

(CO)HOMOLOGY OF POSET LIE ALGEBRAS

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Abstract

We investigate (co)homological properties of Lie algebras that are constructed from a finite poset: the solvable class \mathfrak{gl}^{\succeq} and the nilpotent class \mathfrak{gl}^{\preceq} . We confirm the conjecture [8, 1.16(1), p. 141] that says: every prime power $p^r \leq n - 2$ appears as torsion in $H_*(\mathfrak{nil}_n; \mathbb{Z})$, and every prime power $p^r \leq n - 1$ appears as torsion in $H_*(\mathfrak{sol}_n; \mathbb{Z})$. If \preceq is a bounded poset, then the (co)homology of \mathfrak{gl}^{\succeq} is *torsion-convex*, i.e., if it contains p -torsion, then it also contains p' -torsion for every prime $p' < p$.

We obtain new explicit formulas for the (co)homology of some families over arbitrary fields. Among them are the solvable non-nilpotent analogs of the Heisenberg Lie algebras from [2], the 2-step Lie algebras from [1], strictly block-triangular Lie algebras, etc. The combinatorics of how the resulting generating functions are obtained are interesting in their own right.

All this is done by using AMT (algebraic Morse theory [9, 12, 8]). This article serves as a source of examples of how to construct useful acyclic matchings, each of which in turn induces compelling combinatorial problems and solutions. It also enables graph theory to be used in homological algebra.

Background

Let $\mathfrak{nil}_n \leq \mathfrak{sol}_n \leq \mathfrak{gl}_n$ be Lie algebras of all ((strictly) triangular) $n \times n$ matrices over a field K . By the Ado-Iwasawa theorem, every finite-dimensional Lie K -algebra \mathfrak{g} admits an embedding into some \mathfrak{gl}_n . By Engel's theorem, \mathfrak{g} is nilpotent iff it admits an embedding into some \mathfrak{nil}_n . By Lie's theorem, \mathfrak{g} is solvable iff it admits an embedding into some \mathfrak{sol}_n , when K is algebraically closed of characteristic 0 or more than $\dim \mathfrak{g}$. Lie algebras \mathfrak{gl}_n , \mathfrak{sol}_n , \mathfrak{nil}_n over \mathbb{C} give rise to the classical Lie groups $GL_n = \{A \in \mathbb{C}^{n \times n}; \det A \neq 0\}$ and $SOL_n = \{A \in GL_n; A_{ij} = 0 \text{ for } i > j\}$ and $NIL_n = \{A \in SOL_n; A_{ii} = 1\}$ via the exponential map $X \mapsto \sum_{k \in \mathbb{N}} \frac{X^k}{k!}$.

Nilpotent and solvable Lie algebras are less wild than general ones. For instance, there are 9 cases and 2 infinite families of nonisomorphic Lie algebras of dimension ≤ 3

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over \mathbb{R} or \mathbb{C} , by the Bianchi classification (i.e., continuum many isomorphism classes). But there are only 2 cases and 2 families of nonisomorphic solvable Lie algebras of dimension ≤ 3 over *any* field, by [5]. Also, there are just 9 cases of nonisomorphic nilpotent Lie algebras of dimension ≤ 5 over *any* field, by [6].

In this article, we concentrate on those Lie subalgebras of \mathfrak{nil}_n and \mathfrak{sol}_n over a commutative unital ring R which admit a basis consisting of matrix units e_{ij} . These are precisely \mathfrak{gl}_n^{\prec} and $\mathfrak{gl}_n^{\succeq}$ that arise from posets, and they are combinatorial in nature. Their structure constants are ± 1 or 0. Special cases of these families include \mathfrak{nil}_n , \mathfrak{sol}_n , block strictly triangular matrices, the Heisenberg family from [2], the 2-step family from [1], etc. We are interested in the (co)homology of \mathfrak{gl}_n^{\prec} and $\mathfrak{gl}_n^{\succeq}$ from a combinatorial viewpoint.

Results

In this article, the following facts are proved. Every convex subposet of $([n], \preceq)$ induces a direct summand on (co)homology of \mathfrak{gl}_n^{\prec} and $\mathfrak{gl}_n^{\succeq}$. A poset and its opposite induce the same (co)homology. Poincaré duality $H^k(\mathfrak{gl}_n^{\prec}; \mathbb{Z}) \cong H_{\dim-k}(\mathfrak{gl}_n^{\succeq}; \mathbb{Z})$ holds for all k , but $H^k(\mathfrak{gl}_n^{\succeq}; \mathbb{Z}) \not\cong H_{\dim-k}(\mathfrak{gl}_n^{\prec}; \mathbb{Z})$. There holds $C_*(\mathfrak{gl}_n^{\prec}) = \bigoplus_{w \in \mathbb{Z}^n} [w]$ and $C_{*,p}(\mathfrak{gl}_n^{\prec}) = \bigoplus_{w \in (p\mathbb{Z})^n} [w]$, where $[w]$ is the subcomplex spanned by all wedges with weight vector w . This is very useful for computational purposes: it enabled us to obtain $H_*(\mathfrak{sol}_8; \mathbb{Z})$ on a home computer, even though $\dim \Lambda^* \mathfrak{sol}_8 = 2^{\binom{8}{2}} = 68\,719\,476\,736$.

For any $a, b \in [n]$ and any $m \in \mathbb{N}$ such that the interval $[a, b]$ in \preceq contains more than m elements, there is a direct summand \mathbb{Z}_m in $H_*(\mathfrak{gl}_n^{\prec}; \mathbb{Z})$. If \preceq has a largest and a least element, then $H_*(\mathfrak{gl}_n^{\prec}; \mathbb{Z})$ contains p -torsion iff $p < n$. There is a direct summand \mathbb{Z}_{n-2} in $H_*(\mathfrak{nil}_n; \mathbb{Z})$. If $H_*(\mathfrak{gl}_n^{\prec}; \mathbb{Z})$ does not contain 2-torsion, then its Hasse diagram is a forest of height ≤ 1 , hence $H_*(\mathfrak{gl}_n^{\prec}; \mathbb{Z})$ is torsion-free. If chain complex $C_{*,p}(\mathfrak{gl}_n^{\prec})$ is nonempty, then $C_{*,p'}(\mathfrak{gl}_n^{\prec})$ is nonempty for every prime $p' < p$.

For five families of posets \preceq of height 1 or 2, the generating functions for graded K -modules $H_*(\mathfrak{gl}_n^{\prec}; K)$ are computed in Sections 5 and 6. Our approach has an advantage: we obtain explicit generators for (co)homology, not just a formula for dimension. This enabled us to compute the cohomology algebras w.r.t. cup products.

Conventions

Throughout this article, R will denote a commutative unital ring. Letter p will always denote a prime number, and $[n] = \{1, 2, \dots, n\}$. We omit the wedge signs \wedge in the basis elements of $\Lambda^k \mathfrak{g}$. For any partial ordering \preceq we denote by Γ_{\preceq} its Hasse diagram (acyclic digraph) and by $\bar{\Gamma}_{\preceq}$ the undirected version (simple graph). For a Lie R -algebra \mathfrak{g} we denote its Chevalley chain complex by $C_*(\mathfrak{g}; R) = \Lambda^* \mathfrak{g}$ with boundary $\partial x_1 \cdots x_k = \sum_{r < s} (-1)^{r+s} [x_r, x_s] x_1 \cdots \widehat{x}_r \cdots \widehat{x}_s \cdots x_k$. The associated digraph is denoted by Γ_{C_*} (its vertices are the basis elements of modules $C_k(\mathfrak{g}; R)$, and edges correspond to nonzero entries in the boundary matrices). This digraph is used to define Morse matchings and use AMT (algebraic Morse theory, see [9, 1.1]). We use the notations from that formulation.

1. Posets

In this section, we define poset Lie algebras and explore their general properties.

Let \sim be a transitive relation on a set I (so \sim is a subset of $I \times I$). Let $\mathfrak{gl}^\sim(R)$ be the free R -module on symbols e_{ij} for $i \sim j$ (i.e., on the elements of \sim). Then $[e_{ij}, e_{kl}] = \delta_{jk}e_{il} - \delta_{il}e_{kj}$ is a well-defined bracket that makes \mathfrak{gl}^\sim a Lie R -algebra. It is isomorphic to the Lie subalgebra of $\mathfrak{gl}_I(R)$ (all column-finitary $I \times I$ matrices with the commutator bracket) spanned by basis matrices e_{ij} with $i \sim j$. If I has finite cardinality n , then we denote it by \mathfrak{gl}_n^\sim . Subalgebras of \mathfrak{gl}_n^\sim that admit a module basis

$$\mathcal{B} \subseteq \{e_{ij}; i, j \in [n]\}$$

are precisely the algebras \mathfrak{gl}_n^\sim . Every \mathfrak{gl}_n^\sim is a \mathbb{Z} -graded Lie algebra via $\deg e_{ij} = i - j$.

When the relation \sim is antisymmetric and reflexive, we denote it by \preceq ; when \sim is asymmetric and irreflexive, we denote it by \prec . Any isomorphism of posets $(I, \preceq) \rightarrow (J, \sqsubseteq)$ induces isomorphisms of Lie algebras $\mathfrak{gl}^\preceq \rightarrow \mathfrak{gl}^\sqsubseteq$ and $\mathfrak{gl}^\prec \rightarrow \mathfrak{gl}^\sqsubset$ that send $e_{ij} \mapsto e_{f(i)f(j)}$. Any poset morphism, however, does not canonically induce a Lie algebra morphism: if $I = \{a, b, c, d\}$ and $J = \{1, 2, 3\}$ are linearly ordered and f sends $a \mapsto 1, b \mapsto 2, c \mapsto 2, d \mapsto 3$, then $f[e_{ab}, e_{cd}] = 0 \neq e_{13} = [f(e_{ab}), f(e_{cd})]$.

For any poset \preceq , our \mathfrak{gl}_n^\preceq is an ideal of $\mathfrak{gl}_n^\sqsubseteq$ and thus induces an exact sequence

$$0 \longrightarrow \mathfrak{gl}_n^\prec \xrightarrow{\sqsubseteq} \mathfrak{gl}_n^\preceq \xrightarrow{\diagup} \mathfrak{dgn}_n \longrightarrow 0,$$

where \mathfrak{dgn}_n is the abelian Lie algebra of diagonal matrices. Thus the Hochschild-Serre spectral sequence can be used to obtain information of \mathfrak{gl}_n^\preceq from $\mathfrak{gl}_n^\sqsubseteq$. Additionally, there is an exact sequence

$$0 \longrightarrow \mathfrak{sl}_n^\preceq \xrightarrow{\sqsubseteq} \mathfrak{gl}_n^\preceq \xrightarrow{\text{tr}} R \longrightarrow 0,$$

and the ideal \mathfrak{sl}_n^\preceq of matrices with zero trace was recently studied in [4].

