TWISTED SIMPLICIAL GROUPS AND TWISTED HOMOLOGY OF CATEGORIES

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Abstract

Let A be either a simplicial complex K or a small category $\mathcal{C}$ with $V(A)$ as its set of vertices or objects. We define a twisted structure on A with coefficients in a simplicial group G as a function

$$\delta \colon V(A) \longrightarrow \operatorname{End}(G), \quad v \mapsto \delta_v,$$

such that $\delta_v \circ \delta_w = \delta_w \circ \delta_v$ if there exists an edge in A joining v with w or an arrow either from v to w or from w to v. We give a canonical construction of twisted simplicial groups as well as twisted homology for A with a given twisted structure. Also we determine the homotopy type of this simplicial group as the loop space over certain twisted smash product.

1. Introduction

In 1959 there appeared the works of Barratt, Gugenheim and Moore [3] and Brown [5]. In the first, the authors studied the notion of simplicial fibre-bundle. In the second, twisted tensor products were introduced and studied. Since that time a twisting of standard structures of algebraic topology appeared in various contexts and in many works. In particular, Witten [26], Rohm and Witten [23] introduced a variation $(d + H)$ of the De Rham differential d. The cohomology that is obtained is called twisted De Rham cohomology now. Recently some new versions of homology (cohomology) theory and combinatorial homotopy theory of graphs and similar objects were defined [1,2,7,9–12,24,25]. The classical simplicial homology was varied by considering (co-)homology with local coefficient system for colored posets [9,25] or by inserting δ-factors as numerical data on vertices in the differentials of simplicial chain complexes [10,11].

On the other hand in 1955 James [15] gave a reduced product construction $J(X)$ of a topological space X with given basepoint which is the free monoid generated by X. For a connected CW complex X, the James reduced product $J(X)$ has the same homotopy type as $\Omega\Sigma X$, the loop space of the suspension of X. In 1984 Carlsson generalized the James construction for the case of G-spaces, where G is a discrete group.

The purpose of this paper is to give a canonical twisted construction of simplicial groups as well as twisted homology for a simplicial complex and a small category with a given twisted structure. In particular, our twisted homology (twisted cohomology) is a generalization of the δ-homology (δ-cohomology) introduced in [**10**, **11**] and our constructions are different from the twisted Cartesian product introduced by Barratt, Gugenheim and Moore [**3**], and Brown [**5**] and the balanced product introduced by Carlsson [**6**].

Categories considered in the paper are small and we omit mentioning it. All necessary definitions will be given later in the text of the paper.

Let A be either a simplicial complex K or a category $\mathcal{C}$. We are considering category from the geometrical point of view, using the terminology of underlying quiver, for example we are using the word "vertices" instead of objects. However, composition of morphisms and identity morphisms are involved in our constructions. Let $V(A)$ be the vertex set of A, and G a simplicial group. We denote by $\mathrm{End}(G)$ the monoid of simplicial endomorphisms of G, namely $\mathrm{End}(G)$ consists of graded endomorphisms $f = \{f_n \colon G_n \to G_n\}_{n \geqslant 0}$ such that $d_i \circ f = f \circ d_i$ and $s_i \circ f = f \circ s_i$ for the face homomorphisms $d_i \colon G_n \to G_{n-1}$ and the degeneracy homomorphisms $s_i \colon G_n \to G_{n+1}$ for $0 \leqslant i \leqslant n$. A *twisted structure* on A with coefficients in a simplicial group G is a function

$$\delta \colon V(A) \longrightarrow \mathrm{End}(G), \quad v \mapsto \delta_v,$$

satisfying the following commuting rule:

$$\delta_v \circ \delta_w = \delta_w \circ \delta_v \tag{1}$$

if there exists an edge in A (if A is a simplicial complex) joining v with w or an arrow either from v to w or from w to v (if A is a category). A twisted structure on A with coefficients in G is called *regular* if $\delta_v \colon G \to G$ is a simplicial automorphism for each $v \in V(A)$.

Now let us suppose that a simplicial complex K has a total order. There is a canonical associated simplicial set $S(K)$ by allowing the repeating of vertices in the definition of elements of $S(K)$. For a category $\mathcal{C}$ take $S = S(\mathcal{C})$ to be a nerve of $\mathcal{C}$. The simplicial set $S = S(A)$ has the following nature: the set S_n of n-paths (n-simplices) has a form of the sequences

$$v_0 \alpha_1 v_1 \cdots \alpha_n v_n$$

with $v_i \in V(\mathcal{C})$ and $\alpha_i \colon v_{i-1} \to v_i$ a morphism of the category $\mathcal{C}$ or the sequences $(v_0, v_1, \ldots, v_n)$ for $v_i \in V(K)$ with $v_i \leqslant v_j$ for $i \leqslant j$ under a total order on $V(K)$ and $\{v_0, \ldots, v_n\}$ being a simplex in K. The face operation $d_i \colon S_n \to S_{n-1}$, $0 \leqslant i \leqslant n$, is given by removing the vertex v_i followed by the composition $\alpha_i \alpha_{i+1} \colon v_{i-1} \to v_{i+1}$, and the degeneracy operation $s_i \colon S_n \to S_{n+1}$, $0 \leqslant i \leqslant n$, is given by doubling the vertex v_i with inserting the identity arrow $e_{v_i} \colon v_i \to v_i$ if it is the case of category.

It follows from (1) that the functions δ_{v_i} and δ_{v_j} satisfy the commuting rule on all pairs (v_i, v_j) of vertices of arbitrary n-path.

We construct a Δ-group and a simplicial group $F_\delta^G[A]$ depending on the twisted structure δ. This Δ-group seems to be new. If the δ-structure is given by $\delta_v \equiv \mathrm{id}_G$ for $v \in V(A)$, then $F_\delta^G[A]$ is connected with Carlsson's $J_G(X)$-construction [**6**] or Quillen's tensor product [**21**, Chapter II, Section 3, page 3.7] with its geometric

realization having the homotopy type of $\Omega(B|G| \wedge |A|)$, where $|X|$ is the geometric realization of a simplicial set X.

As a consequence of the main result (Theorem 4.5) in this work we determine the homotopy type of $F_\delta^G[A]$ as the loop space of the δ-*twisted smash product* $B|G| \wedge_\delta |A|$ (see Section 4 for the definition of δ-twisted smash product). Observe that the homotopy type of Carlsson's $J_G(X)$-construction is the loop space of the reduced Borel construction of G-action on $|X|$ [**6**, Theorem 9]. The construction $F_\delta^G[A]$ gives a new type of simplicial group.

One of the points of the main result (Theorem 4.5) is to give a topological interpretation of the δ-homology (δ-cohomology) introduced in [**10**, **11**].

The results of this work can be generalized to the category of digraphs following the lines of the papers [**11**, **12**].

The article is organized as follows. In Section 2, we give a review on the path complexes of categories. The twisted construction of the Δ-groups and simplicial groups is given in Section 3 by examining the Δ-identity and simplicial identities. The homotopy type of the twisted simplicial group $F_\delta^G[A]$ is studied in Section 4, where the main result is Theorem 4.5. In Section 5, we explore the twisted homology of simplicial complexes as well as categories.

Acknowledgments

2. Path complexes of categories and simplicial set extension of simplicial complexes

2.1. Path complexes of categories

Recall that a quiver is a quadruple $\mathcal{Q} = (V(\mathcal{Q}), E(\mathcal{Q}), s, t)$, where the set V is called the set of vertices of $\mathcal{Q}$, the set E is called the set of edges of $\mathcal{Q}$, s and t are two maps $s \colon E \to V$, giving the start or source of the edge, and $t \colon E \to V$, giving the target of the edge. The elements in $E(\mathcal{Q})$ we denote by *arrows* $\alpha \colon v \to w$, or $v\alpha w$.

Recall also that there is a forgetful functor from small categories to quivers

$$\mathbf{Cat} \to \mathbf{Quiv}.$$

By an n-path for a category $\mathcal{C}$ we mean the sequence $v_0 \alpha_1 \cdots v_{n-1} \alpha_n v_n$ with $n \geqslant 1$. There exists a trivial path (the identity arrow) $e_v \colon v \to v$ for each vertex $v \in V(\mathcal{C})$. Let $S_n(\mathcal{C})$ be the set of all n-paths with $n \geqslant 1$ and $S_0(\mathcal{C}) = V(\mathcal{C})$.

The face operations in $S(\mathcal{C})$ are defined as follows: d_0 removes the initial vertex together with the first arrow and the last face, d_n removes the last vertex together with

the last arrowhighlight, and for $1 \leqslant i \leqslant n-1$ the operation d_i is given by removing a vertex followed by the composition of arrows in the following sense:

$$d_i(v_0\alpha_1v_1\alpha_2v_2\cdots\alpha_nv_n) = v_0\alpha_1v_1\cdots v_{i-1}(\alpha_{i+1}\circ\alpha_i)v_{i+1}\cdots\alpha_nv_n.$$

The *degeneracy operations*

$$s_i\colon S_n(\mathcal{C}) \longrightarrow S_{n+1}(\mathcal{C}),$$

for $0 \leqslant i \leqslant n$ are defined by doubling the i-vertex with inserting the trivial path. Then $S(\mathcal{C}) = \{S_n(\mathcal{C})\}_{n\geqslant 0}$ is the classifying simplicial set (or *nerve*) of the category $\mathcal{C}$. Choose a vertex $a_0 \in V(\mathcal{C})$ as a basepoint. Consider the constant n-path $a_0^n = a_0 e_{a_0} a_0 \cdots e_{a_0} a_0$ as a basepoint for the set $S_n(\mathcal{C})$. Then $S(\mathcal{C})$ is a pointed simplicial set [**27**, Chapter 2].

