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ČECH COMPLEXES FOR COVERS OF SMALL CATEGORIES

KOHEI TANAKA

(*communicated by J.F. Jardine*)

Abstract

We present a combinatorial analogue of the nerve theorem for covers of small categories, using the Grothendieck construction. We apply our result to prove the inclusion–exclusion principle for the Euler characteristic of a finite category.

1. Introduction

For an open cover $\mathcal{U} = \{U_a\}_{a \in A}$ of a space X, the Čech nerve $\check{C}(\mathcal{U})$ of $\mathcal{U}$ is a simplicial space defined by

$$\check{C}(\mathcal{U})_n = \coprod_{a_0,\ldots,a_n \in A} U_{a_0 \ldots a_n},$$

where $U_{a_0 \ldots a_n}$ denotes the intersection $U_{a_0} \cap \cdots \cap U_{a_n}$. Here, the face maps omit indices, and the degeneracy maps insert copies of indices. This is a simplicial space augmented over X, and its geometric realization is called the Čech complex of $\mathcal{U}$. Segal showed that the canonical map $|\check{C}(\mathcal{U})| \to X$ is a homotopy equivalence for a suitable cover $\mathcal{U}$ [**Seg68**]. On the other hand, Dugger and Isaksen proved that the natural map $\operatorname{hocolim} \check{C}(\mathcal{U}) \to X$ from the homotopy colimit of the Čech nerve to the original space is a weak homotopy equivalence for any open cover [**DI04**]. In this paper, a combinatorial analogue of their result is provided for small categories. Let C be a small category, and let $\mathcal{D} = \{D_a\}_{a \in A}$ be a collection of subcategories of C satisfying $C = \bigcup_{a \in A} D_a$. We can define the *Čech nerve* $\check{C}(\mathcal{D})$ of $\mathcal{D}$ as a simplicial object in the category of small categories, similarly to the above. In the context of small categories, the Grothendieck construction is known as a model of the homotopy colimit of a diagram in small categories [**Tho79**]. The Grothendieck construction $\mathrm{Gr}(\check{C}(\mathcal{D}))$ of the Čech nerve is also equipped with the natural functor $\mathrm{Gr}(\check{C}(\mathcal{D})) \to C$. A natural question to ask is when this induces a homotopy equivalence on classifying spaces.

Unfortunately, it does not hold for every cover. For example, let C be a small category, formed as follows:

$$x \rightrightarrows y \longrightarrow z,$$

with the terminal object z. If D_1 is the full subcategory with $\mathrm{ob}(D_1) = \{x, y\}$, and D_2 is the full subcategory with $\mathrm{ob}(D_2) = \{y, z\}$, then $C = D_1 \cup D_2$. The classifying space

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of C is contractible; however, the classifying space of the Grothendieck construction for the Čech nerve of the cover $\{D_1, D_2\}$ is homotopy equivalent to a circle.

In order to solve the problem above, we introduce two classes of covers of small categories consisting of *ideals* and *filters*, respectively. These are generalized notions of ideals and filters of posets [**Sta12, Zap98**]. An ideal D of a small category C is a full subcategory such that an object y of C belongs to D, whenever $C(y,x) \neq \emptyset$ for some object x of D. A cover $\mathcal{D} = \{D_a\}_{a\in A}$ of a small category C is called an *ideal cover*, if D_a is an ideal of D for any $a \in A$. We can also define the dual notions; *filters* and *filter covers*.

Theorem 1.1. *Let $\mathcal{D}$ be an ideal cover or a filter cover of a small category C. The natural functor* $\mathrm{Gr}(\check{C}(\mathcal{D})) \to C$ *induces a homotopy equivalence between their classifying spaces.*

As an application of this result, we will show that we can obtain the inclusion–exclusion principle of the Euler characteristics for finite categories with ideal covers or filter covers. The Euler characteristic $\chi(C)$ of a finite category C is introduced by Leinster in [**Lei08**]. He calculates the Euler characteristic of the Grothendieck construction for a diagram in finite categories. By applying this to the (reduced) Čech nerve of a cover for a finite category, we obtain the following formula.

Theorem 1.2 (Inclusion–exclusion principle)**.** *Let $\mathcal{D} = \{D_a\}_{a\in A}$ be an ideal cover or a filter cover of a finite category C, indexed by a totally ordered finite set A. If each intersection $D_{a_0\ldots a_i}$ and C have Euler characteristics, then we have*

$$\chi(C) = \sum_{i=0} \sum_{a_0<\ldots<a_i} (-1)^i \chi(D_{a_0\ldots a_i}).$$

Indeed, Leinster has demonstrated the inclusion–exclusion principle of the Euler characteristic for finite sets, as Example 3.4(d) in [**Lei08**]. The theorem above is a generalization of that result.

