n(g) = gv_n - v_n$. Now, if $0 \leq N \leq M, n > M, r \in F_0$ then

$$\begin{aligned} b_n(a^N r) &= -\sum_{k=N}^{n+N} \left(1 - \frac{k-N}{n}\right) \mathfrak{2}_{x_k, x_{k+1}} + \sum_{k=0}^n \left(1 - \frac{k}{n}\right) \mathfrak{2}_{x_k, x_{k+1}} \\ &= \sum_{k=0}^{N-1} \left(1 - \frac{k}{n}\right) \mathfrak{2}_{x_k, x_{k+1}} - \sum_{k=N}^n \frac{N}{n} \mathfrak{2}_{x_k, x_{k+1}} - \sum_{k=n+1}^{n+N} \left(1 - \frac{k-N}{n}\right) \mathfrak{2}_{x_k, x_{k+1}}. \end{aligned}$$

So, observing that $b(a^N r) = \sum_{k=0}^{N-1} \mathfrak{2}_{x_k, x_{k+1}}$ and that $\mathfrak{2}_{x_k, x_{k+1}}$ is orthogonal to $\mathfrak{2}_{x_{k'}, x_{k'+1}}$ for $k \neq k'$:

$$\begin{aligned} &\|b(a^N r) - b_n(a^N r)\|^2 \\ &= \left\| \sum_{k=0}^{N-1} \mathfrak{2}_{x_k, x_{k+1}} - \sum_{k=0}^{N-1} \left(1 - \frac{k}{n}\right) \mathfrak{2}_{x_k, x_{k+1}} \right. \\ &\quad \left. + \sum_{k=N}^n \frac{N}{n} \mathfrak{2}_{x_k, x_{k+1}} + \sum_{k=n+1}^{n+N} \left(1 - \frac{k-N}{n}\right) \mathfrak{2}_{x_k, x_{k+1}} \right\|^2 \end{aligned}$$

$$\begin{aligned}
 &= \left\| \sum_{k=0}^{N-1} \frac{k}{n} \mathfrak{Z}_{x_k, x_{k+1}} + \frac{N}{n} \sum_{k=N}^n \mathfrak{Z}_{x_k, x_{k+1}} + \sum_{k=n+1}^{n+N} \left(1 - \frac{k-N}{n}\right) \mathfrak{Z}_{x_k, x_{k+1}} \right\|^2 \\
 &= 2\ell(a) \left[\frac{1}{n^2} \sum_{k=1}^{N-1} k^2 + \frac{N^2}{n^2} (n-N+1) + \frac{1}{n^2} \sum_{k=n+1}^{n+N} (n+N-k)^2 \right] \\
 &\leq \frac{2\ell(a)}{n^2} (M^3 + M^2n + M^3) \\
 &\leq \frac{6\ell(a)M^2}{n}.
 \end{aligned}$$

This part is concluded by observing that, if K is an arbitrary finite subset of G , then for $N \gg 0$, the set $a^N K$ is contained in a finite set F of the above form. Defining the 1-cocycle c_n as $c_n = b - b_n$, we then have, for $h \in F$:

$$c_n(a^{-N}h) = a^{-N}(c_n(h) - c_n(a^N)),$$

so by the triangle inequality: $\|c_n(g)\| \leq \sqrt{\frac{12\ell(a)}{n}} M \xrightarrow{n \rightarrow \infty} 0$ for every $g \in K$.

Case 2: G does not have a fixed point in $\partial\mathcal{T}$ but preserves $\{\xi_0, \xi_1\} \subset \partial\mathcal{T}$.

Let $G_0 = G \cap \text{Aut}(\mathcal{T})_{\xi_0} \cap \text{Aut}(\mathcal{T})_{\xi_1}$ and observe that G_0 has index 2 in G . By the first case, $[b]$ is trivial in $\overline{H}^1(G_0, \ell^2(\mathcal{E}))$. By Lemma 3.8 just below, $[b]$ is also trivial in $\overline{H}^1(G, \ell^2(\mathcal{E}))$. □

Lemma 3.8. *Let H be a finite index subgroup in the discrete group G . For any unitary G -module M , the restriction map $\text{Rest}_G^H: \overline{H}^1(G, M) \rightarrow \overline{H}^1(H, M)$ is injective.*

Proof. Let g_1, \dots, g_N be representatives for the left cosets of H in G . Let $b \in Z^1(G, M)$ be a 1-cocycle such that $b|_H$ is a limit of coboundaries. We must show that b is a limit of coboundaries. Passing to the associated affine action $\alpha(g)v = gv + b(g)$ ($g \in G, v \in M$): under the assumption that there is a sequence $(v_k)_{k>0} \in M$ such that $\lim_{k \rightarrow \infty} \|\alpha(h)v_k - v_k\| = 0$ for every $h \in H$, we must show the existence of a sequence $(w_k)_{k>0} \in M$ such that $\lim_{k \rightarrow \infty} \|\alpha(g)w_k - w_k\| = 0$ for every $g \in G$. So, fix $g \in G$. There exists a permutation σ of $\{1, 2, \dots, N\}$ and elements $h_1, \dots, h_N \in H$ such that $gg_i = g_{\sigma(i)}h_i$ for every $i = 1, \dots, N$. Set $w_k = \frac{1}{N} \sum_{i=1}^N \alpha(g_i)v_k$. Then, using the fact that $\alpha(s)x - \alpha(s)y = s(x - y)$ for every $s \in G, x, y \in M$:

$$\begin{aligned}
 \alpha(g)w_k - w_k &= \frac{1}{N} \left(\sum_{i=1}^N \alpha(gg_i)v_k \right) - w_k = \frac{1}{N} \sum_{i=1}^N \alpha(g_{\sigma(i)}h_i)v_k - \frac{1}{N} \sum_{i=1}^N \alpha(g_{\sigma(i)})v_k \\
 &= \frac{1}{N} \sum_{i=1}^N g_{\sigma(i)}(\alpha(h_i)v_k - v_k).
 \end{aligned}$$

