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# THE DECOMPOSABILITY OF A SMASH PRODUCT OF $\mathbf{A}_n^2$-COMPLEXES

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*Abstract*

In this paper, we determine the decomposability of the smash product of two indecomposable $\mathbf{A}_n^2$-complexes, i.e., $(n-1)$-connected finite CW-complexes with dimension at most $n+2$ $(n \geqslant 3)$.

## 1. Introduction

For a finite CW-complex $X$ with co-H-space structure, if $X \simeq X_1 \vee X_2$ with non-contractible $X_1$ and $X_2$, then $X$ is called decomposable; otherwise $X$ is called indecomposable. One of the basic problems in homotopy theory is to classify indecomposable homotopy types. Although it is impossible to find all indecomposable homotopy types, it is indeed possible to solve this problem in special situations. Let $\mathbf{A}_n^k$ $(n \geqslant k+1)$ be the homotopy category consisting of $(n-1)$-connected finite CW-complexes with dimension at most $n+k$. Any complex in $\mathbf{A}_n^k$ is a suspension and thus a co-H-space. Let $\mathbf{F}_n^k$ be the full subcategory of $\mathbf{A}_n^k$ consisting of complexes with torsion-free homology groups, $\mathbf{F}_{n(2)}^k$ the full subcategory of $\mathbf{A}_n^k$ consisting of complexes with 2-torsion-free homology groups, and $\mathbf{F}_{n(2,3)}^k$ the full subcategory of $\mathbf{A}_n^k$ consisting of complexes with 2- and 3-torsion-free homology groups.

In 1950, Chang classified the indecomposable homotopy types in $\mathbf{A}_n^2$ $(n \geqslant 3)$ [**6**], that is

(i) Spheres: $S^n$, $S^{n+1}$, $S^{n+2}$;

(ii) Elementary Moore spaces: $M_{p^r}^n$, $M_{p^r}^{n+1}$ where $p$ is a prime, $r \in \mathbb{Z}^+$ and $M_{p^r}^k$ denotes $M(\mathbb{Z}/p^r, k)$;

(iii) Elementary Chang complexes: $C_\eta^{n+2}$, $C^{(n+2),s}$, $C_r^{(n+2)}$, $C_r^{(n+2),s}$ (see Section 2.1) where $r, s \in \mathbb{Z}^+$, where $\mathbb{Z}^+$ denotes the set of positive integers.

A classification of indecomposable homotopy types of $\mathbf{A}_n^3$ $(n \geqslant 4)$ was given by Baues and Hennes in [**5**] and, for $k \geqslant 4$, the classification of indecomposable homotopy types of $\mathbf{A}_n^k$ $(n \geqslant k+1)$ is wild in the sense similar to that in the representation theory of finite dimensional algebras. Classifications of indecomposable homotopy types of $\mathbf{F}_n^k$ $(n \geqslant k+1)$ for $k = 4, 5, 6$ are given by Baues and Drozd in [**3, 4, 8**] and it is wild for $k \geqslant 7$. Based on these earlier results, in our previous papers [**11, 12**], we

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classify indecomposable homotopy types of $\mathbf{F}^k_{n(2,3)}$ $(n \geqslant k+1)$ for $k \leqslant 6$ and $\mathbf{F}^4_{n(2)}$ $(n \geqslant 5)$.

As pointed out by Wu [**18**], starting from an explicit space $X$, one obtains more indecomposable spaces from the self-smash products of $X$ since self-smash products of co-H-spaces admit decompositions. This motivates us to consider the decomposability of smash products of different indecomposable complexes. There are only a few results for this problem. The decomposability of $M(\mathbb{Z}/p^r, n) \wedge M(\mathbb{Z}/p^s, n)$ is well known [**9**]. Wu [**16**] proved that $M(\mathbb{Z}/2, n) \wedge C^{n+2}_\eta$ and $C^{n+2}_\eta \wedge C^{n+2}_\eta$ are indecomposable. As a main result in this paper, we determine the decomposability of all remaining smash products of two indecomposable complexes in $\mathbf{A}^2_n$ $(n \geqslant 3)$ and give the decomposition whenever possible. Since the suspension functor $\Sigma\colon \mathbf{A}^2_n \to \mathbf{A}^2_{n+1}$ is an equivalence for $n \geqslant 3$, it suffices to deal with the case $n = 3$.

**Theorem 1.1** (**Main theorem**). *For* $r, s, r', s', u \in \mathbb{Z}^+$,

- $M^3_{2^u} \wedge C^5_\eta$, $C^5_r \wedge C^{5,s}$, $C^5_r \wedge C^5_{r'}$, $C^{5,s} \wedge C^{5,s'}$, $C^5_\eta \wedge C^5_r$, $C^5_\eta \wedge C^{5,s}$, $C^5_\eta \wedge C^{5,s}_r$ *are indecomposable;*
- $M^3_{2^u} \wedge C^5_r$ *and* $M^3_{2^u} \wedge C^{5,r}$ *are*
  - ◇ *indecomposable for* $u > r$*;*
  - ◇ *homotopy equivalent to* $M^3_{2^u} \wedge C^5_\eta \vee M^7_{2^u}$ *for* $r \geqslant u$*;*
- $M^3_{2^u} \wedge C^{5,s}_r$ *is homotopy equivalent to*
  - ◇ $C^{8,s}_r \vee C^{9,s}_r$ *for* $u > r, s$*;*
  - ◇ $M^3_{2^u} \wedge C^5_r \vee M^7_{2^u}$ *for* $r < u \leqslant s$*;*
  - ◇ $M^3_{2^u} \wedge C^{5,s} \vee M^7_{2^u}$ *for* $s < u \leqslant r$*;*
  - ◇ $M^3_{2^u} \wedge C^5_\eta \vee M^7_{2^u} \vee M^7_{2^u}$ *for* $u \leqslant r$ *and* $u \leqslant s$*;*
- $C^5_u \wedge C^{5,s}_r$ *is*
  - ◇ *homotopy equivalent to* $C^{9,s}_r \vee C^5_\eta \wedge C^{5,s}_r$ *for* $u \geqslant r$ *and* $u \geqslant s$*;*
  - ◇ *homotopy equivalent to* $C^{9,r}_s \vee C^5_\eta \wedge C^{5,s}_s$ *for* $u = s < r$*;*
  - ◇ *indecomposable, otherwise;*
- $C^{5,u} \wedge C^{5,s}_r$ *is*
  - ◇ *homotopy equivalent to* $C^{9,s}_r \vee C^5_\eta \wedge C^{5,s}_r$ *for* $u \geqslant r$ *and* $u \geqslant s$*;*
  - ◇ *homotopy equivalent to* $C^{9,r}_s \vee C^5_\eta \wedge C^{5,r}_r$ *for* $u = r < s$*;*
  - ◇ *indecomposable, otherwise;*
- $C^{5,s}_r \wedge C^{5,s'}_{r'}$
  - ◇ *if* $s \geqslant r, r', s'$

$$C^{5,s}_r \wedge C^{5,s'}_{r'} \simeq \begin{cases} C^{9,s}_r \vee C^{9,s}_r \vee C^5_\eta \wedge C^{5,s}_r, & s = s' = r' > r; \\ C^{9,s}_r \vee C^5_{r'} \wedge C^{5,s}_r, & s = s' > r' > r; \\ C^{9,s}_r \vee C^{5,s'} \wedge C^{5,s}_r, & s = r' > s' > r; \\ C^{9,s'}_{r'} \vee C^{9,s'}_{r'} \vee C^5_\eta \wedge C^{5,s'}_{r'}, & s \geqslant r \geqslant r', s'; \\ C^{9,s'}_{r'} \vee C^{9,r'}_{s'} \vee C^5_\eta \wedge C^{5,r}_r, & s \geqslant r' > s' = r; \\ C^{9,s'}_{r'} \vee C^5_r \wedge C^{5,s'}_{r'}, & \textit{otherwise}; \end{cases}$$

  - ◇ *if* $r \geqslant r', s'$ *and* $r > s$

$$C_r^{5,s} \wedge C_{r'}^{5,s'} \simeq \begin{cases} C_r^{9,s} \vee C_r^{9,s} \vee C_\eta^5 \wedge C_r^{5,s}, & r = r' = s' > s; \\ C_r^{9,s} \vee C^{5,s'} \wedge C_r^{5,s}, & r = r' > s' > s; \\ C_r^{9,s} \vee C_{r'}^5 \wedge C_r^{5,s}, & r = s' > r' > s; \\ C_{r'}^{9,s'} \vee C_{r'}^{9,s'} \vee C_\eta^5 \wedge C_{r'}^{5,s'}, & r > s \geqslant r', s'; \\ C_{r'}^{9,s'} \vee C_{s'}^{9,r'} \vee C_\eta^5 \wedge C_s^{5,s}, & r \geqslant s' > r' = s; \\ C_{r'}^{9,s'} \vee C^{5,s} \wedge C_{r'}^{5,s'}, & \textit{otherwise.} \end{cases}$$

*Remark 1.2.* For $M_{p^r}^3$, prime $p \neq 2$, it is easy to check that $M_{p^r}^3 \wedge C_\eta^5 \simeq M_{p^r}^6 \vee M_{p^r}^8$; $M_{p^r}^3 \wedge C_{r'}^5 \simeq M_{p^r}^8$; $M_{p^r}^3 \wedge C^{5,s} \simeq M_{p^r}^6$; $M_{p^r}^3 \wedge C_r^{5,s} \simeq *$; $M_{p^r}^3 \wedge M_{q^r}^3 \simeq *$ for prime $q \neq p$ ($*$ is the point space). We will not discuss these cases any more in the following.

It is known from Theorem 1.2 of [**14**] that for any $p$-local CW-complex, there is a functorial decomposition $\Omega\Sigma X \simeq A^{min}(X) \times \Omega(\bigvee_{n=2}^\infty Q_n^{max}(X))$ which is useful to calculate homotopy groups of $\Sigma X$. $Q_n^{max}(X)$ is a wedge summand of $\Sigma X^{(n)}$, where $X^{(n)}$ is an $n$-fold self-smash product of $X$. In order to determine the homotopy type of $Q_n^{max}(X)$, it is significant to decompose $X^{(n)}$ to a wedge of indecomposable spaces. The decomposition of self-smash products is easy for $M_{p^r}^n$ $(p > 2)$ and $M_{2^s}^n$ $(s > 1)$. The decomposition of self-smash products is obtained by Wu [**16**] for $M_2^n$ and $C_\eta^n$. In a sequel we will study the decomposition of self-smash products for Chang complexes.

Section 2 contains necessary notation and lemmas. Related results of elementary Moore spaces and Chang-complexes are stated in Section 3. In Section 5, we prove the last part of Theorem 1.1 by determining the decomposition of $C_r^{5,s} \wedge C_{r'}^{5,s'}$ while the proof of other cases in Theorem 1.1 is given in Section 4.

## 2. Preliminaries

### 2.1. Some notation

- All spaces are suspensions of simply connected finite CW-complexes.
- $|G|$ denotes the order of a group $G$ and $|g|$ denotes the order of an element $g$ in group $G$. If $G$ is an abelian group with decomposition $G \cong C_1 \oplus C_2 \oplus \cdots \oplus C_m$, where $C_t$ is a cyclic group with order infinity or a power of a prime for $t = 1, \cdots, m$, then define $\dim G := m$.
- If $X$ is a subspace of $L$, $Y \simeq L/X$, then $i$ denotes the canonical inclusion $X \hookrightarrow L$, $q$ denotes the canonical projection $L \twoheadrightarrow Y$. Especially for Moore spaces $M_{2^k}^n$, sometimes we denote $i\colon S^n \hookrightarrow M_{2^k}^n$ by $i_n$ and $q\colon M_{2^k}^n \twoheadrightarrow S^{n+1}$ by $q_n$.
- Denote by $H_*X := H_*(X;\mathbb{Z})$ and $H^*(X;\mathbb{Z}/2)$ the **reduced** homology groups and cohomology groups of the space $X$, respectively.
- Let $\mathbf{C}_f$ be the mapping cone of a map $f$. Denote by $[\mathbf{C}_f, \mathbf{C}_{f'}]_\beta^\alpha$ the set of homotopy classes of maps $h$ which satisfy the following homotopy commutative diagram:

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \longrightarrow & \mathbf{C}_f & \longrightarrow & \Sigma X \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle\Sigma\alpha} \\ X' & \xrightarrow{f'} & Y' & \longrightarrow & \mathbf{C}_{f'} & \longrightarrow & \Sigma X'. \end{array}$$

- For $X_i$ and $Y_j$ ($i = 1, \cdots, t,\ j = 1, \cdots, s$) objects in an additive category (such as $\mathbf{A}_n^k$ ($k \geqslant n+2$) or the category of abelian groups), we denote by $f := (f_{ij}) = \begin{pmatrix} f_{11} & \cdots & f_{1t} \\ \cdots & \cdots & \cdots \\ f_{s1} & \cdots & f_{st} \end{pmatrix} : \bigoplus_{i=1}^t X_i \to \bigoplus_{j=1}^s Y_j$ a morphism such that $p_{Y_i} f j_{X_j} = f_{ij}$, where $j_{X_j}$ and $p_{Y_i}$ are canonical inclusions and projections, respectively. Sometimes, $(f_{ij})$ is written graphically as follows to indicate the domain and codomain:

$$\begin{array}{c|cccc|} & X_1 & X_2 & \cdots & X_t \\ \hline Y_1 & f_{11} & f_{12} & \cdots & f_{1t} \\ Y_2 & f_{21} & f_{22} & \cdots & f_{2t} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ Y_s & f_{s1} & f_{s2} & \cdots & f_{st} \\ \hline \end{array}.$$

  To simplify the text, we fix names for the following elementary transformations:
  (i) $-\mathbf{r}_n$ ($-\mathbf{c}_n$): composing $n$-th row (column) with $-id$;
  (ii) $\mathbf{c}_m f + \mathbf{c}_n$: adding the $m$-th column, composed with map $f$, to the $n$-th column;
  (iii) $g\mathbf{r}_m + \mathbf{r}_n$: adding the $m$-th row, composed with map $g$, to the $n$-th row;
  (iv) $k\mathbf{r}_m + \mathbf{r}_n$ ($k\mathbf{c}_m + \mathbf{c}_n$): adding $k$ times the $m$-th row (column) to the $n$-th row (column), where $k \in \mathbb{Z}^+$.
- In the rest of the paper, $S_\omega^k = S^k$ for $\omega \in \{1, 2, 3, 4, a, b\}$.
- $\eta = \eta_n \in [S^{n+1}, S^n]$ is a Hopf map for $n \geqslant 3$ and $\varrho = \varrho_n \in [S^{n+3}, S^n] \cong \mathbb{Z}/24$ is a fixed generator for $n \geqslant 5$.
- $\kappa, \kappa', \varepsilon, \varepsilon' \in \{0, 1\}$.
- $k$ times the identity map $id$ of $\Sigma X$ is written as $k\colon \Sigma X \to \Sigma X$ for nonzero integer $k$ (hence, $1 = id$).
- For $k \geqslant 5$, and $r, s \in \mathbb{Z}^+$, let
  $C^{k,s} = (S^{k-2} \vee S^{k-1}) \bigcup_{\binom{\eta}{2^s}} \mathbf{C}S^{k-1}$; $\quad C_r^k = S^{k-2} \bigcup_{(2^r, \eta)} \mathbf{C}(S^{k-2} \vee S^{k-1})$;
  $C_r^{k,s} = (S^{k-2} \vee S^{k-1}) \bigcup_{\begin{pmatrix} 2^r, & \eta \\ 0\ , & 2^s \end{pmatrix}} \mathbf{C}(S^{k-2} \vee S^{k-1})$; $\quad C_\eta^k = S^{k-2} \bigcup_\eta \mathbf{C}S^{k-1}$.

## 2.2. Spanier–Whitehead duality

If $\mathbf{A}_n^k$ is in the stable range, i.e., $[X, Y] \xrightarrow{\Sigma^m} [\Sigma^m X, \Sigma^m Y]$ is isomorphic for any $X, Y \in \mathbf{A}_n^k$ and any $m \in \mathbb{Z}^+$, then there is a contravariant isomorphism of additive categories

$$D = D_{2n+k}\colon \mathbf{A}_n^k \to \mathbf{A}_n^k,$$

which is called Spanier–Whitehead duality (or $(2n+k)$-duality).