The rest of this paper is organized as follows. Section 2 describes the homotopy theory of Čech complexes, for covers of small categories. Subsequently, we focus on the inclusion–exclusion principle for Euler characteristics of finite categories in Section 3.

2. The Čech nerve for covers of small categories

Let C be a small category, and let $\mathcal{D} = \{D_a\}_{a\in A}$ be a collection of subcategories of C indexed by a set A. The intersection $\bigcap_{a\in A} D_a$ is a subcategory of C, whose set of objects is $\bigcap_{a\in A} \mathrm{ob}(D_a)$ and set of morphisms is $\bigcap_{a\in A} D_a(x,y)$, for each pair of objects x and y.

On the other hand, the union $\bigcup_{a\in A} \mathcal{D}_a$ is a subcategory of C, whose set of objects is $\bigcup_{a\in A} \mathrm{ob}(D_a)$ and the set of morphisms is generated by compositions of morphisms of D_a. That is:

$$\left(\bigcup_{a\in A} D_a\right)(x,y) = \{f_n \ldots f_0 \mid f_i \in D_{a_i}(z_i, z_{i+1}), z_0 = x, z_{n+1} = y\},$$

for each pair of objects x and y.

Definition 2.1. Let C be a small category. A collection of subcategories,

$$\mathcal{D} = \{D_a\}_{a \in A},$$

of C is called a *cover* if $C = \bigcup_{a \in A} D_a$.

Definition 2.2 (Čech nerve)**.** Let $\mathcal{D} = \{D_a\}_{a \in A}$ be a cover of a small category C. The *Čech nerve* $\check{C}(\mathcal{D})$ is a simplicial object in the category **Cat** of small categories, defined as

$$\check{C}(\mathcal{D})_n = \coprod_{a_0, \dots, a_n \in A} D_{a_0 \dots a_n},$$

where $D_{a_0 \dots a_n}$ denotes the intersection $D_{a_0} \cap \cdots \cap D_{a_n}$. Let us denote an object (a morphism) x of $\check{C}(\mathcal{D})_n$ belonging to $D_{a_0 \dots a_n}$ by $x_{a_0 \dots a_n}$. The face map $d_i \colon \check{C}(\mathcal{D})_n \to \check{C}(\mathcal{D})_{n-1}$ is the functor omitting the index a_i. That is:

$$d_i(x_{a_0 \dots a_n}) = x_{a_0 \dots \hat{a_i} \dots a_n},$$

on objects and morphisms. Similarly, the degeneracy map $s_j \colon \check{C}(\mathcal{D})_n \to \check{C}(\mathcal{D})_{n+1}$ is the functor inserting the index a_j between a_j and a_{j+1}. That is:

$$s_j(x_{a_0 \dots a_n}) = x_{a_0 \dots a_j a_j a_{j+1} \dots a_n},$$

on objects and morphisms.

In other words, $\check{C}(\mathcal{D})$ is a functor from Δ^{op} to the category **Cat**, where Δ consists of totally ordered finite sets $[n] = \{0, \dots, n\}$ and order preserving maps. It takes $[n]$ to $\check{C}(\mathcal{D})_n$, and an order preserving map $\varphi \colon [m] \to [n]$ in Δ to the functor $\varphi^* \colon \check{C}(\mathcal{D})_n \to \check{C}(\mathcal{D})_m$. Here, φ^* is given by $\varphi^*(x_{a_0 \dots a_n}) = x_{b_0 \dots b_m}$ on objects and morphisms, satisfying $b_j = a_{\varphi(j)}$ for each $j \in \{0, \dots, m\}$.

A (pseudo-)functor from a small category to **Cat** provides a new small category, called the Grothendieck construction.

Definition 2.3 (Grothendieck construction)**.** Let C be a small category, and let $F \colon C \to$ **Cat** be a functor. The *Grothendieck construction* $\mathrm{Gr}(F)$ is a small category defined by the following:

- The set of objects consists of pairs (c, x) of an object c of C and an object x of $F(c)$.
- The set of morphisms $\mathrm{Gr}(F)((c, x), (d, y))$ consists of pairs of morphisms (f, g), where $f \in C(c, d)$ and $g \in F(d)(F(f)(x), y)$.
- For each composable pair of morphisms

  $$(f_1, g_1) \in \mathrm{Gr}(F)((c, x), (d, y))$$

  and

  $$(f_2, g_2) \in \mathrm{Gr}(F)((d, y), (e, z)),$$

  the composition is given by $(f_2, g_2)(f_1, g_1) = (f_2 f_1, g_2(F(f_2)(g_1)))$.

