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ON THE DIMENSION OF CLASSIFYING SPACES FOR FAMILIES OF ABELIAN SUBGROUPS

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Abstract

We show that a finitely generated abelian group G of torsion-free rank $n \geqslant 1$ admits an $(n+r)$-dimensional model for $E_{\mathfrak{F}_r}G$, where $\mathfrak{F}_r$ is the family of subgroups of torsion-free rank less than or equal to $r \geqslant 0$.

1. Introduction

In this note we consider classifying spaces $E_{\mathfrak{F}}G$ for a family of subgroups $\mathfrak{F}$ of G. We are particularly interested in the minimal dimension, denoted $\operatorname{gd}_{\mathfrak{F}} G$, such a space can have.

Let G be a group. We say a collection of subgroups $\mathfrak{F}$ is a family if it is closed under conjugation and taking subgroups. A G-CW-complex X is said to be a classifying space $E_{\mathfrak{F}}G$ for the family $\mathfrak{F}$ if, for each subgroup $H \leqslant G$, $X^H \simeq \{*\}$ if $H \in \mathfrak{F}$, and $X^H = \varnothing$ otherwise.

The spaces $\underline{E}G = E_{\mathfrak{F}}G$ for $\mathfrak{F} = \mathfrak{F}in$ the family of finite subgroups and $\underline{\underline{E}}G = E_{\mathfrak{F}}G$ for $\mathfrak{F} = \mathcal{V}cyc$ the family of virtually cyclic subgroups have been widely studied for their connection with the Baum-Connes and Farrell-Jones conjectures respectively. For a first introduction into the subject see, for example, the survey [**3**].

We consider finitely generated abelian groups G of finite torsion-free rank $r_0(G) = n$ and families $\mathfrak{F}_r$ of subgroups of torsion-free rank less than or equal to $r < n$. Note that for $r = 0$, $\mathfrak{F}_0 = \mathfrak{F}in$ and that it is a well known fact, see, for example, [**3**], that $\mathbb{R}^n$ is a model for $\underline{E}G$ and that $\operatorname{gd}_{\mathfrak{F}_0} G = n$. For $r = 1$, $\mathfrak{F}_1 = \mathcal{V}cyc$ and it was shown in [**5**, Proposition 5.13(iii)] that $\operatorname{gd}_{\mathfrak{F}_1} G = n + 1$.

The main idea is to use the method developed by Lück and Weiermann [**5**] to build models of $E_{\mathfrak{F}_r}G$ from models for $E_{\mathfrak{F}_{r-1}}G$. We begin by recalling those results in [**5**] that we need for our construction. Let $\mathfrak{F}$ and $\mathfrak{G}$ be families of subgroups of a given group G such that $\mathfrak{F} \subseteq \mathfrak{G}$.

Definition 1.1. [**5**, (2.1)] Let $\mathfrak{F}$ and $\mathfrak{G}$ be families of subgroups of a given group G such that $\mathfrak{F} \subseteq \mathfrak{G}$. Let $\sim$ be an equivalence relation on $\mathfrak{G} \backslash \mathfrak{F}$ satisfying:

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- For $H, K \in \mathfrak{G}\backslash\mathfrak{F}$ with $H \leqslant K$ we have $H \sim K$.
- Let $H, K \in \mathfrak{G}\backslash\mathfrak{F}$ and $g \in G$, then $H \sim K \iff gHg^{-1} \sim gKg^{-1}$.

Such a relation is called a strong equivalence relation. Denote by $[\mathfrak{G}\backslash\mathfrak{F}]$ the equivalence classes of $\sim$ and define for all $[H] \in [\mathfrak{G}\backslash\mathfrak{F}]$ the following subgroup of G:

$$N_G[H] = \{g \in G \,|\, [gHg^{-1}] = [H]\}.$$

Now define a family of subgroups of $N_G[H]$ by

$$\mathfrak{G}[H] = \{K \leqslant N_G[H] \,|\, K \in \mathfrak{G}\backslash\mathfrak{F}\,,\ [K] = [H]\} \cup (\mathfrak{F} \cap N_G[H]).$$

Here $\mathfrak{F} \cap N_G[H]$ is the family of subgroups of $N_G[H]$ belonging to $\mathfrak{F}$.