$D$ satisfies the following properties from [**2, 7, 13**]:

**Proposition 2.1.**

*(i)* $D^2$ *is equal to the identity functor;*

*(ii)* $[X, Y] \xrightarrow{D(\cong)} [DY, DX] \cong [DY \wedge X, S^{2n+k}]$*;*

*(iii)* $[S^{n+q}, DX] \cong [X, S^{n+k-q}]$ *and* $[S^{n+q}, X] \cong [DX, S^{n+k-q}]$*for* $n \leqslant q \leqslant n+k$*;*

*(iv)* $D(X\vee Y)\simeq DX\vee DY$; $D(X\wedge Y)\simeq DX\wedge DY$, *that is for* $X\in\mathbf{A}_n^k, Y\in\mathbf{A}_m^l$, *then* $X\wedge Y\in\mathbf{A}_{n+m}^{k+l}$ *and* $D_{2(n+m)+k+l}(X\wedge Y)\simeq D_{2n+k}X\wedge D_{2m+l}Y$.

*(vi)* *Let* $\{X,Y\}:=\lim\limits_{m\to+\infty}[\Sigma^m X,\Sigma^m Y]$. *Then for any CW-complex* $Z$,

$$\{X\wedge Y,Z\}\cong\{X\wedge S^{2n+k},DY\wedge Z\}.\qquad\square$$

*Note*: It follows from (i) and (iv) above that $X$ is indecomposable if and only if $DX$ is indecomposable.

*Example 2.2.* (Page 49 of [**1**]) For the Spanier–Whitehead duality $D\colon\mathbf{A}_n^2\to\mathbf{A}_n^2$ $(n\geqslant 3)$, we have $DS^n=S^{n+2}$, $DS^{n+1}=S^{n+1}$, $DM_{p^r}^n=M_{p^r}^{n+1}$, $DC_\eta^{n+2}=C_\eta^{n+2}$, $DC_r^{(n+2),s}=C_s^{(n+2),r}$, $DC_r^{n+2}=C^{(n+2),r}$.

### 2.3. Some lemmas

**Lemma 2.3.** *([**7**]) For a cofibre sequence* $X\xrightarrow{f}Y\xrightarrow{i}\mathbf{C}_f$,

$$X\wedge Z\xrightarrow{f\wedge id}Y\wedge Z\xrightarrow{i\wedge id}\mathbf{C}_f\wedge Z$$

*is also a cofibre sequence. That is* $\mathbf{C}_{f\wedge id}\simeq\mathbf{C}_f\wedge Z$. $\square$

**Lemma 2.4.** *(Lemma 14.30 of [**13**]) For*

$$X\xrightarrow{f}U\xrightarrow{i}\mathbf{C}_f,\qquad Y\xrightarrow{g}V\xrightarrow{i}\mathbf{C}_g,$$

$\mathbf{C}_f\wedge\mathbf{C}_g=(U\wedge V)\bigcup_\mu\mathbf{C}(X\wedge V\vee U\wedge Y)\bigcup_\nu\mathbf{CC}(X\wedge Y)$ *where* $\mu=(f\wedge id,id\wedge g)$ *and* $(\mathbf{C}_f\wedge\mathbf{C}_g)/(U\wedge V)\simeq(\Sigma(X\wedge V)\vee\Sigma(U\wedge Y))\bigcup_{\nu'}\mathbf{C}\Sigma(X\wedge Y)$, *where*

$$\nu'=\begin{pmatrix}\Sigma id\wedge g\\ -\Sigma f\wedge id\end{pmatrix}.\qquad\square$$

**Lemma 2.5.** *Let* $A\in\mathbf{A}_6^4$, *with homology groups of the form* $\mathbb{Z}^r\oplus\mathbb{Z}/2^{r_1}\oplus\cdots\oplus\mathbb{Z}/2^{r_\varsigma}$ *for some non-negative integers* $r,r_1,\cdots,r_\varsigma$. *Suppose that*

*(i)* $\dim H_9A+\dim H_{10}A=1$ *and* $H^6(A;\mathbb{Z}/2)\cong H^{10}(A;\mathbb{Z}/2)\cong\mathbb{Z}/2$ *with generators* $a_6$ *and* $a_{10}$, *respectively, satisfying* $Sq^4a_6=a_{10}$;

*(ii)* $\dim H^8(A;\mathbb{Z}/2)\geqslant 2$ *and there are nonzero elements* $a_8\neq a_8'\in H^8(A;\mathbb{Z}/2)$ *such that* $Sq^2a_8=Sq^2a_8'=a_{10}$;

*(iii)* $Sq^2a_6=a_8+a_8'+a_8''\neq 0$ *for some* $a_8''\in H^8(A;\mathbb{Z}/2)$ *such that* $Sq^2a_8''=0$.

*If* $A\simeq X\vee Y$ *and* $H_6X\neq 0$, *then* $H_tX\cong H_tA$ *for* $t=6,9,10$ *and* $\dim H^8(X;\mathbb{Z}/2)\geqslant 2$, *hence* $\dim H_7X+\dim H_8X\geqslant 2$.

*Proof.* We have

$$H^*(X;\mathbb{Z}/2)\underset{j_1^*}{\overset{p_1^*}{\rightleftarrows}}H^*(A;\mathbb{Z}/2)\underset{j_2^*}{\overset{p_2^*}{\leftrightarrows}}H^*(Y;\mathbb{Z}/2)\ ,$$

where $j_1,j_2,p_1,p_2$ are the canonical inclusions and projections and $j_u^*p_u^*=id$ which implies that $p_u^*$ is injective and $j_u^*$ is surjective for $u=1,2$.

Since $H_6X\neq 0$, we get that $H_6X\cong H_6A$, $H_6Y=0$ and $H^6(p_1;\mathbb{Z}/2)$ is isomorphic, hence there is $0\neq x_6\in H^6(X;\mathbb{Z}/2)$ such that $p_1^*(x_6)=a_6$. It follows from

$p_1^*(Sq^4x_6) = Sq^4p_1^*(x_6) = a_{10} \neq 0$ that $0 \neq Sq^4x_6 \in H^{10}(X;\mathbb{Z}/2)$ which implies $H_9X \cong H_9A$ and $H_{10}X \cong H_{10}A$ by (i).

By $j_1^*(a_{10}) = j_1^*(p_1^*(Sq^4x_6)) = Sq^4x_6 \neq 0$ and $Sq^2j_1^*(a_8) = Sq^2j_1^*(a_8') = j_1^*(a_{10})$, we get $j_1^*(a_8) \neq 0$ and $j_1^*(a_8') \neq 0$ in $H^8(X;\mathbb{Z}/2)$.

Since $Sq^2a_6 = a_8 + a_8' + a_8'' \neq 0$ and $Sq^2a_8'' = 0$, $p_1^*(Sq^2x_6) = Sq^2(a_6) = a_8 + a_8' + a_8'' \neq 0$ and $p_1^*Sq^2Sq^2x_6 = 2a_{10} = 0$, thus $Sq^2x_6 \neq 0$ and $Sq^2Sq^2x_6 = 0$. But $Sq^2(j_1^*(a_8)) = j_1^*(a_{10}) \neq 0$, we have $j_1^*(a_8) \neq Sq^2x_6$, thus $\dim H^8(X;\mathbb{Z}/2) \geqslant 2$.

It follows from $H^8(X;\mathbb{Z}/2) = Hom(H_8X,\mathbb{Z}/2) \oplus Ext(H_7X,\mathbb{Z}/2)$ that $\dim H_7X + \dim H_8X \geqslant 2$. □

A complex $X$ is called 2-local if all homotopy groups, or equivalently all homology groups, of $X$ are finitely generated $\mathbb{Z}_{(2)}$-modules, where $\mathbb{Z}_{(2)}$ is the 2-localization of $\mathbb{Z}$. Let $X_{(2)}$ be the 2-localization of $X$ and denote by $X \simeq_{(2)} Y$ if $X_{(2)} \simeq Y_{(2)}$.

**Lemma 2.6.**

*(i) Let $X_1 = S^m \cup_{f_1} \mathbf{C}X_1'$, $X_2 = S^m \cup_{f_2} \mathbf{C}X_2'$ be two (resp. 2-local) complexes, where $X_1'$ and $X_2'$ are $m$-connected. If $X_1 \simeq X_2$ and $H_mX_1 = \mathbb{Z}/2^s$ for some $s \in \mathbb{Z}^+$, then $X_1/S^m \simeq_{(2)} X_2/S^m$ (resp. $X_1/S^m \simeq X_2/S^m$), i.e., $\Sigma X_1' \simeq_{(2)} \Sigma X_2'$ (resp. $\Sigma X_1' \simeq \Sigma X_2'$).*

*(ii) Let $X_1 = X_1^{(n-1)} \cup_{g_1} \mathbf{C}S^{n-1}$, $X_2 = X_2^{(n-1)} \cup_{g_2} \mathbf{C}S^{n-1}$ be two (resp. 2-local) complexes, where $X_1^{(n-1)}$ and $X_2^{(n-1)}$ are $(n-1)$-skeleton of $X_1$ and $X_2$, respectively. If $X_1 \simeq X_2$ and $[X_1, S^n] = \mathbb{Z}/2^t$ for some $t \in \mathbb{Z}^+$, then $X_1^{(n-1)} \simeq_{(2)} X_2^{(n-1)}$ (resp. $X_1^{(n-1)} \simeq X_2^{(n-1)}$).*

*Proof.* It suffices to prove when $X_1$ and $X_2$ are 2-local.

We only prove (i), since the proof of (ii) is dual to the proof of (i).

There are cofibre sequences

$$X_j' \xrightarrow{f_j} S^m \xrightarrow{i_j} X_j \xrightarrow{q_j} \Sigma X_j' \xrightarrow{\Sigma f_j} S^{m+1}, \qquad j = 1, 2.$$

Given a homotopy equivalence $\alpha\colon X_1 \xrightarrow{\simeq} X_2$, it induces

$$\alpha_*\colon \pi_mX_1 \xrightarrow{\cong} \pi_mX_2 \cong \mathbb{Z}/2^s.$$

Since $i_j$ is a generator of $\pi_mX_j$ for $j = 1, 2$, there is an odd integer $k$ such that $\alpha_*ki_1 = i_2 \in \pi_mX_2$. Since $X_1$ and $X_2$ are 2-local, $k\alpha\colon X_1 \to X_2$ is also a homotopy equivalence. Take $\alpha' \in [\Sigma X_1', \Sigma X_2']^{id}_{k\alpha}$, then we have $\alpha'\colon \Sigma X_1' \xrightarrow{\simeq} \Sigma X_2'$. □

## 3. Moore spaces and Chang-complexes

In this section, we will collect some basic facts about Chang-complexes and Moore spaces.

### 3.1. Moore spaces

We list some results of maps between Moore spaces from [**5**].

$$[M^n_{2^r}, M^n_{2^t}] = \begin{cases} \mathbb{Z}/4\langle B(\chi)\rangle, & r = t = 1; \\ \mathbb{Z}/2^{min(r,t)}\langle B(\chi)\rangle \oplus \mathbb{Z}/2\langle i\eta q\rangle, & \text{otherwise}, \end{cases} \qquad (n \geqslant 3)$$

where $B(\chi)$ is given by Proposition 2.3 of [**5**], which satisfies

$$H_nB(\chi) = \chi\colon \mathbb{Z}/2^r \to \mathbb{Z}/2^t, \qquad \chi(1) = 1.$$

$B(\chi) \in [M^n_{2^r}, M^n_{2^t}]^{2^{r-t}}_{id}$ for $r \geqslant t$ and $B(\chi) \in [M^n_{2^r}, M^n_{2^t}]^{id}_{2^{r-t}}$ for $r \leqslant t$; If $r = t$, then $B(\chi) = id$ and if $r = t = 1$, then $i\eta q = 2B(\chi) = 2$.

$$[S^{n+1}, M^n_{2^t}] = \mathbb{Z}/2\langle i\eta\rangle, \quad [M^n_{2^t}, S^n] = \mathbb{Z}/2\langle \eta q\rangle. \qquad (n \geqslant 3)$$

$$[M^{n+1}_{2^t}, S^n] = \begin{cases} \mathbb{Z}/4\langle \eta^1\rangle, & t = 1; \\ \mathbb{Z}/2\langle \eta^t\rangle \oplus \mathbb{Z}/2\langle \eta\eta q\rangle, & t > 1. \end{cases} \qquad (n \geqslant 3)$$

$$[S^{n+2}, M^n_{2^t}] = \begin{cases} \mathbb{Z}/4\langle \xi_1\rangle, & t = 1; \\ \mathbb{Z}/2\langle \xi_t\rangle \oplus \mathbb{Z}/2\langle i\eta\eta\rangle, & t > 1. \end{cases} \qquad (n \geqslant 4)$$

Here we choose a generator $\xi_1$ and set $\xi_t = B(\chi)\xi_1$. The generator $\eta^1$ is the dual map of $\xi_1$ and $\eta^t = \eta^1 B(\chi)$, and $q\xi_t = \eta$, $\eta^t i = \eta$ for $t \geqslant 1$.

$$[M^{n+1}_{2^s}, M^n_{2^r}] = \begin{cases} \mathbb{Z}/2\langle \xi^1_1\rangle \oplus \mathbb{Z}/2\langle \eta^1_1\rangle, & s = r = 1; \\ \mathbb{Z}/4\langle \xi^s_1\rangle \oplus \mathbb{Z}/2\langle \eta^s_1\rangle, & s > 1 = r; \\ \mathbb{Z}/2\langle \xi^1_r\rangle \oplus \mathbb{Z}/4\langle \eta^1_r\rangle, & s = 1 < r; \\ \mathbb{Z}/2\langle \xi^s_r\rangle \oplus \mathbb{Z}/2\langle \eta^s_r\rangle \oplus \mathbb{Z}/2\langle i\eta\eta q\rangle, & \text{otherwise}, \end{cases} \qquad (n \geqslant 4)$$

where $\xi^s_r = B(\chi)\xi_1 \in [M^{n+1}_{2^s}, M^n_{2^r}]^\eta_0$, $\eta^s_r = i\eta^1 B(\chi) \in [M^{n+1}_{2^s}, M^n_{2^r}]^0_\eta$. Note that

$$2\xi^s_1 = i\eta\eta q \ (s > 1); \qquad 2\eta^1_r = i\eta\eta q \ (r > 1).$$

Let $\lambda_{11} := \xi^s_r + \eta^s_r$, then $[M^{n+1}_{2^s}, M^n_{2^r}]^\eta_\eta = \{\lambda_{11}, \lambda_{11} + i\eta\eta q\}$.

The group $[S^{n+3}, M^n_{2^r}]$ is given by the following lemma.

**Lemma 3.1.** *Let $n \geqslant 5$. Then*

$$[S^{n+3}, M^n_2] = \mathbb{Z}/2\langle i\varrho\rangle \oplus \mathbb{Z}/2\langle \rho_1\rangle;\ [S^{n+3}, M^n_{2^r}] = \mathbb{Z}/4\langle i\varrho\rangle \oplus \mathbb{Z}/2\langle \rho_r\rangle\ (r > 1),$$

*where $\rho_r$ is some element of the group $[S^{n+3}, M^n_{2^r}]$ such that $q\rho_r = \eta\eta$ for $r \geqslant 1$.*

*Proof.* The group $[S^{n+3}, M^n_2]$ is obtained from Lemma 5.2 and Theorem 5.11 of [**16**]. For $r > 1$ there are two exact sequences

$$[S^k, S^n] \xrightarrow{(2^r)^*} [S^k, S^n] \to [S^k, M^n_{2^r}] \xrightarrow{q^*} [S^k, S^{n+1}] \xrightarrow{(2^r)^*} [S^k, S^{n+1}],$$
$$k = n + 2, n + 3.$$

The following commutative diagram is induced by $\eta\colon S^{n+3} \to S^{n+2}$:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z}/2 & \longrightarrow & [S^{n+2}, M^n_{2^r}] & \underset{\sigma}{\overset{q^*}{\rightleftarrows}} & \mathbb{Z}/2 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle \eta^*} & & \| & & \\ 0 & \longrightarrow & \mathbb{Z}/4 & \longrightarrow & [S^{n+3}, M^n_{2^r}] & \xrightarrow{q^*} & \mathbb{Z}/2 & \longrightarrow & 0. \end{array}$$

It is known that the upper exact sequence splits. If $\sigma$ is the section of $q^*$, then $\eta^*\sigma$ is the section of the lower $q^*$. □

### 3.2. Chang-complexes

Firstly, since $C^{k,1}_1$ and $\Sigma^{k-4}M^1_2 \wedge M^1_2$ are both indecomposable $\mathbf{A}^2_{k-2}$-complexes with the same homology groups, thus $C^{k,1}_1 \simeq \Sigma^{k-4}M^1_2 \wedge M^1_2$ $(k \geqslant 5)$.