We can take the Grothendieck construction of the Čech nerve $\check{C}(\mathcal{D}) \colon \Delta^{\mathrm{op}} \to$ **Cat** for a cover $\mathcal{D}$ of a small category C. Let us see the details of this category. The set of objects of $\mathrm{Gr}(\check{C}(\mathcal{D}))$ consists of objects $x_{a_0\dots a_n}$, indexed by $a_0, \dots, a_n \in A$ and $n \geqslant 0$. The set of morphisms $\mathrm{Gr}(\check{C}(\mathcal{D}))(x_{a_0\dots a_n}, y_{b_0\dots b_m})$ consists of pairs $(\varphi, f_{b_0\dots b_m})$ of an order preserving map $\varphi\colon [m] \to [n]$ such that $b_j = a_{\varphi(j)}$ for each j, and a morphism $f_{b_0\dots b_m}\colon x_{b_0\dots b_m} \to y_{b_0\dots b_m}$ of $D_{b_0\dots b_m}$. It is equipped with the canonical functor $\rho\colon \mathrm{Gr}(\check{C}(\mathcal{D})) \to C$ that eliminates indices. That is, $\rho(x_{a_0\dots a_n}) = x$ on objects, and $\rho(\varphi, f_{a_0\dots a_n}) = f$ on morphisms. A natural question is to ask when the functor ρ induces a homotopy equivalence on classifying spaces. The classifying space BC of a small category C is constructed as the geometric realization of the nerve of C (see [**Seg68, Qui73, Hir03**], for homotopical properties of classifying spaces). For a cover of a small category, the classifying space of the Grothendieck construction for the Čech nerve is called the *Čech complex*.

Definition 2.4. Let C be a small category. An *ideal* D of C is a full subcategory such that an object y of C belongs to D, whenever $C(y, x) \neq \emptyset$ for some object x of D. Dually, a *filter* D of C is a full subcategory such that an object y of C belongs to D, whenever $C(x, y) \neq \emptyset$ for some object x of D. A cover $\mathcal{D} = \{D_a\}_{a\in A}$ of C is called an *ideal* (a *filter*) *cover* if D_a is an ideal (a filter) for any $a \in A$.

The notions above are generalizations of ideals and filters of posets [**Sta12, Zap98**]. Let D be a full subcategory of a small category C. The category of complement $C\backslash D$ is defined as the full subcategory whose set of objects is $\mathrm{ob}(C)\backslash\mathrm{ob}(D)$. If D is an ideal (a filter), then $C\backslash D$ is a filter (an ideal). In other words, the ideal D yields a functor $C \to \mathcal{P}$, where $\mathcal{P}$ is the poset formed of $0 < 1$. The functor sends $\mathrm{ob}(D)$ to $\{0\}$ and $\mathrm{ob}(C\backslash D)$ to $\{1\}$. Conversely, for a functor $F\colon C \to \mathcal{P}$, the category of fiber $F^{-1}(0)$ is an ideal, and $F^{-1}(1)$ is a filter of C.

Theorem 2.5. *Let $\mathcal{D} = \{D_a\}_{a\in A}$ be an ideal cover or a filter cover of a small category C. Then, the natural functor $\rho\colon \mathrm{Gr}(\check{C}(\mathcal{D})) \to C$ induces a homotopy equivalence between their classifying spaces.*

Proof. We first consider the case in which $\mathcal{D}$ is an ideal cover. We use Quillen's theorem A [**Qui73**] for ρ. In order to apply this, we examine the left homotopy fiber $\rho{\downarrow}x$ of ρ, over an object x of C. The set of objects consists of pairs $(y_{b_0\dots b_n}, f)$ of an object $y_{b_0\dots b_n}$ of $\mathrm{Gr}(\check{C}(\mathcal{D}))$, and a morphism $f\colon y \to x$ of C. The set of morphisms $(\rho{\downarrow}x)((y_{b_0\dots b_n}, f), (z_{c_0\dots c_m}, g))$ consists of morphisms $(\varphi, h_{c_0\dots c_m})\colon y_{b_0\dots b_n} \to z_{c_0\dots c_m}$ of $\mathrm{Gr}(\check{C}(\mathcal{D}))$, such that $g \circ h = f$.

On the other hand, we choose and fix an index $a \in A$ such that $x \in \mathrm{ob}(D_a)$. The set of objects of the over category $\mathrm{Gr}(\check{C}(\mathcal{D})){\downarrow}x_a$ consists of morphisms $(v_j, f_a)\colon y_{a_0\dots a_n} \to x_a$ of $\mathrm{Gr}(\check{C}(\mathcal{D}))$. Here, v_j is the map $[0] \to [n]$ choosing $j \in \{0, \dots, n\}$, and f_a is a morphism from $y_{a_j} = y_a$ to x_a of D_a.