The *opposite* poset of \preceq is the poset \preceq^{op} with the same underlying set, but $i \preceq^{\text{op}} j$ iff $i \succeq j$. A *subposet* of (J, \sqsubseteq) is a subset $I \subseteq J$ with the induced ordering $\preceq = \sqsubseteq \cap (I \times I)$. A subposet (I, \preceq) of (J, \sqsubseteq) is *convex* when $a, b \in I$ and $x \in J$ and $a \sqsubseteq x \sqsubseteq b$ implies $x \in I$, i.e., if it is closed for taking intervals $[a, b] = \{x \in J; a \sqsubseteq x \sqsubseteq b\}$.

Proposition 1.1. **a)** $H_*(\mathfrak{gl}^{\preceq^{\text{op}}}) \cong H_*(\mathfrak{gl}^\preceq)$.

b) If \preceq is a convex subposet of \sqsubseteq , then $C_*(\mathfrak{gl}^\sqsubseteq) = C_*(\mathfrak{gl}^\preceq) \oplus \dots$

Thus a poset and its opposite induce isomorphic homologies, and every convex subposet induces a direct summand on homologies.

Proof. **a)** The transposition map $\mathfrak{gl}^{\preceq^{\text{op}}} \rightarrow \mathfrak{gl}^\preceq$ that sends $e_{ji} \mapsto e_{ji}^t = e_{ij}$ is a module morphism that satisfies $[\mathbf{a}, \mathbf{b}]^t = -[\mathbf{a}^t, \mathbf{b}^t]$ for all $\mathbf{a}, \mathbf{b} \in \mathfrak{gl}_n^\preceq$. The induced module morphisms $\Lambda^k \mathfrak{gl}^{\preceq^{\text{op}}} \rightarrow \Lambda^k \mathfrak{gl}^\preceq$ anticommute with the boundary morphisms. Hence they induce morphisms on all homology modules. The same can be said for the transposition map in the other direction, which produces the inverse morphisms.

b) For any wedge $v = e_{a_1 b_1} \cdots e_{a_k b_k} \in \Lambda^k \mathfrak{gl}^\sqsubseteq$ the set of indices $\epsilon(v) = \{a_1, b_1, \dots, a_k, b_k\}$ is a subposet of (I, \sqsubseteq) . Since the only nonzero brackets are of the form $[e_{ab}, e_{bc}] = e_{ac}$, we see that the set of maximal and minimal indices of v equals the set of maximal and minimal indices of every summand of $\partial(v)$. Thus the whole chain complex $C_*(\mathfrak{gl}^\sqsubseteq)$ is a direct sum of subcomplexes spanned by wedges

with fixed sets of maximal and minimal indices. In particular, when (I, \preceq) is convex in (J, \sqsubseteq) , the wedges $\{v; \epsilon(v) \subseteq I\} \subseteq C_*(\mathfrak{gl}^{\sqsubseteq})$ span the chain complex $C_*(\mathfrak{gl}^{\preceq})$. \square

The analogous two statements for Lie algebras $\mathfrak{gl}_n^{\preceq}$ also hold.

Example 1.2. Let \preceq be the total order on $[n]$, so that $\mathfrak{gl}_n^{\preceq} = \mathfrak{nil}_n$ is the Lie algebra of all upper triangular matrices. The convex subposets of the interval $[1, n]$ are smaller intervals $[i, j]$. For $i, j \in [n]$ let $C_{i,j}$ be the subcomplex of $C_*(\mathfrak{nil}_n)$ spanned by $\{v; \min \epsilon(v) = i, \max \epsilon(v) = j\}$. Notice that $C_{i,j} \cong C_{1,j-i+1}$. We have $C_*(\mathfrak{nil}_n) = \bigoplus_{i \leq j} C_{i,j} = \frac{C_{1,n} \oplus \bigoplus_{1 < i \leq j} C_{i,j} \oplus \bigoplus_{i \leq j < n} C_{i,j}}{\bigoplus_{1 < i \leq j < n} C_{i,j}}$. Therefore

$$H_*(\mathfrak{nil}_n) \cong \frac{H_*(C_{1,n}) \oplus H_*(\mathfrak{nil}_{n-1})^2}{H_*(\mathfrak{nil}_{n-2})},$$

where $C_{1,n}$ is the subcomplex of all wedges that contain both 1 and n as indices. The analogous statement for the Lie algebra $\mathfrak{gl}_n^{\preceq} = \mathfrak{so}_n$ also holds.

Remark 1.3. Disjoint unions of Hasse diagrams induce tensor products:

$$C_*(\mathfrak{gl}^{\preceq \sqcup \preceq'}) \cong C_*(\mathfrak{gl}^{\preceq}) \otimes C_*(\mathfrak{gl}^{\preceq'}) \quad \text{and} \quad C_*(\mathfrak{gl}^{\preceq \sqcup \preceq'}) \cong C_*(\mathfrak{gl}^{\preceq}) \otimes C_*(\mathfrak{gl}^{\preceq'}).$$

Thus from now on, we only consider connected Hasse diagrams, and ignore all mirror images. Note that nonisomorphic posets may induce isomorphic Lie algebras, e.g.,

$$\Gamma_{\preceq}: \begin{array}{cc} 2 & 4 \\ | & | \\ 1 & 3 \end{array} \quad \text{and} \quad \Gamma_{\sqsubseteq}: \begin{array}{ccc} & 2 & 3 \\ & \backslash & / \\ & 1 & \end{array}$$

induce $\mathfrak{gl}_4^{\preceq}(R) \cong \mathfrak{gl}_3^{\sqsubseteq}(R) \cong R^2$, though $\mathfrak{gl}_4^{\preceq}(R) \not\cong \mathfrak{gl}_3^{\sqsubseteq}(R)$, and

$$\Gamma_{\preceq}: \begin{array}{ccc} 2 & & 4 \\ | & / & | \\ 1 & & 3 \end{array} \quad \text{and} \quad \Gamma_{\sqsubseteq}: \begin{array}{ccc} 2 & 3 & 4 \\ & \backslash & | & / \\ & & 1 & \end{array}$$

induce $H_*(\mathfrak{gl}_4^{\preceq}; R) \cong H_*(\mathfrak{gl}_4^{\sqsubseteq}; R)$ and $H_*(\mathfrak{gl}_4^{\preceq}; R) \cong H_*(\mathfrak{gl}_4^{\sqsubseteq}; R)$.

Proposition 1.4. $H^k(\mathfrak{gl}_n^{\preceq}; \mathbb{Z}) \cong H_{N-k}(\mathfrak{gl}_n^{\preceq}; \mathbb{Z})$ for all k , where $N = |\prec| = \dim \mathfrak{gl}_n^{\preceq}$.

However, $H^k(\mathfrak{gl}_n^{\preceq}; \mathbb{Z}) \not\cong H_{N-k}(\mathfrak{gl}_n^{\preceq}; \mathbb{Z})$ for all k , where $N = |\preceq| = \dim \mathfrak{gl}_n^{\preceq} > n$, since by [9, 5.8] the free part is $\mathbb{Z}^{\binom{n}{k}}$, which is symmetric iff $N = n$ iff $\preceq = \{(i, i); i \in [n]\}$.

Proof. By Hazewinkel's theorem [7], it suffices to prove that $\mathfrak{gl}_n^{\preceq}$ is unimodular, i.e., for every $a \in \mathfrak{gl}_n^{\preceq}$ the linear map $[a, -]: \mathfrak{gl}_n^{\preceq} \rightarrow \mathfrak{gl}_n^{\preceq}$ has trace 0. Since $a = \sum_{r \prec s} \alpha_{rs} e_{rs}$ and $\text{tr}[a, -] = \sum_{r \prec s} \alpha_{rs} \text{tr}[e_{rs}, -]$, it suffices to prove $\text{tr}[e_{rs}, -] = 0$. Indeed, in the basis $\{e_{ij}; i \prec j\}$, the only elements that are not annihilated are e_{st} and e_{tr} , but they are mapped to e_{rt} and $-e_{ts}$, so the coefficient of $[e_{rs}, e_{ij}]$ at e_{ij} is 0. \square

Thus $H_*(\mathfrak{gl}_n^{\preceq})$ obeys Poincaré duality: free part appears symmetrically and torsion appears symmetrically shifted by one, $FH_k \cong FH_{N-k}$, $T_p H_k \cong T_p H_{N-k-1}$.

2. Generating functions

Let K be a field and $M = \bigoplus_{k \in \mathbb{N}} M_k$ a graded K -module such that every M_k is finite dimensional. The *Hilbert-Poincaré series* of M is the formal power series

$$\text{HP}_M(t) = \sum_k \dim_K M_k t^k \in \mathbb{Z}[[t]].$$

There holds $\text{HP}_{M/N} = \text{HP}_M - \text{HP}_N$, $\text{HP}_{M \oplus N} = \text{HP}_M + \text{HP}_N$, $\text{HP}_{M \otimes N} = \text{HP}_M \cdot \text{HP}_N$.

For a Lie K -algebra \mathfrak{g} we denote the Hilbert-Poincaré series of its homology by $\text{HP}_{C_*(\mathfrak{g})}(t) = \sum_k \dim_K H_k(\mathfrak{g}) t^k \in \mathbb{Z}[[t]]$, when $\dim_K H_k(\mathfrak{g}) < \infty$ for all k . Thus HP contains all information about the (co)homology of \mathfrak{g} over K . There always holds $\text{HP}_{C_*(\mathfrak{g})}(-1) = \sum_k (-1)^k \dim_K H_k(\mathfrak{g}) = \sum_k (-1)^k \dim_K \Lambda^k(\mathfrak{g}) = \sum_k (-1)^k \binom{\dim \mathfrak{g}}{k} = (1-1)^{\dim \mathfrak{g}} = 0$. If $K \leq K'$ is a field extension, then $\text{HP}_{C_*(\mathfrak{g};K)} = \text{HP}_{C_*(\mathfrak{g};K')}$ by the universal coefficient theorem, so it suffices to compute over fields \mathbb{Q} and \mathbb{Z}_p .

We obtained some information about HP for Lie algebras $\mathfrak{gl}_n^{\preceq}$ and \mathfrak{gl}_n^{\succ} . If K has characteristic 0 or more than n , then by [9, 5.8] there holds $\text{HP}_{C_*(\mathfrak{gl}_n^{\preceq})}(t) = (1+t)^n$. Also, $\text{HP}_{C_*(\mathfrak{gl}^{\preceq \cup \preceq'})} = \text{HP}_{C_*(\mathfrak{gl}^{\preceq})} \cdot \text{HP}_{C_*(\mathfrak{gl}^{\preceq'})}$ and $\text{HP}_{C_*(\mathfrak{gl}^{\prec \cup \succ})} = \text{HP}_{C_*(\mathfrak{gl}^{\prec})} \cdot \text{HP}_{C_*(\mathfrak{gl}^{\succ})}$.