**Cofibre sequences for Chang-complexes**

For $C \in \{C_r^k, C^{k,s}, C_r^{k,s} \mid k \geqslant 5, r, s \in \mathbb{Z}^+\}$, it can be written as mapping cones of different maps, that is, there are different cofibre sequences for $C$.

- The cofibre sequence for $C_\eta^k$

**Cof** : $S^{k-1} \xrightarrow{\eta} S^{k-2} \xrightarrow{i_\eta} C_\eta^k \xrightarrow{q_\eta} S^k \to S^{k-1}$;

- The cofibre sequences for $C_r^k$

**Cof1** : $S^{k-2} \vee S^{k-1} \xrightarrow{(2^r,\eta)} S^{k-2} \xrightarrow{i_S} C_r^k \xrightarrow{q_S} S^{k-1} \vee S^k \to S^{k-1}$;

**Cof2** : $S^{k-1} \xrightarrow{i\eta} M_{2^r}^{k-2} \xrightarrow{i_M} C_r^k \xrightarrow{q_M} S^k \to M_{2^r}^{k-1}$;

**Cof3** : $S^{k-2} \xrightarrow{i_\eta 2^r} C_\eta^k \xrightarrow{i_C} C_r^k \xrightarrow{q_C} S^{k-1} \to C_\eta^{k+1}$;

- The cofibre sequences for $C^{k,s}$

**Cof1** : $S^{k-1} \xrightarrow{\binom{\eta}{2^s}} S^{k-2} \vee S^{k-1} \xrightarrow{i_S} C^{k,s} \xrightarrow{q_S} S^k \to S^{k-1} \vee S^k$;

**Cof2** : $M_{2^s}^{k-2} \xrightarrow{\eta q} S^{k-2} \xrightarrow{i_M} C^{k,s} \xrightarrow{q_M} M_{2^s}^{k-1} \to S^{k-1}$;

**Cof3** : $C_\eta^{k-1} \xrightarrow{2^s q_\eta} S^{k-1} \xrightarrow{i_C} C^{k,s} \xrightarrow{q_C} C_\eta^k \to S^k$;

- The cofibre sequences for $C_r^{k,s}$

**Cof1** : $S^{k-2} \vee S^{k-1} \xrightarrow{\begin{pmatrix} 2^r, \eta \\ 0\ , 2^s \end{pmatrix}} S^{k-2} \vee S^{k-1} \xrightarrow{i_S} C_r^{k,s} \xrightarrow{q_S} S^{k-1} \vee S^k \to S^{k-1} \vee S^k$;

**Cof2** : $M_{2^s}^{k-2} \xrightarrow{i\eta q} M_{2^r}^{k-2} \xrightarrow{i_M} C_r^{k,s} \xrightarrow{q_M} M_{2^s}^{k-1} \to M_{2^r}^{k-1}$;

**Cof3** : $S^{k-2} \vee M_{2^s}^{k-2} \xrightarrow{(2^r,\eta q)} S^{k-2} \xrightarrow{i_{\overline{M}}} C_r^{k,s} \xrightarrow{q_{\overline{M}}} S^{k-1} \vee M_{2^s}^{k-1} \to S^{k-1}$;

**Cof4** : $S^{k-1} \xrightarrow{\binom{i\eta}{2^s}} M_{2^r}^{k-2} \vee S^{k-1} \xrightarrow{i_M} C_r^{k,s} \xrightarrow{q_M} S^k \to M_{2^r}^{k-1} \vee S^k$;

**Cof5** : $C_r^{k-1} \xrightarrow{2^s p_1 q_S} S^{k-1} \xrightarrow{i_C} C_r^{k,s} \xrightarrow{q_C} C_r^k \to S^k$, where $2^s p_1 q_S$ is the composition of $C_r^{k-1} \xrightarrow{q_S} S^{k-1} \vee S^{k-2} \xrightarrow{p_1} S^{k-1} \xrightarrow{2^s} S^{k-1}$;

**Cof6** : $S^{k-2} \xrightarrow{i_S j_1 2^r} C^{k,s} \xrightarrow{i_{\overline{C}}} C_r^{k,s} \xrightarrow{q_{\overline{C}}} S^{k-1} \to C^{(k+1),s}$, where $i_S j_1 2^r$ is the composition of $S^{k-2} \xrightarrow{2^r} S^{k-2} \xrightarrow{j_1} S^{k-2} \vee S^{k-1} \xrightarrow{i_S} C^{k,s}$.

**Homologies and cohomologies**

$$H_* C^{k,s} = \begin{cases} \mathbb{Z}, & * = k-2; \\ \mathbb{Z}/2^s, & * = k-1; \\ 0, & \text{otherwise}; \end{cases} \qquad H_* C_r^k = \begin{cases} \mathbb{Z}/2^r, & * = k-2; \\ \mathbb{Z}, & * = k; \\ 0, & \text{otherwise}; \end{cases}$$

$$H_* C_r^{k,s} = \begin{cases} \mathbb{Z}/2^r, & * = k-2; \\ \mathbb{Z}/2^s, & * = k-1; \\ 0, & \text{otherwise}; \end{cases} \qquad H_* C_\eta^k = \begin{cases} \mathbb{Z}, & * = k-2; \\ \mathbb{Z}, & * = k; \\ 0, & \text{otherwise}; \end{cases}$$

$$H^*(C_r^k;\mathbb{Z}/2) = H^*(C^{k,s};\mathbb{Z}/2) = \begin{cases} \mathbb{Z}/2, & * = k-2, k-1, k; \\ 0, & \text{otherwise}; \end{cases}$$

$$H^*(C_r^{k,s};\mathbb{Z}/2) = \begin{cases} \mathbb{Z}/2, & * = k-2, k; \\ \mathbb{Z}/2 \oplus \mathbb{Z}/2, & * = k-1; \\ 0, & \text{otherwise}; \end{cases} \quad H^*(C_\eta^k;\mathbb{Z}/2) = \begin{cases} \mathbb{Z}/2, & * = k-2; \\ \mathbb{Z}/2, & * = k; \\ 0, & \text{otherwise}; \end{cases}$$

- $Sq^2\colon H^{k-2}(C;\mathbb{Z}/2)\to H^k(C;\mathbb{Z}/2)$ is isomorphic for $C=C^{k,s},C_r^k,C_r^{k,s},C_\eta^k$;
- $Sq^1=0\colon H^{k-2}(C^{k,s};\mathbb{Z}/2)\to H^{k-1}(C^{k,s};\mathbb{Z}/2)$;
- $Sq^1\colon H^{k-1}(C^{k,s};\mathbb{Z}/2)\to H^k(C^{k,s};\mathbb{Z}/2)$ is $\begin{cases}0, & s>1;\\ \text{isomorphic}, & s=1;\end{cases}$
- $Sq^1\colon H^{k-2}(C_r^k;\mathbb{Z}/2)\to H^{k-1}(C_r^k;\mathbb{Z}/2)$ is $\begin{cases}0, & r>1;\\ \text{isomorphic}, & r=1;\end{cases}$
- $Sq^1=0\colon H^{k-1}(C_r^k;\mathbb{Z}/2)\to H^k(C_r^k;\mathbb{Z}/2)$;
- $Sq^1$ on $H^*(C_r^{k,s};\mathbb{Z}/2)$ is given by the following lemma.

**Lemma 3.2.** *Let $v_{k-2},v_k$ be generators of $H^{k-2}(C_r^{k,s};\mathbb{Z}/2)$ and $H^k(C_r^{k,s};\mathbb{Z}/2)$, respectively. Then there are generators $v_{k-1},\overline{v}_{k-1}$ of $H^{k-1}(C_r^{k,s};\mathbb{Z}/2)$ such that*

$$Sq^1v_{k-2}=\begin{cases}v_{k-1}, & r=1;\\ 0, & r>1;\end{cases}\qquad Sq^1\overline{v}_{k-1}=\begin{cases}v_k, & s=1;\\ 0, & s>1;\end{cases}\qquad Sq^1v_{k-1}=0.$$

*Proof.* To simplify the notation, take $k=5$. Using **Cof2** and **Cof3** of $C_r^{5,s}$, we have a split exact sequence

$$0\to H^4(M_{2^s}^4;\mathbb{Z}/2)\xrightarrow{q_M^*}H^4(C_r^{5,s};\mathbb{Z}/2)\xrightarrow{i_M^*}H^4(M_{2^r}^3;\mathbb{Z}/2)\to 0$$

and an isomorphism

$$H^4(S^4;\mathbb{Z}/2)\oplus H^4(M_{2^s}^4;\mathbb{Z}/2)\xrightarrow{(p_1^*,p_2^*)\ \cong}H^4(S^4\vee M_{2^s}^4;\mathbb{Z}/2)\xrightarrow{q_{\overline{M}}^*\ \cong}H^4(C_r^{5,s};\mathbb{Z}/2),$$

where $p_l$ is the canonical projection for $l=1,2$. Let $u_S^4,u_M^4$ be the generators of $H^4(S^4;\mathbb{Z}/2)$ and $H^4(M_{2^s}^4;\mathbb{Z}/2)$, respectively. Note that $p_2q_{\overline{M}}=q_M$.

Take $\overline{v}_4:=q_{\overline{M}}^*p_2^*(u_M^4)=q_M^*(u_M^4)$, $\qquad v_4:=\begin{cases}Sq^1v_3, & r=1,\\ q_{\overline{M}}^*p_1^*(u_S^4), & r>1,\end{cases}$

then we can easily get this lemma. ☐

**Homotopy groups and cohomotopy groups**

For $k\geqslant 5$

- $\pi_{k-1}C_r^k=0$;
- $\pi_kC_r^k=\mathbb{Z}/2\langle(j_1\eta)_S^-\rangle\oplus\mathbb{Z}\langle(2j_2)_S^-\rangle$, where $(j_1\eta)_S^-=q_{S*}^{-1}(j_1\eta)$ and $(2j_2)_S^-=q_{S*}^{-1}(2j_2)$;
- $\pi_{k-1}C^{k,s}=\mathbb{Z}/2^{s+1}\langle i_Sj_2\rangle$, with $i_Sj_1\eta=2^si_Sj_2$;
- $\pi_kC^{k,s}=\mathbb{Z}/2\langle i_Sj_2\eta\rangle$;
- $\pi_{k-1}C_r^{k,s}=\mathbb{Z}/2^{s+1}\langle i_{\underline{M}}j_2\rangle$, with $i_{\underline{M}}j_1i\eta=2^si_{\underline{M}}j_2$ or $\pi_{k-1}C_r^{k,s}=\mathbb{Z}/2^{s+1}\langle i_Sj_2\rangle$, with $i_Sj_1i\eta=2^si_Sj_2$;
- $\pi_kC_r^{k,s}=\mathbb{Z}/2\langle i_{\underline{M}}j_1\xi_r\rangle\oplus\mathbb{Z}/2\langle i_{\underline{M}}j_2\eta\rangle$;
- $[C_r^k,S^{k-2}]=\mathbb{Z}/2\langle\eta p_1q_S\rangle$;
- $[C_r^k,S^{k-1}]=\mathbb{Z}/2^{r+1}\langle p_1q_S\rangle$, with $\eta p_2q_S=2^rp_1q_S$;
- $[C^{k,s},S^{k-2}]=\mathbb{Z}\langle(2p_1)_S^-\rangle\oplus\mathbb{Z}/2\langle(\eta p_2)_S^-\rangle$, where $(2p_1)_S^-=(i_S^*)^{-1}(2p_1)$ and $(\eta p_2)_S^-=(i_S^*)^{-1}(\eta p_2)$;
- $[C^{k,s},S^{k-1}]=0$;

- $[C_r^{k,s}, S^{k-2}] = \mathbb{Z}/2\langle(\eta q p_1)^-_M\rangle \oplus \mathbb{Z}/2\langle(\eta p_2)^-_M\rangle$, where $(\eta q p_1)^-_M = (i^*_{\underline{M}})^{-1}(\eta q p_1)$ and $(\eta p_2)^-_M = (i^*_{\underline{M}})^{-1}(\eta p_2)$;
- $[C_r^{k,s}, S^{k-1}] = \mathbb{Z}/2^{r+1}\langle p_1 q_S\rangle$, with $\eta p_2 q_S = 2^r p_1 q_S$ or $[C_r^{k,s}, S^{k-1}] = \mathbb{Z}/2^{r+1}\langle p_1 q_{\overline{M}}\rangle$, with $\eta q p_2 q_{\overline{M}} = 2^r p_1 q_{\overline{M}}$;
- $[C_r^{k,s}, S^k] = \mathbb{Z}/2^s\langle q p_2 q_{\underline{M}}\rangle = \mathbb{Z}/2^s\langle p_1 q_S\rangle$,

where $X_u \xrightarrow{j_u} X_1 \vee X_2$ is the canonical inclusion and $X_1 \vee X_2 \xrightarrow{p_u} X_u$ is the canonical projection for $u = 1, 2$.

*Remark 3.3.* $[X, Y]$ for $X, Y$ being indecomposable homotopy types of $\mathbf{A}_n^2$ are given by Part IV of [**1**]. We will use these results directly in the rest of this paper.

## 4. Determination of the decomposability except that of $C_r^{5,s} \wedge C_{r'}^{5,s'}$

### 4.1. Some indecomposable cases

**Theorem 4.1.** *$M^3_{2^u} \wedge C^5_\eta$, $C^5_r \wedge C^{5,s}$, $C^5_r \wedge C^5_{r'}$, $C^{5,s} \wedge C^{5,s'}$, $C^5_\eta \wedge C^5_r$, $C^5_\eta \wedge C^{5,s}$ and $C^5_\eta \wedge C_r^{5,s}$ are indecomposable for any $u, r, r', s, s' \in \mathbb{Z}^+$.*

*Proof.* The proofs for all cases in Theorem 4.1 are similar. We give a proof only for the case $C^5_r \wedge C^5_{r'}$.

Let $u_k, u'_k$ be generators of $H^k(C^5_r; \mathbb{Z}/2)$ and $H^k(C^5_{r'}; \mathbb{Z}/2)$, respectively, for $k = 3, 4, 5$.

$$H^*(C^5_r \wedge C^5_{r'}; \mathbb{Z}/2) = \begin{cases} \mathbb{Z}/2\{u_3 \otimes u'_3\}, & * = 6, \\ \mathbb{Z}/2\{u_3 \otimes u'_4, u_4 \otimes u'_3\}, & * = 7, \\ \mathbb{Z}/2\{u_3 \otimes u'_5, u_4 \otimes u'_4, u_5 \otimes u'_3\}, & * = 8, \\ \mathbb{Z}/2\{u_4 \otimes u'_5, u_5 \otimes u'_4\}, & * = 9, \\ \mathbb{Z}/2\{u_5 \otimes u'_5\}, & * = 10, \\ 0, & \text{otherwise.} \end{cases}$$

By the Steenrod operation action on Chang-complexes given in Section 3.2, we get

(i) $Sq^4(u_3 \otimes u'_3) = u_5 \otimes u'_5$;

(ii) $Sq^2(u_3 \otimes u'_5) = Sq^2(u_5 \otimes u'_3) = u_5 \otimes u'_5$;

(iii) $Sq^2(u_3 \otimes u'_3) = \begin{cases} u_3 \otimes u'_5 + u_4 \otimes u'_4 + u_5 \otimes u'_3, & r = r' = 1, \\ u_3 \otimes u'_5 + u_5 \otimes u'_3, & \text{otherwise,} \end{cases}$

(iv) $Sq^2(u_4 \otimes u'_4) = 0$;

(v) $Sq^2(u_3 \otimes u'_4) = u_5 \otimes u'_4, Sq^2(u_4 \otimes u'_3) = u_4 \otimes u'_5$.