The set of morphisms

$$\left(\mathrm{Gr}(\check{C}(\mathcal{D})){\downarrow}x_a\right)((v_j, f_a), (v_k, g_a))$$

consists of morphisms (φ, h) of $\mathrm{Gr}(\check{C}(\mathcal{D}))$, such that $(v_k, g_a) \circ (\varphi, h) = (v_j, f_a)$.

A functor

$$F\colon \rho{\downarrow}x \longrightarrow \mathrm{Gr}(\check{C}(\mathcal{D})){\downarrow}x_a$$

is defined by $F(y_{a_0\dots a_n}, f) = (v_{n+1}, f_a)\colon y_{a_0\dots a_n a} \to x_a$ on objects. Note that the object y belongs to D_a, since D_a is an ideal. A morphism

$$(\varphi, h_{b_0\dots b_m})\colon (y_{a_0\dots a_n}, f) \to (z_{b_0\dots b_m}, g)$$

of $\rho{\downarrow}x$ is sent to a morphism $(\widetilde{\varphi}, h_{b_0\dots b_m a})\colon (v_{n+1}, f_a) \to (v_{m+1}, g_a)$ of $\mathrm{Gr}(\check{C}(\mathcal{D})){\downarrow}x_a$, where $\tilde{\varphi}\colon [m+1] \to [n+1]$ is the extension of $\varphi\colon [m] \to [n]$ with $\tilde{\varphi}(m+1) = n+1$. Conversely, a functor

$$G\colon \mathrm{Gr}(\check{C}(\mathcal{D})){\downarrow}x_a \longrightarrow \rho{\downarrow}x$$

is defined by $G(v_j, g_a) = (y_{a_0\dots a_n}, g)$, for an object $(v_j, g_a)\colon y_{a_0\dots a_n} \to x_a$ of the over category $\mathrm{Gr}(\check{C}(\mathcal{D})){\downarrow}x_a$. There exists a natural transformation $t\colon GF \Rightarrow \mathrm{id}$ defined by

$$t(y_{a_0\dots a_n}, f) = (d_{n+1}, \mathrm{id}_{a_0\dots a_n})\colon (y_{a_0\dots a_n a}, f) \longrightarrow (y_{a_0\dots a_n}, f).$$

Note that the classifying space of the over category $\mathrm{Gr}(\check{C}(\mathcal{D})){\downarrow}x_a$ is contractible, since it has a terminal object. We conclude that the classifying spaces of $\mathrm{Gr}(\check{C}(\mathcal{D})){\downarrow}x_a$ and $\rho{\downarrow}x$ are homotopy equivalent and contractible. Quillen's theorem A completes the proof for the case of ideal covers.

If $\mathcal{D}$ is a filter cover of C, the opposite cover $\mathcal{D}^{\mathrm{op}} = \{D_a^{\mathrm{op}}\}_{a\in A}$ is an ideal cover of C^{op}. Hence, the classifying space of the left homotopy fiber $\rho'{\downarrow}x$ is contractible for the natural functor $\rho'\colon \mathrm{Gr}(\check{C}(\mathcal{D}^{\mathrm{op}})) \to C^{\mathrm{op}}$, and any object x of C^{op}. The left homotopy fiber $\rho'{\downarrow}x$ of ρ' coincides with the right homotopy fiber $x{\downarrow}\rho$ of ρ for any object $x \in \mathrm{ob}(C) = \mathrm{ob}(C^{\mathrm{op}})$. Again, Quillen's theorem A implies the result. □

Thomason's homotopy colimit theorem [**Tho79**] tells us that the Čech complex for a cover $\mathcal{D}$ of a small category is homotopy equivalent to the homotopy colimit $\mathrm{hocolim}\, B\check{C}(\mathcal{D})$ of the diagram $B\check{C}(\mathcal{D})\colon \Delta^{\mathrm{op}} \to \mathbf{Top}$, in the category **Top** of spaces. The diagram $B\check{C}(\mathcal{D})$ is a Reedy cofibrant simplicial space [**Hir03**], with respect to the Strøm model structure on **Top**, since the degenerate part (the latching object) is a direct summand with open complement in each dimension. The Bousfield–Kan map $\mathrm{hocolim}\, B\check{C}(\mathcal{D}) \to |B\check{C}(\mathcal{D})|$ is a homotopy equivalence by Theorem 18.7.4 of [**Hir03**]. Consequently, we have a homotopy equivalence

$$|B\check{C}(\mathcal{D})| \simeq BC,$$

for an ideal cover or a filter cover $\mathcal{D}$ of a small category C.

If the index set A is equipped with a total order, then we can consider the ordered Čech nerve.