Since

$$\lim_{k \rightarrow \infty} \|\alpha(h_i)v_k - v_k\| = 0$$

for $i = 1, \dots, N$, we deduce

$$\lim_{k \rightarrow \infty} \|\alpha(g)w_k - w_k\| = 0. \quad \square$$

4. Proof of Theorem 1.3 and one application

4.1. Proof of Theorem 1.3(ii)

We first assume that $\iota: \prod_{v \in V} M^{G_v} \rightarrow \prod_{e \in A} M^{G_e}$ has dense image. Now, observing the explicit formula for $\tilde{\partial}$ we see that $\partial: \prod_{e \in A} M^{G_e} \rightarrow H^1(G, M)$ is continuous for the product topology. Furthermore, the assumption of property (T) for the vertex groups implies that $H^1(G_v, M) = 0$ for each v . This means that ∂ is onto, by the Mayer-Vietoris sequence (1). We therefore have that $\tilde{\partial}: \prod_{e \in A} M^{G_e} \rightarrow Z^1(G, M)$ is onto. Furthermore, $\text{im}(\iota) = \ker(\partial)$ is dense in $\prod_{e \in A} M^{G_e}$, which means $\tilde{\partial}|_{\ker(\partial)}$ has dense image in $B^1(G, M)$. This of course means that $B^1(G, M)$ is dense in $Z^1(G, M)$ and hence $\overline{H}^1(G, M) = 0$.

Conversely, assume that $\overline{H}^1(G, M) = 0$. Continuing to assume that all vertex groups have property (T), the Mayer-Vietoris sequence yields that $\tilde{\partial}: \prod_{e \in A} M^{G_e} \rightarrow Z^1(G, M)$ is onto and that $\text{im}(\iota) = \ker(\partial)$. Therefore, choosing $\omega \in \prod_{e \in A} M^{G_e}$, we must show that ω can be approximated by elements in the image of ι . By assumption, there exists a sequence $(m_k)_{k \geq 1}$ of vectors in M with $\tilde{\partial}\omega(g) = \lim_{k \rightarrow \infty} gm_k - m_k$. By definition of $\tilde{\partial}\omega|_{G_v}$, the sequence $(-m_k + \int_{v_0}^v \omega)_{k \geq 1}$ is almost G_v -invariant, for all $v \in V$. Denote by P_v the orthogonal projection of M onto M^{G_v} , and define $f_k \in \prod_{v \in V} M^{G_v}$ by $(f_k)_v = P_v(-m_k + \int_{v_0}^v \omega)$ (for $v \in V$).

Claim. $\lim_{k \rightarrow \infty} \|(f_k)_v - (-m_k + \int_{v_0}^v \omega)\| = 0$ for every $v \in V$.

Suppose this does not hold for some $v \in V$. Passing to a subsequence, we may assume that $\|(f_k)_v - (-m_k + \int_{v_0}^v \omega)\|$ is bounded below by a positive constant. But $(f_k)_v - (-m_k + \int_{v_0}^v \omega)$ belongs to the orthogonal complement $(M^{G_v})^\perp$. So the sequence $\left(\frac{(f_k)_v - (-m_k + \int_{v_0}^v \omega)}{\|(f_k)_v - (-m_k + \int_{v_0}^v \omega)\|} \right)_{k \geq 1}$ is an almost G_v -invariant sequence of unit vectors in $(M^{G_v})^\perp$, which clearly has no non-zero G_v -invariant vector. This contradicts property (T) for G_v , establishing the claim.

The proof of the theorem is then finished by showing that $\omega = \lim_{k \rightarrow \infty} \iota(f_k)$. But, for $e \in A$:

$$\begin{aligned} \omega_e - \iota(f_k)_e &= \omega_e - (f_k)_{e_+} + t_e^{-1}(f_k)_{e_-} \\ &= \omega_e - (-m_k + \int_{v_0}^{e_+} \omega) + t_e^{-1}(-m_k + \int_{v_0}^{e_-} \omega) \\ &\quad + [(-m_k + \int_{v_0}^{e_+} \omega) - (f_k)_{e_+}] + t_e^{-1}[(f_k)_{e_-} - (-m_k + \int_{v_0}^{e_-} \omega)]. \end{aligned}$$

By the claim, the two terms in brackets go to 0 for $k \rightarrow \infty$. It remains to show that $\lim_{k \rightarrow \infty} \|\omega_e - (-m_k + \int_{v_0}^{e_+} \omega) + t_e^{-1}(-m_k + \int_{v_0}^{e_-} \omega)\| = 0$. But

$$\begin{aligned} \omega_e - (-m_k + \int_{v_0}^{e_+} \omega) + t_e^{-1}(-m_k + \int_{v_0}^{e_-} \omega) \\ = \omega_e - t_e^{-1}(m_k - t_e m_k) + t_e^{-1}(\int_{v_0}^{e_-} \omega) - \int_{v_0}^{e_+} \omega \end{aligned}$$

$$\begin{aligned} &= \omega_e - t_e^{-1}(m_k - t_e m_k) - t_e^{-1}(\tilde{\partial}\omega(t_e)) - \omega_e \\ &= -t_e^{-1}[\tilde{\partial}\omega(t_e) + (m_k - t_e m_k)] \rightarrow 0, \end{aligned}$$

where the last line converges to 0 by assumption. This concludes the proof. \square

4.2. The case of HNN-extensions

Let $G = HNN(\Gamma, A, \theta)$ be an HNN-extension, where A is a subgroup of Γ and $\theta: A \rightarrow \Gamma$ is a monomorphism. Recall from [Ser77] that G can be seen as the fundamental group of a graph of groups with one vertex, with group Γ , and one edge, with group A . Let t be the stable letter in G corresponding to the unique edge, satisfying $tat^{-1} = \theta(a)$, for every $a \in A$. If $A = \Gamma$ and θ is an automorphism of Γ , then G is the semi-direct product $\Gamma \rtimes_{\theta} \mathbb{Z}$.