In the Chevalley chain complex $C_*(\mathfrak{gl}_n^{\preceq})$, the *weight vector* of a wedge v of basis matrices e_{ij} is the vector $w_v = (w_1, \dots, w_n) \in \mathbb{Z}^n$ in which w_i is the number of times i appears as a right index $e_{ai} \in v$ minus the number of times it appears as a left index $e_{ib} \in v$. In symbols, $w_i = |\{a \in [n]; e_{ai} \in v\}| - |\{b \in [n]; e_{ib} \in v\}|$. Thus $-n < w_i < n$ for all i , and if i is a minimal (resp. maximal) element of \preceq , then $w_i \leq 0$ (resp. $w_i \geq 0$). If our field K has characteristic p , then by [9, 6.1] we have

$$\text{HP}_{C_*(\mathfrak{gl}_n^{\preceq})}(t) = (1+t)^n \cdot \text{HP}_{C_{*,p}(\mathfrak{gl}_n^{\preceq})}(t),$$

where p -complex $C_{*,p}(\mathfrak{gl}_n^{\preceq})$ is the chain subcomplex (direct summand) of $C_*(\mathfrak{gl}_n^{\preceq})$, spanned by all wedges v with $w_v \in (p\mathbb{Z})^n$, i.e., weights are noninvertible in $\mathbb{Z}_p = \frac{\mathbb{Z}}{p\mathbb{Z}}$. Let $\tilde{C}_{*,p}(\mathfrak{gl}_n^{\preceq})$ be spanned by all $v \neq \emptyset$ with $w_v \in (p\mathbb{Z})^n$.

Since every edge in the digraph $\Gamma_{C_*(\mathfrak{gl}^{\prec})}$ is of the form $\begin{matrix} \cdots e_{ax} e_{xb} \cdots \\ \cdots e_{ab} \cdots \end{matrix}$, edges preserve the weight vector, i.e., all summands of $\partial(v)$ have the same weight vector w_v .

Corollary 2.1. $C_*(\mathfrak{gl}_n^{\preceq}) = \bigoplus_{w \in \mathbb{Z}^n} [w]$ and $C_{*,p}(\mathfrak{gl}_n^{\preceq}) = \bigoplus_{w \in (p\mathbb{Z})^n} [w]$, where $[w]$ is the subcomplex spanned by all wedges with weight vector w .

This holds over any ring, and it is very useful for computational purposes, since each boundary ∂_k is a huge matrix in block diagonal form, and we only have to deal with the very many much smaller blocks. Even if we are working with sparse matrices, it is much more effective to deal with each $[w]$ separately.

Similarly, $C_*(\mathfrak{gl}_n^{\succ}) = \bigoplus_{w \in \mathbb{Z}^n} [w]$ and $C_{*,p}(\mathfrak{gl}_n^{\succ}) = \bigoplus_{w \in \mathbb{Z}^n} [w]$. The isolated vertices of Γ_{C_*} (wedges of diagonals e_{ii} , which by [9, 5.8] generate the free part in $H_*(\mathfrak{gl}_n^{\preceq}; \mathbb{Z})$) constitute the subcomplex $[0]$, and for any $w = (w_1, \dots, w_n) \neq 0$ we notice that

$$\gcd(w_1, \dots, w_n) = 1 \text{ implies } [w] \simeq 0,$$

by [9, 6.1]. Thus many $[w]$ can be omitted from the direct sum without changing the homology of the complex.

Of course, the usefulness of this only becomes apparent when we are able to generate each $[w]$ (without first generating the whole $C_*(\mathfrak{gl}_n^{\preceq})$ and selecting wedges).

Remark 2.2. Recall that the *height* of a poset \preceq is $\max\{r; \exists x_0 \prec x_1 \prec \dots \prec x_r\}$, i.e., the length of the longest chain in the poset. In sections 5 and 6, we will see that when height is 1 or 2, the p -complex is small enough to allow direct computation. There, we shall also be using a lemma from the theory of discrete Fourier transforms: given $\sum_{k \in \mathbb{N}} c_k t^k = f(t) \in \mathbb{C}[[t]]$ and $p \in \mathbb{N}$, the sum of every p -th term is $\sum_{k \in p\mathbb{N}} c_k t^k = \frac{1}{p} \sum_{i \in [p]} f(\epsilon^i t)$ where $\epsilon = e^{2\pi i/p}$. More generally, $\sum_{k \in p\mathbb{N}} c_k t^{jk+l} = \frac{t^l}{p} \sum_{i \in [p]} f(\epsilon^i t^j)$.

Remark 2.3. Over a field, any chain complex C_* admits a decomposition

$$C_* = C'_* \oplus C''_* \text{ with } \partial' = 0 \text{ and } C''_* \simeq 0,$$

so $H_*(C_*) = C'_*$. Indeed, any submodule is a direct summand, $\text{Im} \partial_{k+1} \leq \text{Ker} \partial_k \leq C_k$ implies there exist $D'_k, D''_k \leq C_k$ with $\text{Im} \partial_{k+1} \oplus D'_k = \text{Ker} \partial_k$ and $\text{Im} \partial_{k+1} \oplus D'_k \oplus D''_k = C_k$, hence $\partial|_{D'_k} = 0$ and $\partial: D''_k \xrightarrow{\cong} \text{Im} \partial_k$, so $C'_k = D'_k$ and $C''_k = \text{Im} \partial_{k+1} \oplus D''_k$ suffice.

Finding the submodule C'_* can make the computation of cup products possible. If C_* is the Chevalley complex, then for $\alpha \in H^i(\mathfrak{g})$ and $\beta \in H^j(\mathfrak{g})$ the rule

$$(\alpha \smile \beta)(x_1 \cdots x_{i+j}) = \sum_{\substack{\pi \in S_{i+j}, \pi_1 < \dots < \pi_i, \\ \pi_{i+1} < \dots < \pi_{i+j}}} \text{sgn} \pi \alpha(x_{\pi_1} \cdots x_{\pi_i}) \beta(x_{\pi_{i+1}} \cdots x_{\pi_{i+j}})$$

makes $H^*(\mathfrak{g})$ a graded-commutative algebra. If $C'_* = H_*(C_*)$ admits a finite basis consisting of wedges $\{v_i; i \in I\}$, then $C'^* = H^*(C_*)$ has a dual basis $\mathcal{B} = \{\chi_{v_i}; i \in I\}$, and there holds $\chi_{v_i} \smile \chi_{v_j} = \chi_{v_i \wedge v_j}$ with $\chi_v = 0$ iff $v \notin \mathcal{B}$.

Example 2.4. Let $\text{HP}_{n,p}$ be the generating function for (co)homology of $\mathfrak{nil}_n(\mathbb{Z}_p)$, and $C_{1,n}$ the subcomplex from Example 1.2. Thus $\text{HP}_{n,p} = \text{HP}_{C_{1,n}} + 2\text{HP}_{n-1,p} - \text{HP}_{n-2,p}$. On $C_{1,n}$ we define a Morse matching

$$\mathcal{M} = \left\{ \begin{array}{l} e_{1i} e_{in} \cdots \\ \downarrow \\ v = e_{1n} \cdots \end{array} ; \begin{array}{l} i \text{ is maximal such} \\ \text{that } e_{1i}, e_{in} \notin v \end{array} \right\}.$$

Then $\mathring{\mathcal{M}} = \{v; e_{1n} \in v, \forall i: e_{1i} \in v \text{ or } e_{in} \in v\} \cup \{v; e_{1n} \notin v, \forall i: e_{1i} \notin v \text{ or } e_{in} \notin v\}$ are the critical vertices, which span the direct summands C_n and C'_n of $C_{1,n}$, and the new boundary is the restriction of the old one: $\mathring{\partial} = \partial$. Let $e = \bigwedge_{i < j} e_{ij}$ be the wedge of all elements and $N = \binom{n}{2} = \dim \mathfrak{nil}_n$. The map $C_{*,n} \rightarrow C'_{N-*,n}$ that sends $v \mapsto e \setminus v$ is an isomorphism of chain complexes, hence

$$\text{HP}_{C_{1,n}} = \text{HP}_{C'_n}(t) + t^N \text{HP}_{C'_n}\left(\frac{1}{t}\right).$$

Thus $C'_n = \langle v; e_{1n} \notin v, \forall i: e_{1i} \notin v \text{ or } e_{in} \notin v \rangle$ and its p -subcomplex are at the heart of the problem for computing HP for $\mathfrak{nil}_n(\mathbb{Z}_p)$ and $\mathfrak{sol}_n(\mathbb{Z}_p)$.

3. Homology tables

It is always desirable to have concrete examples of objects under inspection. We provide two tables for the homology of families that are studied in later sections. In each of the two cases, the first row is a sequence of Hasse diagrams Γ_{\preceq} , and under each of them is the graded module $H_*(\mathfrak{g}^{\preceq}; \mathbb{Z})$.

4. Torsion properties

Proposition 4.1. *For any $a, b \in [n]$ and any $m \in \mathbb{N}$ such that the interval $[a, b]$ in \preceq contains more than m elements, there is a direct summand \mathbb{Z}_m in $H_*(\mathfrak{gl}_n^{\preceq}; \mathbb{Z})$.*

Therefore the (co)homology of $\mathfrak{gl}_n^{\preceq}$ contains p -torsion for all primes p that are smaller than the size of the largest interval in the poset \preceq .

Proof. Given $a \prec x_1, \dots, x_{m-1} \prec b$, we define $v = e_{ab} \bigwedge_{i=1}^{m-1} e_{ax_i} e_{x_i b}$, which has weights $w_v = (-m, 0, \dots, 0, m)$. Thus $v \in \text{Ker} \partial$ and $mv \in \text{Im} \partial$, so it remains to show that $lv \in \text{Im} \partial$ implies $l \in m\mathbb{Z}$. All the edges in Γ_{C_*} ending in v are listed below (obtained by splitting e_{ab} or some e_{ax_i} or some $e_{x_i b}$), and we define \mathcal{M} as the set of dotted edges:

$$\begin{array}{l}
 e_{aa}v = e_{aa}e_{ab} \bigwedge_i e_{ax_i} e_{x_i b} \\
 e_{bb}v = e_{bb}e_{ab} \bigwedge_i e_{ax_i} e_{x_i b} \\
 v'_x := e_{ax}e_{xb} \bigwedge_i e_{ax_i} e_{x_i b} \\
 v'_{ry} := e_{ab}e_{ay}e_{y x_r} e_{x_r b} \bigwedge_{i \neq r} e_{ax_i} e_{x_i b} \\
 v'_{sz} := e_{ab}e_{ax_s} e_{x_s z} e_{z b} \bigwedge_{i \neq s} e_{ax_i} e_{x_i b}
 \end{array}
 \begin{array}{l}
 \xrightarrow{\quad} \\
 \xrightarrow{\quad} \\
 \xrightarrow{\quad} \\
 \xrightarrow{\quad} \\
 \xrightarrow{\quad}
 \end{array}
 \begin{array}{l}
 e_{ab}e_{ax}e_{xb} \bigwedge_{i \neq 1} e_{ax_i} e_{x_i b} =: v_x \\
 e_{ab} \bigwedge_i e_{ax_i} e_{x_i b} = v \\
 e_{ab}e_{ay}e_{yb} \bigwedge_{i \neq r} e_{ax_i} e_{x_i b} =: v_{ry} \\
 e_{ab}e_{az}e_{zb} \bigwedge_{i \neq s} e_{ax_i} e_{x_i b} =: v_{sz}
 \end{array}$$