2.2. Simplicial set extension of simplicial complexes

Let K be a simplicial complex with its vertex set $V(K)$ which has a total order. Recall that the simplicial set $S(K)$ induced by K is given as follows:

1. The set $S(K)_n$ consists of sequences $(v_0, v_1, \ldots, v_n)$ of vertices of K such that $v_0 \leqslant v_1 \leqslant \cdots \leqslant v_n$ and $\{v_0, \ldots, v_n\}$ forms a simplex in K.

2. The face operation $d_i\colon S(K)_n \to S(K)_{n-1}$, $0 \leqslant i \leqslant n$, is given by

$$d_i(v_0, v_1, \ldots, v_n) = (v_0, v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n). \tag{2}$$

3. The degeneracy operation $s_i\colon S(K)_n \to S(K)_{n+1}$, $0 \leqslant i \leqslant n$, is given by

$$s_i(v_0, v_1, \ldots, v_n) = (v_0, v_1, \ldots, v_{i-1}, v_i, v_i, v_{i+1}, \ldots, v_n).$$

The geometric realization of the simplicial set $S(K)$ is homeomorphic to the polyhedron $|K|$ (see, for example, [**8**]). We choose a basepoint in $S(K)_n$ by $a_0^n = (a_0, a_0, \ldots, a_0)$ for a vertex $a_0 \in V(K)$ so that $S(K)$ is a pointed simplicial set.

3. The twisted construction of Δ-groups and simplicial groups

A Δ-*set* is a sequence of sets $X = \{X_n, n \geqslant 0\}$ with maps called *faces*

$$d_i\colon X_n \to X_{n-1}, \ 0 \leqslant i \leqslant n,$$

such that the following condition holds

$$d_id_j = d_jd_{i+1},$$

for $i \geqslant j$. It is called the Δ-*identity* [**27**]. Roughly speaking a Δ-set is a simplicial set without degeneracies.

A Δ-set $G = \{G_n\}_{n\geqslant 0}$ is called a Δ-*group* if each G_n is a group, and each face d_i is a group homomorphism [**27**].

We carry out the following steps for constructing a Δ-group and a simplicial group depending on the twisted structure δ on A:

1) Choose a vertex $a_0 \in V(A)$ as a basepoint. Consider the constant n-path $a_0^n = a_0 e_{a_0} a_0 \cdots e_{a_0} a_0$ as a basepoint for the set S_n. Define $F^G[A]_n$ to be the free product of the group G_n with indices running over all non-basepoint elements in $S_n(A)$. More precisely, let $(G_n)_x$ be a copy of G_n labeled by an element

$x \in S_n(A)$. Then $F^G[A]_n$ is the quotient group of the free product $*_{x \in S_n}(G_n)_x$ subject to the relations that $(G_n)_{a_0^n} = \{1\}$.

2) Let g_x denote the element $g \in G_n$ in its copy group $(G_n)_x$ labeled by x. Define a (twisted) face operation

$$d_i^\delta \colon F^G[A]_n \to F^G[A]_{n-1}, \ 0 \leqslant i \leqslant n$$

to be the (unique) group homomorphism such that d_i^δ restricted to each copy $(G_n)_{v_0 \alpha_1 v_1 \cdots \alpha_n v_n}$ is given by the twisted formula

$$d_i^\delta|_{(G_n)_{v_0\alpha_1 v_1 \cdots \alpha_n v_n}}(g_{v_0\alpha_1 v_1 \cdots \alpha_n v_n}) = (\delta_{v_i}(d_i(g)))_{d_i(v_0\alpha_1 v_1 \cdots \alpha_n v_n)}. \qquad (3)$$

Namely the element $d_i g \in (G_{n-1})_{d_i(v_0\alpha_1 v_1 \cdots \alpha_n v_n)}$ is twisted by the endomorphism $\delta_{v_i} \colon G_{n-1} \to G_{n-1}$ By Proposition 3.1 below, the sequence of groups $F_\delta^{G,\Delta}[A] = \{F^G[A]_n\}_{n \geqslant 0}$ with the twisted face operations forms a Δ-group, where the commuting rule (1) assures the Δ-identity.

3) Suppose that the δ-structure is regular. We define a twisted degeneracy operation $s_i^\delta \colon F^G[A]_n \to F^G[A]_{n+1}, 0 \leqslant i \leqslant n$, to be the (unique) group homomorphism such that s_i^δ restricted to each copy $(G_n)_{v_0\alpha_1 v_1 \cdots \alpha_n v_n}$ is given by the twisted formula

$$s_i^\delta|_{(G_n)_{v_0\alpha_1 v_1 \cdots \alpha_n v_n}}(g_{v_0\alpha_1 v_1 \cdots \alpha_n v_n}) = ((\delta_{v_i})^{-1}(s_i(g)))_{s_i(v_0\alpha_1 v_1 \cdots \alpha_n v_n)}. \qquad (4)$$

By Proposition 3.1, the sequence of groups $F_\delta^G[A] = \{F^G[A]_n\}_{n \geqslant 0}$ with the twisted face operations d_i^δ in step 2 and the above twisted degeneracy operations s_i^δ forms a simplicial group [**27**, Chapter 2], where the commuting rule (1) and invertibility of δ_v are essential for assuring the simplicial identities.

The Δ-group $F_\delta^{G,\Delta}[A]$ is a generalization of δ-homology [**10**, **11**] in the following sense. Let G be any abelian group. Let $\delta \colon V(A) \to \mathrm{End}(G)$ be a function satisfying the commuting rule (1). We consider G as a discrete simplicial group with $G_n = G$ and the faces and degeneracies given by the identity. Then we have the δ-structure on A. By taking the abelianization of the (non-commutative) Δ-group $F_\delta^{G,\Delta}[A]$ in step 2, we have an abelian Δ-group with the twisted faces given in (3) on chains

$$(F_\delta^G[A]_n)^{\mathrm{ab}} = \left(\bigoplus_{x \in S_n} G_x\right)/(G)_{a_0^n} \qquad (5)$$

having the differentials given by

$$\partial_n(g_{v_0\alpha_1 v_1 \cdots \alpha_n v_n}) = \sum_{i=0}^n (-1)^i (\delta_{v_i}(d_i(g)))_{d_i(v_0\alpha_1 v_1 \cdots \alpha_n v_n)}. \qquad (6)$$

In the case where G is a commutative ring R and δ_v is the translation given by an element $\delta_v \in R$, the above formula on the differentials is exactly given by inserting the factor δ_v as described in [**10**, **11**].

Proposition 3.1. *Let S be $S(A)$ where A is either a simplicial complex K or a category $\mathcal{C}$ with twisted structure in a simplicial group G. Then*

1. *The sequence of groups $F_\delta^{G,\Delta}[A] = \{F^G[A]_n\}_{n \geqslant 0}$ with the twisted face operations defined by formula 3 forms a Δ-group.*

2. *Suppose that the twisted structure δ is regular. Then the sequence of groups $F_\delta^G[A] = \{F^G[A]_n\}_{n \geqslant 0}$ with the twisted face operations d_i^δ given by formula 3 and the twisted degeneracy operations s_i^δ given by formula 4 forms a simplicial group.*

Proof. The proof is given by examining the Δ-identity for assertion (1), and the simplicial identities for assertion (2). Since the examination on the Δ-identity for assertion (1) is part of the simplicial identities for assertion (2), we only need to prove assertion (2) with paying attention that the invertibility of δ_v is not required in the Δ-identity for face operations.

For the basepoint a_0^n in S_n, we see that each d_i^δ sends $(G_n)_{a_0^n}$ into $(G_{n-1})_{a_0^{n-1}}$, so

$$d_i^\delta \colon F_\delta^G[A]_n \to F_\delta^G[A]_{n-1}$$

is a well-defined homomorphism. The same is true for

$$s_i^\delta \colon F_\delta^G[A]_n \to F_\delta^G[A]_{n+1}.$$

Now we show that the simplicial-identity holds for the face and degeneracy operations. For simplicity of notation, we only consider the case $S = S(K)$. For the case $S(\mathcal{C})$ of a category $\mathcal{C}$, the proof follows along the same lines, keeping in mind the following rules:

1) When a vertex v_i is removed from the sequence $(v_0, \ldots, v_n)$, the arrow from $v_{i-1} \to v_{i+1}$ in the sequence $(v_0, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n)$ is given by the composition $v_{i-1} \to v_i \to v_{i+1}$.