The map $\iota: M^{\Gamma} \rightarrow M^A$ is given by $m \mapsto (1 - t^{-1})m$.

Proof of Corollary 1.4. First observe that $Ker(1 - t) \cap M^{\Gamma} = Ker(1 - t^{-1}) \cap M^{\Gamma} = M^G$, as $\Gamma \cup \{t\}$ generates G . In other words, 1 is not an eigenvalue of $t|_{M^{\Gamma}}$ if and only if $M^G = 0$.

Claim. *Let M be a unitary G -module; if $M^G \neq 0$, then $\overline{H}^1(G, M) \neq 0$ (in particular, $H^1(G, M) \neq 0$).*

To see it, let M^{\perp} be the orthogonal of M^G in M ; from the decomposition $M = M^G \oplus M^{\perp}$ we get a decomposition $\overline{H}^1(G, M) = \overline{H}^1(G, M^G) \oplus \overline{H}^1(G, M^{\perp})$, and it is enough to check that $\overline{H}^1(G, M^G) \neq 0$. But $\overline{H}^1(G, M^G) = H^1(G, M^G) = \text{hom}(G, M^G)$, which is non-zero as G maps onto \mathbb{Z} .

We may now prove the first statement of Corollary 1.4. Assume first that $H^1(G, M) = 0$. By Theorem 1, the map $(1 - t^{-1})|_{M^{\Gamma}}$ is then onto. By the previous claim, it follows that $M^G = 0$ and so $\iota = (1 - t^{-1})|_{M^{\Gamma}}$ is also injective and therefore invertible, meaning that 1 is not a spectral value of $t|_{M^{\Gamma}}$. Conversely, if 1 is not a spectral value of $t|_{M^{\Gamma}}$, then $(1 - t^{-1})|_{M^{\Gamma}}$ is invertible, in particular, it is onto, so $H^1(G, M) = 0$.

We now pass to the second statement of Corollary 1.4. If $\overline{H}^1(G, M) = 0$, then by the claim, 1 is not an eigenvalue of $t|_{M^{\Gamma}}$. Conversely, if 1 is not an eigenvalue, then $(\text{Im}((1 - t^{-1})|_{M^{\Gamma}}))^{\perp} = \text{Ker}((1 - t^{-1})|_{M^{\Gamma}}) = 0$, i.e. $\text{Im}((1 - t^{-1})|_{M^{\Gamma}})$ is dense, so $\overline{H}^1(G, M) = 0$ by Theorem 1.3. \square

5. The first ℓ^2 -Betti number

5.1. Computing ℓ^2 -Betti numbers

Let G be a countably infinite group acting without inversion and co-compactly on a tree \mathcal{T} , with quotient graph $X = (V, E) = G \backslash \mathcal{T}$.

Let EG be a contractible CW-complex endowed with a proper, free G -action. For a G -CW-complex Z , we may define

$$\overline{H}_{(2)}^i(Z; G) := \overline{H}_{(2)}^i(Z \times EG; G)$$

(see [CG86, Proposition 2.2]) using the fact that the action of G on $Z \times EG$ is now free. We denote by $\beta^i(G) := \dim_G \overline{H}_{(2)}^i(EG, G)$ the i -th L^2 -Betti number.

Let Y be the geometric realization of \mathcal{T} , so that Y is a contractible, 1-dimensional CW-complex. Let Y' be the set of vertices of \mathcal{T} , viewed as a subcomplex of Y . Recall that the relative L^2 -cohomology $\overline{H}_{(2)}^i(Y, Y'; G)$ is the cohomology of the complex

$$C_{(2)}^*(Y, Y'; G) := \ker[C_{(2)}^*(Y \times EG) \xrightarrow{\text{Rest}} C_{(2)}^*(Y' \times EG)].$$

Lemma 5.1. *i) $\dim_G \overline{H}_{(2)}^i(Y; G) = \beta^i(G)$ for $i \geq 0$.*

ii) $\dim_G \overline{H}_{(2)}^i(Y', G) = \sum_{v \in V} \beta^i(G_v)$ for $i \geq 0$.

iii) $\overline{H}_{(2)}^0(Y, Y'; G) = 0$ and $\dim_G \overline{H}_{(2)}^i(Y, Y'; G) = \sum_{e \in A} \beta^{i-1}(G_e)$ for $i \geq 1$.

Proof. i) Since Y is contractible, $Y \times EG$ is a contractible CW-complex on which G acts properly freely. By uniqueness of EG , the space $Y \times EG$ is G -equivariantly homotopic to EG . So

$$\dim_G \overline{H}_{(2)}^i(Y; G) = \dim_G \overline{H}_{(2)}^i(Y \times EG; G) = \beta^i(G).$$

ii) Choosing a vertex \tilde{v} in each G -orbit of Y' , we get

$$\begin{aligned} \dim_G \overline{H}_{(2)}^i(Y', G) &= \sum_{v \in V} \dim_G \overline{H}_{(2)}^i(G \cdot \tilde{v}; G) \\ &= \sum_{v \in V} \dim_{G_{\tilde{v}}} \overline{H}_{(2)}^i(\tilde{v}; G_{\tilde{v}}) = \sum_{v \in V} \beta^i(G_v), \end{aligned}$$

where the previous to last equality is [CG86, Proposition 2.5].