Suppose that $C^5_r \wedge C^5_{r'} = X \vee Y$ and $H_6 X \neq 0$, $C^5_r \wedge C^5_{r'}$ satisfies the conditions in Lemma 2.6,

$$H_t X \cong H_t(C^5_r \wedge C^5_{r'}), \quad t = 6, 9, 10; \qquad \dim H_7 X + \dim H_8 X \geqslant 2.$$

If $Y$ is not contractible, then it follows from the homology groups of $C^5_r \wedge C^5_{r'}$ that $Y \simeq M^7_{2^x}$ or $Y \simeq M^8_{2^x}$ for some positive integer $x$. This contradicts the isomorphism $Sq^2\colon H^7(C^5_r \wedge C^5_{r'}; \mathbb{Z}/2) \to H^9(C^5_r \wedge C^5_{r'}; \mathbb{Z}/2)$. □

In the rest of this subsection we will give a cell structure of spaces $M_{2^r}^n \wedge C_\eta^{n+2}$ $(n \geqslant 3)$ and $C_\eta^5 \wedge C_r^{5,s}$ which will be used later.

**Lemma 4.2.** *$M_{2^r}^n \wedge C_\eta^{n+2} \simeq S^{2n} \cup_{h^r} \mathbf{C}C^{(2n+2),r} \simeq C_r^{2n+2} \cup_{h_r} \mathbf{C}S^{2n+2}$ $(n \geqslant 3)$, where $h^r$ is determined by $h^r i_S = (2^r, \eta)$, i.e.,*

$$\begin{array}{ccccccc}
C^{(2n+2),r} & \xrightarrow{h^r} & S^{2n} & \xrightarrow{i_{\overline{C}}} & M_{2^r}^n \wedge C_\eta^{n+2} & \xrightarrow{q_{\overline{C}}} & C^{(2n+3),r}\ ; \\
\uparrow{\scriptstyle i_S} & \nearrow{\scriptstyle h^r i_S=(2^r,\eta)} & & & & & \\
S^{2n} \vee S^{2n+1} & & & & & &
\end{array}$$

*$h_r$ is determined by $q_S h_r = \binom{\eta}{2^r}$, i.e.,*

$$\begin{array}{ccccccc}
S^{2n+2} & \xrightarrow{h_r} & C_r^{2n+2} & \xrightarrow{i_{\underline{C}}} & M_{2^r}^n \wedge C_\eta^{n+2} & \xrightarrow{q_{\underline{C}}} & S^{2n+3}\ . \\
& \searrow{\scriptstyle q_S h_r=\binom{\eta}{2^r}} & \downarrow{\scriptstyle q_S} & & & & \\
& & S^{2n+1} \vee S^{2n+2} & & & &
\end{array}$$

*The top rows are cofibre sequences in each commutative diagram.*

*Moreover,*

$$\pi_{2n+1}(M_{2^r}^n \wedge C_\eta^{n+2}) = [M_{2^r}^n \wedge C_\eta^{n+2}, S^{2n+2}] = 0;$$

$$\mathbb{Z}/2^r \cong \frac{\pi_{2n+2} C_r^{2n+2}}{\langle (j_1\eta)_S^-, 2^{r-1}(2j_2)_S^- \rangle} \xrightarrow[(i_{\underline{C}})_*]{\cong} \pi_{2n+2}(M_{2^r}^n \wedge C_\eta^{n+2});$$

$$\mathbb{Z}/2^r \cong \frac{[C^{(2n+3),r}, S^{2n+1}]}{\langle (\eta p_2)_S^-, 2^{r-1}(2p_1)_S^- \rangle} \xrightarrow[(q_{\underline{C}})^*]{\cong} [M_{2^r}^n \wedge C_\eta^{n+2}, S^{2n+1}];$$

$$\frac{\pi_{2n+3} S^{2n}}{\langle 2^r \varrho_{2n}, \eta^{(3)} \rangle} \xrightarrow[(i_{\overline{C}})_*]{\cong} \pi_{2n+3}(M_{2^r}^n \wedge C_\eta^{n+2}); \quad \frac{[S^{2n+3}, S^{2n}]}{\langle 2^r \varrho_{2n}, \eta^{(3)} \rangle} \xrightarrow[(q_{\overline{C}})^*]{\cong} [M_{2^r}^n \wedge C_\eta^{n+2}, S^{2n}];$$

*Proof.* From Lemma 2.4

$$(M_{2^r}^n \wedge C_\eta^{n+2})/S^{2n} \simeq (S^{2n+1} \vee S^{2n+2}) \cup_{\binom{\eta}{-2^r}} \mathbf{C}S^{2n+2} \simeq C^{(2n+3),r};$$

$$(M_{2^r}^n \wedge C_\eta^{n+2})^{(2n+2)} \simeq S^{2n} \cup_{(2^r,\eta)} \mathbf{C}(S^{2n} \vee S^{2n+1}) \simeq C_r^{2n+2}.$$

So there are cofibre sequences

$$\begin{array}{ccccc}
C^{(2n+2),r} & \xrightarrow{h^r} & S^{2n} & \xrightarrow{i_{\overline{C}}} & M_{2^r}^n \wedge C_\eta^{n+2}\ ; \\
\uparrow{\scriptstyle i_S} & \nearrow{\scriptstyle h^r i_S=(a,x\eta)} & & & \\
S^{2n} \vee S^{2n+1} & & & &
\end{array}
\quad
\begin{array}{ccccc}
S^{2n+2} & \xrightarrow{h_r} & C_r^{2n+2} & \xrightarrow{i_{\underline{C}}} & M_{2^r}^n \wedge C_\eta^{n+2}, \\
& \searrow{\scriptstyle q_S h_r=\binom{y\eta}{b}} & \downarrow{\scriptstyle q_S} & & \\
& & S^{2n+1} \vee S^{2n+2} & &
\end{array}$$

where $a, b \in \mathbb{Z}$ and $x, y \in \{0, 1\}$. It follows from the monomorphisms $[C^{(2n+2),r}, S^{2n}] \xrightarrow{i_S^*} [S^{2n} \vee S^{2n+1}, S^{2n}]$ and $[S^{2n+2}, C_r^{2n+2}] \xrightarrow{(q_S)_*} [S^{2n+2}, S^{2n+1} \vee S^{2n+2}]$ that $h^r$ and $h_r$ are determined by $h^r i_S$ and $q_S h_r$, respectively.

By $H_{2n}(M^n_{2^r}\wedge C^{n+2}_\eta)=\mathbb{Z}/2^r$ and $\pi_{2n+1}(M^n_{2^r}\wedge C^{n+2}_\eta)\cong\pi_{2n+1}(C^{2n+2}_r)=0$,

$$a=2^r,\ x=1.$$

By $H_{2n+2}(M^n_{2^r}\wedge C^{n+2}_\eta)=\mathbb{Z}/2^r$ and $[M^n_{2^r}\wedge C^{n+2}_\eta,S^{2n+2}]\cong[C^{(2n+3),r},S^{2n+2}]=0$

$$b=2^r,\ y=1.$$

Thus we prove the first part of this lemma. Now $\pi_*(M^n_{2^r}\wedge C^{n+2}_\eta)$ $(*=2n+1,2n+2,2n+3)$ and $[M^n_{2^r}\wedge C^{n+2}_\eta,S^m]$ $(m=2n,2n+1,2n+2)$ are easily obtained. □

**Lemma 4.3.** *$C^5_\eta\wedge C^{5,s}_r$ is homotopy equivalent to the mapping cone of map $S^9\xrightarrow{\binom{i_{\overline{C}}\varrho_6}{h_s}}M^3_{2^r}\wedge C^5_\eta\vee C^9_s$ ($i_{\overline{C}}$ and $h_s$ are defined in Lemma 4.2) and $\pi_9(C^5_\eta\wedge C^{5,s}_r)\cong\begin{cases}\mathbb{Z}/2^{s+1}\oplus\mathbb{Z}/2, & r>1,\\ \mathbb{Z}/2^s\oplus\mathbb{Z}/2, & r=1.\end{cases}$*

*Proof.* Apply Lemma 2.4 to the cofibre sequences

$$S^4\xrightarrow{\eta}S^3_a\to C^5_\eta;\qquad S^3\vee M^3_{2^s}\xrightarrow{f=(2^r,\eta q)}S^4_b\to C^{5,s}_r;$$

$$(C^5_\eta\wedge C^{5,s}_r)/S^6\simeq(\Sigma(S^4\wedge S^3_b)\vee\Sigma(S^3_a\wedge(S^3\vee M^3_{2^s})))\cup_{\mathcal{A}}\mathbf{C}\Sigma S^4\wedge(S^3\vee M^3_{2^s}),$$

where $\mathcal{A}=\binom{\Sigma1\wedge f}{-\Sigma\eta\wedge1}$, i.e.,

$$(C^5_\eta\wedge C^{5,s}_r)/S^6\simeq(S^8\vee S^7\vee M^7_{2^s})\cup_{\mathcal{A}}\mathbf{C}(S^8\vee M^8_{2^s}),$$

where $\mathcal{A}=\begin{array}{c|cc|} & S^8 & M^8_{2^s}\\ \hline S^8 & 2^r & \eta q\\ S^7 & \eta & 0\\ M^7_{2^s} & 0 & \eta\wedge1\\ \hline\end{array}\xrightarrow[q\mathbf{r}_3+\mathbf{r}_1]{\cong}\begin{array}{c|cc|} & S^8 & M^8_{2^s}\\ \hline S^8 & 2^r & 0\\ S^7 & \eta & 0\\ M^7_{2^s} & 0 & \eta\wedge1\\ \hline\end{array}$. Hence

$$(C^5_\eta\wedge C^{5,s}_r)/S^6\simeq M^4_{2^s}\wedge C^5_\eta\vee C^{9,r}.$$

Apply Lemma 2.4 to the cofibre sequences

$$S^4_1\xrightarrow{\eta}S^3\to C^5_\eta;\qquad S^4_2\xrightarrow{g=\binom{i\eta}{2^s}}M^3_{2^r}\vee S^4_a\to C^{5,s}_r;$$

$$(C^5_\eta\wedge C^{5,s}_r)^{(9)}\simeq(M^6_{2^r}\vee S^7)\cup_{\mathcal{B}}\mathbf{C}(M^7_{2^r}\vee S^8\vee S^7),$$

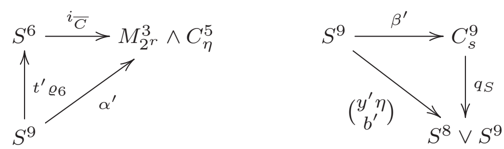
where $\mathcal{B}=\begin{array}{c|ccc|} & M^7_{2^r} & S^8 & S^7\\ \hline M^6_{2^r} & \eta\wedge1 & 0 & i\eta\\ S^7 & 0 & \eta & 2^s\\ \hline\end{array}\xrightarrow[\mathbf{c}_1i+\mathbf{c}_3]{\cong}\begin{array}{c|ccc|} & M^7_{2^r} & S^8 & S^7\\ \hline M^6_{2^r} & \eta\wedge1 & 0 & 0\\ S^7 & 0 & \eta & 2^s\\ \hline\end{array}$. Thus

$$(C^5_\eta\wedge C^{5,s}_r)^{(9)}\simeq M^3_{2^r}\wedge C^5_\eta\vee C^9_s.$$

There is a cofibre sequence

$$S^9\xrightarrow{\binom{\alpha'}{\beta'}}M^3_{2^r}\wedge C^5_\eta\vee C^9_s\to C^5_\eta\wedge C^{5,s}_r\to S^{10}\xrightarrow{\binom{\Sigma\alpha'}{\Sigma\beta'}}M^4_{2^r}\wedge C^5_\eta\vee C^{10}_s,\tag{1}$$

where $\alpha'=i_{\overline{C}}(t'\varrho_6)$, $t'=1$ for $r=1$ and $t'\in\{1,2\}$ for $r>1$ since $\pi_9(M^3_{2^r}\wedge C^5_\eta)=\begin{cases}\mathbb{Z}/2\langle i_{\overline{C}}\varrho_6\rangle, & r=1\\ \mathbb{Z}/4\langle i_{\overline{C}}\varrho_6\rangle, & r\geqslant2\end{cases}$ and $C^5_\eta\wedge C^{5,s}_r$ is indecomposable. $\beta'$ is determined by $q_S\beta' = \binom{y'\eta}{b'}$ for some $y' \in \{0,1\}$ and $b' \in \mathbb{Z}$ since $(q_S)_*\colon \pi_9 C_s^9 \to \pi_9(S^8 \vee S^9)$ is injective.




By $H_9(C_\eta^5 \wedge C_r^{5,s}) = \mathbb{Z}/2^s$, $b' = 2^s$. From $[C_\eta^5 \wedge C_r^{5,s}, S^9] \cong [(C_\eta^5 \wedge C_r^{5,s})/S^6,\ S^9] = [M_{2^s}^4 \wedge C_\eta^5 \vee C^{9,r}, S^9] \cong \mathbb{Z}/2^r$ and the exact sequence

$$0 \to \frac{[S^{10}, S^9]}{\langle y'\eta \rangle} \to [C_\eta^5 \wedge C_r^{5,s}, S^9] \to \mathbb{Z}/2^r \to 0$$

we have $y' = 1$. Thus $\beta' = h_s$. So for $r = 1$,

$$\begin{aligned}\pi_9(C_\eta^5 \wedge C_r^{5,s}) &\cong \frac{\mathbb{Z}/2\langle i_{\overline{C}}\varrho_6\rangle \oplus \mathbb{Z}/2\langle (i_1\eta)^-\rangle \oplus \mathbb{Z}\langle (2i_2)^-\rangle}{\langle (t' i_{\overline{C}}\varrho_6, (i_1\eta)^-, 2^{s-1}(2i_2)^-)\rangle}\\ &\cong \frac{\mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}}{\langle (1,1,2^{s-1})\rangle} \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2^s.\end{aligned}$$

Next we will determine $t'$ for $r > 1$.