2) When a vertex v_i is doubled from the sequence $(v_0, \ldots, v_n)$, we insert the trivial path $e_{v_i} \colon v_i \to v_i$ in the sequence

$$(v_0, \ldots, v_{i-1}, v_i, v_i, v_{i+1}, \ldots, v_n).$$

Let $g_{(v_0, \ldots, v_n)} \in (G_n)_{(v_0, \ldots, v_n)}$, and let $i \geqslant j$ with $0 \leqslant j \leqslant n-1$ and $0 \leqslant i \leqslant n$. Then

$$
\begin{aligned}
d_i^\delta \circ d_j^\delta(g_{(v_0,\ldots,v_n)}) &= d_i^\delta(\delta_{v_j}^{n-1}(d_j g)_{(v_0,\ldots,v_{j-1},v_{j+1},\ldots,v_n)}) \\
&= d_i^\delta(d_j(\delta_{v_j}^n(g))_{(v_0,\ldots,v_{j-1},v_{j+1},\ldots,v_n)}) \\
&= (\delta_{v_{i+1}}^{n-2}(d_i(d_j(\delta_{v_j}^n(g)))))_{(v_0,\ldots,v_{j-1},v_{j+1},\ldots,v_i,v_{i+2},\ldots,v_n)} \\
&= (d_i(d_j(\delta_{v_{i+1}}^n(\delta_{v_j}^n(g)))))_{(v_0,\ldots,v_{j-1},v_{j+1},\ldots,v_i,v_{i+2},\ldots,v_n)} \\
&= (d_j(d_{i+1}(\delta_{v_j}^n(\delta_{v_{i+1}}^n(g)))))_{(v_0,\ldots,v_{j-1},v_{j+1},\ldots,v_i,v_{i+2},\ldots,v_n)} \\
&= (d_j(\delta_{v_j}^{n-1}(d_{i+1}(\delta_{v_{i+1}}^n(g)))))_{d_j d_{i+1}(v_0,\ldots,v_n)} \\
&= (\delta_{v_j}^{n-2}(d_j(\delta_{v_{i+1}}^{n-1}(d_{i+1}(g)))))_{d_j d_{i+1}(v_0,\ldots,v_n)} \\
&= d_j^\delta \circ d_{i+1}^\delta(g_{(v_0,\ldots,v_n)}),
\end{aligned}
$$

where $\delta_{v_{i+1}}^n \circ \delta_{v_j}^n = \delta_{v_j}^n \circ \delta_{v_{i+1}}^n$ because $v_j v_{i+1}$ forms a 1-simplex in K. (In the case of $S = S(\mathcal{C})$, $\delta_{v_{i+1}}^n \circ \delta_{v_j}^n = \delta_{v_j}^n \circ \delta_{v_{i+1}}^n$ because there is an arrow given by a composition from v_j to v_{i+1}.)

Let $i \geqslant j$, and let $g_{(v_0,\ldots,v_n)} \in (G_n)_{(v_0,\ldots,v_n)}$. We shorten the notation s_i^δ as s_i. Then

$$
\begin{aligned}
s_j s_i (g_{(v_0,\ldots,v_n)}) &= s_j((\delta_{v_i}^{n+1})^{-1}(s_i(g))_{(v_0,\ldots,v_i,v_i,\ldots,v_n)}) \\
&= s_j(s_i((\delta_{v_i}^n)^{-1}(g))_{(v_0,\ldots,v_i,v_i,\ldots,v_n)}) \\
&= (\delta_{v_j}^{n+2})^{-1}(s_j s_i((\delta_{v_i}^n)^{-1}(g)))_{(v_0,\ldots,v_j,v_j,\ldots,v_i,v_i,\ldots,v_n)} \\
&= (s_j s_i(\delta_{v_j}^n)^{-1}((\delta_{v_i}^n)^{-1}(g)))_{(v_0,\ldots,v_j,v_j,\ldots,v_i,v_i,\ldots,v_n)} \\
&= (s_j s_i(\delta_{v_i}^n)^{-1}((\delta_{v_j}^n)^{-1}(g)))_{(v_0,\ldots,v_j,v_j,\ldots,v_i,v_i,\ldots,v_n)} \\
&= (\delta_{v_i}^{n+2})^{-1}(s_{i+1}((\delta_{v_j}^{n+1})^{-1}s_j(g)))_{(v_0,\ldots,v_j,v_j,\ldots,v_i,v_i,\ldots,v_n)} \\
&= s_{i+1} s_j(g_{(v_0,\ldots,v_n)}).
\end{aligned}
$$

We shorten the notation d_i^δ as d_i. Let $i < j$, and let $g_{(v_0,\ldots,v_n)} \in (G_n)_{(v_0,\ldots,v_n)}$. Then

$$
\begin{aligned}
d_i s_j (g_{(v_0,\ldots,v_n)}) &= d_i((\delta_{v_j}^{n+1})^{-1}(s_j(g))_{(v_0,\ldots,v_j,v_j,\ldots,v_n)}) \\
&= d_i((s_j(\delta_{v_j}^n)^{-1}(g))_{(v_0,\ldots,v_j,v_j,\ldots,v_n)}) \\
&= \delta_{v_i}^n(d_i(s_j(\delta_{v_j}^n)^{-1}(g)))_{(v_0,\ldots,v_{i-1},v_{i+1},\ldots,v_j,v_j,\ldots,v_n)} \\
&= (d_i s_j \delta_{v_i}^n(\delta_{v_j}^n)^{-1}(g))_{(v_0,\ldots,v_{i-1},v_{i+1},\ldots,v_j,v_j,\ldots,v_n)} \\
&= (s_{j-1} d_i(\delta_{v_j}^n)^{-1}(\delta_{v_i}^n)(g))_{(v_0,\ldots,v_{i-1},v_{i+1},\ldots,v_j,v_j,\ldots,v_n)} \\
&= ((\delta_{v_j}^n)^{-1}(s_{j-1}(\delta_{v_i}^{n-1}(d_i(g)))))_{(v_0,\ldots,v_{i-1},v_{i+1},\ldots,v_j,v_j,\ldots,v_n)} \\
&= s_{j-1} d_i(g_{(v_0,\ldots,v_n)}).
\end{aligned}
$$

Let $i = j$, and let $g_{(v_0,\ldots,v_n)} \in (G_n)_{(v_0,\ldots,v_n)}$. Then

$$
\begin{aligned}
d_j s_j (g_{(v_0,\ldots,v_n)}) &= d_j((\delta_{v_j}^{n+1})^{-1}(s_j(g))_{(v_0,\ldots,v_j,v_j,\ldots,v_n)}) \\
&= d_j((s_j(\delta_{v_j}^n)^{-1}(g))_{(v_0,\ldots,v_j,v_j,\ldots,v_n)}) \\
&= (\delta_{v_j}^n(d_j s_j(\delta_{v_j}^n)^{-1}(g)))_{(v_0,\ldots,v_n)} \\
&= (d_j s_j(\delta_{v_j}^n(\delta_{v_j}^n)^{-1}(g)))_{(v_0,\ldots,v_n)} \\
&= g_{(v_0,\ldots,v_n)}.
\end{aligned}
$$

Let $i = j + 1$, and let $g_{(v_0,\ldots,v_n)} \in (G_n)_{(v_0,\ldots,v_n)}$. We remind that the rule is such that we add the twisting of the vertex we are removing. When we apply d_{j+1}, we should add twisting δ_{v_j} because the second v_j is located in the position $(j+1)$ under removing. Then we have

$$
\begin{aligned}
d_{j+1} s_j (g_{(v_0,\ldots,v_n)}) &= d_{j+1}((\delta_{v_j}^{n+1})^{-1}(s_j(g))_{(v_0,\ldots,v_j,v_j,\ldots,v_n)}) \\
&= d_{j+1}((s_j(\delta_{v_j}^n)^{-1}(g))_{(v_0,\ldots,v_j,v_j,\ldots,v_n)}) \\
&= \delta_{v_j}^n(d_{j+1} s_j(\delta_{v_j}^n)^{-1}(g))_{(v_0,\ldots,v_n)} \\
&= (d_{j+1} s_j(\delta_{v_j}^n(\delta_{v_j}^n)^{-1}(g)))_{(v_0,\ldots,v_n)} \\
&= g_{(v_0,\ldots,v_n)}.
\end{aligned}
$$

Let $i > j + 1$, and let $g_{(v_0,\ldots,v_n)} \in (G_n)_{(v_0,\ldots,v_n)}$. Then

$$
\begin{aligned}
d_i s_j (g_{(v_0,\ldots,v_n)}) &= d_i((\delta_{v_j}^{n+1})^{-1}(s_j(g))_{(v_0,\ldots,v_j,v_j,\ldots,v_n)}) \\
&= d_i((s_j(\delta_{v_j}^n)^{-1}(g))_{(v_0,\ldots,v_j,v_j,\ldots,v_n)}) \\
&= \delta_{v_{i-1}}^n(d_i(s_j(\delta_{v_j}^n)^{-1}(g)))_{(v_0,\ldots,v_j,v_j,\ldots,v_{i-2},v_i,\ldots,v_n)} \\
&= (d_i s_j \delta_{v_{i-1}}^n(\delta_{v_j}^n)^{-1}(g))_{(v_0,\ldots,v_j,v_j,\ldots,v_{i-2},v_i,\ldots,v_n)} \\
&= (s_j d_{i-1}(\delta_{v_j}^n)^{-1}(\delta_{v_{i-1}}^n)(g))_{(v_0,\ldots,v_j,v_j,\ldots,v_{i-2},v_i,\ldots,v_n)} \\
&= ((\delta_{v_j}^n)^{-1}(s_j(\delta_{v_{i-1}}^{n-1}(d_{i-1}(g)))))_{(v_0,\ldots,v_j,v_j,\ldots,v_{i-2},v_i,\ldots,v_n)} \\
&= s_j d_{i-1}(g_{(v_0,\ldots,v_n)}). \qquad \square
\end{aligned}
$$

4.   Homotopy type of $F_\delta^{\mathcal{G}}[S]$

4.1.   The δ-twisted Cartesian products

We generalize now the notion of twisted structure for a simplicial set, not necessarily a simplicial group G. Let Y be a simplicial set. We denote by $\mathrm{End}(Y)$ the monoid of self-simplicial maps of Y. Let A be a simplicial complex or a category with vertex set $V(A)$. A *twisted structure* on A in a simplicial set Y is a function

$$\delta\colon V(A) \longrightarrow \mathrm{End}(Y), \quad v \mapsto \delta_v,$$

such that the commuting rule

$$\delta_v \circ \delta_w = \delta_w \circ \delta_v \tag{7}$$

holds if there is an edge joint v and w in the case of simplicial complexes and arrow between v and w in the case of category. A twisted structure on A is called *regular* if $\delta_v\colon Y \to Y$ is a simplicial isomorphism for each $v \in V(A)$.