iii) In degree 0, we have $C_{(2)}^0(Y, Y'; G) = 0$, as $Y \times EG$ and $Y' \times EG$ have the same vertices. In degree $i \geq 1$, denote by $Z^{(i)}$ the set of i -cells of the CW-complex Z . Observe that

$$(Y \times EG)^{(i)} = \coprod_{k=0}^i (Y^{(k)} \times EG^{(i-k)}) = (Y^{(0)} \times EG^{(i)}) \amalg (Y^{(1)} \times EG^{(i-1)})$$

as Y is 1-dimensional. So

$$C_{(2)}^i(Y, Y'; G) = \ell^2(Y^{(1)} \times EG^{(i-1)}).$$

and the co-boundary operator $d^{(i)}: C_{(2)}^i(Y, Y'; G) \rightarrow C_{(2)}^{i+1}(Y, Y'; G)$ coincides with $1 \otimes d^{(i-1)}$. So $\dim_G \overline{H}_{(2)}^i(Y, Y'; G) = \dim_G \ker(1 \otimes d^{(i-1)}) = \sum_{e \in A} \beta^{i-1}(G_e)$ by an argument similar to Part (ii) above (by choosing one representative for each G -orbit in $Y^{(1)}$). \square

The second part of Proposition 5.2 below, on amenable vertex-groups, was first obtained by Schafer [Sch03, Corollary 3.12, (ii)].

Proposition 5.2. *Assume that, for every vertex v of \mathcal{T} the stabilizer G_v satisfies $\beta^i(G_v) = 0$ for $i \geq 1$. Then*

$$\beta^1(G) = \sum_{e \in A} \frac{1}{|G_e|} - \sum_{v \in V} \frac{1}{|G_v|}$$

and $\beta^i(G) = \sum_{e \in A} \beta^{i-1}(G_e)$ for $i \geq 2$. In particular, if G_v is amenable for every $v \in V$, then $\beta^i(G) = 0$ for $i \geq 2$.

Proof. According to [CG86, Lemma 2.3], the relative L^2 -cohomology sequence:

$$\begin{aligned} 0 \longrightarrow \overline{H}_{(2)}^0(Y, Y'; G) \longrightarrow \overline{H}_{(2)}^0(Y; G) \longrightarrow \overline{H}_{(2)}^0(Y'; G) \\ \longrightarrow \overline{H}_{(2)}^1(Y, Y'; G) \longrightarrow \overline{H}_{(2)}^1(Y; G) \longrightarrow \overline{H}_{(2)}^1(Y'; G) \longrightarrow \dots \end{aligned}$$

is weakly exact. Then by the rank theorem for von Neumann G -dimension, whenever some space has G -dimension 0, the alternate sum of the G -dimensions of the previous terms vanishes. The first statement then follows immediately from Lemma 5.1. If all vertex-groups are amenable, then so are all edge-groups, hence $\beta^{i-1}(G_e) = 0$ for $i \geq 2$ and $e \in A$. \square

Example 5.3. a) The Baumslag-Solitar group $BS(1, 2)$ is the solvable group with presentation

$$BS(1, 2) = \langle a, b \mid aba^{-1} = b^2 \rangle.$$

Consider then the group H with presentation

$$H = \langle a_0, a_1, a_2 \mid a_0 a_1 a_0^{-1} = a_1^2; a_1 a_2 a_1^{-1} = a_2^2 \rangle.$$

Clearly H is the amalgamated product of two copies of $BS(1, 2)$ over \mathbb{Z} :

$$H = BS(1, 2) *_Z BS(1, 2),$$

so $\beta^i(H) = 0$ for every $i \geq 0$ by Proposition 5.2.

b) Consider the famous Higman group H_4 (see [Hig51]) with its presentation on 4 generators and 4 relations:

$$H_4 = \langle a_0, a_1, a_2, a_3 \mid a_i a_{i+1} a_i^{-1} = a_{i+1}^2, i \in \mathbb{Z}/4\mathbb{Z} \rangle.$$

Then the subgroups $\langle a_0, a_1, a_2 \rangle$ and $\langle a_2, a_3, a_0 \rangle$ are both isomorphic to the group H above, while $\langle a_0, a_2 \rangle$ is free of rank 2, and H_4 is an amalgamated product of two copies of H over the free group \mathbb{F}_2 :

$$H_4 = H *__{\mathbb{F}_2} H.$$

By Proposition 5.2 we get $\beta^i(H_4) = \beta^{i-1}(\mathbb{F}_2)$ for $i \geq 2$, i.e.

$$\beta^i(H_4) = \begin{cases} 0 & \text{if } i \neq 2, \\ 1 & \text{if } i = 2. \end{cases}$$

5.2. Sufficient conditions for vanishing and non-vanishing

The following result will be important for the treatment of reduced graphs of groups in the next section. Note that the assumption is satisfied if vertex groups have property (T).

Proposition 5.4. *Let G be a graph of groups with at least one edge, such that all vertex groups satisfy $H^1(G_v, \ell^2(G)) = 0$. The following are true:*

1. *If for every edge e we have $|G_e| = \infty$, then $H^1(G, \ell^2(G)) = 0$ and $\beta^1(G) = 0$.*
2. *If there is an edge $e \in A$ such that $|G_e| < \infty$ and $e_+ = e_-$ then $H^1(G, \ell^2(G)) \neq 0$. If, moreover, G is non-amenable, then $\beta^1(G) > 0$.*
3. *If there is an edge $e \in A$ such that $|G_e| < \infty$, $e_+ \neq e_-$, $[G_{e_+} : G_e] \geq 2$ and $[G_{e_-} : t_e G_e t_e^{-1}] \geq 2$, then $H^1(G, \ell^2(G)) \neq 0$. If, moreover, G is non-amenable, then $\beta^1(G) > 0$.*

Proof. (1) By Chiswell’s sequence (1), we have that $H^1(G, \ell^2(G)) = 0$ if and only if the map $\iota: \prod_{v \in V} \ell^2(G)^{G_v} \rightarrow \prod_{e \in A} \ell^2(G)^{G_e}$ is onto.

If for every $e \in A$ we have that $|G_e| = \infty$ then $\ell^2(G)^{G_e} = \{0\}$ and ι is onto.