By computing the exact sequence $\pi_9$ of cofibre sequence (1), we get

$$\pi_9(C_\eta^5 \wedge C_r^{5,s}) \cong \frac{\mathbb{Z}/4 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}}{\langle (t', 1, 2^{s-1})\rangle} \cong \begin{cases} \mathbb{Z}/4 \oplus \mathbb{Z}/2^s, & t' = 2,\\ \mathbb{Z}/2 \oplus \mathbb{Z}/2^{s+1}, & t' = 1.\end{cases} \tag{2}$$

On the other hand, $\pi_9(C_\eta^5 \wedge C_r^{5,s}) \cong [C_\eta^{14}, C_r^{13,s}] \cong [C_\eta^7, C_r^{6,s}]$ ($[C_\eta^{k+1}, C_r^{k,s}]$ is stable for $k \geqslant 6$). There is an exact sequence

$$[C_\eta^7, S^5] \xrightarrow{\binom{i\eta}{2^s}_*} [C_\eta^7, M_{2^r}^4 \vee S^5] \to [C_\eta^7, C_r^{6,s}] \to [C_\eta^7, S^6] = 0.$$

By Lemma 3.1, we know that $\mathbb{Z}/12 \cong [C_\eta^7, S^4] \xrightarrow{i_*} [C_\eta^7, M_{2^r}^4] \cong \mathbb{Z}/4$ is an epimorphism. By $[C_\eta^7, C_\eta^6] \cong \frac{[C_\eta^7, S^4]}{\eta_*[C_\eta^7, S^5]} \cong \frac{[S^7, C_\eta^6]}{\eta^*[S^6, C_\eta^6]} \cong \mathbb{Z}/6$ [**15**, Proposition 2.6 (iii)], we get $\mathbb{Z} = [C_\eta^7, S^5] \xrightarrow{\eta_* = 6} [C_\eta^7, S^4] = \mathbb{Z}/12$, which implies

$$(i\eta)_* = 2\colon \mathbb{Z} = [C_\eta^7, S^5] \xrightarrow{\eta_*} [C_\eta^7, S^4] \xrightarrow{i_*} [C_\eta^7, M_{2^r}^4] \cong \mathbb{Z}/4. \tag{3}$$

Hence

$$\pi_9(C_\eta^5 \wedge C_r^{5,s}) \cong [C_\eta^7, C_r^{6,s}] \cong \frac{\mathbb{Z}/4 \oplus \mathbb{Z}}{\langle (2, 2^s)\rangle} \cong \mathbb{Z}/2^{s+1} \oplus \mathbb{Z}/2. \tag{4}$$

Together with (2), $t' = 1$. □

**4.2.** $M_{2^u}^3 \wedge C$**,** $C \in \{C_r^5, C^{5,s}, C_r^{5,s} \mid r, s \in \mathbb{Z}^+\}$

(1) $M_{2^u}^3 \wedge C_r^5$ and $M_{2^u}^3 \wedge C^{5,r}$

There is the following commutative diagram:

$$\begin{array}{ccccc} M^3_{2^u}\wedge(S^3\vee S^4) & \xrightarrow{1\wedge(2^r,\eta)} & M^3_{2^u}\wedge S^3 & \longrightarrow & M^3_{2^u}\wedge C^5_r\ , \\ \Vert & \nearrow_{(1\wedge 2^r,1\wedge\eta)} & & & \\ M^3_{2^u}\wedge S^3\vee M^3_{2^u}\wedge S^4 & & & & \end{array}$$

where the top row is a cofibre sequence. Hence

$$M^3_{2^u}\wedge C^5_r \simeq M^3_{2^u}\wedge S^3 \cup_{(1\wedge 2^r,1\wedge\eta)} \mathbf{C}(M^3_{2^u}\wedge S^3\vee M^3_{2^u}\wedge S^4).$$

- If $r \geqslant u$ and $r>1$, then $1\wedge 2^r\simeq 0$, hence $M^3_{2^u}\wedge C^5_r\simeq M^7_{2^u}\vee M^3_{2^u}\wedge C^5_\eta$.
- If $r=u=1$, $1\wedge\eta\in[M^7_{2^u},M^6_{2^u}]^\eta_\eta$ implies $1\wedge\eta=\lambda_{11}+\varepsilon_u i\eta\eta q$ for some $\varepsilon_u \in\{0,1\}$ (for $u=1$, $\varepsilon_1=0$, i.e., $\lambda_{11}=1\wedge\eta$), then

$$M^3_2\wedge C^5_1\simeq M^6_2\cup_{\mathcal{M}=(i_6\eta q_6,\lambda_{11})}\mathbf{C}(M^6_2\vee M^7_2).$$

From $\begin{array}{c|cc|} & M^6_2 & M^7_2 \\ \hline M^6_2 & i_6\eta q_6 & \lambda_{11} \\ \hline \end{array}\ \begin{array}{c|cc|} & M^6_2 & M^7_2 \\ \hline M^6_2 & id & 0 \\ M^7_2 & i_7q_6 & id \\ \hline \end{array} = \begin{array}{c|cc|} & M^6_2 & M^7_2 \\ \hline M^6_2 & 0 & \lambda_{11} \\ \hline \end{array}$, we get $\mathcal{M}\cong$ $(0,\lambda_{11})$, thus $M^3_2\wedge C^5_1\simeq M^7_2\vee M^3_2\wedge C^5_\eta$.
- If $r<u$, apply Lemma 2.4 to the cofibre sequences

$$S^3_1\xrightarrow{2^u}S^3_a\to M^3_{2^u};\qquad S^3_2\vee S^4\xrightarrow{(2^r,\ \eta)}S^3_b\to C^5_r,$$

then we can easily get

$$(M^3_{2^u}\wedge C^5_r)/S^6\simeq S^7\vee C^{9,u}_r. \tag{5}$$

Suppose that $M^3_{2^u}\wedge C^5_r\simeq X\vee Y$ are decomposable and $H_6(X)\neq 0$. Since $Sq^2$ on $H^6(M^3_{2^u}\wedge C^5_r;\mathbb{Z}/2)$ and $H^7(M^3_{2^u}\wedge C^5_r;\mathbb{Z}/2)$ are nontrivial, $M^3_{2^u}\wedge C^5_r$ has no direct summands $M^6_{2^r}$ and $M^8_{2^u}$. Thus $M^3_{2^u}\wedge C^5_r\simeq X\vee M^7_{2^r}$. By Lemma 2.6 and (5), $(X/S^6)\vee M^7_{2^r}\simeq S^7\vee C^{9,u}_r$ which contradicts the uniqueness of the decomposability of $\mathbf{A}^3_n$-complexes [**5**]. So $M^3_{2^u}\wedge C^5_r$ is indecomposable for $r<u$.

In summary, $M^3_{2^u}\wedge C^5_r$ is homotopy equivalent to $M^3_{2^u}\wedge C^5_\eta\vee M^7_{2^u}$ for $r\geqslant u$ and indecomposable for $u>r$. Thus $M^3_{2^u}\wedge C^{5,r}\simeq D(M^4_{2^u}\wedge C^5_r)$ is homotopy equivalent to $M^3_{2^u}\wedge C^5_\eta\vee M^7_{2^u}$ for $r\geqslant u$ and indecomposable for $u>r$.

(2) $M^3_{2^u}\wedge C^{5,s}_r$

If $u>r$ and $u>s$ there is a cofibre sequence

$$S^3\wedge C^{5,s}_r\xrightarrow{2^u\wedge 1}S^3\wedge C^{5,s}_r\to M^3_{2^u}\wedge C^{5,s}_r,$$

$[C^{k,s}_r,C^{k,s}_r]\cong\mathbb{Z}/2^{max(r,s)+1}\oplus\mathbb{Z}/2^{m_{sr}}\oplus\mathbb{Z}/2^{m_{sr}+1}$ for $k\geqslant 5$ which implies $2^u\wedge 1=0\in[S^3\wedge C^{5,s}_r,S^3\wedge C^{5,s}_r]$. Thus $M^3_{2^u}\wedge C^{5,s}_r\simeq C^{8,s}_r\vee C^{9,s}_r$.

If $u\leqslant r$ and $r>1$, by $2^r=0\in[M^6_{2^u},M^6_{2^u}]$ and the following cofibre sequence

$$M^3_{2^u}\wedge(S^3\vee M^3_{2^s})\xrightarrow{1\wedge(2^r,\eta q)}M^3_{2^u}\wedge S^3\to M^3_{2^u}\wedge C^{5,s}_r,$$

we get $M^3_{2^u}\wedge C^{5,s}_r\simeq M^7_{2^u}\vee\mathbf{C}_{1\wedge\eta q}\simeq M^3_{2^u}\wedge C^{5,s}\vee M^7_{2^u}$. So

$$M^3_{2^u}\wedge C^{5,s}_r\simeq\begin{cases} M^3_{2^u}\wedge C^{5,s}\vee M^7_{2^u}, & s<u\leqslant r,\\ M^3_{2^u}\wedge C^5_\eta\vee M^7_{2^u}\vee M^7_{2^u}, & u\leqslant r,s \text{ and } r>1.\end{cases}$$

If $u \leqslant s$ and $s > 1$, $M_{2^u}^3 \wedge C_r^{5,s} \simeq D(M_{2^u}^4 \wedge C_s^{5r})$, we get

$$M_{2^u}^3 \wedge C_r^{5,s} \simeq \begin{cases} M_{2^u}^3 \wedge C_r^5 \vee M_{2^u}^7, & r < u \leqslant s, \\ M_{2^u}^3 \wedge C_\eta^5 \vee M_{2^u}^7 \vee M_{2^u}^7, & u \leqslant r, s \text{ and } s > 1. \end{cases}$$

If $u = r = s = 1$, then from Corollary 3.7 of [**16**],

$$M_2^3 \wedge C_1^{5,1} \simeq \Sigma M_2^3 \wedge M_2^1 \wedge M_2^1 \simeq M_2^3 \wedge C_\eta^5 \vee M_2^7 \vee M_2^7.$$

**4.3. $C_u^5 \wedge C_r^{5,s}$ and $C^{5,u} \wedge C_r^{5,s}$ for $u, r, s \in \mathbb{Z}^+$**

We only prove the case $C_u^5 \wedge C_r^{5,s}$ since $C^{5,u} \wedge C_r^{5,s} \simeq D(C_u^5 \wedge C_s^{5,r})$.

Let $m_{uv}$ be the minimum of non-negative integers $u$ and $v$.

The following corollary is easy to get from Lemma 2.5.

**Corollary 4.4.** *If $C_u^5 \wedge C_r^{5,s} \simeq X \vee Y$ is decomposable and $H_6(X) \neq 0$, then $X$ is indecomposable and $Y \simeq C_l^{9,t}$ for some $t \in \{m_{us}, r\}, l \in \{m_{ur}, m_{us}\}$.* □

Firstly, we study $(C_u^5 \wedge C_r^{5,s})/S^6$.

Applying Lemma 2.4 to the following cofibre sequences

$$S_1^3 \vee S^4 \xrightarrow{(2^u,\eta)} S_a^3 \to C_u^5; \qquad S_2^3 \vee M_{2^s}^3 \xrightarrow{(2^r,\eta q)} S_b^3 \to C_r^{5,s},$$

$$(C_u^5 \wedge C_r^{5,s})/S^6 \simeq (S^7 \vee S^8 \vee S^7 \vee M_{2^s}^7) \cup_{\mathcal{A}} \mathbf{C}(S^7 \vee S^8 \vee M_{2^s}^7 \vee M_{2^s}^8),$$

$$\mathcal{A} = \begin{array}{c|cccc|} & S^7 & S^8 & M_{2^s}^7 & M_{2^s}^8 \\ \hline S^7 & 2^r & 0 & \eta q & 0 \\ S^8 & 0 & 2^r & 0 & \eta q \\ S^7 & -2^u & \eta & 0 & 0 \\ M_{2^s}^7 & 0 & 0 & -2^u & \eta \wedge 1 \\ \hline \end{array}.$$

Note that $q(\eta \wedge 1) = \eta q$ for $u \geqslant s$; $2^u = \begin{cases} 0, & u > 1 \\ i\eta q, & u = s = 1 \end{cases}$ in $[M_{2^s}^7, M_{2^s}^7]$.

For $s \leqslant u < r$, by transformations $i\mathbf{r}_1 + \mathbf{r}_4$ if $u = 1$ (otherwise, omitting this one); $q\mathbf{r}_4 + \mathbf{r}_2$; $2^{r-u}\mathbf{r}_3 + \mathbf{r}_1$ and $-\mathbf{r}_3$,

$$\mathcal{A} \cong \begin{array}{c|cccc|} & S^7 & S^8 & M_{2^s}^7 & M_{2^s}^8 \\ \hline S^7 & 0 & 0 & \eta q & 0 \\ S^8 & 0 & 2^r & 0 & 0 \\ S^7 & 2^u & \eta & 0 & 0 \\ M_{2^s}^7 & 0 & 0 & 0 & \eta \wedge 1 \\ \hline \end{array},$$

$$(C_u^5 \wedge C_r^{5,s})/S^6 \simeq C_u^{9,r} \vee C^{9,s} \vee M_{2^s}^4 \wedge C_\eta^5 \quad (s \leqslant u < r). \tag{6}$$

Similarly,

$$(C_u^5 \wedge C_r^{5,s})/S^6 \simeq C_r^{9,s} \vee C^{9,r} \vee M_{2^s}^4 \wedge C_\eta^5 \quad (s \leqslant u \text{ and } r \leqslant u), \tag{7}$$

$$(C_u^5 \wedge C_r^{5,s})/S^6 \simeq C^{9,r} \vee \mathbf{C}_{\mathcal{A}'} \quad (r \leqslant u < s), \tag{8}$$

$$(C_u^5 \wedge C_r^{5,s})/S^6 \simeq \mathbf{C}_{\mathcal{A}''} \quad (r \leqslant u < s), \tag{9}$$

where $\mathcal{A}' = \begin{array}{c|ccc|} & S^7 & M_{2^s}^7 & M_{2^s}^8 \\ \hline S^7 & 2^r & \eta q & 0 \\ M_{2^s}^7 & 0 & 2^u & \eta \wedge 1 \\ \hline \end{array}$ and $\mathcal{A}'' = \begin{array}{c|cccc|} & S^7 & S^8 & M_{2^s}^7 & M_{2^s}^8 \\ \hline S^7 & 0 & 0 & \eta q & 0 \\ S^8 & 0 & 2^r & 0 & \eta q \\ S^7 & 2^u & \eta & 0 & 0 \\ M_{2^s}^7 & 0 & 0 & 2^u & \eta \wedge 1 \\ \hline \end{array}$.

Secondly, we study the codimension 1 skeleton $(C_u^5 \wedge C_r^{5,s})^{(9)}$ of $C_u^5 \wedge C_r^{5,s}$.

Using the cofibre sequences

$$S_1^4 \xrightarrow{i\eta} M_{2^u}^3 \to C_u^5; \qquad S_2^4 \xrightarrow{g=\binom{i\eta}{2^s}} M_{2^r}^3 \vee S_b^4 \to C_r^{5,s},$$

we get

$$(C_u^5 \wedge C_r^{5,s})^{(9)} \simeq (M_{2^u}^3 \wedge S_b^4 \vee M_{2^u}^3 \wedge M_{2^r}^3) \cup_{\mathcal{B}} \mathbf{C}(S_1^4 \wedge S_b^4 \vee S_1^4 \wedge M_{2^r}^3 \vee M_{2^u}^3 \wedge S_2^4),$$

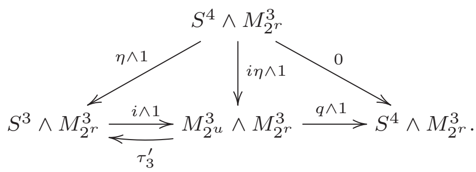
where

$$\mathcal{B} = \begin{array}{c} \\ M_{2^u}^3 \wedge S_b^4 \\ M_{2^u}^3 \wedge M_{2^r}^3 \end{array} \begin{array}{|ccc|} S_1^4 \wedge S_b^4 & S_1^4 \wedge M_{2^r}^3 & M_{2^u}^3 \wedge S_2^4 \\ \hline i\eta \wedge 1 & i\eta \wedge 0 & 1 \wedge 2^s \\ i\eta \wedge 0 & i\eta \wedge 1 & 1 \wedge i\eta \\ \hline \end{array}.$$

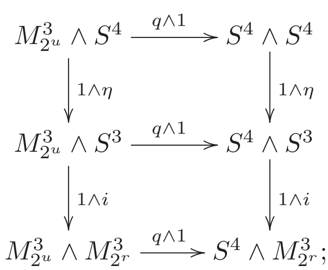
Let

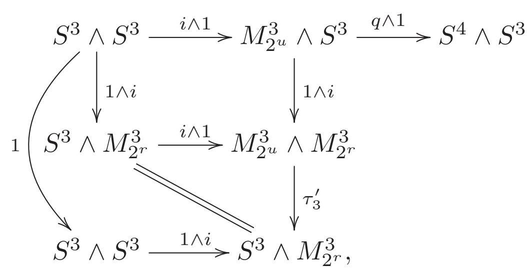
$$\mathcal{B}_1 = \begin{array}{c} \\ M_{2^u}^3 \wedge M_{2^r}^3 \end{array} \begin{array}{|cc|} S_1^4 \wedge M_{2^r}^3 & M_{2^u}^3 \wedge S_2^4 \\ \hline i\eta \wedge 1 & 1 \wedge i\eta \\ \hline \end{array}.$$

For $u \geqslant r$ and $u > 1$, by Lemma 6.2.1 of [**10**], there is a retraction $\tau_3'$ of $i \wedge 1$ yielding the following commutative diagram:




From the following commutative diagrams:




we get $(q \wedge 1)(1 \wedge i\eta) = i\eta q \in [M_{2^u}^7, M_{2^r}^7]$; $\tau_3'(1 \wedge i) \in [M_{2^u}^6, M_{2^r}^6]_1^{2^{u-r}}$.