Let $S = S(A)$ be a simplicial set defined in Section 2. Given any twisted structure δ on A in a simplicial set Y, we can carry out the following construction of δ-*twisted Cartesian product*:

1. The sequence of sets is given by $Y_n \times S_n$ for $n \geqslant 0$.

2. The faces $d_i\colon Y_n \times S_n \to Y_{n-1} \times S_{n-1}$, $0 \leqslant i \leqslant n$, are defined by the formula

$$d_i(y, v_0\alpha_1 v_1 \cdots \alpha_n v_n) = (\delta_{v_i}(d_i y), d_i(v_0\alpha_1 v_1 \cdots \alpha_n v_n)). \tag{8}$$

3. Suppose that the twisted structure δ is regular. We define the degeneracy operation $s_i\colon Y_n \times S_n \to Y_{n+1} \times S_{n+1}$, $0 \leqslant i \leqslant n$, by the formula

$$s_i(y, v_0\alpha_1 v_1 \cdots \alpha_n v_n) = (\delta_{v_i}^{-1}(s_i y), s_i(v_0\alpha_1 v_1 \cdots \alpha_n v_n)). \tag{9}$$

Remark 4.1. The definition of our δ-twisted Cartesian product is different from that of the classical twisted Cartesian product in simplicial theory [3, 8].

Proposition 4.2. *Let $S = S(A)$ for A with a twisted structure δ in a simplicial set Y. Then*

1. *The sequence of sets $Y \times_{\delta,\Delta} S = \{Y_n \times S_n\}_{n \geqslant 0}$ with the face operations defined by (8) forms a Δ-set.*

2. *Suppose that the twisted structure δ is regular. Then the sequence of sets $Y \times_\delta S = \{Y_n \times S_n\}_{n \geqslant 0}$ with the face operations defined in (8) and the degeneracy operations defined by (9) forms a simplicial set.*

Proof. The assertions follow along the lines in the proof of Proposition 3.1 for examining the Δ-identity and the simplicial identities.   □

The first assertion of the above proposition is sufficient for having homology (cohomology) with twisted coefficients. Let A have a twisted structure δ in a simplicial set Y. Let G be an abelian group. The *twisted homology* (*twisted cohomology*) of A with coefficients in G is defined by

$$H_n^{\mathrm{twist}}(A; G) = H_n(G(Y \times_{\delta,\Delta} S(A))), \ H_{\mathrm{twist}}^n(A; G) = H^n(G(Y \times_{\delta,\Delta} S(A))),$$

for $n \geqslant 0$, where $G(X)$ for a simplicial set X is given by $G(X) = \oplus_{x \in X}(G)_x$ with $(G)_x$ a copy of G labeled by x. A consequence of (1) of Proposition (4.2) is the following corollary.

Corollary 4.3. *Let A have a twisted structure δ in a simplicial set Y. Then the twisted homology $H_*^{\mathrm{twist}}(A; G)$ (twisted cohomology $H_{\mathrm{twist}}^*(A; G)$) is the homology (cohomology) of the chain complex $C_*^{\mathrm{twist}}(A)$ with coefficients in G, where*

$$C_n^{\mathrm{twist}}(A) = \mathbb{Z}(Y_n) \otimes \mathbb{Z}(S(A)_n),$$

with the differential $\partial_n^{\mathrm{twist}} \colon C_n^{\mathrm{twist}}(A) \to C_{n-1}^{\mathrm{twist}}(A)$ given by the formula

$$\partial_n(y \otimes v_0\alpha_1 v_1 \cdots \alpha_n v_n) = \sum_{i=0}^{n} (-1)^i \delta_{v_i}(d_i y) \otimes d_i(v_0\alpha_1 v_1 \cdots \alpha_n v_n),$$

for $n \geqslant 0$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Connections between cartesian product and tensor product is usual in Algebraic Topology, they start with the Eilenberg-Zilber theorem.

From the second assertion of Proposition 4.2, there is a connection between δ-twisted Cartesian products and simplicial fibre bundles which is described as follows. Let $S = S(A)$ have a regular twisted structure δ in a simplicial set Y. Observe that the coordinate projection

$$p \colon Y \times_\delta S \to S, \quad (g, v_0\alpha_1 v_1 \cdots \alpha_n v_n) \mapsto v_0\alpha_1 v_1 \cdots \alpha_n v_n$$

is a simplicial map. Recall from [**8**, page 155] that a simplicial map $p \colon E \to B$ is called a *simplicial fibre bundle* with fibre F if for each $b \in B_n$, there is a commutative diagram

$$
\begin{array}{ccccc}
\Delta[n] \times F & \xrightarrow[\cong]{\alpha_b} & E' & \xrightarrow{\tilde{f}_b} & E \\
\downarrow{\scriptstyle\mathrm{proj.}} & & \downarrow{\scriptstyle\mathrm{pull\text{-}back}} & & \downarrow \\
\Delta[n] & =\!=\!=\!= & \Delta[n] & \xrightarrow{f_b} & B,
\end{array}
\qquad (10)
$$

where $\Delta[n]$ is the standard simplicial n-simplex, $f_b \colon \Delta[n] \to B$ is the representing map of $b \in B_n$ and α_b is a simplicial isomorphism.

Theorem 4.4. *Let $S = S(A)$ have a regular twisted structure δ in a simplicial set Y. Then the coordinate projection map*

$$p \colon Y \times_\delta S \to S$$

is a simplicial fibre bundle with fibre Y.

Proof. Let $\sigma = (0, 1, \ldots, n) \in \Delta[n]_n$ be the standard non-degenerate n-simplex. For each $b = v_0\alpha_1 v_1 \cdots \alpha_n v_n \in S_n$, we have $f_b(\sigma) = b$ by definition. By taking iterated faces, we have $f_b(i) = v_i$. The twisted structure on A induces a function

$$\delta \colon V(\Delta[n]) = \{(0), (1), \ldots, (n)\} \longrightarrow \mathrm{End}(Y)$$

given by

$$\delta_{(i)} = \delta_{v_i},$$

for $0 \leqslant i \leqslant n$. For $0 \leqslant i < j \leqslant n$, since there is an edge between v_i and v_j or an

arrow $v_i \to v_j$, we have

$$\delta_{(i)} \circ \delta_{(j)} = \delta_{(j)} \circ \delta_{(i)},$$

by the commuting rule $\delta_{v_i} \circ \delta_{v_j} = \delta_{v_j} \circ \delta_{v_i}$ and so a twisted structure on $\Delta[n]$ in Y. From the definition of δ-twisted Cartesian product, we have the simplicial map

$$(\mathrm{id}, f_b) \colon Y \times_\delta \Delta[n] \longrightarrow E = Y \times_\delta S$$

inducing a fibrewise simplicial isomorphism

$$
\begin{array}{ccc}
Y \times_\delta \Delta[n] & \longrightarrow & E' \\
\downarrow{\scriptstyle p} & & \downarrow \\
\Delta[n] & =\!=\!=\!=\!= & \Delta[n],
\end{array}
$$

where E' is given by the pull-back in diagram (10) with $E = Y \times_\delta S$ and $B = S$. Now we show that there is a fibrewise simplicial isomorphism

$$
\begin{array}{ccc}
Y \times \Delta[n] & \overset{\alpha_b}{\underset{\cong}{\longrightarrow}} & Y \times_\delta \Delta[n] \\
\downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} \\
\Delta[n] & =\!=\!=\!=\!= & \Delta[n].
\end{array}
$$

The simplicial isomorphism α_b is constructed by untwisting the twisted faces and degeneracies. More precisely, observe that, each element w in $\Delta[n]_q$ can be uniquely expressed as a monotone sequence

$$w = (\overbrace{j_1(w), \ldots, j_1(w)}^{l_1(w)}, \ldots, \overbrace{j_{t_w}(w), \ldots, j_{t_w}(w)}^{l_{t_w}(w)}),$$

with $l_i(w) \geqslant 1$ for $1 \leqslant i \leqslant t$, $0 \leqslant j_1(w) < j_2(w) < \cdots < j_{t_w}(w) \leqslant n$ and $\sum_{i=1}^{t_w} l_i(w) = q + 1$. Let

$$\{i_1(w), \ldots, i_{s_w}(w)\} = \{0, 1, \ldots, n\} \smallsetminus \{j_1(w), j_2(w), \ldots, j_{t_w}(w)\},$$

with $i_1(w) < i_2(w) < \cdots < i_{s_w}(w)$ be the set of missing vertices in w. Define the function

$$\alpha_b \colon Y_q \times \Delta[n]_q \longrightarrow Y_q \times \Delta[n]_q$$

by setting

$$\alpha_b(y, w) = (\delta_{(j_{t_w}(w))}^{-l_{t_w}(w)+1} \cdots \delta_{(j_1(w))}^{-l_1(w)+1} \delta_{(i_1(w))} \cdots \delta_{(i_{s_w}(w))}(y), w).$$