(2) Assume that for every $e \in A$ we have $|G_e| < \infty$ and $e_+ = e_-$. By contradiction assume that $H^1(G, \ell^2(G)) = \{0\}$ then ι is onto by Chiswell’s sequence (1), and, in particular, there is an f such that $\iota(f)_e = \chi_{G_e}$. Let $v := e_+ = e_-$. Then, we have that $f_v(x) - f_v(t_e x) = \chi_{G_e}(x)$, in particular, if $x \notin G_e$ then $f_v(x) - f_v(t_e x) = 0$, i.e. $f_v(x) = f_v(t_e x)$. Taking $x = t_e^{\pm n}$ for $n \in \mathbb{N}$, a straightforward induction shows that if $n \geq 1$ then $f_v(t_e^n) = f_v(t_e)$ and $f_v(t_e^{-n}) = f_v(1)$. Since $f_v \in \ell^2(G)$ and $\langle t_e \rangle$ is an infinite subgroup of G , we conclude that $f_v(1) = f_v(t_e) = 0$.

On the other hand, $f_v(1) - f_v(t_e) = \chi_{G_e}(1) = 1$ which means that either $f_v(1) \neq 0$ or $f_v(t_e) \neq 0$, a contradiction.

(3) Assume that there is $e \in A$ such that $|G_e| < \infty$, $e_+ \neq e_-$ and $[G_{e_+} : G_e] \geq 2$ and $[G_{e_-} : t_e G_e t_e^{-1}] \geq 2$.

Fix such an e and set $v = e_+$ and $u = e_-$. We may then take e to be in the maximal spanning tree of the quotient graph so that $G_e \leq G_v \cap G_u$.

Observe that $\iota^{-1}(\ell^2(G)^{G_e}) = \ell^2(G)^{G_v} \oplus \ell^2(G)^{G_u}$. By contradiction, assume that $\iota(f_v, f_u) = f_v - f_u = \chi_{G_e}$ for some $(f_v, f_u) \in \ell^2(G)^{G_v} \oplus \ell^2(G)^{G_u}$. This means that $f_v(x) = f_u(x)$ for every $x \notin G_e$.

By assumption, there is a $g_v \in G_v \setminus G_e$ and a $g_u \in G_u \setminus G_e$. Then, $g_v g_u$ is a hyperbolic isometry of the tree (from which the graph of groups decomposition comes). This means that for each $n \in \mathbb{N}$ the element $(g_v g_u)^n \in G$ is distinct and not in G_e . We claim that $f_v((g_v g_u)^{-n}) = f_v(1)$ for every $n \in \mathbb{N}$. Assume $n = 1$. Then, since $(g_v g_u)^n \notin G_e$, and f_u and f_v are G_u and G_v -invariant respectively, we have that

$$\begin{aligned} f_v(g_u^{-1} g_v^{-1}) &= f_u(g_u^{-1} g_v^{-1}) \\ &= f_u(g_v^{-1}) \\ &= f_v(g_v^{-1}) \\ &= f_v(1). \end{aligned}$$

Assume that $f_v((g_v g_u)^{-n}) = f_v(1)$. Then, again, we have that $(g_v g_u)^{n+1} \notin G_e$ and so

$$\begin{aligned} f_v(g_u^{-1} g_v^{-1} (g_v g_u)^{-n}) &= f_u(g_u^{-1} g_v^{-1} (g_v g_u)^{-n}) \\ &= f_u(g_v^{-1} (g_v g_u)^{-n}) \\ &= f_v(g_v^{-1} (g_v g_u)^{-n}) \\ &= f_v((g_v g_u)^{-n}) \\ &= f_v(1). \end{aligned}$$

Therefore, the set $\{g \in G : f_v(g) = f_v(1)\}$ is infinite. This means that $f_v(1) = 0$. A similar argument shows that $f_u(1) = 0$. But this is impossible as $f_v(1) - f_u(1) = \chi_{G_e}(1) = \chi_{G_e}(1) = 1$, a contradiction. Therefore, ι is not onto.

The statements regarding $\beta^1(G)$ follow from one of the possible definitions for $\beta^1(G)$, namely the von Neumann dimension of $\overline{H}^1(G, \ell^2(G))$ (see Definition 1.30 in [Lüc02]), together with Guichardet’s classical result that $\overline{H}^1(G, \ell^2(G)) = H^1(G, \ell^2(G))$ when G is non-amenable. \square

When vertex stabilizers G_v are non-amenable with $\beta^1(G_v) = 0$, part (1) of Proposition 5.4 appears as Theorem 4.1 in [MV07].

5.3. Reduced graphs of groups

Definition 5.5 ([Her88]). A graph of groups is said to be *reduced* if whenever $e \in E$ such that $e_+ \neq e_-$ we have that $[G_{e_+} : G_e] \geq 2$ and $[G_{e_-} : t_e G_e t_e^{-1}] \geq 2$. Otherwise, it is said to be *unreduced*.

One may pass from an unreduced graph of groups to a reduced one simply by retracting edges $e \in E$ such that $e_+ \neq e_-$ and $G_e = G_{e_+}$ without affecting the isomorphism type of the group. This was proven in successive generality in [Her88, For02, dC09].

We make the important observation that the cases of Proposition 5.4 account for all possibilities whenever the graph of groups is reduced. Indeed either $|G_e| = \infty$ for every $e \in A$ (which is case (1)), or there is an edge $e \in A$ such that $|G_e| < \infty$. Then either $e_+ = e_-$ and we are in case (2) or for every $e \in A$ such that $|G_e| < \infty$ we must have that $e_+ \neq e_-$. For such edge e we must have $[G_{e_+} : G_e] \geq 2$ and $[G_{e_-} : t_e G_e t_e^{-1}] \geq 2$ (which is case (3)) because the graph of groups is reduced.