Definition 2.6 (Ordered Čech nerve)**.** Let $\mathcal{D} = \{D_a\}_{a\in A}$ be a cover of a small category C, indexed by a totally ordered set A. The *ordered Čech nerve* $\check{C}^o(\mathcal{D})$ is a simplicial object in **Cat**, defined as

$$\check{C}^o(\mathcal{D})_n = \coprod_{a_0 \leqslant \dots \leqslant a_n} D_{a_0 a_1 \dots a_n}.$$

The face and degeneracy maps are defined similarly to the ordinary Čech nerve.

The ordered Čech nerve is smaller, and often easier to treat, than the ordinary Čech nerve. There exists an inclusion $\check{C}^o(\mathcal{D}) \to \check{C}(\mathcal{D})$ of simplicial objects in **Cat**. This induces a functor $I\colon \mathrm{Gr}(\check{C}^o(\mathcal{D})) \to \mathrm{Gr}(\check{C}(\mathcal{D}))$ between their Grothendieck constructions.

Proposition 2.7. *Let $\mathcal{D} = \{D_a\}_{a \in A}$ be a cover of a small category C, indexed by a totally ordered set A. The inclusion functor $I\colon \mathrm{Gr}(\check{C}^o(\mathcal{D})) \to \mathrm{Gr}(\check{C}(\mathcal{D}))$ induces a homotopy equivalence between their classifying spaces.*

Proof. For an object $x_{a_0 \dots a_n}$ of $\mathrm{Gr}(\check{C}(\mathcal{D}))$, there exists a permutation $\sigma \in \Sigma_n$, such that $a_{\sigma(0)} \leqslant \dots \leqslant a_{\sigma(n)}$ in A. In the case that $a_i = a_j$ for some $i < j$, we always choose $\sigma(i) < \sigma(j)$. This yields an inverse functor $J\colon \mathrm{Gr}(\check{C}(\mathcal{D})) \to \mathrm{Gr}(\check{C}^o(\mathcal{D}))$, sending $x_{a_0 \dots a_n}$ to $x_{a_{\sigma(0)} \dots a_{\sigma(n)}}$ on objects. For a morphism $(\varphi, f)\colon x_{a_0 \dots a_n} \to y_{b_0 \dots b_m}$ of $\mathrm{Gr}(\check{C}(\mathcal{D}))$, let $\sigma \in \Sigma_n$ and $\tau \in \Sigma_m$ be permutations for the canonical reordering of $a_0 \dots a_n$ and $b_0 \dots b_m$, respectively. For simplicity, denote $\alpha_i = a_{\sigma(i)}$ and $\beta_j = b_{\tau(j)}$, for each i and j. For every $j \in \{0, \dots, m\}$, we have

$$\beta_j = b_{\tau(j)} = a_{\varphi\tau(j)} = \alpha_{\sigma^{-1}\varphi\tau(j)}.$$

Consider the composition of maps $J(\varphi) = \sigma^{-1}\varphi\tau\colon [m] \to [n]$. We can describe

$$b_0 \dots b_m = \overbrace{a_0 \dots a_0}^{\varphi^{-1}(0)} \overbrace{a_1 \dots a_1}^{\varphi^{-1}(1)} \dots \overbrace{a_n \dots a_n}^{\varphi^{-1}(n)}.$$

The permutation τ acts on $b_0 \dots b_m$ as σ:

$$b_{\tau(0)} \dots b_{\tau(m)} = \overbrace{a_{\sigma(0)} \dots a_{\sigma(0)}}^{\varphi^{-1}(\sigma(0))} \overbrace{a_{\sigma(1)} \dots a_{\sigma(1)}}^{\varphi^{-1}(\sigma(1))} \dots \overbrace{a_{\sigma(n)} \dots a_{\sigma(n)}}^{\varphi^{-1}(\sigma(n))}.$$

It follows that $J(\varphi)$ preserves order. The functor J sends (φ, f) to $(J(\varphi), f)$ on morphisms.

Trivially, we have $JI = \mathrm{id}$ on $\mathrm{Gr}(\check{C}^o(\mathcal{D}))$. Let us examine the composition IJ. A functor $K\colon \mathrm{Gr}(\check{C}(\mathcal{D})) \to \mathrm{Gr}(\check{C}(\mathcal{D}))$ is defined by $K(x_{a_0 \dots a_n}) = x_{a_0 \dots a_n a_{\sigma(0)} \dots a_{\sigma(n)}}$ on objects. For a morphism (φ, f) of $\mathrm{Gr}(\check{C}^o(\mathcal{D}))$, define $K(\varphi, f) = (\varphi * J(\varphi), f)$, where $\varphi * J(\varphi)\colon [2m+1] \to [2n+1]$ is given as

$$\begin{cases} \varphi * J(\varphi)(i) = \varphi(i) & \text{if } 0 \leqslant i \leqslant m, \\ \varphi * J(\varphi)(i) = J(\varphi)(i - m - 1) & \text{if } m+1 \leqslant i \leqslant 2m+1. \end{cases}$$