So $\tau_3'(1 \wedge i\eta) \in [M_{2^u}^7, M_{2^r}^6]_\eta^\eta$ for $u = r$ and $\tau_3'(1 \wedge i\eta) \in [M_{2^u}^7, M_{2^r}^6]_\eta^0$ for $u > r$. Since $\eta \wedge 1\colon S^4 \wedge M_{2^r}^3 \to S^3 \wedge M_{2^r}^3$ is also an element in $[M_{2^r}^7, M_{2^r}^6]_\eta^\eta$,

$$\tau_3'(1 \wedge i\eta) = \begin{cases} \eta \wedge 1 + \kappa i\eta\eta q, & r = u > 1, \\ \eta_r^u + \kappa i\eta\eta q, & u > r. \end{cases}$$

Note that, since the composition $M_{2^u}^7 \xrightarrow{B(\chi)} M_{2^r}^7 \xrightarrow{\eta\wedge 1} M_{2^r}^6$ $(u > r)$ is an element in $[M_{2^u}^7, M_{2^r}^6]_\eta^0$, $(\eta \wedge 1)B(\chi) = \eta_r^u + \kappa' i\eta\eta q$, which implies that

$$\mathcal{B}_1 = \begin{array}{c} \\ M_{2^u}^6 \\ M_{2^u}^7 \end{array} \begin{array}{|cc|} M_{2^r}^7 & M_{2^u}^7 \\ \hline \eta_u^r + \kappa i\eta\eta q & 1 \wedge \eta \\ i\eta q & 0 \\ \hline \end{array} \xrightarrow[\mathbf{Tr1}]{\cong} \begin{array}{c} \\ M_{2^u}^6 \\ M_{2^u}^7 \end{array} \begin{array}{|cc|} M_{2^r}^7 & M_{2^u}^7 \\ \hline 0 & 1 \wedge \eta \\ i\eta q & 0 \\ \hline \end{array}, \quad r > u \geqslant 1, \qquad (10)$$

where the invertible transformation **Tr1** is $\mathbf{c}_2(-B(\chi)-(\kappa+\kappa')i\eta q)+\mathbf{c}_1$. Similarly,

$$\mathcal{B}_1=\begin{cases} \begin{array}{c|cc|} & M^7_{2^r} & M^7_{2^u} \\ \hline M^6_{2^r} & \eta\wedge 1 & \eta\wedge 1+\kappa i\eta\eta q \\ M^7_{2^r} & 0 & i\eta q \\ \hline \end{array} \xrightarrow[\mathbf{Tr2}]{\cong} \begin{array}{c|cc|} & M^7_{2^r} & M^7_{2^u} \\ \hline M^6_{2^r} & \eta\wedge 1 & 0 \\ M^7_{2^r} & 0 & i\eta q \\ \hline \end{array}, & r=u>1, \\ \begin{array}{c|cc|} & M^7_{2^r} & M^7_{2^u} \\ \hline M^6_{2^r} & \eta\wedge 1 & \eta^u_r+\kappa i\eta\eta q \\ M^7_{2^r} & 0 & i\eta q \\ \hline \end{array} \xrightarrow[\mathbf{Tr3}]{\cong} \begin{array}{c|cc|} & M^7_{2^r} & M^7_{2^u} \\ \hline M^6_{2^r} & \eta\wedge 1 & 0 \\ M^7_{2^r} & 0 & i\eta q \\ \hline \end{array}, & 1\leqslant r<u, \\ \begin{array}{c|cc|} & M^7_2 & M^7_2 \\ \hline C^{8,1}_1 & 1\wedge i\eta & i\eta\wedge 1 \\ \hline \end{array}, & r=u=1, \end{cases} \tag{11}$$

where the invertible transformations are given by

$$\mathbf{Tr2}:\mathbf{c}_1(1+\kappa i\eta q)+\mathbf{c}_2;\qquad \mathbf{Tr3}:\mathbf{c}_1(-B(\chi)-(\kappa+\kappa')i\eta q)+\mathbf{c}_2.$$

- For $s\geqslant u$, since $2^s=0\in[M^7_{2^u},M^7_{2^u}]$ for $s>1$, take the invertible transformation $\mathbf{c}_1q+\mathbf{c}_3$ on $\mathcal{B}$ for $s=1$ to get $\mathcal{B}\cong \begin{array}{c|c|} & S^8 \\ \hline M^7_{2^u} & i\eta \\ \hline \end{array}\oplus\mathcal{B}_1$, thus

$$(C^5_u\wedge C^{5,s}_r)^{(9)}\simeq\mathbf{C}_{\mathcal{B}}\simeq\mathbf{C}_{i\eta}\vee\mathbf{C}_{\mathcal{B}_1}=C^9_u\vee\mathbf{C}_{\mathcal{B}_1}. \tag{12}$$

**Lemma 4.5.** *The mapping cone of the map* $\begin{array}{c|cc|} & M^7_2 & M^7_2 \\ \hline C^{8,1}_1 & 1\wedge i\eta & i\eta\wedge 1 \\ \hline \end{array}$ *is homotopy equivalent to* $M^3_2\wedge C^5_\eta\vee C^{9,1}_1$.

*Proof.* $\Sigma C^5_1\wedge C^{5,1}_1\simeq C^5_1\wedge M^3_2\wedge M^1_2\simeq(M^3_2\wedge C^5_\eta\ \vee M^7_2)\ \wedge M^1_2\ \simeq\ C^{6,1}_1\wedge\ C^5_\eta\vee C^{10,1}_1$, hence $C^5_1\wedge C^{5,1}_1\simeq C^{5,1}_1\wedge C^5_\eta\vee C^{9,1}_1$. Together with $(C^{5,1}_1\wedge C^5_\eta)^{(9)}\simeq M^3_2\wedge C^5_\eta\vee C^9_1$ (from Lemma 4.3), we have $(C^5_1\wedge C^{5,1}_1)^{(9)}\simeq M^3_2\wedge C^5_\eta\vee C^9_1\vee C^{9,1}_1$. On the other hand, from (12), $(C^5_1\wedge C^{5,1}_1)^{(9)}\simeq C^9_1\vee C^{8,1}_1\cup_{(1\wedge i\eta,i\eta\wedge 1)}\mathbf{C}(M^7_2\vee M^7_2)$. So by Lemma 2.6, $C^{8,1}_1\cup_{(1\wedge i\eta,i\eta\wedge 1)}\mathbf{C}(M^7_2\vee M^7_2)\simeq M^3_2\wedge C^5_\eta\vee C^{9,1}_1$. □

Now, from (10), (11), (12) and Lemma 4.5 we get

$$(C^5_u\wedge C^{5,s}_r)^{(9)}\simeq C^9_u\vee C^{9,r}_u\vee M^3_{2^u}\wedge C^5_\eta\quad(s\geqslant u\text{ and }r\geqslant u); \tag{13}$$

$$(C^5_u\wedge C^{5,s}_r)^{(9)}\simeq C^9_u\vee C^{9,u}_r\vee M^3_{2^r}\wedge C^5_\eta\quad(s\geqslant u>r). \tag{14}$$

- For $u>s$, it is easy to get

$$(C^5_u\wedge C^{5,s}_r)^{(9)}\simeq M^3_{2^r}\wedge C^5_\eta\vee\mathbf{C}_{\mathcal{B}'}\quad(u>s\text{ and }u\geqslant r); \tag{15}$$

$$(C^5_u\wedge C^{5,s}_r)^{(9)}\simeq\mathbf{C}_{\mathcal{B}''}\quad(r>u>s),$$

where $\mathcal{B}'=\begin{array}{c|cc|} & S^8 & M^7_{2^u} \\ \hline M^7_{2^u} & i\eta & 2^s \\ M^7_{2^r} & 0 & i\eta q \\ \hline \end{array}$ and $\mathcal{B}''=\begin{array}{c|ccc|} & S^8 & M^7_{2^r} & M^7_{2^u} \\ \hline M^7_{2^u} & i\eta & 0 & 2^s \\ M^6_{2^u} & 0 & \eta^r_u+\kappa i\eta\eta q & 1\wedge\eta \\ M^7_{2^u} & 0 & i\eta q & 0 \\ \hline \end{array}$.

The decomposability of $C^5_u\wedge C^{5,s}_r$ can be obtained from the structure of $(C^5_u\wedge C^{5,s}_r)/S^6$ and $(C^5_u\wedge C^{5,s}_r)^{(9)}$ now.

(i) For $s<u<r$, $C^5_u\wedge C^{5,s}_r$ is indecomposable, otherwise by Corollary 4.4 and (6), $C^5_u\wedge C^{5,s}_r\simeq X\vee C^{9,r}_u$. However, $\pi_8C^{9,r}_u\cong\mathbb{Z}/2^{r+1}$ is not a direct summand of $\pi_8(C^5_u\wedge C^{5,s}_r)\cong[C^{5,u},C^{5,s}_r]\cong\mathbb{Z}/2^{s+1}\oplus\mathbb{Z}/2^r$.

(ii) For $u < s, u < r$, suppose $C_u^5 \wedge C_r^{5,s}$ is decomposable, by Corollary 4.4 and (13), $C_u^5 \wedge C_r^{5,s} \simeq X \vee C_u^{9,r}$. There is a cofibre sequence, $S^7 \vee S^8 \vee M_{2^s}^7 \vee M_{2^s}^8 \xrightarrow{\mathcal{A}''} S^7 \vee S^8 \vee S^7 \vee M_{2^s}^7 \to (C_u^5 \wedge C_r^{5,s})/S^6 \to S^8 \vee S^9 \vee M_{2^s}^8 \vee M_{2^s}^9 \xrightarrow{\Sigma\mathcal{A}''} S^8 \vee S^9 \vee S^8 \vee M_{2^s}^8$, where $\mathcal{A}''$ is the map in (9). So,

$$\mathbb{Z} \oplus \mathbb{Z}/2^s \xrightarrow{[\Sigma\mathcal{A}'', S^9]} \mathbb{Z} \oplus \mathbb{Z}/2^s \oplus \mathbb{Z}/2 \to [(C_u^5 \wedge C_r^{5,s})/S^6, S^9] \to 0,$$

where $[\Sigma\mathcal{A}'', S^9] = \begin{array}{c|cc|} & \mathbb{Z} & \mathbb{Z}/2^s \\ \mathbb{Z} & 2^r & 0 \\ \mathbb{Z}/2^s & 0 & 2^u \\ \mathbb{Z}/2 & 1 & 1 \end{array}$ ($k\colon A_1 \to A_2$ denotes the homomorphism of abelian groups defined by multiplication by $k$). Thus

$$[C_u^5 \wedge C_r^{5,s}, S^9] \cong [(C_u^5 \wedge C_r^{5,s})/S^6, S^9] \cong \frac{\mathbb{Z} \oplus \mathbb{Z}/2^s \oplus \mathbb{Z}/2}{\langle (2^r, 0, 1), (0, 2^u, 1) \rangle} \cong \mathbb{Z}/2^{r+1} \oplus \mathbb{Z}/2^u,$$

which is a contradiction since $[C_u^{9,r}, S^9] \cong \mathbb{Z}/2^r$. Hence $C_u^5 \wedge C_r^{5,s}$ is indecomposable for $u < s$ and $u < r$.

(iii) For $r \leqslant u < s$, the indecomposability of $C_u^5 \wedge C_r^{5,s}$ is easily obtained from the following lemma.

**Lemma 4.6.** *The wedge summand $\mathbf{C}_{\mathcal{A}'}$ in (8) is indecomposable.*

*Proof.* Assume that $\mathbf{C}_{\mathcal{A}'} \simeq Z_1 \vee Z_2$ is decomposable and $H_9 Z_1 \neq 0$. It follows from the homology groups of $\mathbf{C}_{\mathcal{A}'}$ and $Sq^2$ on $H^*(\mathbf{C}_{\mathcal{A}'}; \mathbb{Z}/2)$ that $Z_2 = C_u^{9,u}$ or $C_r^{9,u}$, which contradicts $[\mathbf{C}_{\mathcal{A}'}, S^9] \cong \mathbb{Z}/2^{u+1}$. □

(iv) For $u \geqslant s, u \geqslant r$ or $u = s < r$.

**Lemma 4.7.** *In* (15), $\mathbf{C}_{\mathcal{B}'} \simeq C_r^{9,s} \vee C_s^9$.

*Proof.* Let $W := (M_{2^u}^7 \vee M_{2^r}^7) \cup_{\begin{pmatrix} i\eta & 2^s i \\ 0 & 0 \end{pmatrix}} \mathbf{C}(S^8 \vee S^7) = M_{2^r}^7 \vee U$, where $U := M_{2^u}^7 \cup_{(i\eta, 2^s i)} \mathbf{C}(S^8 \vee S^7)$. From $U^{(8)} = M_{2^u}^7 \cup_{2^s i} \mathbf{C}S^7 \simeq M_{2^s}^7 \vee S^8$ and $H_8 U \cong \pi_8 U = \mathbb{Z}$ we get $U \simeq S^8 \vee C_s^9$, $W \simeq M_{2^r}^7 \vee S^8 \vee C_s^9$.
There is a cofibre sequence

$$S^8 \xrightarrow{\begin{pmatrix} yi\eta \\ b \\ 0 \end{pmatrix}} W \simeq M_{2^r}^7 \vee S^8 \vee C_s^9 \to \mathbf{C}_{\mathcal{B}'} \to S^9,$$

for some $y \in \{0, 1\}$ and $b \in \mathbb{Z}$. From $H_8(\mathbf{C}_{\mathcal{B}'}) = \mathbb{Z}/2^s$, $b = 2^s$. Since $Sq^2$ is isomorphic on both $H^7((C_u^5 \wedge C_r^{5,s})^{(9)}; \mathbb{Z}/2)$ and $H^7(\mathbf{C}_{\mathcal{B}'}; \mathbb{Z}/2)$, $y = 1$. Hence $\mathbf{C}_{\mathcal{B}'} \simeq C_r^{9s} \vee C_s^9$. □

From (7), (14), (15), together with Lemma 4.7, for $u \geqslant s, u \geqslant r$,

$$(C_u^5 \wedge C_r^{5,s})/S^6 \simeq C_r^{9,s} \vee C^{9,r} \vee M_{2^s}^4 \wedge C_\eta^5;$$
$$(C_u^5 \wedge C_r^{5,s})^{(9)} \simeq M_{2^r}^3 \wedge C_\eta^5 \vee C_s^9 \vee C_r^{9,s}.$$

Now we can get the following decomposition from $(C_u^5 \wedge C_r^{5,s})/S^6$ and $(C_u^5 \wedge$

$C_r^{5,s})^{(9)}$

$$C_u^5 \wedge C_r^{5,s} \simeq \begin{cases} C_r^{9,s} \vee C_\eta^5 \wedge C_r^{5,s}, & u \geqslant r \text{ and } u \geqslant s, \\ C_s^{9,r} \vee C_\eta^5 \wedge C_r^{5,r}, & u = r < s. \end{cases}$$

The proof, which we will omit, is similar to the proof of the case $C_r^{5,s} \wedge C_{r'}^{5,s'}$ $(s = r' > s' > r)$, given in the next section. For details, see our preprint [**17**].

## 5. Decomposition of $C_r^{5,s} \wedge C_{r'}^{5,s'}$, $r, r', s, s' \in \mathbb{Z}^+$

### 5.1. Preliminaries

Similar to Corollary 4.4 we have

**Lemma 5.1.** *If $C_r^{5,s} \wedge C_{r'}^{5,s'}$ is decomposable, then $C_r^{5,s} \wedge C_{r'}^{5,s'} \simeq X \vee C_l^{9,k}$ or $C_r^{5,s} \wedge C_{r'}^{5,s'} \simeq X \vee C_l^{9,k} \vee C_{l'}^{9,k'}$, where $X$ is indecomposable, $H_6X \neq 0$ and $\{k, k'\} \subset \{m_{r,s'}, m_{s,r'}, m_{r,r'}\}$, $\{l, l'\} \subset \{m_{s,s'}, m_{r,s'}, m_{s,r'}\}$.*

By $C_r^{5,s} \wedge C_{r'}^{5,s'} \simeq D(C_s^{5,r} \wedge C_{s'}^{5,r'})$, we can assume $s = \max\{r, s, r', s'\}$.