We now turn to the proof of Theorem 1.1. We recall the theorem (and slightly rephrase one of the items):

Theorem 5.6. *Let X be a graph, T a maximal tree in X , (\mathcal{G}, X) a reduced graph of groups, and $G = \pi_1(\mathcal{G}, X, T)$. Assume that $\beta^1(G_v) = 0$ for every $v \in V$ and $\sum \frac{1}{|G_e|} < \infty$. Then $\beta^1(G) = 0$ if and only if G belongs to one of the following cases:*

1. *The graph X is a single vertex. Then $G = G_v$.*
2. *The graph X is a single loop and $G = \mathbb{Z} \rtimes G_v$.*
3. *The graph X is a single edge with $|G_e| < \infty$, $e_+ \neq e_-$ and $[G_{e_\pm} : G_e] = 2$.*
4. *Every edge group is infinite.*

Proof. If $X = \{v\}$ is a single vertex then $G = G_v$ so $\beta^1(G) = 0$.

Assume X is a single loop with $G = \mathbb{Z} \rtimes G_v$. Then, $G_e \cong G_v$ and we have that $\beta^1(G) = 0$ by Proposition 5.2.

If X is a single edge then $G = G_{e_+} *_{G_e} G_{e_-}$ is an amalgamated product. Assuming that $|G_e| < \infty$ and $[G_{e_\pm} : G_e] = 2$. Then we may again apply Proposition 5.2 to deduce that

$$\beta^1(G) = \frac{1}{|G_e|} - \frac{1}{|G_{e_+}|} - \frac{1}{|G_{e_-}|} = 0.$$

Finally, suppose that every edge group is infinite. It then follows that all vertex groups are infinite as well, and hence by Proposition 5.2, we conclude that $\beta^1(G) = 0$.

Conversely, suppose that $\beta^1(G) = 0$. We may assume that we are not in case (1), i.e. X has at least one edge. If G is non-amenable, then by Proposition 5.4 all edge groups are infinite, i.e. we are in case (4). So assume G is amenable, and let G act without inversion on the universal cover \mathcal{T} of the graph of groups (\mathcal{G}, X) . By the main result of [PV91], the action of G on \mathcal{T} is elementary. If G fixes a vertex v , then $X = \{v\}$ as X is reduced, and we excluded this. If G fixes two boundary points of \mathcal{T} , then by Lemma 18 of [dC09], either X is a loop and G is a semi-direct product

$G = \mathbb{Z} \rtimes G_v$ (and we are in case (2)), or X is a segment and G is an amalgamated product with both indices $[G_{e_{\pm}} : G_e]$ being equal to 2 (and we are in case (3)). If G fixes exactly one boundary point of \mathcal{T} , then by Lemma 17 of [dC09], X is a loop and G is an ascending HNN-extension $G = HNN(G_v, \theta)$, where $\theta: G_v \rightarrow G_v$ is injective but not surjective. This of course implies that G_v is infinite. Since $G_e \simeq G_v$, we are in case (4). \square

In Theorem 5.6, let \mathcal{T} be the universal cover of the graph of groups (\mathcal{G}, X) . Assume that \mathcal{T} is locally finite and G acts properly on \mathcal{T} . Then the assumption $\sum \frac{1}{|G_e|} < \infty$ guarantees that G is a lattice in the locally compact group $Aut(\mathcal{T})$.

6. Large groups of automorphisms of \mathcal{T}

In this section, we are concerned with closed subgroups G of the automorphism group of a locally finite tree \mathcal{T} , acting transitively on the boundary $\partial\mathcal{T}$. It is known (see Proposition I.10.2 in [FTN91]) that G has one or two orbits on the set V of vertices of \mathcal{T} , so that \mathcal{T} is either regular or bi-regular. We denote by \widehat{G} the dual of G , i.e. the set of irreducible unitary representations of G , up to unitary equivalence.

Pointwise stabilizers in G of finite subtrees of \mathcal{T} , form a basis of compact open neighborhoods of the identity in G ; for J a finite subtree, let G_J be its pointwise stabilizer in G . For $\pi \in \widehat{G}$, let $P_{\pi,J}$ be the orthogonal projection from the Hilbert space of π , onto the subspace of $\pi(G_J)$ -fixed vectors. We denote by ℓ_{π} the minimum cardinality of (the vertex set of) a finite subtree J such that $P_{\pi,J} \neq 0$. Following [FTN91], we say that:

- π is *spherical* if $\ell_{\pi} = 1$;
- π is *special* if $\ell_{\pi} = 2$;
- π is *cuspidal* if $\ell_{\pi} > 2$.

Note that π is spherical if and only if π is a spherical representation with respect to the Gelfand pair (G, G_a) , where G_a is the stabilizer of an arbitrary vertex $a \in V$.

Our aim, in this section, is to give a new proof of a result of Nebbia [Neb12] describing $H^1(G, \pi)$, for $\pi \in \widehat{G}$; a feature of our proof is that Nebbia appeals to Delorme’s theorem [Del75] for the vanishing of the first cohomology of a non-trivial spherical representation associated with an arbitrary Gelfand pair. In our situation, we bypass the use of Delorme’s result thanks to the concrete description of spherical representations from [FTN91].

6.1. The case of two orbits on V

If G has two orbits on V , then G acts without inversion on \mathcal{T} , with fundamental domain an edge $e = [a, b]$, so G appears as an amalgamated product $G = G_a *_G G_b$. (Examples are provided by $G = PSL_2(F)$, where F is a non-archimedean local field; or by $G = Aut^+(\mathcal{T})$, the subgroup generated by elliptic automorphisms.) In this case G has a unique special representation σ (see Theorem III.2.6 in [FTN91], and the comments following the proof).

We now turn to the proof of Theorem 1.2:

Proof. Let M_π be the Hilbert space of π . If $\ell_\pi > 2$, then $M_\pi^{G_e} = \{0\}$, so the result follows from Theorem 1.3.