Out of v'_{ry} (resp. v'_{sz}) an edge goes only to v and v_{ry} (resp. v_{sz}). There is an edge from v'_x to v_{ry} (resp. v_{sz}) iff $x = y$ (resp. $x = z$). Notice that for any choices $a \prec x \prec b$, $a \prec y \prec x_r \prec b$, $a \prec x_s \prec z \prec b$ the vertices $v_{ry}, v_{sz}, v'_x, v'_{ry}, v'_{sz}$ are pairwise distinct, but $v_x = v_{ry}$ (resp. $v_x = v_{sz}$) iff $1 = r$ (resp. $1 = s$) and $x = y$ (resp. $x = z$). For any $a \prec y \prec x_1$ (resp. $x_1 \prec z \prec b$) we remove $v'_{1y} \rightarrow v_{1y}$ (resp. $v'_{1z} \rightarrow v_{1z}$). Then

$$\mathcal{M} = \{v'_x \rightarrow v_x, v'_{ry} \rightarrow v_{ry}, v'_{sz} \rightarrow v_{sz}; r \neq 1 \neq s\}$$

is a Morse matching. From v'_{1y} (resp. v'_{1z}) to v there are m zig-zag paths of same sign, so $\partial(v'_{1y}) = mv$ (resp. $\partial(v'_{1z}) = -mv$) and the result follows. For different choices of $a, b \in [n]$ and $m < |[a, b]|$, the union of resulting \mathcal{M} 's is a Morse matching on the whole chain complex, so we get distinct direct summands \mathbb{Z}_m in $H_*(\mathfrak{gl}_n^{\preceq}; \mathbb{Z})$. \square

A poset is *bounded* when it contains a largest and a smallest element.

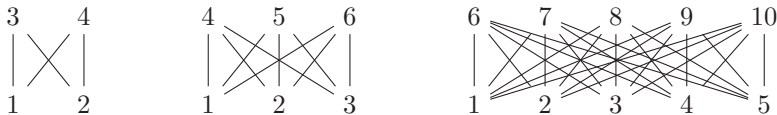
Corollary 4.2. *If \preceq is bounded, then $H_*(\mathfrak{gl}_n^{\preceq}; \mathbb{Z})$ contains p -torsion iff $p < n$.*

Proof. If a is the largest element and b the smallest element, then interval $[a, b]$ has n elements and by Proposition 4.1 $H_*(\mathfrak{gl}_n^{\preceq}; \mathbb{Z})$ contains p -torsion for every $p < n$. Conversely, for any prime $p \geq n$ all nonzero weights of indices are smaller than p , so $\tilde{C}_{*,p}(\mathfrak{gl}_n^{\preceq}; \mathbb{Z}) \cong 0$, hence $H_*(\mathfrak{gl}_n^{\preceq}; \mathbb{Z})$ does not contain p -torsion. \square

Thus for bounded posets (and we suspect for all posets), $H_*(\mathfrak{gl}_n^{\preceq}; \mathbb{Z})$ is *torsion-convex*. This is not true for solvable Lie algebras which do not come from posets, e.g., the nilpotent example $L_{6,19}(p)$ from [6, p. 647] contains only p -torsion.

Remark 4.3. If poset \preceq is not bounded, i.e., contains several minimal and/or several maximal elements, then the largest torsion can be much bigger than the largest interval. For example, if the Hasse diagram of the poset is the p -complete bipartite graph (it has $2p$ vertices and height 1), then the largest interval has 2 elements but

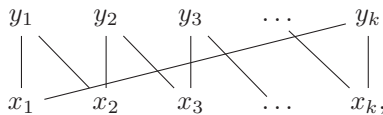
$H_*(\mathfrak{gl}_n^{\leftarrow}; \mathbb{Z})$ contains p -torsion, generated by the wedge of all nondiagonals. Indeed, for $v = \bigwedge_{i \prec j} e_{ij}$ we have $\partial(v) = 0$ and $\partial(e_{ii}v) = \pm p v$ for any i . For example, in



the first case induces 2-torsion, second induces 2,3-torsion, third induces 2,3,5-torsion. This shows the difference between $H_*(\mathfrak{gl}_n^{\leftarrow}; \mathbb{Z})$ and $H_*(\mathfrak{gl}_n^{\leftarrow}; \mathbb{Z})$: the former can have a lot of torsion even if all intervals are small, but the latter in the case of bipartite Hasse diagrams is torsion-free (since differentials are zero).

Corollary 4.4. *If $H_*(\mathfrak{gl}_n^{\leftarrow}; \mathbb{Z})$ does not contain 2-torsion, then its Hasse diagram is a forest of height ≤ 1 , hence $H_*(\mathfrak{gl}_n^{\leftarrow}; \mathbb{Z})$ is torsion-free.*

Proof. If $H_*(\mathfrak{gl}_n^{\leftarrow}; \mathbb{Z})$ contains no 2-torsion, then by Proposition 4.1 the poset does not contain intervals with 3 or more elements, so its height is at most 1. If $\bar{\Gamma}_{\prec}$ contained a cycle

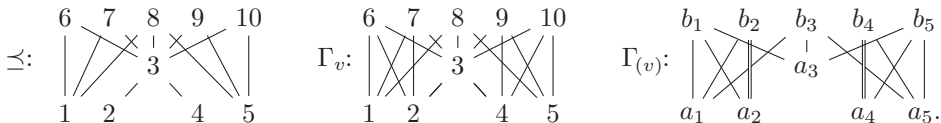


then wedge $\bigwedge_{i \in \mathbb{Z}_k} e_{x_i y_i} e_{x_i y_{i-1}} \in C_{*,2}(\mathfrak{gl}^{\leftarrow})$ would generate 2-torsion, a contradiction. Therefore $\bar{\Gamma}_{\prec}$ is a forest. Hence every wedge of e_{xy} 's (viewed as a subgraph of $\bar{\Gamma}_{\prec}$) is a forest, so it has leaves (indices of weight ± 1). Thus $\tilde{C}_{*,p}(\mathfrak{gl}^{\leftarrow}) \cong 0$ for any p . \square

Conjecture 4.5. *If $H_*(\mathfrak{gl}_n^{\leftarrow}; \mathbb{Z})$ contains p -torsion, then it contains p' -torsion for every prime $p' < p$.*

Proposition 4.6. *If $\tilde{C}_{*,p}(\mathfrak{gl}_n^{\leftarrow}) \not\cong 0$, then $\tilde{C}_{*,p'}(\mathfrak{gl}_n^{\leftarrow}) \not\cong 0$ for every $p' < p$.*

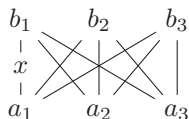
Proof. By assumption, there exists a wedge $\emptyset \neq v \in C_*(\mathfrak{gl}_n^{\leftarrow})$ with weights $w_v \in (p\mathbb{Z})^n$. Let Γ_v be a digraph, whose edges $i \rightarrow j$ are the elements $e_{ij} \in v$. The indices appearing in v are partitioned into three sets: those with positive/negative/zero weights. Let $\Gamma_{(v)}$ be the bipartite multigraph with vertices the indices in v with positive/negative weights, denoted a_i and b_j . Given a and b with $w_a < 0 < w_b$, and given a directed path γ from a to b in Γ_v , we remove all edges of γ from Γ_v and add an edge $a \rightarrow b$ to $\Gamma_{(v)}$. We keep doing this until Γ_v has no edges. For example:



In general, $\Gamma_{(v)}$ is a bipartite multigraph whose vertices have degrees in $p\mathbb{N}$. We wish to obtain a graph whose vertices all have degree p . We create a new multigraph $\Gamma'_{(v)}$:

instead of every vertex a of degree pk , we draw k copies of it, each of which has p edges to those b 's to which a was connected. We create a new multigraph $\Gamma''_{(v)}$: instead of every vertex b of degree pl , we draw l copies of it, each of which has p edges to those a 's to which b was connected. Now $\Gamma''_{(v)}$ is a p -regular bipartite multigraph. By König's theorem [3, 10.8, p. 257], the edges of $\Gamma''_{(v)}$ can be colored with only p colors. Let Γ'' be the subgraph consisting of those edges colored by the first p' colors, so it is p' -regular. Now we reverse the process: we join b 's that came from an b of degree pl into a single vertex of degree $p'l$, and then join a 's that came from an a of degree pk into a single vertex of degree $p'k$. The result is a subgraph of Γ_v , so it corresponds to a subwedge $\emptyset \neq v' \in C_*(\mathfrak{gl}_n^{\leftarrow})$ with weights $w_{v'} \in (p'\mathbb{Z})^n$. \square

Remark 4.7. Chain complex $\tilde{C}_{*,p}$ can be nonempty but still contractible. E.g., for



our $C_{*,3}$ has a basis consisting of $u = e_{a_1x}e_{xb_1} \wedge_{(i,j) \neq (1,1)} e_{a_i b_j}$ and $v = \wedge_{(i,j)} e_{a_i b_j}$ and \emptyset , with $\partial(u) = \pm v$ and $\partial(v) = 0$, so $\tilde{C}_{*,3}$ is contractible. In fact, if the poset consists of elements $a_1, \dots, a_p, b_1, \dots, b_p, x_1, \dots, x_n$ with $n < p$ and relations $a_i \prec b_j$ and $a_i \prec x_i \prec b_i$ for all i and j , then $C_{*,p}$ is isomorphic to the simplicial chain complex of the n -ball, and is therefore homotopy equivalent to $R \leftarrow 0 \leftarrow 0 \leftarrow 0 \leftarrow \dots$.

Proposition 4.8. $H_*(\mathfrak{nil}_n; \mathbb{Z})$ contains a direct summand \mathbb{Z}_{n-2} .

Determining what torsion appears in the general $H_*(\mathfrak{gl}_n^{\leftarrow}; \mathbb{Z})$ is more difficult. Thus tables $(H_*(\mathfrak{nil}_n; \mathbb{Z}))_{n \in \mathbb{N}}$ and $(H_*(\mathfrak{so}_n; \mathbb{Z}))_{n \in \mathbb{N}}$ contain all prime powers \mathbb{Z}_{p^r} .