Clearly $\alpha_b \colon Y_q \times \Delta[n]_q \longrightarrow Y_q \times \Delta[n]_q$ is bijective for $q \geqslant 0$. It suffices to show that α_b is a simplicial map. We shorten the notation $l_i(w)$ as l_i, and similarly for other

functions on w. Note that, for $0 \leqslant i \leqslant q$,

$$
\begin{aligned}
d_i^\delta(\alpha_b(y, w)) &= d_i^\delta(\delta_{(j_t)}^{-l_t+1} \cdots \delta_{(j_1)}^{-l_1+1} \delta_{(i_1)} \cdots \delta_{(i_s)}(y), w) \\
&= (\delta_{(i')}(d_i \delta_{(j_t)}^{-l_t+1} \cdots \delta_{(j_1)}^{-l_1+1} \delta_{(i_1)} \cdots \delta_{(i_s)}(y)), d_i(w)) \\
&= (\delta_{(i')} \delta_{(j_t)}^{-l_t+1} \cdots \delta_{(j_1)}^{-l_1+1} \delta_{(i_1)} \cdots \delta_{(i_s)}(d_i(y)), d_i(w)),
\end{aligned}
$$

where (i') is the i-th vertex of the q-simplex w. Let k be the minimal integer such that

$$
l_1 + \cdots + l_k - 1 \geqslant i.
$$

Then $i' = j_k$ and

$$
d_i^\delta(\alpha_b(y, w)) = (\delta_{(j_t)}^{-l_t+1} \cdots \delta_{j_k}^{-l_k+2} \cdots \delta_{(j_1)}^{-l_1+1} \delta_{(i_1)} \cdots \delta_{(i_s)}(d_i(y)), d_i(w)).
$$

Note that

$$
d_i w = (\overbrace{j_1(w), \ldots, j_1(w)}^{l_1(w)}, \ldots, \overbrace{j_k(w), \ldots, j_k(w)}^{l_k(w)-1}, \ldots, \overbrace{j_{t_w}(w), \ldots, j_{t_w}(w)}^{l_{t_w}(w)}).
$$

From the definition of α_b, we have

$$
\alpha_b(d_i(y), d_i(w)) = d_i^\delta(\alpha_b(y, w)).
$$

Consider the degeneracy operation s_i, $0 \leqslant i \leqslant q$. We have

$$
s_i w = (\overbrace{j_1(w), \ldots, j_1(w)}^{l_1(w)}, \ldots, \overbrace{j_k(w), \ldots, j_k(w)}^{l_k(w)+1}, \ldots, \overbrace{j_{t_w}(w), \ldots, j_{t_w}(w)}^{l_{t_w}(w)}).
$$

Then

$$
\begin{aligned}
s_i^\delta(\alpha_b(y, w)) &= s_i^\delta(\delta_{(j_t)}^{-l_t+1} \cdots \delta_{(j_1)}^{-l_1+1} \delta_{(i_1)} \cdots \delta_{(i_s)}(y), w) \\
&= (\delta_{(j_k)}^{-1}(s_i \delta_{(j_t)}^{-l_t+1} \cdots \delta_{(j_1)}^{-l_1+1} \delta_{(i_1)} \cdots \delta_{(i_s)}(y)), s_i(w)) \\
&= (\delta_{(j_k)}^{-1} \delta_{(j_t)}^{-l_t+1} \cdots \delta_{(j_1)}^{-l_1+1} \delta_{(i_1)} \cdots \delta_{(i_s)}(s_i(y)), s_i(w)) \\
&= (\delta_{(j_t)}^{-l_t+1} \cdots \delta_{(j_k)}^{-l_k} \cdots \delta_{(j_1)}^{-l_1+1} \delta_{(i_1)} \cdots \delta_{(i_s)}(s_i(y)), s_i(w)) \\
&= \alpha_b(s_i(y), s_i(w)).
\end{aligned}
$$
$\square$

4.2. The δ-twisted smash products

Now we consider the pointed constructions. Let Y be a pointed simplicial set with the base-point $* = \{s_0^n *\}_{n \geqslant 0}$. A *reduced twisted structure* on A in Y is a function

$$
\delta \colon V(K) \longrightarrow \mathrm{End}_*(Y), \quad v \mapsto \delta_v,
$$

such that the commuting rule (7) holds, where $\mathrm{End}_*(Y)$ is the monoid of base-point-preserving self-simplicial maps of Y. A reduced twisted structure on A is called *regular* if $\delta_v \colon Y \to Y$ is a simplicial isomorphism for each $v \in V(A)$.

Suppose that $S = S(A)$ has regular reduced twisted structure δ in a pointed simplicial set Y. Since $\delta_v(*) = *$ for $v \in V(A)$ there is a canonical inclusion

$$
* \times S = * \times_\delta S \hookrightarrow Y \times_\delta S.
$$

Let a_0 be a vertex of A treated as the base-point with the induced δ-structure by the

restriction. Then there is a canonical simplicial inclusion

$$Y \times_\delta a_0 \longrightarrow Y \times_\delta S.$$

The δ-twisted smash product $Y \wedge_\delta S$ is then defined as the simplicial quotient set

$$Y \wedge_\delta S := (Y \times_\delta S)/((Y \times_\delta a_0) \cup (* \times_\delta S)). \tag{11}$$

4.3. The classifying spaces of the twisted simplicial groups

Recall that there is a functor $\bar{W}$ from simplicial groups to simplicial sets, which plays the role of the classifying space functor in simplicial theory. The construction of the functor $\bar{W}$ is briefly reviewed using the terminology of categorical digraphs as follows.

Let G be a group (without assuming simplicial structure). Let $\mathcal{G}$ be the category with a single vertex a and a collection of arrows labeled by the elements $g \in G$, where the identity $1 = e_a \colon a \to a$ is considered as the trivial path of the vertex a. Then $\mathcal{G}$ is a categorical digraph with the composition operation induced by the multiplication of the group G. Let $W(G) = S(\mathcal{G})$ defined in Section 2.

Now let $G = \{G_n\}_{n \geqslant 0}$ be a simplicial group. The face homomorphisms $d_i \colon G_n \to G_{n-1}$ and the degeneracy homomorphisms $s_i \colon G_n \to G_{n+1}$ induce simplicial face maps $W(d_i) \colon W(G_n) \to W(G_{n-1})$ and simplicial degeneracy maps $W(s_i) \colon W(G_n) \to W(G_{n+1})$ for $0 \leqslant i \leqslant n$ so that $\{W(G_n)\}_{n \geqslant 0}$ is a bi-simplicial set. We recall that a bi-simplicial set is a simplicial object in the category of simplicial sets [**14**]. The simplicial set $\bar{W}(G)$ is then defined as the diagonal simplicial set of the bi-simplicial set $W(G_*)_*$.

Theorem 4.5. *Let $S = S(A)$ have a regular twisted structure in a simplicial group G. Then there is a natural simplicial map*

$$\theta \colon \bar{W}(G) \wedge_\delta S \longrightarrow \bar{W}(F_\delta^G[S]),$$

which is a homotopy equivalence after geometric realization, where the twisted structure of A in the simplicial set $\bar{W}(G)$ is induced from its twisted structure in G through the functor $\bar{W}$.

Proof. Let v be a vertex of A. By applying the functor $\bar{W}$ to the simplicial isomorphism $\delta_v \colon G \to G$, we have a sequence of isomorphisms of pointed simplicial sets

$$\bar{W}(\delta_v^n) \colon \bar{W}(G_n) \longrightarrow \bar{W}(G_n)$$

and the commutative diagram

$$
\begin{array}{ccccc}
\bar{W}(G_{n-1}) & \xleftarrow{\bar{W}(d_i)} & \bar{W}(G_n) & \xrightarrow{\bar{W}(s_i)} & \bar{W}(G_{n+1}) \\
\downarrow{\scriptstyle \bar{W}(\delta_v^{n-1})} & & \downarrow{\scriptstyle \bar{W}(\delta_v^n)} & & \downarrow{\scriptstyle \bar{W}(\delta_v^{n+1})} \\
\bar{W}(G_{n-1}) & \xleftarrow{\bar{W}(d_i)} & \bar{W}(G_n) & \xrightarrow{\bar{W}(s_i)} & \bar{W}(G_{n+1}),
\end{array}
$$

for $0 \leqslant i \leqslant n$ and $v \in V(A)$.