Assume $\ell_\pi = 1$; if π is the trivial representation, then $H^1(G, \pi) = 0$, as G is generated by the union of two compact subgroups (so every homomorphism $G \rightarrow \mathbb{C}$ is trivial). So we may assume that π is non-trivial, and appeal to the realization of π as a boundary representation, as in Chapter II of [FTN91]: the space M_π is then a suitable completion of the space of locally constant functions on $\partial\mathcal{T}$, and there exists $s \in]0, 1[\cup (\frac{1}{2} + i\mathbb{R})$ such that the G -action is given by

$$\pi(g)\xi(\omega) = P(g, \omega)^s \xi(g^{-1}\omega),$$

where $P(g, \omega)$ is the Radon-Nikodym derivative $\frac{d\nu_{g\omega}}{d\nu_\omega}(\omega)$, where ν_x , for $x \in V$, is the unique G_x -invariant probability measure on $\partial\mathcal{T}$.

Let $\partial\mathcal{T} = \partial\mathcal{T}_a \cup \partial\mathcal{T}_b$ be the partition of $\partial\mathcal{T}$ induced by the edge e : so $\partial\mathcal{T}_a$ is the set of ends ω such that the ray $[a, \omega[$ does not contain b , and vice-versa. Then $M_\pi^{G_a}$ is 1-dimensional (it consists of constant functions on $\partial\mathcal{T}$), $M_\pi^{G_e}$ is 2-dimensional (it consists of functions constant on $\partial\mathcal{T}_a$ and $\partial\mathcal{T}_b$), and $M_\pi^{G_b}$ is 1-dimensional: the latter consists of functions ξ constant on $\partial\mathcal{T}_a$ and $\partial\mathcal{T}_b$, which, moreover, satisfy:

$$\xi|_{\partial\mathcal{T}_b} = q^{2s} \xi|_{\partial\mathcal{T}_a},$$

where $q + 1$ is the degree of the vertex a ; this follows from the computation of $\frac{d\nu_y}{d\nu_x}$ in Section II.1 of [FTN91]. It is then clear that our map $\iota: M_\pi^{G_a} \oplus M_\pi^{G_b} \rightarrow M_\pi^{G_e}$ is onto, so by Theorem 1.3 we have $H^1(G, \pi) = 0$.

Finally we deal with the special representation σ . Then $M_\sigma^{G_a} = M_\sigma^{G_b} = \{0\}$ (since $\ell_\sigma = 2$), so by Theorem 1.3 we have: $H^1(G, \sigma) \simeq M_\sigma^{G_e}$. By Proposition III.2.3 of [FTN91]⁴ we have $\dim M_\sigma^{G_e} = 1$, completing the proof. \square

6.2. The case of one orbit on V

In this case \mathcal{T} is a $(q + 1)$ -regular tree, on which G acts with inversions and transitively on the vertex set of \mathcal{T} . Examples of this situation are provided by $G = \text{Aut}(\mathcal{T})$, or $G = \text{PGL}_2(F)$, with F a non-archimedean local field; less classical examples appear in [Ama96].

Since G acts with inversions on \mathcal{T} , Theorem 1.3 does not apply immediately. To remedy this, we pass to the first barycentric subdivision \mathcal{T}_1 of \mathcal{T} , where the assumptions of Theorem 1.3 hold.

The new action of G on \mathcal{T}_1 has a single edge as a quotient with vertex set $\{a, b\}$ and edge set $\{e\}$. Say that a corresponds to some vertex \tilde{a} of \mathcal{T} , and b corresponds to some edge \tilde{e} of \mathcal{T} , with $\tilde{a} \in \tilde{e}$.

With this notation, we have that $G_a = \text{stab}_G(\tilde{a})$ and $G_b = \text{stab}_G(\tilde{e})$ and $G_e = G_a \cap G_b$. Here, $\text{stab}_G(\tilde{e})$ denotes those elements of G which preserve \tilde{e} as a set whereas G_e corresponds to the point-wise stabilizer of \tilde{e} in G , so $[\text{stab}_G(\tilde{e}) : G_e] \leq 2$. And so

$$G = G_a *_{G_e} G_b.$$

In this case, the pair (G, G_b) is a Gelfand pair (see Lemma II.4.1 in [FTN91]).

⁴Strictly speaking, this deals with groups having one orbit on vertices, but the comments following Theorem III.2.6 in [FTN91] show how to modify it for two orbits.

Moreover, up to unitary equivalence, G has two special representations σ^+ , σ^- , distinguished by the fact that σ^+ is a spherical representation for the Gelfand pair (G, G_b) , while σ^- is not (see Theorem III.2.6 in [FTN91]).

The following result has been obtained by Amann [Ama03] for $G = \text{Aut}(\mathcal{T})$, and by Nebbia [Neb12] in the general case.

Theorem 6.1. *Let G be a closed subgroup of $\text{Aut}(\mathcal{T})$, acting transitively on $\partial\mathcal{T}$ and V . If $\pi \in \widehat{G} \setminus \{\sigma^-\}$, then $H^1(G, \pi) = 0$; on the other hand $H^1(G, \sigma^-)$ is 1-dimensional.*

Proof. If $\ell_\pi > 2$ or if π is the trivial 1-dimensional representation, the proof is the same as for the corresponding cases in Theorem 1.2.

If π is non-trivial and $\ell_\pi = 1$, the proof is analogous to the corresponding case in Theorem 1.2: using the realization of π as a boundary representation, we have that $M_\pi^{G_a}$ is the 1-dimensional space of constant functions on $\partial\mathcal{T}$, that $M_\pi^{G_e}$ is the 2-dimensional space of functions constant on $\partial\mathcal{T}_a$ and $\partial\mathcal{T}_b$. The only change is that $M_\pi^{G_b}$ is now the 1-dimensional space of functions ξ constant on $\partial\mathcal{T}_a$ and $\partial\mathcal{T}_b$, such that

$$\xi|_{\partial\mathcal{T}_b} = q^s \xi|_{\partial\mathcal{T}_a}.$$

So $\iota: M_\pi^{G_a} \oplus M_\pi^{G_b} \rightarrow M_\pi^{G_e}$ is onto, and the result follows from Theorem 1.3.