Proof. Let $[w]$ be the direct summand subcomplex of $C_*(\mathfrak{nil}_n; \mathbb{Z})$, spanned by all wedges with weight vector $w = (-1, 3 - n, 1, \dots, 1) \in \mathbb{Z}^n$. For any $v \in [w]$ there holds: v either contains all $e_{23}, e_{24}, \dots, e_{2n}$ (iff $v = e_{12} \wedge_{3 \leq i \leq n} e_{2i} =: \alpha$) or exactly one of them is missing; v contains exactly one e_{1x} and one e_{yn} (which may be equal); $e_{1n} \in v$ iff $v = e_{1n} \wedge_{3 \leq i < n} e_{2i} =: \beta$. If $e_{2n} \in v \neq \alpha$, then $e_{2i} \notin v$ for a unique $i < n$ and $e_{in} \notin v$. If $e_{2n} \notin v \neq \beta$, then $\wedge_{3 \leq i < n} e_{2i} \subseteq v$ and $e_{in} \in v$ for a unique $i > 2$. Thus

$$\mathcal{M} = \{e_{2i}e_{in} \cdots \longrightarrow e_{2n} \cdots\}$$

is a matching. There does not exist a pair of distinct edges $e_{2i}e_{in} \cdots \xrightarrow{e_{2n} \cdots} e_{2n} \cdots$ (because e_{in} is unique in a wedge, so we must bracket it with e_{2i} to obtain e_{2n}), hence \mathcal{M} is a Morse matching and every zig-zag path contains at most one zig-zag. The critical vertices are $\dot{\mathcal{M}} = \{\alpha, \beta\}$. From α to β there is one direct edge $\alpha \xrightarrow{(-1)^n} \beta$ and $n - 3$ zig-zag paths $\alpha \xrightarrow{(-1)^r} e_{1r} \wedge_{i \neq r} e_{2i} \xleftarrow{(-1)^{n+r}} e_{1r}e_{rn} \wedge_{i \neq n} e_{2i} \xrightarrow{-1} \beta$ which have sign $(-1)^{r+n+r+1-1} = (-1)^n$, hence $\dot{\partial}(\alpha) = \pm(n-2)\beta$ and we are finished. \square

Notice that Propositions 1.1 and 4.8 confirm the conjecture [8, 1.16.(1), p. 141].

5. Posets of height 1

In this section, all posets are assumed to have height 1, so Hasse diagrams correspond to bipartite graphs. A wedge in the p -complex $C_{*,p}(\mathfrak{gl}_n^{\prec})$ in this case corresponds to a subset of edges of the Hasse diagram Γ_{\preceq} , so that those edges constitute a subgraph of $\bar{\Gamma}_{\preceq}$ in which all vertices have degrees in $p\mathbb{Z}$.

This notion is similar to p -regular subgraphs (which are those in which every vertex has degree p), so we shall call such full subgraphs p^+ -regular.

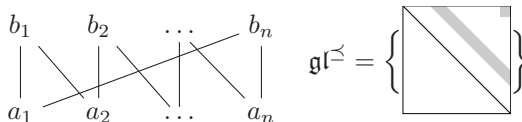
Height 1 also means that \mathfrak{gl}_n^{\prec} has all brackets zero, so we just need to count all p^+ -regular subgraphs with k edges to obtain $H_k(\mathfrak{gl}_n^{\prec}; \mathbb{Z}_p)$. This can still be quite complex, since subgraphs of different sizes contribute to different degrees.

Corollary 5.1. $\dim H_k(\mathfrak{gl}_n^{\prec}; \mathbb{Z}_2) = \sum_i \binom{n}{k-i} |\{\text{Eulerian subgraphs in } \bar{\Gamma}_{\preceq} \text{ of size } i\}|$.

This is because 2^+ -regular graphs are those in which every vertex has even degree, i.e., Eulerian graphs. We look at a few particular cases.

5.1. Path and cycle posets

Let our poset \preceq be given by the Hasse diagram below which is an n -cycle. Thus $\dim \mathfrak{gl}_{2n}^{\preceq} = 4n$.



Proposition 5.2. $\text{HP}_{C_*(\mathfrak{gl}^{\preceq}; \mathbb{Z}_2)} = (1+t)^{2n}(1+t^{2n})$ and $\text{HP}_{C_*(\mathfrak{gl}^{\preceq}; \mathbb{Z}_p)} = (1+t)^{2n}$ if $p > 2$.

Proof. In the given Hasse diagram, there are only two p^+ -regular subgraphs: the empty subgraph (wedge \emptyset) and the whole subgraph (wedge $\bigwedge_{i \in \mathbb{Z}_n} e_{a_i b_i} e_{a_i b_{i-1}}$). \square

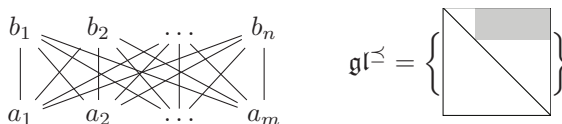
The case when Γ_{\preceq} is a path, or more generally a tree (of height 1), is trivial:

Proposition 5.3. If Γ_{\preceq} is a tree, then $H_k(\mathfrak{gl}_n^{\preceq}; \mathbb{Z}) \cong \mathbb{Z}^{\binom{n}{k}}$.

Proof. For any \preceq we have $H_k(\mathfrak{gl}_n^{\preceq}; \mathbb{Q}) \cong \mathbb{Q}^{\binom{n}{k}}$. In the case of a tree, any subgraph of $\bar{\Gamma}_{\preceq}$ is a forest, hence for any prime p there are no p^+ -regular subgraphs. Thus $H_k(\mathfrak{gl}_n^{\preceq}; \mathbb{Z})$ does not contain any torsion, and the result follows. \square

5.2. Complete bipartite posets

Let our poset \preceq be given by the Hasse diagram below which is a complete bipartite graph. Thus $\dim \mathfrak{gl}_{m+n}^{\preceq} = m + n + mn$.



Proposition 5.4. $\dim H_k(C_{*,p}(\mathfrak{gl}_n^{\leq}))$ equals the number of all $m \times n$ matrices with k entries 1 and $mn - k$ entries 0, such that the sum of every row/column is in $p\mathbb{Z}$.

Proof. The edges in $\bar{\Gamma}_{\geq}$ correspond to elements of $[m] \times [n]$, hence the wedges of $\bar{\Gamma}_{\geq}$ correspond to subsets of $[m] \times [n]$, or equivalently, to 0/1-matrices of size $m \times n$. The p^+ -regular subgraphs of $\bar{\Gamma}_{\geq}$ correspond to those 0/1-matrices in which the sum of every column and the sum of every row is a multiple of p . \square

Let $\text{HP}_{m,n,p}$ denote the generating function for the homology of our $\mathfrak{gl}_{m+n}^{\leq}(\mathbb{Z}_p)$. Thus $\text{HP}_{m,n,p} = \text{HP}_{n,m,p}$ and $\text{HP}_{1,n,p} = (1+t)^{n+1}$ and $\text{HP}_{m,n,p} = (1+t)^{m+n}$ for $p > \min\{m, n\}$. Let $\epsilon = e^{2\pi i/p} \in \mathbb{C}$ be the p -th primitive root of unity.

Proposition 5.5. $\text{HP}_{p,n,p} = (1+t)^{p+n} \sum_{i \in p\mathbb{N}} \binom{n}{i} t^{pi} = \frac{(1+t)^{p+n}}{p} \sum_{j \in [p]} (1 + \epsilon^j t^p)^n$.

Proof. If $m = p$, then in a p^+ -regular subgraph, every b_i has degree p or 0. Thus obtaining a p^+ -regular subgraph means that out of b_1, \dots, b_n we must choose $i \in p\mathbb{N}$ elements. Such a subgraph has pi edges. The second equality follows from Remark 2.2. \square

Proposition 5.6. $\text{HP}_{m,n,2} = \frac{(1+t)^{m+n}}{2^{m+n}} \sum_{i,j} \binom{m}{i} \binom{n}{j} (1+t)^{(m-i)(n-j)+ij} (1-t)^{(m-i)j+(n-j)i}$, therefore $\dim H_k(\mathfrak{gl}_{m+n}^{\leq}; \mathbb{Z}_2) = \sum_{i,j,l} \frac{\binom{m}{i} \binom{n}{j} \binom{(m-i)(n-j)+ij+m+n}{l} \binom{(m-i)j+(n-j)i}{k-l} (-1)^{k-l}}{2^{m+n}}$.

This formula and its proof are due to Richard Stanley (personal communication).

Proof. We call a 0/1-matrix E *even* when the sum of every column and row is even.

$$\begin{aligned} \text{HP}_{m,n,2} &= (1+t)^{m+n} \sum_{E \in \{0,1\}^{m \times n} \text{ even}} t^{\sum_{ij} E_{ij}} \\ &\stackrel{(1)}{=} \frac{(1+t)^{m+n}}{2^{m+n}} \sum_{E \in \{0,1\}^{m \times n}} t^{\sum_{ij} E_{ij}} \prod_i (1 + (-1)^{\sum_j E_{ij}}) \prod_j (1 + (-1)^{\sum_i E_{ij}}) \\ &\stackrel{(2)}{=} \frac{(1+t)^{m+n}}{2^{m+n}} \sum_{E \in \{0,1\}^{m \times n}} t^{\sum_{ij} E_{ij}} \left(\sum_{x \in \{-1,1\}^m} \prod_i x_i^{\sum_j E_{ij}} \right) \left(\sum_{y \in \{-1,1\}^n} \prod_j y_j^{\sum_i E_{ij}} \right) \\ &\stackrel{(3)}{=} \frac{(1+t)^{m+n}}{2^{m+n}} \sum_{E \in \{0,1\}^{m \times n}} t^{\sum_{ij} E_{ij}} \sum_{x \in \{-1,1\}^m, y \in \{-1,1\}^n} \left(\prod_i x_i^{\sum_j E_{ij}} \right) \left(\prod_j y_j^{\sum_i E_{ij}} \right) \\ &\stackrel{(4)}{=} \frac{(1+t)^{m+n}}{2^{m+n}} \sum_{x \in \{-1,1\}^m, y \in \{-1,1\}^n} \sum_{E \in \{0,1\}^{m \times n}} \prod_{i,j} (x_i y_j t)^{E_{ij}} \\ &\stackrel{(5)}{=} \frac{(1+t)^{m+n}}{2^{m+n}} \sum_{x \in \{-1,1\}^m, y \in \{-1,1\}^n} \prod_{i,j} (1 + x_i y_j t) \\ &\stackrel{(6)}{=} \frac{(1+t)^{m+n}}{2^{m+n}} \sum_{i,j} \binom{m}{i} \binom{n}{j} (1+t)^{(m-i)(n-j)+ij} (1-t)^{(m-i)j+(n-j)i}. \end{aligned}$$

Equation (1) holds, since $\prod_i (1 + (-1)^{\sum_j E_{ij}}) \prod_j (1 + (-1)^{\sum_i E_{ij}}) = 0$ whenever the sum of a row $\sum_j E_{ij}$ or column $\sum_i E_{ij}$ is odd. Furthermore, (2) is by the multibinomial theorem, (3) is just distributivity, (4) is just rearranging the sums and products. Equation (5) holds, because writing $\prod_{i,j} (1 + x_i y_j t)$ as a sum, there are 2^{mn} choices (i.e., 0/1-matrix) of picking 1 or $x_i y_j t$ as a factor. Equation (6) holds, since if we choose i out of m minuses for x and j out of n minuses for y , then $x_i y_j = 1$ for $(m-i)(n-j) + ij$ pairs and $x_i y_j = -1$ for $(m-i)j + (n-j)i$ pairs. \square