Theorem 1.2. [**5**, Theorem 2.3] *Let $\mathfrak{F} \subseteq \mathfrak{G}$ and $\sim$ be as in Definition 1.1. Denote by I a complete set of representatives of the conjugacy classes in $[\mathfrak{G}\backslash\mathfrak{F}]$. Then the G-CW-complex given by the cellular G push-out*

$$\begin{array}{ccc} \sqcup_{[H]\in I} G \times_{N_G[H]} E_{\mathfrak{F}\cap N_G[H]}(N_G[H]) & \xrightarrow{\ i\ } & E_{\mathfrak{F}}(G) \\ {\scriptstyle \sqcup_{[H]\in I} id_G \times_{N_G[H]} f_{[H]}} \downarrow & & \downarrow \\ \sqcup_{[H]\in I} G \times_{N_G[H]} E_{\mathfrak{G}[H]}(N_G[H]) & \longrightarrow & X, \end{array}$$

where either i or the $f_{[H]}$ are inclusions, is a model for $E_{\mathfrak{G}}(G)$.

The condition on the two maps being inclusions is not that strong a restriction, as one can replace the spaces by the mapping cylinders, see [**5**, Remark 2.5]. Hence one has:

Corollary 1.3. [**5**, Remark 2.5] *Suppose there exists an n-dimensional model for $E_{\mathfrak{F}}G$ and, for each $H \in I$, an $(n-1)$-dimensional model for $E_{\mathfrak{F}\cap N_G[H]}(N_G[H])$ and an n-dimensional model for $E_{\mathfrak{G}[H]}(N_G[H])$. Then there is an n-dimensional model for $E_{\mathfrak{G}}G$.*

Corollary 1.3 gives us a tool to find an upper bound for $\mathrm{gd}_{\mathfrak{G}}\, G$. A very useful tool to find a lower bound for $\mathrm{gd}_{\mathfrak{G}}\, G$ is the following Mayer-Vietoris sequence [**4**], which is an immediate consequence of Theorem 1.2, see also [**1**, Proposition 7.1] for the Bredon-cohomology version.

Corollary 1.4. *With the notation as in Theorem 1.2 we have following long exact cohomology sequence:*

$$\cdots \to H^i(G\backslash E_{\mathfrak{G}}G) \to \Big(\prod_{[H]\in I} H^i(N_G[H]\backslash E_{\mathfrak{G}[H]}N_G[H])\Big) \oplus H^i(G\backslash E_{\mathfrak{F}}G) \to$$
$$\prod_{[H]\in I} H^i(N_G[H]\backslash E_{\mathfrak{F}\cap N_G[H]}N_G[H]) \to H^{i+1}(G\backslash E_{\mathfrak{G}}G) \to \cdots.$$

This note will be devoted to proving the following Theorem:

Main Theorem. *Let G be a finitely generated abelian group of finite torsion-free rank $n \geqslant 1$, and denote by $\mathfrak{F}_r$ the family of subgroups of torsion-free rank less than or equal to $r \geqslant 0$. Then*

$$\mathrm{gd}_{\mathfrak{F}_r}\, G \leqslant n + r.$$

The case for more general classes of groups G is going to be dealt with, using different methods, by the second author in his Ph.D. thesis.

2. The construction

Throughout, let G denote a finitely generated abelian group of torsion-free rank $r_0(G) = n$.

The idea is to construct models for $E_{\mathfrak{F}_r}G$ in terms of models for $E_{\mathfrak{F}_{r-1}}G$ using the push-out of Theorem 1.2 inductively. As a first step we shall define an equivalence relation in the sense of Definition 1.1.

Lemma 2.1. *Let $\sim$ denote the following relation on $\mathfrak{F}_r \backslash \mathfrak{F}_{r-1}$:*

$$H \sim K \iff rk(H \cap K) = r.$$

Then $\sim$ is a strong equivalence relation.

Proof. We show that $\sim$ is transitive: If $H \sim K$ and $K \sim L$, this implies that both $H \cap K$ and $K \cap L$ are finite index subgroups of K. Hence also $H \cap K \cap L$ is a finite index subgroup of K, and, in particular, of $K \cap L$ and thus of L. Hence $H \cap L$ is finite index in both H and L. The rest is easily checked. □

Definition 2.2. We say a subgroup M of G is maximal if it is not properly contained in a subgroup of G of the same torsion-free rank as M.