Recall from the definition that the group $(F_\delta^G[S])_n = F^G[S]_n$ is the quotient group of the free product $*_{x \in S_n}(G_n)_x$ subject to the relations $(G_n)_{a_0^n} = \{1\}$. For $x \in S_n$,

the inclusion $(G_n)_x \hookrightarrow F^G[S]_n$ induces a simplicial map

$$j_x \colon \bar{W}((G_n)_x) \longrightarrow \bar{W}((F_\delta^G[S])_n)$$

and so a simplicial map

$$\tilde{\theta}_n \colon \bar{W}(G_n) \times S_n \longrightarrow \bar{W}((F_\delta^G[S])_n) \quad (y, x) \mapsto j_x(y),$$

where we consider S_n as a discrete simplicial set with $(S_n)_q = S_n$ for $q \geqslant 0$ and faces and degeneracies being given by the identity map. Since $(G_n)_{a_0^n} = \{1\}$, we have

$$\tilde{\theta}_n(\bar{W}(G_n) \times \{a_0^n\}) = *.$$

Since $j_x(*) = *$ for any $x \in S_n$, we have

$$\tilde{\theta}_n(* \times S_n) = *$$

and so $\tilde{\theta}$ factors through the smash product. Let

$$\theta_n \colon \bar{W}(G_n) \wedge S_n \longrightarrow \bar{W}((F_\delta^G[S])_n)$$

be the resulting simplicial map. Now we let n be varied. Let

$$x = v_0 \alpha_1 v_1 \cdots \alpha_n v_n \in S_n = (S_n)_q \text{ and}$$

$$y = a g_1 a g_2 a \cdots g_q a \in \bar{W}(G_n)_q,$$

with $g_1, \ldots, g_q \in G_n$. Then

$$\theta_n(y \wedge x) = a(g_1)_x a(g_2)_x a \cdots (g_q)_x a.$$

Under the twisted faces in the simplicial group $F_\delta^G[S]$, we have

$$
\begin{aligned}
\bar{W}(d_i^\delta)(\theta_n(y \wedge x)) &= \bar{W}(d_i^\delta)(a(g_1)_x a(g_2)_x a \cdots (g_q)_x a \\
&= a(d_i^\delta((g_1)_x))a(d_i^\delta((g_2)_x))a \cdots (d_i^\delta((g_q)_x))a \\
&= a((\delta_{v_i}^{n-1}(d_i(g_1)))_{d_i x})a \cdots ((\delta_{v_i}^{n-1}(d_i(g_q)))_{d_i x})a \\
&= \theta_{n-1}((\bar{W}(\delta_{v_i}^{n-1})(d_i(y))) \wedge d_i(x)),
\end{aligned}
$$

for $0 \leqslant i \leqslant n$. Similarly, under the twisted faces in the simplicial group $F_\delta^G[S]$, we have

$$\bar{W}(s_i^\delta)(\theta_n(y \wedge x)) = \theta_{n+1}((\bar{W}((\delta_{v_i}^{n+1})^{-1})(s_i(y))) \wedge s_i(x)).$$

We define a δ-twisted structure on the sequence of simplicial sets

$$\{\bar{W}(G_n) \wedge S_n\}_{n \geqslant 0},$$

by the formulae

$$d_i^\delta(y \wedge x) = (\bar{W}(\delta_{v_i}^{n-1})(d_i(y))) \wedge d_i(x),$$

$$s_i^\delta(y \wedge x) = (\bar{W}((\delta_{v_i}^{n+1})^{-1})(s_i(y))) \wedge s_i(x),$$

for $0 \leqslant i \leqslant n$, $x = v_0 \alpha_1 v_1 \cdots \alpha_n v_n \in S_n = (S_n)_q$ and $y \in \bar{W}(G_n)_q$. Since each θ_n is injective and $\bar{W}((F_\delta^G[S])_*)_*$ is a bi-simplicial set, the simplicial identities hold for d_i^δ and s_i^δ in $T_{*,*} = \{\bar{W}(G_n) \wedge S_n\}_{n \geqslant 0}$ so that $T_{*,*}$ is a bi-simplicial set with

$$\theta_* \colon T_{*,*} \longrightarrow \bar{W}((F_\delta^G[S])_*)_*$$

being a morphism of bi-simplicial sets.

Recall that, by the Whitehead Theorem [**18**, Proposition 4.3], the canonical inclusion

$$\bar{W}(G') \vee \bar{W}(G'') \to \bar{W}(G' * G'')$$

is a homotopy equivalence after geometric realization for any simplicial groups G' and G''. It follows that

$$\theta_n \colon \bar{W}(G_n) \wedge S_n \longrightarrow \bar{W}((F_\delta^G[S])_n)$$

is a homotopy equivalence after geometric realization for $n \geqslant 0$. According to [**4**], the geometric realization

$$|\theta_*| \colon |T_{*,*}| \longrightarrow |\bar{W}((F_\delta^G[S])_*)_*|$$

is a homotopy equivalence. Moreover, by [**4**], the geometric realization of the diagonal simplicial set associated to a bi-simplicial set $X_{*,*}$ is homotopy equivalent to $|X_{*,*}|$. From the definition of the functor $\bar{W}$, the simplicial set $\bar{W}(F_\delta^G[S])$ is given by the diagonal simplicial set of the bi-simplicial set $\bar{W}((F_\delta^G[S])_*)_*$. By the definition of δ-twisted smash product, it is evident that the diagonal simplicial set of $T_{*,*}$ is $\bar{W}(G) \wedge_\delta S$. $\square$

Let $|X|$ denote the geometric realization of a simplicial set X. By Theorem 4.5, we have the following.

Corollary 4.6. *Let $S = S(A)$ have a regular twisted structure in a simplicial group G. Then the topological group $|F_\delta^G[S]|$ has the homotopy type of $\Omega|\bar{W}(G) \wedge_\delta S|$.*

If a twisted structure is trivial then the realization of our construction gives the same homotopy type as the Carlsson' $J_G(X)$-construction.

Corollary 4.7. *Let $S = S(A)$ with twisted structure in a simplicial group G given by $\delta_v \equiv \mathrm{id}_G$ for $v \in V(A)$. Then the topological group $|F_\delta^G[S]|$ has the homotopy type of $\Omega|B(G) \wedge S|$. If in this case $G = \mathbb{Z}$, the we have Milnor's $F[K]$ construction.*

Proof. The assertion follows from Corollary 4.6 and the definition of δ-twisted smash product. $\square$

Example. Let G be a simplicial group $\mathbb{Z}$ concentrated on dimension 0. Let S be the simplicial circle S^1 given as the boundary of a 2-simplex. Let δ be involution for every vertex. Then the δ-twisted Cartesian product $\bar{W}(G) \times_\delta S^1$ is the Klein bottle. From (11), there is a cofibre sequence

$$S^1 \vee S^1 \longrightarrow \bar{W}(G) \times_\delta S^1 \longrightarrow \bar{W}(G) \wedge_\delta S^1$$

and so $\bar{W}(G) \wedge_\delta S^1 \simeq S^2$. Thus $F_\delta^G[S^1]$ is a simplicial group model for ΩS^2.

5. Twisted homology of simplicial complexes and categories

5.1. Remarks on twisted homology

In this subsection, we give some remarks on the twisted Δ-groups $F_\delta^{G,\Delta}[S]$, where $S = S(A)$ with a twisted structure δ in a simplicial group G. The construction $F_\delta^{G,\Delta}[S]$ seems to be general, which only requires the commuting rule (1), and so one has the

homology groups from the chain complexes given by $\mathbb{Z}(F_\delta^{G,\Delta}[S])$ and $(F_\delta^{G,\Delta}[S])^{\mathrm{ab}}$ (as well as $\mathbb{Z}(\bar{W}(G) \times_{\delta,\Delta} S)$ in Section 4.1) for any twisted structure. On the other hand, the Δ-groups $F_\delta^{G,\Delta}[S]$ could be very wild in general. For instance, if δ_v is the trivial endomorphism of G for any vertex v, all twisted face operations in $F_\delta^{G,\Delta}[S]$ become trivial, which concludes that the differentials in $\mathbb{Z}(F_\delta^{G,\Delta}[S])$, $(F_\delta^{G,\Delta}[S])^{\mathrm{ab}}$ and $\mathbb{Z}(\bar{W}(G) \times_{\delta,\Delta} S)$ are zero maps and so their homology is given by the chains themselves. This indicates that the homology groups from the chain complexes given by $\mathbb{Z}(F_\delta^{G,\Delta}[S])$, $(F_\delta^{G,\Delta}[S])^{\mathrm{ab}}$ and $\mathbb{Z}(\bar{W}(G) \times_{\delta,\Delta} S)$ may not be homotopy invariants for general twisted structures δ. If the twisted structure is regular, the homology groups from the chain complexes given by $\mathbb{Z}(F_\delta^{G,\Delta}[S])$ and $\mathbb{Z}(\bar{W}(G) \times_{\delta,\Delta} S)$ coincide with the homology groups from the chain complexes given by the simplicial abelian groups $\mathbb{Z}(F_\delta^{G}[S])$ and $\mathbb{Z}(\bar{W}(G) \times_\delta S)$ by [**8**, Section 5] and so Theorems 4.4 and 4.5 assure that these homology groups only depend on the homotopy type of the corresponding fibre bundles.

Some simplicial techniques can be used for understanding the twisted Δ-groups $F_\delta^{G,\Delta}[S]$.

Proposition 5.1. *Let $S = S(K)$ for a simplicial complex K with a twisted structure in a simplicial group G. Let K_1 and K_2 be simplicial sub-complexes of K such that $K_1 \cup K_2 = K$ and $K_1 \cap K_2 \neq \emptyset$. Let δ_{K_i} be the restriction of the twisted structure δ_K on K_i. We choose the basepoint in $K_1 \cap K_2$. Then*

1. *The Δ-group*

$$F_\delta^{G,\Delta}[S(K)] = F_\delta^{G,\Delta}[S(K_1)] *_{F_\delta^{G,\Delta}[S(K_1 \cap K_2)]} F_\delta^{G,\Delta}[S(K_2)]$$

   is the free product with amalgamation.