Remark 5.7. Denote $[\epsilon] = \{\epsilon^i; i \in [p]\}$. Since $\sum_{r \in [p]} \epsilon^{rs} = \begin{cases} p; & \text{if } s \in p\mathbb{N} \\ 0; & \text{if } s \notin p\mathbb{N} \end{cases}$, we have

$$\begin{aligned}
\text{HP}_{m,n,p} &= (1+t)^{m+n} \sum_{p^+-\text{regular } E \in \{0,1\}^{m \times n}} t^{\sum_{ij} E_{ij}} \\
&= \frac{(1+t)^{m+n}}{p^{m+n}} \sum_{E \in \{0,1\}^{m \times n}} t^{\sum_{ij} E_{ij}} \prod_i \left(\sum_{r \in [p]} \epsilon^r \sum_j E_{ij} \right) \prod_j \left(\sum_{r \in [p]} \epsilon^r \sum_i E_{ij} \right) \\
&= \frac{(1+t)^{m+n}}{p^{m+n}} \sum_{E \in \{0,1\}^{m \times n}} t^{\sum_{ij} E_{ij}} \left(\sum_{x \in [\epsilon]^m} \prod_i x_i^{\sum_j E_{ij}} \right) \left(\sum_{y \in [\epsilon]^n} \prod_j y_j^{\sum_i E_{ij}} \right) \\
&= \frac{(1+t)^{m+n}}{p^{m+n}} \sum_{E \in \{0,1\}^{m \times n}} t^{\sum_{ij} E_{ij}} \sum_{x \in [\epsilon]^m, y \in [\epsilon]^n} \left(\prod_i x_i^{\sum_j E_{ij}} \right) \left(\prod_j y_j^{\sum_i E_{ij}} \right) \\
&= \frac{(1+t)^{m+n}}{p^{m+n}} \sum_{x \in [\epsilon]^m, y \in [\epsilon]^n} \sum_{E \in \{0,1\}^{m \times n}} \prod_{i,j} (x_i y_j t)^{E_{ij}} \\
&= \frac{(1+t)^{m+n}}{p^{m+n}} \sum_{x \in [\epsilon]^m, y \in [\epsilon]^n} \prod_{i,j} (1 + x_i y_j t) = \frac{(1+t)^{m+n}}{p^{m+n}} \sum_{x \in \mathbb{Z}_p^m, y \in \mathbb{Z}_p^n} \prod_{i,j} (1 + \epsilon^{x_i + y_j} t).
\end{aligned}$$

Remark 5.8. We were pleasantly surprised that such beautiful and striking formulas can be derived. Moreover, by using the transfer matrix method [13, 4.7, p. 500], Matjaž Konvalinka obtained an alternative formula:

$$\text{HP}_{m,n,2} = \frac{(1+t)^{m+n}}{2^{m+n}} \sum_k \binom{n}{k} \left(\sum_i \left(-\binom{n}{2i} + 2 \sum_j \binom{k}{2j} \binom{n-k}{2i-2j} \right) t^{2i} \right)^m.$$

In fact, the transfer matrix of this 2-dimensional sequence $\text{HP}_{m,n,2}$ possesses a number of nice properties (symmetry, orthogonal set of eigenvectors, etc.).

Remark 5.9. In particular, the first few generating functions are

$$\begin{aligned}
\text{HP}_{2,n,2} &= \frac{(1+t)^{n+2} \left((1-t^2)^n + (1+t^2)^n \right)}{2}, \\
\text{HP}_{3,n,2} &= \frac{(1+t)^{n+3} \left(3(1-t^2)^n + (1+3t^2)^n \right)}{4}, \\
\text{HP}_{4,n,2} &= \frac{(1+t)^{n+4} \left(3(1-t^2)^{2n} + 4(1-t^4)^n + (1+6t^2+t^4)^n \right)}{8}, \\
\text{HP}_{5,n,2} &= \frac{(1+t)^{n+5} \left(10(1-t^2)^{2n} + 5(1+2t^2-3t^4)^n + (1+10t^2+5t^4)^n \right)}{16}.
\end{aligned}$$

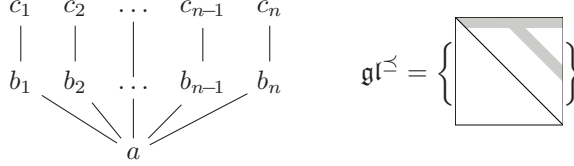
Since these are not of the form $\prod_i (1+t^{d_i})$, it follows that the cohomology is not isomorphic to any exterior polynomial algebra (it is a proper quotient, there are nontrivial relations). In fact, by Remark 2.3 the cup product corresponds to the wedge product, so multiplying two generators (i.e., 0/1-matrices E and E') in $H^*(\mathfrak{gl}_{m+n}^{\leftarrow}; \mathbb{Z}_p)$ gives either 0 (when they both have value 1 at some same entry: $\exists i, j: E_{ij} = 1 = E'_{ij}$) or their sum (when the matrices have disjoint supports: $\forall i, j: E_{ij} E'_{ij} = 0$). An interesting problem is to obtain a presentation, or at least the minimal number of generators for the cohomology algebra: how many even 0/1-matrices does one need so that every even 0/1-matrix is a product (i.e., disjoint sum) of these?

6. Posets of height 2

In this section, all posets are assumed to have height 2. A wedge in $C_{*,p}(\mathfrak{gl}_n^{\leftarrow})$ in this case corresponds to a subset of edges of the Hasse diagram Γ_{\leq} , together with the set of all directed paths of length 2 (two paths are identified when they have the same endpoints). For a given family of Hasse diagrams indexed by $n \in \mathbb{N}$, we denote by $\text{HP}_{n,p}$ the generating function for the (co)homology of $\mathfrak{gl}_n^{\leftarrow}(\mathbb{Z}_p)$.

6.1. Fork posets

Let our poset \preceq be given by the Hasse diagram below left. Thus $\dim \mathfrak{gl}_{2n+1}^{\preceq} = 5n + 1$. A specific faithful representation is shown below right.

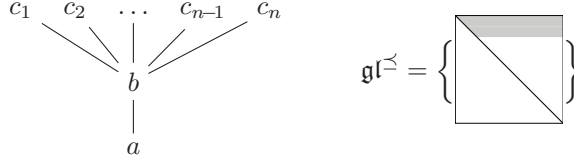


Proposition 6.1. $\text{HP}_{n,2} = (1+t)^{2n+1}(1+t^3)^n$ and $\text{HP}_{n,p} = (1+t)^{2n+1}$ for $p \geq 3$.

Proof. Only index a can have weight more than ± 2 , so $\tilde{C}_{*,p} \cong 0$ for $p \geq 3$. Every b_i has weight 0 or ± 1 , so $C_{*,2}$ is spanned by $\{\bigwedge_{i \in \sigma} e_{ab_i} e_{b_i c_i} e_{ac_i}; \sigma \subseteq [n]\}$, and the boundary of these wedges is 0, hence $\text{HP}_{C_{*,2}} = \sum_{k \in \mathbb{N}} \binom{n}{k} t^{3k} = (1+t^3)^n$. \square

6.2. Umbrella posets

Let our poset \preceq be given by the Hasse diagram below left. Thus $\dim \mathfrak{gl}_{n+2}^{\preceq} = 3n + 3$. An example of a faithful representation is below right.



Notice that \mathfrak{gl}^{\preceq} is the family of 2-step algebras from [1].

Proposition 6.2. $\text{HP}_{n,2} = \frac{(1+t)^{n+3}(1+t^2)^n + (1+t)^{n+1}(1-t^2)^{n+1}}{2}$.

Proof. Only indices a and b can have weight more than ± 2 , so $\tilde{C}_{*,p} \cong 0$ for $p \geq 3$. In a wedge $v \in C_{*,2}$, an index c_i has weight 0 (iff $e_{ac_i}, e_{bc_i} \notin v$) or 2 (iff $e_{ac_i}, e_{bc_i} \in v$), and for any choice $\sigma \subseteq [n]$ that determines which c_i appear in the wedge, b has weight $-|\sigma|$ (iff $e_{ab} \notin v$) or $1 - |\sigma|$ (iff $e_{ab} \in v$). Therefore the complex $C_{*,2}$ admits a basis

$\{e''_{\sigma} := \bigwedge_{i \in \sigma} e_{ac_i} e_{bc_i}; \sigma \subseteq [n], |\sigma| \in 2\mathbb{N}\} \cup \{e'_{\sigma} := e_{ab} \bigwedge_{i \in \sigma} e_{ac_i} e_{bc_i}; \sigma \subseteq [n], |\sigma| + 1 \in 2\mathbb{N}\}$,
with $\partial(e''_{\sigma}) = 0$ and $\partial(e'_{\sigma}) = 0$. Thus

$$\dim H_{2k}(\mathfrak{gl}^{\preceq}; \mathbb{Z}_2) = |\{\sigma \subseteq [n]; |\sigma| = k \in 2\mathbb{N}\}| \quad \text{and} \\ \dim H_{2k+1}(\mathfrak{gl}^{\preceq}; \mathbb{Z}_2) = |\{\sigma \subseteq [n]; |\sigma| = k \in 2\mathbb{N} + 1\}|.$$

Since $\sum_{k \in 2\mathbb{N}} t^k \binom{n}{k} + \sum_{k \in 2\mathbb{N}+1} t^k \binom{n}{k} = (1+t)^n$ and $\sum_{k \in 2\mathbb{N}} t^k \binom{n}{k} - \sum_{k \in 2\mathbb{N}+1} t^k \binom{n}{k} = (1-t)^n$, we have $\sum_{k \in 2\mathbb{N}} t^k \binom{n}{k} = \frac{(1+t)^n + (1-t)^n}{2}$ and $\sum_{k \in 2\mathbb{N}+1} t^k \binom{n}{k} = \frac{(1+t)^n - (1-t)^n}{2}$. Therefore

$$\text{HP}_{C_{*,2}(\mathfrak{gl}_n^{\preceq})} = \sum_{k \in 2\mathbb{N}} t^{2k} \binom{n}{k} + \sum_{k \in 2\mathbb{N}+1} t^{2k+1} \binom{n}{k}$$

$$= \frac{(1+t^2)^n + (1-t^2)^n}{2} + t \frac{(1+t^2)^n - (1-t^2)^n}{2} = \frac{(1+t)(1+t^2)^n + (1-t)(1-t^2)^n}{2}.$$

$$\text{Then } \text{HP}_{n,2} = (1+t)^{n+2} \text{HP}_{C_{*,2}} = \frac{(1+t)^{n+3}(1+t^2)^n + (1+t)^{n+1}(1-t^2)^{n+1}}{2}. \quad \square$$

Corollary 6.3. *If $\mathbb{Z}_2 \leq R$, then there is an isomorphism of graded R -algebras*

$$\begin{aligned} H^*(\mathfrak{gl}^{\preceq}; R) &\cong \Lambda[x_a, x_b, x_i, y_\omega, z_i; i \in [n], \omega \in \binom{[n]}{2}] / I, \\ I &= \langle y_{\{i,j\}} y_{\{j,k\}}, y_{\{i,j\}} y_{\{k,l\}} - y_{\{i,k\}} y_{\{j,l\}}, y_{\{i,j\}} z_i, y_{\{i,j\}} z_k - y_{\{i,k\}} z_j, z_i z_j \rangle, \end{aligned}$$

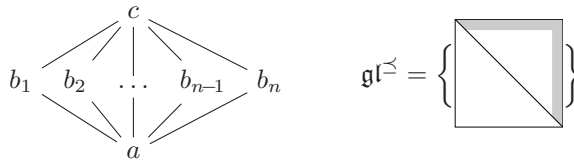
where $x_a \equiv e_{aa}$, $x_b \equiv e_{bb}$, $x_i \equiv e_{c_i c_i}$, $y_{\{i,j\}} \equiv e_{a c_i} e_{b c_i} e_{a c_j} e_{b c_j}$, $z_i \equiv e_{ab} e_{a c_i} e_{b c_i}$.