For σ^+ , we have $M_{\sigma^+}^{G_a} = \{0\}$ (since $\ell_{\sigma^+} = 2$), and $M_{\sigma^+}^{G_b}$ is 1-dimensional (since σ^+ is spherical for (G, G_b)), and $M_{\sigma^+}^{G_e}$ is 1-dimensional (by Proposition III.2.3 in [FTN91]); so ι is onto and $H^1(G, \sigma^+) = 0$.

Finally, for σ^- we have $M_{\sigma^-}^{G_a} = \{0\}$ (since $\ell_{\sigma^-} = 2$) and $M_{\sigma^-}^{G_b} = \{0\}$ (as σ^- is not spherical for (G, G_b)); so $H^1(G, M_{\sigma^-}) \simeq M_{\sigma^-}^{G_e}$ by Theorem 1. But $M_{\sigma^-}^{G_e}$ is 1-dimensional, by Proposition III.2.3 in [FTN91]. \square

References

- [Ama96] Olivier Éric Amann. *Sur les représentations unitaires du groupe des automorphismes de l'arbre homogène*. Thesis (Master), Univ. de Lausanne, 1996.
- [Ama03] Olivier Éric Amann. *Groups of tree-automorphisms and their unitary representations*. Thesis (Ph.D.), ETH Zurich, 2003.
- [BdlHV08] Bachir Bekka, Pierre de la Harpe, and Alain Valette. *Kazhdan's property (T)*, volume 11 of *New Mathematical Monographs*. Cambridge University Press, Cambridge, 2008.
- [BH99] Martin R. Bridson and André Haefliger. *Metric Spaces of Non-positive Curvature*, volume 319 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1999.
- [Bie75] Robert Bieri. Mayer-Vietoris sequences for HNN-groups and homological duality. *Math. Z.*, 143(2):123–130, 1975.
- [Bro94] Kenneth S. Brown. *Cohomology of Groups*, volume 87 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1994. Corrected reprint of the 1982 original.

- [CG86] Jeff Cheeger and Mikhael Gromov. L_2 -cohomology and group cohomology. *Topology*, 25(2):189–215, 1986.
- [Chi76] I.M. Chiswell. Exact sequences associated with a graph of groups. *J. Pure Appl. Algebra*, 8(1):63–74, 1976.
- [dC09] Yves de Cornulier. Infinite conjugacy classes in groups acting on trees. *Groups Geom. Dyn.*, 3(2):267–277, 2009.
- [Del75] Patrick Delorme. Sur la 1-cohomologie des représentations des groupes de Lie semi-simples. Analyse fonctionnelle. *C. R. Acad. Sci. Paris Sér. A–B*, 280:Ai, A1101–A1103, 1975.
- [For02] Max Forester. Deformation and rigidity of simplicial group actions on trees. *Geom. Topol.*, 6:219–267 (electronic), 2002.
- [FTN91] Alessandro Figà-Talamanca and Claudio Nebbia. *Harmonic Analysis and Representation Theory for Groups Acting on Homogeneous Trees*, volume 162 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1991.
- [FVM12] Talia Fernós, Alain Valette, and Florian Martin. Reduced 1-cohomology and relative property (T). *Math. Z.*, 270(3–4):613–626, 2012.
- [Gui72] Alain Guichardet. Sur la cohomologie des groupes topologiques. II. *Bull. Sci. Math. (2)*, 96:305–332, 1972.
- [Her88] Frank Herrlich. Graphs of groups with isomorphic fundamental group. *Arch. Math. (Basel)*, 51(3):232–237, 1988.
- [Hig51] Graham Higman. A finitely generated infinite simple group. *J. London Math. Soc.*, 26:61–64, 1951.
- [Lüc02] Wolfgang Lück. L^2 -invariants: Theory and Applications to Geometry and K -theory, volume 44 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, 2002.
- [MV07] Florian Martin and Alain Valette. On the first L^p -cohomology of discrete groups. *Groups Geom. Dyn.*, 1(1):81–100, 2007.
- [MV10] Florian Martin and Alain Valette. Free groups and reduced 1-cohomology of unitary representations. In *Quanta of Maths*, volume 11 of *Clay Mathematics Proceedings*, pages 459–463. American Mathematical Society, Providence, RI, 2010.
- [Neb12] Claudio Nebbia. Cohomology for groups of isometries of regular trees. *Expo. Math.*, 30(1):1–10, 2012.
- [PV91] Isabelle Pays and Alain Valette. Sous-groupes libres dans les groupes d’automorphismes d’arbres. *Enseign. Math. (2)*, 37(1–2):151–174, 1991.
- [Sch03] James A. Schafer. Graphs of groups and von Neumann dimension. *J. Pure Appl. Algebra*, 180(3):285–297, 2003.
- [Ser77] Jean-Pierre Serre. *Arbres, amalgames, SL_2* . Société Mathématique de France, Paris, 1977. Avec un sommaire anglais, Rédigé avec la collaboration de Hyman Bass, Astérisque, No. 46.

- [**Sha00**] Yehuda Shalom. Rigidity of commensurators and irreducible lattices. *Invent. Math.*, 141(1):1–54, 2000.
- [**Swa69**] Richard G. Swan. Groups of cohomological dimension one. *J. Algebra*, 12:585–610, 1969.
- [**Wat82**] Yasuo Watatani. Property T of Kazhdan implies property FA of Serre. *Math. Jpn.*, 27(1):97–103, 1982.

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