Lemma 2.3. *G satisfies $(M_{\mathfrak{F}_{r-1} \subseteq \mathfrak{F}_r})$, i.e. every subgroup $H \in \mathfrak{F}_r \backslash \mathfrak{F}_{r-1}$ is contained in a unique $H_{max} \in \mathfrak{F}_r \backslash \mathfrak{F}_{r-1}$, which is maximal.*

Proof. The existence follows from [**6**]. As regards uniqueness, suppose H is included in two different maximal elements $K, L \in \mathfrak{F}_r \backslash \mathfrak{F}_{r-1}$: then $H \leqslant KL$. Note that, since $H \sim L$ and $H \leqslant L$, it follows that $|L : H| < \infty$. Hence

$$|KL : K| = |L : K \cap L| \leqslant |L : H| < \infty$$

implies $KL \in \mathfrak{F}_r \backslash \mathfrak{F}_{r-1}$, contrary to the maximality of K and L. □

Note that we always have maximal elements in $\mathfrak{F}_r \backslash \mathfrak{F}_{r-1}$ as long as the ambient group is polycyclic [**6**], but uniqueness already fails for the Klein-bottle group K, which is non-abelian but contains a free abelian subgroup of rank 2 as an index 2 subgroup. Denote

$$K = \langle a, b \,|\, aba^{-1} = b^{-1} \rangle$$

and consider $\mathfrak{F}_1$ the family of cyclic subgroups. Since $a^2 = (ab^{-1})^2$, in follows that $\langle a^2 \rangle \leqslant \langle ab^{-1} \rangle$ as well as $\langle a^2 \rangle \leqslant \langle a \rangle$, both of which are maximal.

For $M \leqslant G$ a subgroup of G we denote by $\mathfrak{All}(M)$ the family of all subgroups of M.

Lemma 2.4. *Let M be a maximal subgroup of G of torsion-free rank r. Then $\mathbb{R}^{n-r}$ is a model for $E_{\mathfrak{All}(M)}G$, and $\mathrm{gd}_{\mathfrak{All}(M)}\, G = n - r$.*

Proof. Since M is maximal it follows that G/M is torsion-free of rank $n - r$ and hence $\mathbb{R}^{n-r}$ is a model for $E(G/M)$. The action of G given by the projection $G \twoheadrightarrow G/M$ now yields the claim. □

Lemma 2.5. *Let $\mathfrak{F}$ and $\mathfrak{G}$ be two families of subgroups of G. Then*

$$\mathrm{gd}_{\mathfrak{F}\cup\mathfrak{G}}\, G \leqslant \max\{\mathrm{gd}_{\mathfrak{F}}\, G, \mathrm{gd}_{\mathfrak{G}}\, G, \mathrm{gd}_{\mathfrak{F}\cap\mathfrak{G}}\, G+1\}.$$

Proof. By the universal property of classifying spaces for families, there are maps, unique up to G-homotopy, $E_{\mathfrak{F}\cap\mathfrak{G}}G \to E_{\mathfrak{G}}G$ and $E_{\mathfrak{F}\cap\mathfrak{G}}G \to E_{\mathfrak{F}}G$. Now the double mapping cylinder yields a model for $E_{\mathfrak{F}\cup\mathfrak{G}}G$ of the desired dimension. □

Lemma 2.6. *Given $r < n$, suppose there exists a $d \geqslant n$ such that $\mathrm{gd}_{\mathfrak{F}_{r-1}}\, G \leqslant d$ and that for all maximal subgroups N with $r_0(N) > r-1$ we also have $\mathrm{gd}_{\mathfrak{F}_{r-1}\cap\mathfrak{All}(N)}\, G \leqslant d$. Then*

$$\mathrm{gd}_{\mathfrak{F}_r}\, G \leqslant d+1 \quad \text{and} \quad \mathrm{gd}_{\mathfrak{F}_r\cap\mathfrak{All}(M)}\, G \leqslant d+1,$$

for all maximal subgroups M of $r_0(M) > r$.

Proof. We begin by applying Theorem 1.2 to the families $\mathfrak{G} = \mathfrak{F}_r$ and $\mathfrak{F} = \mathfrak{F}_{r-1}$. Lemma 2.3 implies that G satisfies $(M_{\mathfrak{F}_{r-1}\subseteq\mathfrak{F}_r})$. Denote by $\mathcal{N}$ the set of equivalence classes of maximal elements in $\mathfrak{F}_r\backslash\mathfrak{F}_{r-1}$. Then [**5**, Corollary 2.8] gives a push-out:

$$\begin{array}{ccc} \sqcup_{N\in\mathcal{N}} E_{\mathfrak{F}_{r-1}}(G) & \longrightarrow & E_{\mathfrak{F}_{r-1}}(G) \\ \downarrow & & \downarrow \\ \sqcup_{N\in\mathcal{N}} E_{\mathfrak{F}_{r-1}\cup\mathfrak{All}(N)}(G) & \longrightarrow & Y, \end{array}$$

and Y is a model for $E_{\mathfrak{F}_r}G$.