If $\mathbb{Z}_p \leq R$ with $p \geq 3$, then $H^*(\mathfrak{gl}^{\preceq}; R) \cong \Lambda[x_a, x_b, x_i]$.

Proof. In the proof above, we saw that the 2-complex is generated by the wedges $e''_\sigma = \bigwedge_{i \in \sigma} e_{a c_i} e_{b c_i}$ with $|\sigma| \in 2\mathbb{N}$ and $e'_\sigma = e_{ab} \bigwedge_{i \in \sigma} e_{a c_i} e_{b c_i}$ with $|\sigma| \in 2\mathbb{N} + 1$. Hence $H^*(\mathfrak{gl}^{\preceq}; R)$ is generated by the duals of all $e_\tau e''_\sigma$ and $e_\tau e'_\sigma$, where e_τ is the wedge of diagonals, as in [9, 5.10]. By Remark 2.3, cup products correspond to wedges. Thus e_τ is the product of diagonals, e''_σ is a (non-unique) product of those e''_ω with $|\omega| = 2$ (denoted by y_ω), and e'_σ is a (non-unique) product of an $e'_{\{i\}}$ (denoted by z_i) and $e''_{\sigma \setminus \{i\}}$. In this way, our cohomology algebra can be viewed as the subalgebra of the whole exterior algebra $\Lambda[e_{xy}; x \preceq y]$. The relations come from the property that a wedge behaves as a set, and is zero whenever it contains duplicate elements. \square

6.3. Diamond posets

Let our poset \preceq be given by the Hasse diagram below left. Thus $\dim \mathfrak{gl}_{n+2}^{\preceq} = 3n + 3$. A specific embedding into \mathfrak{gl}_n is shown below right.



Notice that \mathfrak{gl}^{\preceq} is the family of Heisenberg algebras, whose homology was determined in [2]. We compute H_* of the solvable analog (added diagonals).

Proposition 6.4. $\text{HP}_{n,2} = (1+t)^{n+2} (1+t^3) \frac{(1+t^2)^{n-1} + (1-t^2)^{n-1}}{2}$.

Proof. In a wedge $v \in C_*(\mathfrak{gl}^{\preceq})$, indices b_i have weight either ± 1 (iff $e_{ab_i} \notin v \ni e_{b_i c}$ or $e_{ab_i} \in v \not\ni e_{b_i c}$) or 0, so if v is contained in $C_{*,p}$, then every b_i appears twice or not at all in v . For any $\sigma \subseteq [n]$ denote $e''_\sigma = \bigwedge_{i \in \sigma} e_{ab_i} e_{b_i c}$ and $e'_\sigma = e_{ac} \bigwedge_{i \in \sigma} e_{ab_i} e_{b_i c}$. Then

$$\{e''_\sigma; |\sigma| \in p\mathbb{N}\} \cup \{e'_\sigma; |\sigma| + 1 \in p\mathbb{N}\}$$

is a basis for $C_{*,p}(\mathfrak{gl}^{\preceq})$, with $\partial(e''_\sigma) = -\sum_{i \in \sigma} e'_{\sigma \setminus \{i\}}$ and $\partial(e'_\sigma) = 0$. We define a Morse matching $\mathcal{M} = \{e''_{\sigma \cup \{n\}} \rightarrow e'_\sigma; n \notin \sigma\}$. Critical vertices are $\mathcal{M} = \{e''_\sigma; n \notin \sigma\} \cup \{e'_\sigma; n \in \sigma\}$.

Zig-zag paths come in pairs:

$$\begin{array}{ccc}
 e''_{\sigma} & \xrightarrow{\quad} & e'_{\sigma \setminus \{i\}} \\
 & \searrow \mathcal{M} & \nearrow \\
 e_{(\sigma \setminus \{i\}) \cup \{n\}} & \xrightarrow{\quad} & e_{(\sigma \setminus \{i,j\}) \cup \{n\}}
 \end{array}
 \quad \text{and} \quad
 \begin{array}{ccc}
 e''_{\sigma} & \xrightarrow{\quad} & e'_{\sigma \setminus \{j\}} \\
 & \searrow \mathcal{M} & \nearrow \\
 e_{(\sigma \setminus \{j\}) \cup \{n\}} & \xrightarrow{\quad} & e_{(\sigma \setminus \{i,j\}) \cup \{n\}}.
 \end{array}$$

Therefore the boundary is $\partial(e''_{\sigma}) = \sum_{i,j \in \sigma} 2 e'_{(\sigma \setminus \{i,j\}) \cup \{n\}}$. Over \mathbb{Z}_2 , this is zero, so

$$\begin{aligned}
 \dim H_{2k}(\mathfrak{gl}^{\prec}; \mathbb{Z}_2) &= |\{\sigma \subseteq [n-1]; |\sigma| = k \in 2\mathbb{N}\}| \quad \text{and} \\
 \dim H_{2k+1}(\mathfrak{gl}^{\prec}; \mathbb{Z}_2) &= |\{\sigma \subseteq [n-1]; |\sigma| = k-1 \in 2\mathbb{N}\}|.
 \end{aligned}$$

Every e''_{σ} contributes $t^{2|\sigma|}$ and every $e'_{\sigma \cup \{n\}}$ contributes $t^{2|\sigma|+3}$. Hence

$$\begin{aligned}
 \text{HP}_{C^{*,2}(\mathfrak{gl}^{\prec})} &= \sum_{k \in 2\mathbb{N}} t^{2k} \binom{n-1}{k} + \sum_{k \in 2\mathbb{N}} t^{2k+3} \binom{n-1}{k} \\
 (1+t^3) \sum_{k \in 2\mathbb{N}} (t^2)^k \binom{n-1}{k} &= (1+t^3) \frac{(1+t^2)^{n-1} + (1-t^2)^{n-1}}{2}.
 \end{aligned}$$

Over \mathbb{Z}_p with $p > 2$, additional matchings are required, but things become messy. \square

Corollary 6.5. *If $\mathbb{Z}_2 \leq R$, then there is an isomorphism of graded R -algebras*

$$H^*(\mathfrak{gl}^{\prec}; R) \cong \Lambda[x_a, x_i, x_c, y_{\omega}, z; i \in [n], \omega \in \binom{[n-1]}{2}] / \langle y_{\{i,j\}} y_{\{j,k\}} \rangle,$$

where $x_a \equiv e_{aa}$, $x_i \equiv e_{b_i b_i}$, $x_c \equiv e_{cc}$, $y_{\{i,j\}} \equiv e''_{\{i,j\}} + e''_{\{i,n\}} + e''_{\{j,n\}}$, $z \equiv e_{ac} e_{ab_n} e_{b_n c}$, where $e''_{\{i,j\}} = e_{ab_i} e_{b_i c} e_{ab_j} e_{b_j c}$ (the length of a wedge determines the degree).

Proof. In the proof above, we saw that the 2-complex is spanned by certain wedges $e''_{\sigma} = \bigwedge_{i \in \sigma} e_{ab_i} e_{b_i c}$ and $e'_{\sigma} = e_{ab} \bigwedge_{i \in \sigma} e_{ab_i} e_{b_i c}$. For $\bar{e}''_{\sigma} := e''_{\sigma} + \sum_{i \in \sigma} e''_{(\sigma \setminus \{i\}) \cup \{n\}}$, the set

$$\{\bar{e}''_{\sigma}, e''_{\tau \cup \{n\}}; \sigma, \tau \subseteq [n-1], |\sigma|, |\tau| + 1 \in 2\mathbb{N}\} \cup \{e'_{\sigma}; \sigma \subseteq [n], |\sigma| \in 2\mathbb{N} + 1\}$$

is a new basis, and after the same matching as above, the critical vertices are

$$\{\bar{e}''_{\sigma}; \sigma \subseteq [n-1], |\sigma| \in 2\mathbb{N}\} \cup \{e'_{\sigma \cup \{n\}}; \sigma \subseteq [n-1], |\sigma| \in 2\mathbb{N}\},$$

but this time there are no zig-zag paths, since $\partial(\bar{e}''_{\sigma}) = 0$. In this way, we obtained a subcomplex with zero boundaries that generates homology. Let $\bar{\chi}''_{\sigma}$ and $\chi'_{\sigma \cup \{n\}}$ be the dual wedges that generate cohomology, so that $\bar{\chi}''_{\sigma}(\bar{e}''_{\sigma}) = \bar{\chi}''_{\sigma}(e''_{\sigma}) = 1$. Since $\bar{e}''_{\sigma} \bar{e}''_{\tau} = \begin{cases} \bar{e}''_{\sigma \cup \tau}; & \text{if } \sigma \cap \tau = \emptyset \\ 0; & \text{if } \sigma \cap \tau \neq \emptyset \end{cases}$ and $\bar{e}''_{\sigma} e'_{\tau \cup \{n\}} = \begin{cases} e'_{\sigma \cup \tau \cup \{n\}}; & \text{if } \sigma \cap \tau = \emptyset \\ 0; & \text{if } \sigma \cap \tau \neq \emptyset \end{cases}$ and $e'_{\sigma} e'_{\tau} = 0$, these relations also hold for cup products of dual wedges, by Remark 2.3. Thus $H^*(\mathfrak{gl}^{\prec}; R)$ is isomorphic to the subalgebra of $\Lambda[e_{xy}; x \preceq y]$ generated by all $e_{ii}, \bar{e}''_{\{i,j\}}, e'_{\{n\}}$. \square