2. *Suppose that G is a simplicial abelian group. Then there is a short exact sequence of the chain complexes*

$$F_\delta^{G,\Delta}[S(K_1 \cap K_2)]^{\mathrm{ab}} \hookrightarrow F_\delta^{G,\Delta}[S(K_1)]^{\mathrm{ab}} \oplus F_\delta^{G,\Delta}[S(K_1)]^{\mathrm{ab}} \twoheadrightarrow F_\delta^{G,\Delta}[S(K)]^{\mathrm{ab}}.$$

Proof. Assertion (2) follows from assertion (1) by taking abelianization. For proving assertion (1), we check that $S(K) = S(K_1) \cup_{S(K_1 \cap K_2)} S(K_2)$. By the definition, the elements in $S(K)_n$ are given by $(v_0, \ldots, v_n)$ with $v_0 \leqslant v_1 \leqslant \cdots \leqslant v_n$ such that $\{v_0, \ldots, v_n\}$ form a simplex in K. Since $K = K_1 \cup K_2$, the simplex $\{v_0, \ldots, v_n\}$ is either in K_1 or K_2. This shows that $S(K) = S(K_1) \cup_{S(K_1 \cap K_2)} S(K_2)$. Assertion (1) then follows from our construction. $\square$

The second assertion in the above proposition assures that the Mayer-Vietoris sequence can be applied in the homology theory given by $H_*(F_\delta^{G,\Delta}[S(K)]^{\mathrm{ab}})$ for twisted simplicial complexes. Similar results hold for the case $S(\mathcal{C})$ for category $\mathcal{C}$ with twisted structure.

Now let us consider the special case of the twisted Δ-groups of the cones of simplicial complexes and categories. For a simplicial complex K, the cone $CK = a * K$ is defined in the usual way. For a category $\mathcal{C}$, the cone $C\mathcal{C} = a * \mathcal{C}$ is the category obtained by adding an initial vertex a with assigning a unique arrow $e_{a,v} \colon a \to v$ for each $v \in V(\mathcal{C})$ and the identical arrow $\mathrm{id}_a \colon a \to a$. Suppose that $\delta \colon V(K) \to \mathrm{End}(G)$ or $V(\mathcal{C}) \to \mathrm{End}(G)$ is a twisted structure in a simplicial group G. A twisted structure δ_{CK} (or $\delta_{C\mathcal{C}}$) is called a *regular extension* of the twisted structure δ_K (or $\delta_\mathcal{C}$) if

1. $\delta_{CK}|_{V(K)} = \delta_K$ (or $\delta_{CC}|_{V(C)} = \delta_C$) and
2. $\delta(a) \in \mathrm{Aut}(G)$.

We recall that, since there is an edge or arrow between a and any other vertex, the commuting rule (1) of the δ-structure forces that $\delta_a \circ \delta_v = \delta_v \circ \delta_a$ for any vertex v.

Recall also that any group G can be considered as a discrete simplicial group with $G_n = G$ and faces and degeneracies being the identity map.

Proposition 5.2. *Let $S = S(a * A)$ where A is a simplicial complex K or a category C with a twisted structure given as a regular extension of a twisted structure on A in a group G. Then the following chain complexes*

$$\mathbb{Z}(F_\delta^{G,\Delta}[S]), \quad F_\delta^{G,\Delta}[S]^{\mathrm{ab}}$$

are contractible.

Proof. Let

$$\Phi \colon (F_\delta^{G,\Delta}[S])_n = F^G[S]_n \longrightarrow (F_\delta^{G,\Delta}[S])_{n+1} = F^G[S]_{n+1}$$

be the (unique) group homomorphism such that

$$\Phi|_{(G_n)_{v_0\alpha_1 v_1 \cdots \alpha_n v_n}}(g_{v_0\alpha_1 v_1 \cdots \alpha_n v_n}) = (\delta_a)^{-1}(g)_{ae_{a,v_0} v_0\alpha_1 v_1 \cdots \alpha_n v_n}.$$

Let us compute $d_i^\delta \circ \Phi$. We shorten the notation d_i^δ as d_i.

$$
\begin{aligned}
d_0\Phi(g_{v_0\alpha_1 v_1 \cdots \alpha_n v_n}) &= d_0((\delta_a)^{-1}(g)_{ae_{a,v_0} v_0\alpha_1 v_1 \cdots \alpha_n v_n}) \\
&= (\delta_a(d_0((\delta_a)^{-1}(g))))_{v_0\alpha_1 v_1 \cdots \alpha_n v_n} \\
&= (\delta_a((\delta_a)^{-1}(g)))_{v_0\alpha_1 v_1 \cdots \alpha_n v_n} \text{ because } G \text{ is discrete} \\
&= g_{v_0\alpha_1 v_1 \cdots \alpha_n v_n}.
\end{aligned}
$$

Thus

$$d_0\Phi = \mathrm{id}.$$

Now let $i > 0$. Then

$$
\begin{aligned}
d_i\Phi(g_{v_0\alpha_1 v_1 \cdots \alpha_n v_n}) &= d_i((\delta_a)^{-1}(g)_{ae_{a,v_0} v_0\alpha_1 v_1 \cdots \alpha_n v_n}) \\
&= (\delta_{v_{i-1}}(d_i(\delta_a)^{-1}(g)))_{ae_{a,v_0} v_0\alpha_1 v_1 \cdots v_{i-2}\alpha_{i-1}\alpha_i v_i \cdots \alpha_n v_n} \\
&= ((\delta_a)^{-1}\delta_{v_{i-1}}g)_{ae_{a,v_0} v_0\alpha_1 v_1 \cdots v_{i-2}\alpha_{i-1}\alpha_i v_i \cdots \alpha_n v_n} \\
&\qquad \text{because } G \text{ is discrete} \\
&= \Phi d_{i-1}(g_{v_0\alpha_1 v_1 \cdots \alpha_n v_n}).
\end{aligned}
$$

Thus we have the following identity:

$$d_i\Phi = \Phi d_{i-1}.$$

It follows that, in the chain complex $\mathbb{Z}(F_\delta^{G,\Delta}[S])$,

$$
\begin{aligned}
\partial_{n+1} \circ \Phi &= \sum_{i=0}^{n+1}(-1)^i d_i\Phi \\
&= \mathrm{id} - \sum_{i=1}^{n+1}(-1)^{i-1}\Phi d_{i-1} \\
&= \mathrm{id} - \Phi \circ \partial_n.
\end{aligned}
$$

and so $\mathbb{Z}(F_\delta^{G,\Delta}[S])$ is contractible. By taking the abelianization, the homomorphisms Φ^{ab} defines a null homotopy for the chain complex $F^{G,\Delta}[S]^{\mathrm{ab}}$. $\qquad\square$

5.2. An application to twisted homology

In this subsection, we consider the abelianization $F_\delta^G[S]^{\mathrm{ab}}$ of the simplicial group $F_\delta^G[S]$, where $S = S(A)$ for A a simplicial complex K or a category $\mathcal{C}$ with a regular twisted structure in an abelian group G. The chain complex of $F_\delta^G[S]^{\mathrm{ab}}$ is specifically given in (5) and (6). A natural question is as follows:

Question 5.3. Does there exist a topological space whose ordinary homology gives the homology of the chain complex $F_\delta^G[S]^{\mathrm{ab}}$?

As an application of Theorems 4.5 and 4.4, we affirmatively answer the above question in some special case. Let $G = \mathbb{Z}^{\oplus k}$ be a free abelian group of rank k. Recall that a regular twisted structure of A with coefficients in $G = \mathbb{Z}^k$ is a function

$$\delta \colon V(A) \longrightarrow \mathrm{Aut}(G) = \mathrm{GL}_k(\mathbb{Z})$$

satisfying the commuting rule (1). Let F_k be a free group of rank k, and let $q \colon \mathrm{Aut}(F_k) \to \mathrm{GL}_k(\mathbb{Z})$ be the canonical homomorphism by sending an automorphism of F_k to the corresponding automorphism of $\mathbb{Z}^k = F_k^{\mathrm{ab}}$. We say that a twisted structure δ of A with coefficients in $G = \mathbb{Z}^{\oplus k}$ is *normal* if δ is regular and there exists a regular twisted structure $\tilde{\delta}$ of A with coefficients in the free group F_k such that the following diagram commutes

$$
\begin{array}{ccc}
 & & \mathrm{Aut}(F_k) \\
 & \overset{\tilde{\delta}}{\nearrow} & \big\downarrow {\scriptstyle q} \\
V(A) & \xrightarrow{\ \delta\ } & \mathrm{Aut}(G) = \mathrm{GL}_k(\mathbb{Z}).
\end{array}
$$

In such a case, we call $\tilde{\delta}$ a *lifting* of the twisted structure δ in the free group F_k.