By assumption we have that $\mathrm{gd}_{\mathfrak{F}_{r-1}}\, G \leqslant d$ and $\mathrm{gd}_{\mathfrak{F}_{r-1}\cap\mathfrak{All}(N)}\, G \leqslant d$ for all $N \in \mathcal{N}$. Furthermore, by Lemma 2.4 we have that $\mathrm{gd}_{\mathfrak{All}(N)}\, G = n - r_0(N) < n$. Lemma 2.5 now implies that $\mathrm{gd}_{\mathfrak{F}_{r-1}\cup\mathfrak{All}(N)}\, G \leqslant d+1$. Applying Corollary 1.3 to the above push-out yields

$$\mathrm{gd}_{\mathfrak{F}_r}\, G \leqslant d+1.$$

The second claim is proved similarly applying Theorem 1.2 to the families $\mathfrak{G} = \mathfrak{F}_r \cap \mathfrak{All}(M)$ and $\mathfrak{F} = \mathfrak{F}_{r-1} \cap \mathfrak{All}(M)$. The argument of Lemma 2.3 applies here as well and hence G satisfies $(M_{(\mathfrak{F}_{r-1}\cap\mathfrak{All}(M))\subseteq(\mathfrak{F}_r\cap\mathfrak{All}(M))})$. We denote by $\mathcal{N}(M)$ the set of equivalence classes of maximal elements in $\mathfrak{F}_r \cap \mathfrak{All}(M)\backslash\mathfrak{F}_{r-1} \cap \mathfrak{All}(M)$. this now gives us a push-out:

$$\begin{array}{ccc} \sqcup_{N\in\mathcal{N}(M)} E_{\mathfrak{F}_{r-1}\cap\mathfrak{All}(M)}(G) & \longrightarrow & E_{\mathfrak{F}_{r-1}\cap\mathfrak{All}(M)}(G) \\ \downarrow & & \downarrow \\ \sqcup_{N\in\mathcal{N}(M)} E_{(\mathfrak{F}_{r-1}\cap\mathfrak{All}(M))\cup\mathfrak{All}(N)}(G) & \longrightarrow & Z, \end{array}$$

and Z is a model for $E_{\mathfrak{F}_r\cap\mathfrak{All}(M)}G$.

Since $N \leqslant M$, it follows that $(\mathfrak{F}_{r-1} \cap \mathfrak{All}(M)) \cap \mathfrak{All}(N) = \mathfrak{F}_{r-1} \cap \mathfrak{All}(N)$ and hence, by assumption $\mathrm{gd}_{(\mathfrak{F}_{r-1}\cap\mathfrak{All}(M))\cap\mathfrak{All}(N)}\, G \leqslant d$ and Lemma 2.5 implies that $\mathrm{gd}_{(\mathfrak{F}_{r-1}\cap\mathfrak{All}(M))\cup\mathfrak{All}(N)}\, G \leqslant d+1$. Now the same argument as above applies and

$$\mathrm{gd}_{\mathfrak{F}_r\cap\mathfrak{All}(M)}\, G \leqslant d+1.$$ □

Proof of Main Theorem: We begin by noting that for $r = 0$ we have that $\mathfrak{F}_r = \mathfrak{F}_0$ is the family of all finite subgroups of G. Then for all maximal subgroups M of rank 1, we have that $\mathfrak{F}_0 = \mathfrak{F}_0 \cap \mathfrak{All}(M)$. Furthermore, it is well known that $\mathrm{gd}_{\mathfrak{F}_0} G = n$, see, for example, [**3**].

Now an induction using Lemma 2.6 yields the claim. □

Question 2.7. *Is the bound of our Main Theorem sharp, i.e. for $n > r$, is*

$$\mathrm{gd}_{\mathfrak{F}_r} G = n + r\ ?$$

Since $\mathrm{gd}_{\mathfrak{F}_0} G = \mathrm{gd}_{\mathfrak{F}_0 \cap \mathfrak{All}(N)} G = n$ for all maximal subgroups N, we can assume equality in the inductive step (assumptions of Lemma 2.6). Then a successive application of the Mayer-Vietoris sequences to the push-outs in Lemmas 2.6 and 2.5, reduces the question to whether the map

$$H^d(G \backslash E_{\mathfrak{F}_{r-1}} G) \to H^d(G \backslash E_{\mathfrak{F}_{r-1} \cap \mathfrak{All}(N)} G)$$

is surjective or not.

We know by [**5**] that $\mathrm{gd}_{\mathfrak{F}_1} G = n + 1$ and it was shown in [**2**] that the question has a positive answer for $G = \mathbb{Z}^3$, i.e. that $\mathrm{gd}_{\mathfrak{F}_2}(\mathbb{Z}^3) = 5$.

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