There exist natural transformations $t\colon K \Rightarrow IJ$ with $t(x_{a_0 \dots a_n}) = (d_0^n, \mathrm{id}_x)$, and $s\colon K \Rightarrow \mathrm{id}$ with $s(x_{a_0 \dots a_n}) = (d_{n+1} \dots d_{2n+1}, \mathrm{id}_x)$, for an object $x_{a_0 \dots a_n}$ of $\mathrm{Gr}(\check{C}(\mathcal{D}))$. Therefore, the induced maps from I and J on classifying spaces are homotopy inverses to each other. □

For a cover $\mathcal{D} = \{D_a\}_{a \in A}$ of a small category C, indexed by a totally ordered set A, the natural functor $\rho^o\colon \mathrm{Gr}(\check{C}^o(\mathcal{D})) \to C$ is given as the composition $\rho^o = \rho I$. Theorem 2.5 and Proposition 2.7 imply the following.

Corollary 2.8. *Let $\mathcal{D} = \{D_a\}_{a \in A}$ be an ideal cover or a filter of a small category C, indexed by a totally ordered set A. The natural functor $\rho^o\colon \mathrm{Gr}(\check{C}^o(\mathcal{D})) \to C$ induces a homotopy equivalence between their classifying spaces.*

Note that the Grothendieck construction of the ordered Čech nerve is independent of the order on the index set.

Proposition 2.9. *Let $\mathcal{D} = \{D_a\}_A$ be a cover of a small category C, and let the index set A be equipped with two total order $\leqslant_1$ and $\leqslant_2$. Denote the ordered Čech nerve induced from $\leqslant_i$ by $\check{C}^o(\mathcal{D})_i$ for $i = 1, 2$, respectively. Then, the Grothendieck constructions $\mathrm{Gr}(\check{C}^o(\mathcal{D})_1)$ and $\mathrm{Gr}(\check{C}^o(\mathcal{D})_2)$ are isomorphic to each other as categories.*

Proof. Similarly to the proof of Proposition 2.7, we obtain a permutation σ for an object $x_{a_0\ldots a_n}$ of $\mathrm{Gr}(\check{C}^o(\mathcal{D})_1)$, such that $x_{a_{\sigma(0)}\cdots a_{\sigma(n)}}$ belongs to $\mathrm{Gr}(\check{C}^o(\mathcal{D})_2)$. This yields a functor $F\colon \mathrm{Gr}(\check{C}^o(\mathcal{D})_1) \to \mathrm{Gr}(\check{C}^o(\mathcal{D})_2)$. We also obtain an inverse functor $G\colon \mathrm{Gr}(\check{C}^o(\mathcal{D})_2) \to \mathrm{Gr}(\check{C}^o(\mathcal{D})_1)$ by reordering the indexes. A straightforward calculation shows that $FG = \mathrm{id}$ and $GF = \mathrm{id}$. □

Next, we define the reduced Čech nerve, smaller than the ordered Čech nerve.

Definition 2.10 (Reduced Čech nerve)**.** Let Δ_{inj} denote the subcategory of Δ, consisting of the same objects as Δ and injective order preserving maps. Let $\mathcal{D} = \{D_a\}_{a\in A}$ be a cover of a small category C, indexed by a totally ordered set A. The *reduced Čech nerve* of $\mathcal{D}$ is a functor $\check{C}^o(\widetilde{\mathcal{D}})\colon \Delta_{inj}^{\mathrm{op}} \to \mathbf{Cat}$, defined by

$$\check{C}^o(\widetilde{\mathcal{D}})_n = \coprod_{a_0<\ldots<a_n} D_{a_0a_1\ldots a_n}.$$

This is the diagram in **Cat** where the degenerate parts of the ordered Čech nerve have been removed. It has only face maps omitting indices. Note that if A is finite, then $\check{C}^o(\widetilde{\mathcal{D}})_n$ is the empty category for n greater than the cardinality $A^\sharp$ of A.

The inclusion $\Delta_{inj} \to \Delta$ induces a functor $L\colon \mathrm{Gr}(\check{C}^o(\widetilde{\mathcal{D}})) \to \mathrm{Gr}(\check{C}^o(\mathcal{D}))$ on their Grothendieck constructions.