**Lemma 5.2.** *If $s > r', s'$, then $C_r^{5,s} \wedge C_{r'}^{5,s'} \simeq C_{r'}^{9,s'} \vee C_r^5 \wedge C_{r'}^{5,s'}$, hence*

$$C_r^{5,s} \wedge C_{r'}^{5,s'} \simeq \begin{cases} C_{r'}^{9,s'} \vee C_{r'}^{9,s'} \vee C_\eta^5 \wedge C_{r'}^{5,s'}, & s > r', s' \text{ and } r \geqslant r', s', \\ C_{r'}^{9,s'} \vee C_{s'}^{9,r'} \vee C_\eta^5 \wedge C_r^{5,r}, & s > r' > s' = r, \\ C_{r'}^{9,s'} \vee C_r^5 \wedge C_{r'}^{5,s'}, & s > r', s' \text{ and } s' \neq r < r' \text{ or } r < s'. \end{cases}$$

*Proof.* Since $|[C_{r'}^{5,s'}, C_{r'}^{5,s'}]| = max\{2^{s'+1}, 2^{r'+1}\}$ [**1**], this lemma is easily obtained from the cofibre sequence $S^4 \wedge C_{r'}^{5,s'} \xrightarrow{\binom{i\eta}{2^s}\wedge 1} (M_{2^r}^3 \vee S^4) \wedge C_{r'}^{5,s'} \to C_r^{5,s} \wedge C_{r'}^{5,s'}$. □

Now by $C_r^{5,s} \wedge C_{r'}^{5,s'} \simeq C_{r'}^{5,s'} \wedge C_r^{5,s}$ and $C_r^{5,s} \wedge C_{r'}^{5,s'} \simeq D(C_s^{5,r} \wedge C_{s'}^{5,r'})$, it suffices to compute the following cases, denoted by **Cases ✠**:

(a) $s = r' > s' > r$  (b) $s = r' > s' = r$;  (c) $s = r' = s' > r$;
(d) $s = r' = s' = r$;  (e) $s = s' > r' > r$;  (f) $s = s' > r' = r$.

We will prove case (a) of **Cases ✠** and omit the proofs of the other cases since they are similar or easier.

### 5.2. $(C_r^{5,s} \wedge C_{r'}^{5,s'})/S^6$ and $(C_r^{5,s} \wedge C_{r'}^{5,s'})^{(9)}$ for $s = r' > s' > r$

**(1) Determining $(C_r^{5,s} \wedge C_{r'}^{5,s'})/S^6$**

Applying Lemma 2.4 to the cofibre sequences

$$S_1^3 \vee M_{2^s}^3 \xrightarrow{f=(2^r,\eta q)} S_a^3 \to C_r^{5,s}; \qquad S_2^3 \vee M_{2^{s'}}^3 \xrightarrow{f'=(2^{r'},\eta q)} S_b^3 \to C_{r'}^{5,s'},$$

with the identification $\Sigma M_{2^s}^3 \wedge M_{2^{s'}}^3 \simeq M_{2^{s'}}^7 \vee M_{2^{s'}}^8$ we have

$$(C_r^{5,s} \wedge C_{r'}^{5,s'})/S^6 \simeq (S^7 \vee M_{2^s}^7 \vee S^7 \vee M_{2^{s'}}^7) \cup_{\mathcal{A}} \mathbf{C}(S^7 \vee M_{2^{s'}}^7 \vee M_{2^s}^7 \vee M_{2^{s'}}^7 \vee M_{2^{s'}}^8),$$

where $\mathcal{A} =$

| | $S^7$ | $M_{2^{s'}}^7$ | $M_{2^s}^7$ | $M_{2^{s'}}^7$ | $M_{2^{s'}}^8$ |
|---|---|---|---|---|---|
| $S^7$ | $2^{r'}$ | $\eta q$ | $0$ | $0$ | $0$ |
| $M_{2^s}^7$ | $0$ | $0$ | $0$ | $i\eta q$ | $\xi_s^{s'} + \kappa i\eta\eta q$ |
| $S^7$ | $-2^r$ | $0$ | $\eta q$ | $0$ | $0$ |
| $M_{2^{s'}}^7$ | $0$ | $-2^r$ | $0$ | $0$ | $\eta \wedge 1$ |

.

Take transformations $2^{r'-r}\mathbf{r}_3+\mathbf{r}_1$; $-\mathbf{r}_3$; $-\mathbf{r}_4$ on $\mathcal{A}$ to get $(C_r^{5,s}\wedge C_{r'}^{5,s'})/S^6\simeq C_r^{9,s}\vee L$, where $L$ is the mapping cone of the map

$$\begin{array}{c|ccc|}
 & M^7_{2^{s'}} & M^7_{2^{s'}} & M^8_{2^{s'}} \\ \hline
S^7 & \eta q & 0 & 0 \\
M^7_{2^s} & 0 & i\eta q & \xi_s^{s'}+\kappa i\eta\eta q \\
M^7_{2^{s'}} & 2^r & 0 & \eta\wedge 1 \\ \hline
\end{array}.$$

**Lemma 5.3.** *$L\simeq C^{9,r}\vee(C_s^{9,s'}\vee C_r^9)\cup_{\binom{\alpha}{\gamma}}\mathbf{C}S^9$ where $\alpha=i_{\underline{M}}\binom{\xi_s}{0}$ and $\gamma$ is determined by $q_S\gamma=\binom{0}{2^{s'}}$ ($i_{\underline{M}}$ and $q_S$ are the maps in **Cof4** of $C_s^{9,s'}$ and **Cof1** of $C_r^9$, respectively).*

*Proof.* It is easy to know $L^{(9)}$ is the mapping cone of the map $\begin{array}{c|ccc|}
 & M^7_{2^{s'}} & M^7_{2^{s'}} & S^8 \\ \hline
S^7 & \eta q & 0 & 0 \\
M^7_{2^s} & 0 & i\eta q & 0 \\
M^7_{2^{s'}} & 2^r & 0 & i\eta \\ \hline
\end{array}$.
Hence $L^{(9)}\simeq C_s^{9,s'}\vee L_1$, where $L_1$ is the mapping cone of $\begin{array}{c|cc|}
 & M^7_{2^{s'}} & S^8 \\ \hline
S^7 & \eta q & 0 \\
M^7_{2^{s'}} & 2^r & i\eta \\ \hline
\end{array}$.
Similarly as in the proof of Lemma 4.7, we get $L_1\simeq C^{9,r}\vee C_r^9$ and $L^{(9)}\simeq C_s^{9,s'}\vee C^{9,r}\vee C_r^9$.

There is a cofibre sequence

$$S^9\xrightarrow{f_L=\begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix}}C_s^{9,s'}\vee C^{9,r}\vee C_r^9\to L\to S^{10}\to C_s^{10,s'}\vee C^{10,r}\vee C_r^{10},$$

$\alpha=i_{\underline{M}}\binom{z\xi_s}{x\eta}$, $\beta=i_S\binom{0}{y\eta}$, $\gamma$ is determined by $q_S\gamma=\binom{w\eta}{a}$, where $x,y,z,w\in\{0,1\}$ and $a\in\mathbb{Z}$.

$$\begin{array}{ccc}
 & & S^9 \\
 & \swarrow^{\binom{z\xi_s}{x\eta}} & \downarrow\alpha \\
M^7_{2^s}\vee S^8 & \xrightarrow[i_{\underline{M}}]{} & C_s^{9,s'}
\end{array};\qquad
\begin{array}{ccc}
 & & S^9 \\
 & \swarrow^{\binom{0}{y\eta}} & \downarrow\beta \\
S^7\vee S^8 & \xrightarrow[i_S]{} & C^{9,r}
\end{array};\qquad
\begin{array}{ccc}
S^9 & & \\
\downarrow\gamma & \searrow^{\binom{w\eta}{a}} & \\
C_r^9 & \xrightarrow[q_S]{} & S^8\vee S^9
\end{array},$$

$a=2^{s'}$ from $[L,S^{10}]\cong\mathbb{Z}/2^{s'}$. From Proposition 2.1, for $s=r'>s'>r$,

$$[(C_r^{5,s}\wedge C_{r'}^{5,s'})/S^6,S^8]\cong[C_r^{5,s}\wedge C_{r'}^{5,s'},S^8]\cong[C_r^{5,s},C_{s'}^{5,r'}]\cong\mathbb{Z}/2^s\oplus\mathbb{Z}/2^{r+1}\oplus\mathbb{Z}/2^{r+1}.$$

Together with $[(C_r^{5,s}\wedge C_{r'}^{5,s'})/S^6,S^8]\cong[C_r^{9,s},S^8]\oplus[L,S^8]$, we get

$$[L,S^8]\cong\mathbb{Z}/2^s\oplus\mathbb{Z}/2^{r+1}.$$

On the other hand, by the cofibre sequence of $L$ above, there is an exact sequence

$$0\to\frac{\mathbb{Z}/2}{\langle x,y,z,w\rangle}\to[L,S^8]\to\mathbb{Z}/2^{s+1}\oplus\mathbb{Z}/2^{r+1}\xrightarrow{(\alpha^*,\gamma^*)}\mathbb{Z}/2,$$

where $\alpha^*(1) = z, \gamma^*(1) = w$. Hence $w = 0, z = 1$. By the following commutative diagram:

$$\begin{array}{ccccc}
S^9 & \xrightarrow{f_L} & C_s^{9,s'} \vee C^{9,r} \vee C_r^9 & \longrightarrow & L \\
\Vert & & \downarrow{\scriptstyle \Theta} & & \vdots{\scriptstyle \hat{\Theta}} \\
S^9 & \xrightarrow{\theta=\Theta f_L} & C_s^{9,s'} \vee C^{9,r} \vee C_r^9 & \longrightarrow & \mathbf{C}_\theta,
\end{array}$$

where $\Theta = \begin{array}{c|ccc|} & C_s^{9,s'} & C^{9,r} & C_r^9 \\ \hline C_s^{9,s'} & \hat{\mu} & 0 & 0 \\ C^{9,r} & \hat{\lambda} & 1 & 0 \\ C_r^9 & 0 & 0 & 1 \\ \hline \end{array}$ is a homotopy equivalence, $\hat{\mu}$ and $\hat{\lambda}$ are induced by the following commutative diagrams:

$$\begin{array}{ccccc}
S^8 & \xrightarrow{\binom{i\eta}{2^{s'}}} & M_{2^s}^7 \vee S^8 & \xrightarrow{i_{\underline{M}}} & C_s^{9,s'} \\
\Vert & & \downarrow{\scriptstyle \binom{1,0}{xq,1}} & & \vdots{\scriptstyle \hat{\mu}} \\
S^8 & \xrightarrow[\binom{i\eta}{2^{s'}}]{} & M_{2^s}^7 \vee S^8 & \xrightarrow[i_{\underline{M}}]{} & C_s^{9,s'}
\end{array}, \qquad
\begin{array}{ccccc}
S^8 & \xrightarrow{\binom{i\eta}{2^{s'}}} & M_{2^s}^7 \vee S^8 & \xrightarrow{i_{\underline{M}}} & C_s^{9,s'} \\
\downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \binom{0,0}{yq,0}} & & \vdots{\scriptstyle \hat{\lambda}} \\
S^8 & \xrightarrow[\binom{\eta}{2^r}]{} & S^7 \vee S^8 & \xrightarrow[i_S]{} & C^{9,r}
\end{array}.$$

Then $\theta = \Theta f_L = \begin{pmatrix} \hat{\mu}\alpha \\ \hat{\lambda}\alpha + \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} \hat{\mu}\alpha \\ 0 \\ \gamma \end{pmatrix}$. Rewrite $\hat{\mu}\alpha$ as $\alpha$,

$$L \simeq \mathbf{C}_\theta \simeq C^{9,r} \vee (C_s^{9,s'} \vee C_r^9) \cup_{\binom{\alpha}{\gamma}} \mathbf{C}S^9$$

$\alpha, \gamma$ satisfy the conditions in the lemma. □

Thus

$$(C_r^{5,s} \wedge C_{r'}^{5,s'})/S^6 \simeq C_r^{9,s} \vee C^{9,r} \vee (C_s^{9,s'} \vee C_r^9) \cup_{\binom{\alpha}{\gamma}} \mathbf{C}S^9, \; s = r' > s' > r. \tag{16}$$

**(2) Determining** $(C_r^{5,s} \wedge C_{r'}^{5,s'})^{(9)}$

Apply Lemma 2.4 to **Cof4** of $C_r^{5,s}$ and $C_{r'}^{5,s'}$ to get

$$(C_r^{5,s} \wedge C_{r'}^{5,s'})^{(9)} \simeq (M_{2^r}^3 \wedge M_{2^{r'}}^3 \vee M_{2^r}^7 \vee M_{2^{r'}}^7 \vee S^8) \cup_{\mathcal{B}} \mathbf{C}(M_{2^r}^7 \vee S^8 \vee M_{2^{r'}}^7 \vee S^8),$$

$$\mathcal{B} = \begin{array}{c|cccc|} & M_{2^r}^7 & S^8 & M_{2^{r'}}^7 & S^8 \\ \hline M_{2^r}^3 \wedge M_{2^{r'}}^3 & 1\wedge i\eta & 0 & i\eta \wedge 1 & 0 \\ M_{2^r}^7 & 2^{s'} & 0 & 0 & i\eta \\ M_{2^{r'}}^7 & 0 & i\eta & 2^s & 0 \\ S^8 & 0 & 2^{s'} & 0 & 2^s \\ \hline \end{array} \cong \begin{array}{c|cccc|} & M_{2^r}^7 & S^8 & M_{2^{r'}}^7 & S^8 \\ \hline M_{2^r}^3 \wedge M_{2^{r'}}^3 & 1\wedge i\eta & 0 & i\eta \wedge 1 & 0 \\ M_{2^r}^7 & 0 & 0 & 0 & i\eta \\ M_{2^{r'}}^7 & 0 & i\eta & 0 & 0 \\ S^8 & 0 & 2^{s'} & 0 & 0 \\ \hline \end{array}$$

by noting that $2^{s'} = 0 \in [M_{2^r}^7, M_{2^r}^7]$ and $2^s = 0 \in [M_{2^{r'}}^7, M_{2^{r'}}^7]$ for $s = r' > s' > r$. From (11), for $r' > r$, $(M_{2^r}^3 \wedge M_{2^{r'}}^3) \cup_{(1\wedge i\eta, i\eta\wedge 1)} \mathbf{C}(M_{2^r}^7 \vee M_{2^{r'}}^7) \simeq M_{2^r}^3 \wedge C_\eta^5 \vee C_r^{9,r'}$. Thus

$$(C_r^{5,s} \wedge C_{r'}^{5,s'})^{(9)} \simeq M_{2^r}^3 \wedge C_\eta^5 \vee C_r^{9,r'} \vee C_{r'}^{9,s'} \vee C_r^9 \quad \text{for } s = r' > s' > r. \tag{17}$$

### 5.3. Decomposition of $C_r^{5,s}\wedge C_{r'}^{5,s'}$ for $s=r'>s'>r$

Denote a column vector by $(\zeta_1,\zeta_2,\cdots,\zeta_s)^T$ in the rest of the paper.

From (17), there is a cofibre sequence

$$S^9\xrightarrow{(\delta_1,\mu,\nu,\omega)^T} M_{2^r}^3\wedge C_\eta^5\vee C_r^{9,s}\vee C_{r'}^{9,s'}\vee C_r^9\to C_r^{5,s}\wedge C_{r'}^{5,s'}.$$

Since $\delta_1=i_{\overline{C}}t_1\varrho_6$ for some $t_1\in\mathbb{Z}$,

$$(C_r^{5,s}\wedge C_{r'}^{5,s'})/S^6\simeq C^{9,r}\vee(C_r^{9,s}\vee C_{r'}^{9,s'}\vee C_r^9)\cup_{(\mu,\nu,\omega)^T}\mathbf{C}S^9.$$

On the other hand, from (16)

$$(C_r^{5,s}\wedge C_{r'}^{5,s'})/S^6\simeq C^{9,r}\vee(C_r^{9,s}\vee C_{r'}^{9,s'}\vee C_r^9)\cup_{(0,\alpha,\gamma)^T}\mathbf{C}S^9.$$

Thus $(C_r^{9,s}\vee C_{r'}^{9,s'}\vee C_r^9)\cup_{(\mu,\nu,\omega)^T}\mathbf{C}S^9\simeq(C_r^{9,s}\vee C_{r'}^{9,s'}\vee C_r^9)\cup_{(0,\alpha,\gamma)^T}\mathbf{C}S^9$, which are 2-local spaces. Consequently, there is a homotopy equivalence $\lambda$ yielding the following homotopy commutative diagram:

$$\begin{array}{ccccccc}
S^9 & \xrightarrow{(\mu,\nu,\omega)^T} & C_r^{9,s}\vee C_{r'}^{9,s'}\vee C_r^9 & \longrightarrow & \mathbf{C}_{(\mu,\nu,\omega)^T} & \longrightarrow & S^{10}\\
\| & & \downarrow{\scriptstyle\lambda'} & & \downarrow{\scriptstyle\lambda\,\simeq} & & \|\\
S^9 & \xrightarrow{(0,\alpha,\gamma)^T} & C_r^{9,s}\vee C_{r'}^{9,s'}\vee C_r^9 & \longrightarrow & \mathbf{C}_{(0,\alpha,\gamma)^T} & \longrightarrow & S^{10},
\end{array}$$

where $\lambda'$ is the restriction of $\lambda$, which is a self-homotopy equivalence of $C_r^{9,s}\vee C_{r'}^{9,s'}\vee C_r^9$.