Lemma 5.4. *Let $\phi \colon G \to \Gamma$ be a simplicial homomorphism between simplicial groups such that G and Γ are free groups in each dimension. Suppose that ϕ is a homotopy equivalence. Then $\phi^{\mathrm{ab}} \colon G^{\mathrm{ab}} \to \Gamma^{\mathrm{ab}}$.*

Proof. For a simplicial group G, consider the bi-simplicial set bar construction $\bar{W}_{*,*}(G) = \{\bar{W}(G_n)\}_{n \geqslant 0}$ with $\Delta \bar{W}_{*,*}(G) = \bar{W}(G)$. The simplicial homomorphism ϕ induces a bi-simplicial map $\bar{W}_{*,*}(\phi) \colon \bar{W}_{*,*}(G) \to \bar{W}_{*,*}(\Gamma)$ with a homotopy equivalence

$$\Delta \bar{W}_{*,*}(\phi) \colon \Delta \bar{W}_{*,*}(G) \longrightarrow \Delta \bar{W}_{*,*}(\Gamma)$$

by assumption. Thus

$$\mathbb{Z}[\Delta \bar{W}_{*,*}(\phi)] \colon \Delta \mathbb{Z}[\bar{W}_{*,*}(G)] \longrightarrow \Delta \mathbb{Z}[\bar{W}_{*,*}(\Gamma)]$$

is a homotopy equivalence. By Quillen [20], there is a spectral sequence with E^2-terms given by $\pi_*(\{\pi_*(\mathbb{Z}[\bar{W}(G_n)])\}_{n \geqslant 0})$ converging to $\pi_*(\Delta \mathbb{Z}[\bar{W}_{*,*}(G)])$. Since each G_n is a free group, $\pi_k(\mathbb{Z}[\bar{W}(G_n)]) = 0$ for $k \neq 1$ and $\pi_1(\mathbb{Z}[\bar{W}(G_n)]) \cong G_n^{\mathrm{ab}}$ with a simplicial isomorphism $G^{\mathrm{ab}} \to \{\pi_1(\mathbb{Z}[\bar{W}(G_n)])\}_{n \geqslant 0}$. The homotopy equivalence property of $\Delta \bar{W}_{*,*}(\phi)$ implies that ϕ^{ab} is a homotopy equivalence. $\qquad\square$

For a reduced simplicial set X, let $G(X)$ denote Kan's G-construction [16], which is a simplicial free group whose geometric realization is homotopy equivalent to $\Omega|X|$.

Let Y be a pointed simplicial set with a basepoint $*$. Let $\mathbb{Z}[Y] = \mathbb{Z}(Y)/\mathbb{Z}(*)$ be the free abelian simplicial group generated by Y modulo the basepoint $*$. Recall [8] that $\pi_n(\mathbb{Z}[Y]) \cong \tilde{H}_n(Y;\mathbb{Z})$ for $n \geqslant 0$.

Theorem 5.5. *Let $S = S(A)$ have a normal twisted structure δ in a free abelian group $G = \mathbb{Z}^k$. Let $\tilde{\delta}$ be a lifting of δ in F_k. Then there is an isomorphism*

$$H_n(F_\delta^G[S]^{\mathrm{ab}}) \cong \tilde{H}_{n+1}(|\bar{W}(F_k) \wedge_{\tilde{\delta}} S|;\mathbb{Z}),$$

for $n \geqslant 0$.

Proof. Let $\theta \colon \bar{W}(F_k) \wedge_{\tilde{\delta}} S \longrightarrow \bar{W}F_{\tilde{\delta}}^{F_k}[S]$ be the simplicial map in 4.5. By [17, Theorem 11.1], the simplicial map θ induces a simplicial homomorphism $\phi \colon G(\bar{W}(F_k) \wedge_{\tilde{\delta}} S) \longrightarrow F_{\tilde{\delta}}^{F_k}[S]$. Moreover, ϕ is a homotopy equivalence because θ is a homotopy equivalence after geometric realization. Since $\tilde{\delta}$ is a lifting of δ, we have $F_{\tilde{\delta}}^{F_k}[S]^{\mathrm{ab}} = F_\delta^G[S]^{\mathrm{ab}}$. By Lemma 5.4,

$$\phi^{\mathrm{ab}} \colon G(\bar{W}(F_k) \wedge_{\tilde{\delta}} S)^{\mathrm{ab}} \longrightarrow F_{\tilde{\delta}}^{F_k}[S]^{\mathrm{ab}} = F_\delta^G[S]^{\mathrm{ab}}$$

is a homotopy equivalence. It follows that

$$\phi_*^{\mathrm{ab}} \colon \pi_*(G(\bar{W}(F_k) \wedge_{\tilde{\delta}} S)^{\mathrm{ab}}) \longrightarrow \pi_*(F_\delta^G[S]^{\mathrm{ab}}).$$

By [8, Formula (10.1a), p.186], we have

$$\pi_*(G(\bar{W}(F_k) \wedge_{\tilde{\delta}} S)^{\mathrm{ab}}) \cong \tilde{H}_{*+1}(W(F_k) \wedge_{\tilde{\delta}} S).$$

From [8, Section 5], the homology of the chain complex $F_\delta^G[S]^{\mathrm{ab}}$ coincides with the homotopy groups of the simplicial abelian groups $F_\delta^G[S]^{\mathrm{ab}}$. The proof is finished. $\square$

Remark 5.6. By Theorem 4.4, the coordinate projection map $\bar{W}(F_k) \times_{\tilde{\delta}} S \to S$ is a simplicial fibre bundle with fibre $\bar{W}(F_k)$. By Quillen's theorem [20], the geometric realization $|\bar{W}(F_k) \times_{\tilde{\delta}} S| \to |S|$ is a Serre fibration with fibre $|\bar{W}(F_k)|$. Since F_k is a free group of rank k, $|\bar{W}(F_k)|$ is homotopy equivalent to $\mathbb{C} \setminus Q_k$, where Q_k is a finite subset of $\mathbb{C}$ of cardinal k. In other words, the space $|\bar{W}(F_k) \times_{\tilde{\delta}} S|$ may be viewed up to homotopy as a quantization of $|S|$ in the sense of a punctured complex line fibration over $|S|$.

References

[1] H. Barcelo and R. Laubenbacher, Perspectives on A-homotopy theory and its applications. *Discrete Math.* **298** (2002), 39–61.

[2] H. Barcelo, X. Kramer, R. Laubenbacher, and C. Weaver, Foundations of a connectivity theory for simplicial complexes. *Adv. Appl. Math.* **26** (2001), 97–128.

[3] M.G. Barratt, V.K.A.M. Gugenheim, and J.C. Moore, On semisimplicial fibre-bundles. *Amer. J. Math.* **81** (1959), 639–657.

[4] A.K. Bousfield and E.M. Friedlander, Homotopy theory of Γ-spaces, spectra, and bisimplicial sets. *Geometric Applications of Homotopy Theory* (Proc. Conf., Evanston, Ill., 1977), II, pp. 80–130. Lecture Notes in Math., vol. 658, Springer, Berlin, 1978.

[5]   E.H. Brown, Jr, Twisted tensor products, I. *Ann. of Math.* **69** (1959), 223–246.

[6]   G. Carlsson, A simplicial group construction for balanced products. *Topology* **23** (1984), 85–89.

[7]   A. Cavicchioli and M. Meschiari, A homology theory for colored graphs. *Discrete Math.* **137** (1995), 99–136.

[8]   E.B. Curtis, Simplicial homotopy theory. *Adv. Math.* **6** (1971), 107–209.

[9]   B. Everitt and P. Turner, Homology of coloured posets: A generalisation of Khovanov's cube construction. *J. Algebra* **322** (2009), 429–448.

[10] A. Grigor'yan, Yu. Muranov, and S.-T. Yau, Graphs associated with simplicial complexes. *Homology Homotopy Appl.* **16** (2014), 295–311.

[11] A. Grigor'yan, Yu. Muranov, and S.-T. Yau, On a cohomology of digraphs and Hochschild cohomology. *J. Homotopy Relat. Struct.* **11** (2016), no. 2, 209–230.

[12] A. Grigor'yan, Y. Lin, Yu. Muranov, and S.-T. Yau, Homologies of path complexes and digraphs, preprint `arXiv:1207.2834`.

[13] A. Grigor'yan, Y. Lin, Yu. Muranov, and S.-T. Yau, Path complexes and their homologies, University of Bielefeld Preprint Number 13055.

[14] P.G. Goerss and J.F. Jardine, *Simplicial Homotopy Theory*. Birkhäuser Verlag, Basel, 2009. xvi+510 pp.

[15] I.M. James, Reduced product spaces. *Ann. of Math.* (2) **62** (1955), 170–197.

[16] D.M. Kan, A combinatorial definition of homotopy groups. *Ann. of Math.* (2) **67** (1958), 282–312.

[17] D.M. Kan, On homotopy theory and c.s.s. groups. *Ann. of Math.* (2) **68** (1958), 38–53.

[18] D.M. Kan and W.P. Thurston, Every connected space has the homology of a $K(\pi, 1)$. *Topology* **15** (1976), 253–258.

[19] J.W. Milnor, On the construction FK. In: J.F. Adams, *Algebraic Topology. Student's Guide*. Cambridge University Press, 1972.

[20] D.G. Quillen. Spectral sequences of a double semi-simplicial group. *Topology* **5** (1966), 155–157.

[21] D.G. Quillen, *Homotopical Algebra*. Lecture Notes in Math., vol. 43. Springer-Verlag, Berlin, New York, 1967. iv+156 pp.

[22] D.G. Quillen, The geometric realization of a Kan fibration is a Serre fibration. *Proc. Amer. Math. Soc.* **19** (1968), 1499–1500.

[23] R. Rohm, and E. Witten, The antisymmetric tensor field in superstring theory. *Ann. Phys.* **170** (1986), no. 2, 454–489.

[24] M.E. Talbi and D. Benayat, Homology theory of graphs. *Mediterr. J. Math.* **11** (2014), 813–828.

[25] P. Turner and E. Wagner, The homology of digraphs as a generalization of Hochschild homology. *J. Algebra Appl.* **11** (2012), 1250031 (13 pages).

[**26**]   E. Witten, Supersymmetry and Morse theory. *J. Differential Geom.* **17** (1982), 661–692.

[**27**]   J. Wu, Simplicial objects and homotopy groups. *Braids*, pp. 31–181. Lect. Notes Ser. Inst. Math. Sci. Natl. Univ. Singap., vol. 19. World Sci. Publ., Hackensack, NJ, 2010.

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