Lemma 2.11. *Let $\mathcal{D} = \{D_a\}_{a\in A}$ be a cover of a small category. For two morphisms $(\varphi, f), (\chi, g)\colon x_{a_0\ldots a_n} \to y_{b_0\ldots b_m}$ of $\mathrm{Gr}(\check{C}(\mathcal{D}))$, if the indices $a_0, \ldots, a_n$ are distinguished, then $\varphi = \chi$.*

Proof. For any $j \in \{0, \ldots, m\}$, the equality $a_{\varphi(j)} = b_j = a_{\chi(j)}$ holds. Since the indices $a_0, \ldots, a_n$ are distinguished, $\varphi(j)$ and $\chi(j)$ must be equal to each other. □

The lemma above also holds for morphisms of $\mathrm{Gr}(\check{C}^o(\mathcal{D}))$ and $\mathrm{Gr}(\check{C}^o(\widetilde{\mathcal{D}}))$, since these are subcategories of $\mathrm{Gr}(\check{C}(\mathcal{D}))$.

Proposition 2.12. *Let $\mathcal{D} = \{D_a\}_{a\in A}$ be a cover of a small category C, indexed by a totally ordered set A. The inclusion functor $L\colon \mathrm{Gr}(\check{C}^o(\widetilde{\mathcal{D}})) \to \mathrm{Gr}(\check{C}^o(\mathcal{D}))$ has a right adjoint functor.*

Proof. For an arbitrary object $x_{a_0\ldots a_n}$ of $\mathrm{Gr}(\check{C}^o(\mathcal{D}))$, we obtain an object $x_{\alpha_0\ldots\alpha_{n'}}$ of $\mathrm{Gr}(\check{C}^o(\widetilde{\mathcal{D}}))$ and a surjective order preserving map $\psi\colon [n] \to [n']$, such that $\alpha_j = a_{\psi(j)}$, by removing duplicate indices. This determines the reduction functor

$$R\colon \mathrm{Gr}(\check{C}^o(\mathcal{D})) \to \mathrm{Gr}(\check{C}^o(\widetilde{\mathcal{D}})),$$

sending $x_{a_0\ldots a_n}$ to $x_{\alpha_0\ldots\alpha_{n'}}$ on objects. For a morphism $(\varphi, f)\colon x_{a_0\ldots a_n} \to y_{b_0\ldots b_m}$ of $\mathrm{Gr}(\check{C}^o(\widetilde{\mathcal{D}}))$, the order preserving map $\varphi\colon [m] \to [n]$ induces $\varphi'\colon [m'] \to [n']$, making the following diagram commutative:

$$\begin{array}{ccc} [m] & \xrightarrow{\varphi} & [n] \\ {\scriptstyle\psi}\downarrow & & \downarrow{\scriptstyle\psi} \\ [m'] & \xrightarrow[\varphi']{} & [n']. \end{array}$$

The reduction functor R sends (φ, f) to (φ', f) on morphisms. We take objects $x_{a_0\dots a_n}$ of $\mathrm{Gr}(\check{C}^o(\widetilde{\mathcal{D}}))$ and $y_{b_0\dots b_m}$ of $\mathrm{Gr}(\check{C}^o(\mathcal{D}))$, respectively. Since $a_0, \dots, a_n$ are distinguished, we have that

$$\begin{aligned} \mathrm{Gr}(\check{C}^o(\widetilde{\mathcal{D}}))(x_{a_0\dots a_n}, R(y_{b_0\dots b_m})) &= \mathrm{Gr}(\check{C}^o(\widetilde{\mathcal{D}}))(x_{a_0\dots a_n}, y_{\beta_0\dots\beta_{m'}}) \\ &\cong D_{\beta_0\dots\beta_{m'}}(x, y) \\ &= D_{b_0\dots b_m}(x, y) \\ &\cong \mathrm{Gr}(\check{C}^o(\mathcal{D}))(L(x_{a_0\dots a_n}), y_{b_0\dots b_m}), \end{aligned}$$

by Lemma 2.11. Therefore, we conclude that the reduction functor R is right adjoint to L. □

The reduced Čech nerve is also equipped with the natural functor

$$\tilde{\rho} = \rho^o L\colon \check{C}^o(\widetilde{\mathcal{D}}) \to C,$$

the same as the ordered Čech nerve and the ordinary Čech nerve. Form Proposition 2.12 and Corollary 2.8, we can deduce the following.

Corollary 2.13. *Let $\mathcal{D} = \{D_a\}_{a\in A}$ be an ideal cover or a filter cover of a small category C, indexed by a totally ordered set A. The natural functor $\tilde{\rho}\colon \check{C}^o(\widetilde{\mathcal{D}}) \to C$ induces a homotopy equivalence between their classifying spaces.*

Definition 2.14. A cover $\mathcal{D} = \{D_a\}_{a\in A}$ of a small category C is called *locally finite* when

$$\{a \in A \mid x \in \mathrm{ob}(D_a)\}^\sharp < \infty,$$

for any object x of C.