Corollary 6.6. *For $F(t) = \frac{t(1+t)^{n+1}}{1+t+t^2} (1 - (\frac{t}{1+2t+t^2})^{\lceil n/2 \rceil})$ and $\epsilon = e^{2\pi i/3}$ we have*

$$\text{HP}_{n,3} = \frac{(1+t)^{n+2}}{3} \sum_{i \in [3]} (1 + \epsilon^i t)(1 + \epsilon^i t^2)^n - \frac{t+1}{t} F(\epsilon^i t^2).$$

Proof. By the derivation of Proposition 6.4, for any prime p the generating function over \mathbb{Z}_p is

$$\begin{aligned}
& (1+t)^{n+2} \left(\sum_{k \in p\mathbb{N}} \binom{n}{k} - \text{rank}_{\mathbb{Z}_p} \partial_k \right) t^{2k} + \sum_{k \in p\mathbb{N}-1} \left(\binom{n}{k} - \text{rank}_{\mathbb{Z}_p} \partial_{k+1} \right) t^{2k+1} \\
&= (1+t)^{n+2} \sum_{k \in p\mathbb{N}} \binom{n}{k} t^{2k} + \left(\binom{n}{k-1} - \text{rank}_{\mathbb{Z}_p} \partial_k \right) t^{2k-1} \\
&= (1+t)^{n+2} \sum_{k \in p\mathbb{N}} \binom{n}{k} t^{2k} + \binom{n}{k-1} t^{2k-1} - \frac{t+1}{t} t^{2k} \text{rank}_{\mathbb{Z}_p} \partial_k \\
&= \frac{(1+t)^{n+2}}{p} \sum_{i \in [p]} (1 + \epsilon^i t^2)^n + t^{-1} \epsilon^i t^2 (1 + \epsilon^i t^2)^n - \frac{t+1}{t} F(\epsilon^i t^2) \text{ by Remark 2.2} \\
&= \frac{(1+t)^{n+2}}{p} \sum_{i \in [p]} (1 + \epsilon^i t)(1 + \epsilon^i t^2)^n - \frac{t+1}{t} F(\epsilon^i t^2),
\end{aligned}$$

where ∂_k is a combinatorial 0/1-matrix, namely the ‘subsets incidence’ matrix of size $\binom{n}{k-1} \times \binom{n}{k}$, where $\sigma\tau$ -th entry is 1 iff $\sigma \subseteq \tau$. By [10], $\text{rank}_{\mathbb{Z}_3} \partial_k = \sum_j \binom{n-2j-1}{k-j-1}$, so

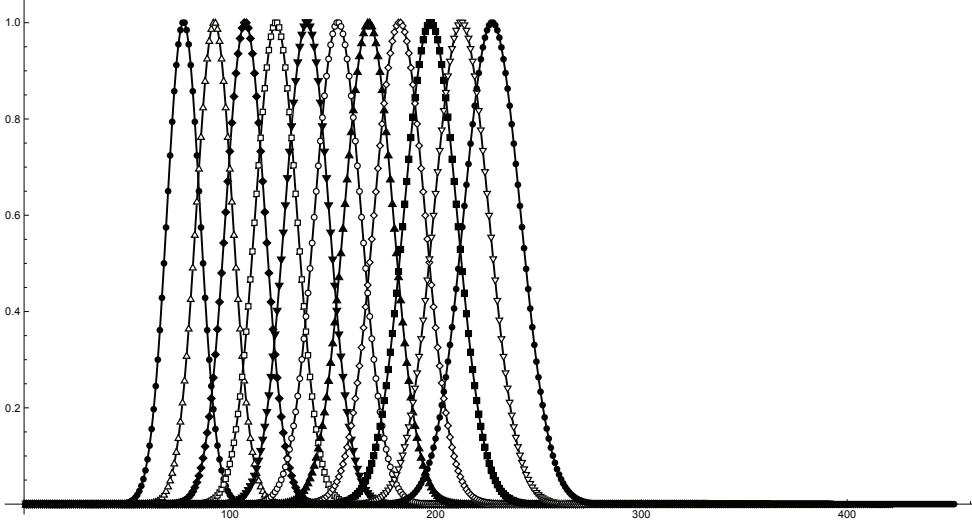
$$\begin{aligned}
F(t) &= \sum_k \text{rank}_{\mathbb{Z}_3} \partial_k t^k = \sum_{k,j \in \mathbb{N}} \binom{n-2j-1}{k-j-1} t^k = \sum_{j,k \in \mathbb{N}} \binom{n-2j-1}{k} t^{k+j+1} \\
&= t \sum_j t^j \sum_k \binom{n-2j-1}{k} t^k = t \sum_{0 \leq n-2j-1} t^j (1+t)^{n-2j-1} \\
&= t(1+t)^{n-1} \sum_{j \leq (n-1)/2} \left(\frac{t}{(1+t)^2} \right)^j = t(1+t)^{n-1} \left(\sum_{j \in \mathbb{N}} - \sum_{j \geq n/2} \right) \\
&= t(1+t)^{n-1} \left(1 - \left(\frac{t}{1+2t+t^2} \right)^{\lceil n/2 \rceil} \right) / \left(1 - \frac{t}{1+2t+t^2} \right) = \frac{t(1+t)^{n+1}}{1+t+t^2} \left(1 - \left(\frac{t}{1+2t+t^2} \right)^{\lceil n/2 \rceil} \right).
\end{aligned}$$

For cases $p \geq 5$ we feel intimidated to try computing the generating function. \square

7. Distributions of coefficients

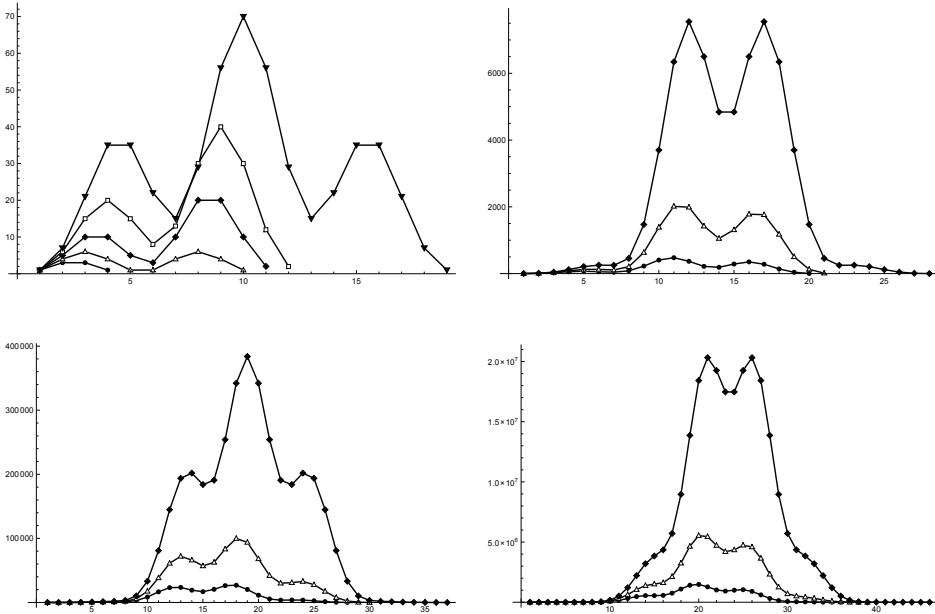
In this section, we observe how the coefficients of the generating functions (i.e., dimensions of homologies, i.e., Betti numbers) we computed are allocated. Some research in this direction was done in [11], where it was shown that the coefficients for nilpotent Lie algebras are often unimodal (i.e., ascending and then descending) or have an M-shape. Here we display much more exotic examples.

For the diamond poset, the weighted coefficients (divided by maximum) of $\text{HP}_{n,2}$ in Proposition 6.4 for $50 \leq n \leq 150$ with $n \in 10\mathbb{N}$ are plotted in the following picture:

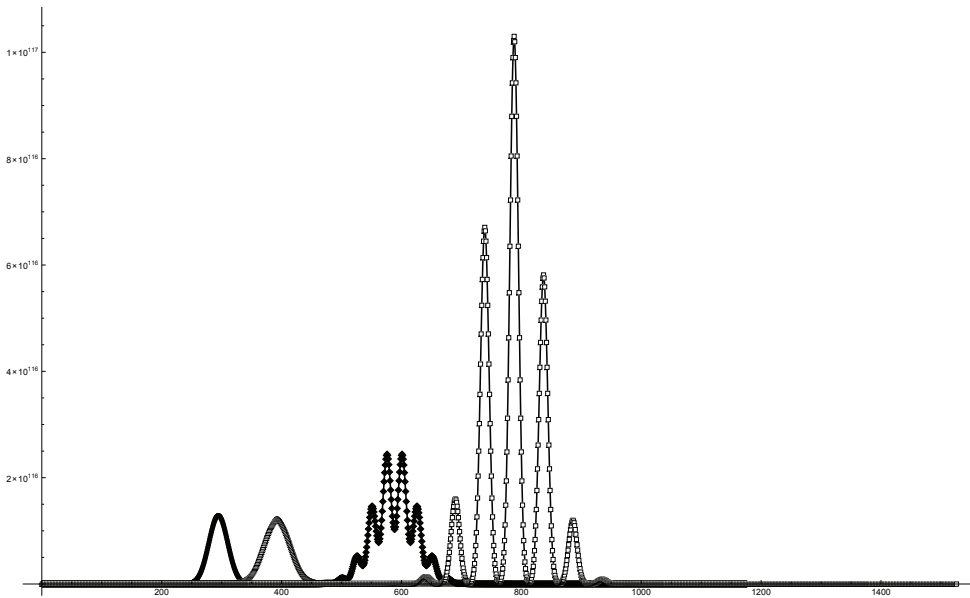


For the diamond poset, the coefficients of $\text{HP}_{n,3}$ in Corollary 6.6 for $1 \leq n \leq 14$ are plotted in the following pictures (though for large n the distribution becomes

binomial):



For the complete bipartite posets, the coefficients of $HP_{p,n,p}$ in Proposition 5.5 for $n = 200$ and $p = 2, 3, 5, 7$ (from left to right) are plotted in the following picture:



We suspect this provides an affirmative answer to the second question in [11, p. 86].

8. Afterword

8.1. Conclusion

We've seen, that AMT enables graph theory and combinatorics to come to the aid of homological algebra. It was unexpected that so much rich problems come from really simple posets of height only 1 and 2.

At present, very few Hilbert-Poincaré series for the (co)homology of Lie algebras are known (we provided six new families), and fewer still are the computed cup products in the literature (we provided three new families, which are quite interesting). The formulas seem to be nicest over rings of characteristic two, partly due to the fact that the primitive root of unity $e^{2\pi i/p} \in \mathbb{C}$ is real iff $p = 2$.

Calculations of $H^*(\mathfrak{gl}_n^{\prec}; \mathbb{Z}_p)$ for posets of larger height are also possible, but then it is easier to compute over large characteristics p . The use of the p -complex $C_{*,p}(\mathfrak{gl}_n^{\prec})$ for the right kind of posets \preceq and primes p can reduce the calculation of cohomology to mere enumerative combinatorics, which is the goal of AMT.

Acknowledgments

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