Let $\widetilde{\Gamma}$ be induced by the following commutative diagram:

$$\begin{array}{ccccc}
S^9 & \xrightarrow{(\delta_1,\mu,\nu,\omega)^T} & M_{2^r}^3\wedge C_\eta^5\vee C_r^{9,s}\vee C_{r'}^{9,s'}\vee C_r^9 & \longrightarrow & C_r^{5,s}\wedge C_{r'}^{5,s'}\\
\| & & \downarrow{\scriptstyle\Gamma} & & \vdots\,{\scriptstyle\widetilde{\Gamma}}\\
S^9 & \xrightarrow{\theta=\Gamma(\delta_1,\mu,\nu,\omega)^T} & M_{2^r}^3\wedge C_\eta^5\vee C_r^{9,s}\vee C_{r'}^{9,s'}\vee C_r^9 & \longrightarrow & \mathbf{C}_\theta.
\end{array}$$

Since $\Gamma=\begin{array}{r|cc|}
 & M_{2^r}^3\wedge C_\eta^5 & C_r^{9,s}\vee C_{r'}^{9,s'}\vee C_r^9\\ \hline
M_{2^r}^3\wedge C_\eta^5 & 1 & 0\\
C_r^{9,s}\vee C_{r'}^{9,s'}\vee C_r^9 & 0 & \lambda'\\ \hline
\end{array}$ is a self-homotopy equivalence, $\widetilde{\Gamma}$ is also a homotopy equivalence.

$$\theta=\Gamma(\delta_1,\mu,\nu,\omega)^T=(\delta_1,\lambda'(\mu,\nu,\omega)^T)^T=(\delta_1,0,\alpha,\gamma)^T.$$

Consequently, for $s=r'>s'>r$,

$$C_r^{5,s}\wedge C_{r'}^{5,s'}\simeq\mathbf{C}_\theta\simeq C_r^{9,s}\vee Q_1,\qquad Q_1=(M_{2^r}^3\wedge C_\eta^5\vee C_{r'}^{9,s'}\vee C_r^9)\cup_{(\delta_1,\alpha,\gamma)^T}\mathbf{C}S^9.$$

**Lemma 5.4.** *$Q_1\simeq C^{5,s'}\wedge C_r^{5,s}$, $s=r'>s'>r$.*

*Proof.* **(1) Determining $\delta_1$, i.e., $t_1$**

By Lemma 5.1, $C_r^{5,s}\wedge C_{r'}^{5,s'}$ cannot split out $M_{2^r}^3\wedge C_\eta^5$, which implies that $\delta_1\neq 0$ in $\pi_9(M_{2^r}^3\wedge C_\eta^5)$. Hence we can assume that $t_1=1$ for $r=1$ and $t_1\in\{1,2\}$ for $r\geqslant 2$. Next we determine $t_1$ for $r\geqslant 2$.

**Lemma 5.5.** *For $s \geqslant r', s'$ and $r \geqslant 2$, there is a short exact sequence of abelian groups*

$$0 \to \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2^{s'+\epsilon_{r'}} \to \pi_9(C_r^{5,s} \wedge C_{r'}^{5,s'}) \to \mathbb{Z}/2 \oplus \mathbb{Z}/2 \to 0,$$

*where $\epsilon_{r'} = 0$ for $r' = 1$ and $\epsilon_{r'} = 1$ for $r' > 1$.*

*Proof.* We only prove the case $r' > 1$

$$\pi_9(C_r^{5,s} \wedge C_{r'}^{5,s'}) \cong [C_s^{5,r} \wedge C_{s'}^{5,r'}, S^7] \cong [C_s^{7,r}, C_{r'}^{6,s'}] \quad \text{(Proposition 2.1).}$$

By **Cof5** of $C_s^{7,r}$, there is an exact sequence

$$[S^7, C_{r'}^{6,s'}] \xrightarrow{(2^r p_1 q_S)^*} [C_s^7, C_{r'}^{6,s'}] \to [C_s^{7,r}, C_{r'}^{6,s'}] \to [S^6, C_{r'}^{6,s'}] \xrightarrow{0} [C_s^7, C_{r'}^{6,s'}].$$

$(2^r p_1 q_S)^*$ is zero for $r \geqslant 2$ and $[C_s^7, C_{r'}^{6,s'}] \cong [S^9, C^{5,s} \wedge C_{r'}^{5,s'}] \cong \pi_9(C_{r'}^{9,s'} \vee C_\eta^5 \wedge C_{r'}^{5,s'})$ $\cong \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2^{s'+1}$ for $s \geqslant r', s'$ (the last isomorphism is from (4)). □

There is a cofibre sequence

$$S^9 \xrightarrow{(\delta_1, \alpha, \gamma)^T} M_{2^r}^3 \wedge C_\eta^5 \vee C_{r'}^{9,s'} \vee C_r^9 \to Q_1 \to S^{10},$$

which implies the following exact sequence

$$\mathbb{Z} \xrightarrow{(\delta_{1*}, \alpha_*, \gamma_*)^T} \mathbb{Z}/4\langle i_{\overline{C}} \varrho_6 \rangle \oplus \mathbb{Z}/2\langle i_{\underline{M}} j_1 \xi_{r'} \rangle \oplus \mathbb{Z}/2\langle i_{\underline{M}} j_2 \eta \rangle \oplus \mathbb{Z}/2\langle (j_1\eta)_S^- \rangle \oplus \mathbb{Z}\langle (2j_2)_S^- \rangle$$
$$\longrightarrow \pi_9 Q_1 \longrightarrow 0,$$

where $\delta_{1*}(1) = t_1 i_{\overline{C}} \varrho_6$, $\alpha_*(1) = i_{\underline{M}} j_1 \xi_{r'}$ and $\gamma_*(1) = 2^{s'-1}(2j_2)_S^-$.

$$\pi_9 Q_1 \cong \frac{\mathbb{Z}/4 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}}{\langle (t_1, 1, 0, 0, 2^{s'-1}) \rangle} \cong \begin{cases} \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2^{s'+1}, & t_1 = 1, \\ \mathbb{Z}/4 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2^{s'}, & t_1 = 2. \end{cases}$$

Since $\pi_9(C_r^{5,s} \wedge C_{r'}^{5,s'}) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \pi_9 Q_1$ induced by $C_r^{5,s} \wedge C_{r'}^{5,s'} \simeq C_r^{9,s} \vee Q_1$ for $s = r' > s' > r$, together with the short exact sequence for $\pi_9(C_r^{5,s} \wedge C_{r'}^{5,s'})$ in Lemma 5.5, we have

$$\pi_9 Q_1 = \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2^{s'+1}, \quad \text{i.e., } t_1 = 1, \quad \delta_1 = i_{\overline{C}} \varrho_6.$$

**(2) The cell structure of $C^{5,s'} \wedge C_r^{5,s}$**

Apply Lemma 2.4 to **Cof1** of $C^{5,s'}$ and **Cof4** of $C_r^{5,s}$ to get

$$(C^{5,s'} \wedge C_r^{5,s})^{(9)} \simeq M_{2^r}^3 \wedge C_\eta^5 \vee C_{r'}^{5,s'} \vee C_r^9.$$

There is a cofibre sequence

$$S^9 \xrightarrow{(\hat{\delta}, \hat{\alpha}, \hat{\gamma})^T} M_{2^r}^3 \wedge C_\eta^5 \vee C_{r'}^{9,s'} \vee C_r^9 \to C^{5,s'} \wedge C_r^{5,s} \to S^{10}, \tag{18}$$

where $\hat{\delta} = i_{\overline{C}} \hat{t} \varrho_6$, $\hat{\alpha} = i_{\underline{M}} \binom{\hat{x}\xi_{r'}}{\hat{y}\eta}$ and $\hat{\gamma}$ is determined by $q_S \binom{\hat{z}\eta}{\hat{a}}$ for $\hat{x}, \hat{y}, \hat{z} \in \{0, 1\}$ and $\hat{t}, \hat{a} \in \mathbb{Z}$, i.e.,

$$\begin{array}{ccc} & & S^9 \\ & \swarrow^{\hat{t}\varrho_6} & \downarrow{\scriptstyle\hat{\delta}} \\ S^6 & \xrightarrow[i_{\overline{C}}]{} & M^3_{2^r}\wedge C^5_\eta \end{array};\quad \begin{array}{ccc} & & S^9 \\ & \swarrow^{\binom{\hat{x}\xi_{r'}}{\hat{y}\eta}} & \downarrow{\scriptstyle\hat{\alpha}} \\ M^7_{2^{r'}}\vee S^8 & \xrightarrow[i_{\underline{M}}]{} & C^{9,s'}_{r'} \end{array};\quad \begin{array}{ccc} S^9 & & \\ \downarrow{\scriptstyle\hat{\gamma}} & \searrow^{\binom{\hat{z}\eta}{\hat{a}}} & \\ C^9_r & \xrightarrow[q_S]{} & S^8\vee S^9, \end{array}$$

$\hat{a} = 2^{s'}$ for $H_9(C^{5,s'} \wedge C^{5,s}_r) \cong \mathbb{Z}/2^{s'}$. Cofibre sequence (18) induces

$$0 \to \frac{\mathbb{Z}/2}{\langle \hat{x}, \hat{y}, \hat{z}\rangle} \to [C^{5,s'} \wedge C^{5,s}_r, S^8] \to \mathbb{Z}/2^{r'+1} \oplus \mathbb{Z}/2^{r+1} \xrightarrow{(\hat{\alpha}^*,\hat{\gamma}^*)} \mathbb{Z}/2,$$

where $\hat{\alpha}^*(1) = \hat{x}$ and $\hat{\gamma}^*(1) = \hat{z}$. Since $C^{5,s'} \wedge C^{5,s}_r$ is indecomposable for $s = r' > s' > r$, $\hat{x} = 1$ or $\hat{y} = 1$, which implies $\frac{\mathbb{Z}/2}{\langle \hat{x},\hat{y},\hat{z}\rangle} = 0$. On the other hand,

$$[C^{5,s'} \wedge C^{5,s}_r, S^8] \cong [C^{5,s}_r, C^5_{s'}] \cong \mathbb{Z}/2^{r+1} \oplus \mathbb{Z}/2^{r'}.$$

Thus $\hat{x} = 1$ and $\hat{z} = 0$. Now from the following commutative diagram:

$$\begin{array}{ccccc} S^9 & \xrightarrow{(\hat{\delta},\hat{\alpha},\hat{\gamma})^T} & M^3_{2^r} \wedge C^5_\eta \vee C^{9,s'}_{r'} \vee C^9_r & \longrightarrow & C^{5,s'} \wedge C^{5,s}_r \\ \| & & \downarrow{\scriptstyle\Theta\ (\simeq)} & & \vdots\,{\scriptstyle\widetilde{\Theta}\ (\simeq)} \\ S^9 & \xrightarrow{\theta=\Theta(\hat{\delta},\hat{\alpha},\hat{\gamma})^T} & M^3_{2^r} \wedge C^5_\eta \vee C^{9,s'}_{r'} \vee C^9_r & \longrightarrow & \mathbf{C}_\theta, \end{array}$$

where the self-homotopy equivalence $\Theta$ is given by

$$\begin{array}{c|ccc|} & M^3_{2^r}\wedge C^5_\eta & C^{9,s'}_{r'} & C^9_r \\ \hline M^3_{2^r}\wedge C^5_\eta & 1 & 0 & 0 \\ C^{9,s'}_{r'} & 0 & \lambda & 0 \\ C^9_r & 0 & 0 & 1 \\ \hline \end{array}, \qquad \begin{array}{ccccc} S^8 & \xrightarrow{\binom{i\eta}{2^{s'}}} & M^7_{2^{r'}} \vee S^8 & \xrightarrow{i_{\underline{M}}} & C^{9,s'}_{r'} \\ \| & & \downarrow{\scriptstyle\binom{1,\,0}{\hat{y}q,\,1}} & & \vdots\,{\scriptstyle\lambda} \\ S^8 & \xrightarrow[\binom{i\eta}{2^{s'}}]{} & M^7_{2^{r'}} \vee S^8 & \xrightarrow[i_{\underline{M}}]{} & C^{9,s'}_{r'}, \end{array}$$

we can assume $\hat{y} = 0$. Next we are going to calculate $\hat{\delta}$, i.e., $\hat{t}$.

$\hat{t} = 1$ for $r = 1$ and $\hat{t} \in \{1, 2\}$ for $r \geqslant 2$ since $\hat{\delta} \neq 0$.

For the case $r \geqslant 2$, similarly as the calculation of $\pi_9 Q_1$, we get

$$\pi_9(C^{5,s'} \wedge C^{5,s}_r) \cong \begin{cases} \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2^{s'+1}, & \hat{t} = 1, \\ \mathbb{Z}/4 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2^{s'}, & \hat{t} = 2. \end{cases} \tag{19}$$

On the other hand, $\pi_9(C^{5,s'} \wedge C^{5,s}_r) \cong [C^7_{s'}, C^{6,s}_r]$ and from **Cof3** of $C^7_{s'}$, we get the following exact sequence

$$[C^8_\eta, C^{6,s}_r] \xrightarrow{(i_\eta 2^{s'})^* = 0} [S^6, C^{6,s}_r] \to [C^7_{s'}, C^{6,s}_r] \to [C^7_\eta, C^{6,s}_r] \xrightarrow{(i_\eta 2^{s'})^*} [S^5, C^{6,s}_r]. \tag{20}$$

There is a commutative diagram for the morphism $(i_\eta 2^{s'})^*$ by (3)

$$\begin{array}{ccccc}
 & & \overset{(i_\eta 2^{s'})^*}{\curvearrowright} & & \\
[C_\eta^7, C_r^{6,s}] & \xrightarrow{i_\eta^*} & [S^5, C_r^{6,s}] & \xrightarrow{(2^{s'})^*} & [S^5, C_r^{6,s}] \\
(i_{\underline{M}})_* \uparrow \cong & & (i_{\underline{M}})_* \uparrow \cong & & (i_{\underline{M}})_* \uparrow \cong \\
\frac{[C_\eta^7, M_{2^r}^4 \vee S^5]}{\binom{i\eta}{2^s}_* [C_\eta^7, S^5]} & \xrightarrow{i_\eta^*} & \frac{[S^5, M_{2^r}^4 \vee S^5]}{\binom{i\eta}{2^s}_* [S^5, S^5]} & \xrightarrow{(2^{s'})^*} & \frac{[S^5, M_{2^r}^4 \vee S^5]}{\binom{i\eta}{2^s}_* [S^5, S^5]} \\
\| & & \| & & \| \\
\frac{\mathbb{Z}/4 \oplus \mathbb{Z}}{\langle (2, 2^s) \rangle} & \xrightarrow{\varphi} & \frac{\mathbb{Z}/2 \oplus \mathbb{Z}}{\langle (1, 2^s) \rangle} & \xrightarrow{(2^{s'})^*} & \frac{\mathbb{Z}/2 \oplus \mathbb{Z}}{\langle (1, 2^s) \rangle},
\end{array}$$

where $\varphi(1,0) = 0$ and $\varphi(0,1) = (0,2)$. Hence $ker(i_\eta 2^{s'})^* \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2^{s'+1}$ in (20). Together with (19), we have

$$\pi_9(C^{5,s'} \wedge C_r^{5,s}) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2^{s'+1}, \quad \text{i.e., } \hat{t} = 1.$$

From the analysis above, we get $\hat{\delta} = \delta$, $\hat{\alpha} = \alpha$ and $\hat{\gamma} = \gamma$, i.e., $Q_1 \simeq C^{5,s'} \wedge C_r^{5,s}$. $\square$

Hence $C_r^{5,s} \wedge C_{r'}^{5,s'} \simeq C_r^{9,s} \vee C^{5,s'} \wedge C_r^{5,s}$ for $s = r' > s' > r$.

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