Proposition 2.15. *Let $\mathcal{D} = \{D_a\}_{a\in A}$ be a locally finite ideal cover of a small category C, indexed by a totally ordered set A. The natural functor $\tilde{\rho}\colon \mathrm{Gr}(\check{C}^o(\widetilde{\mathcal{D}})) \to C$ has a left adjoint functor.*

Proof. A functor $\pi\colon C \to \mathrm{Gr}(\check{C}^o(\widetilde{\mathcal{D}}))$ is defined by $\pi(x) = x_{a_0\dots a_n}$ on objects, where $D_{a_0}, \dots, D_{a_n}$ are whole distinguished ideals in $\mathcal{D}$ that contain x. For a morphism $f\colon x \to y$ of C, the object x belongs to an ideal D if y does. When we describe $\pi(x) = x_{a_0\dots a_n}$ and $\pi(y) = y_{b_0\dots b_m}$, the inclusion relation $\{b_0, \dots b_m\} \subset \{a_0, \dots, a_n\}$ holds. This yields an injection $\varphi\colon [m] \to [n]$, such that $b_j = a_{\varphi(j)}$. We define $\pi(f) = (\varphi, f)\colon \pi(x) \to \pi(y)$ on morphisms.

Take objects x of C and $y_{b_0\dots b_m}$ of $\mathrm{Gr}(\check{C}^o(\widetilde{\mathcal{D}}))$, respectively. Then, by Lemma 2.11,

$$C(x, \tilde{\rho}(y_{b_0\dots b_m})) = C(x, y) = D_{b_0\dots b_m}(x, y) \cong \mathrm{Gr}(\check{C}^o(\widetilde{\mathcal{D}}))(\pi(x), y_{b_0\dots b_m}).$$

Therefore, we conclude that the functor π is left adjoint to $\tilde{\rho}$. □

When $\mathcal{D} = \{D_a\}_{a\in A}$ is a filter cover of a small category C, the opposite cover $\mathcal{D}^{\mathrm{op}} = \{D_a^{\mathrm{op}}\}_{a\in A}$ is an ideal cover of C^{op}. The following corollary then follows immediately.

Corollary 2.16. *Let $\mathcal{D} = \{D_a\}_{a\in A}$ be a locally finite filter cover of a small category C, indexed by a totally ordered set A. The natural functor $\mathrm{Gr}(\check{C}^o(\widetilde{\mathcal{D}}^{\mathrm{op}})) \to C^{\mathrm{op}}$ has a left adjoint functor.*

3. The inclusion–exclusion principle for the Euler characteristics of finite categories

In this section, we focus on the inclusion–exclusion principle for the Euler characteristic of a finite category. The Euler characteristic of a finite category was introduced by Leinster [**Lei08**]. This is a generalization of Möbious inversion for posets [**Rot64**]. Let us briefly review the definition. In this paper, we use the rational numbers $\mathbb{Q}$ as the value of Euler characteristics of finite categories.

Definition 3.1. Suppose that C is a finite category that has finitely many objects and morphisms.

1. The *similarity matrix* of C is the function $\zeta\colon \mathrm{ob}(C) \times \mathrm{ob}(C) \to \mathbb{Q}$, given by the cardinality of the set of morphisms $\zeta(a,b) = C(a,b)^\sharp$.
2. Let $u\colon \mathrm{ob}(C) \to \mathbb{Q}$ denote the column vector with $u(a) = 1$, for any object a of C. A *weighting* on C is a column vector $w\colon \mathrm{ob}(C) \to \mathbb{Q}$ such that $\zeta w = u$. Dually, a *coweighting* on C is a row vector $v\colon \mathrm{ob}(C) \to \mathbb{Q}$, such that $v\zeta = u^*$, where u^* is the transposed matrix of u.

Note that we have

$$\sum_{i\in \mathrm{ob}(C)} w(i) = u^*w = v\zeta w = vu = \sum_{j\in \mathrm{ob}(C)} v(j),$$

if both a weighting and a coweighting exist. Moreover,

$$\sum_{i\in \mathrm{ob}(C)} w(i) = u^*w = v\zeta w = v\zeta w' = u^*w' = \sum_{i\in \mathrm{ob}(C)} w'(i),$$

for two (co)weightings w and w' on C. This guarantees the following definition.

Definition 3.2. Let C be a finite category. We say that C *has Euler characteristic* if it has both a weighting w, and a coweighting v, on C. Then, the *Euler characteristic* of C is defined by

$$\chi(C) = \sum_{i\in \mathrm{ob}(C)} w(i) = \sum_{j\in \mathrm{ob}(C)} v(j).$$

It is well-known that the topological Euler characteristic χ_T has the following inclusion–exclusion formula:

$$\chi_T(A\cup B) = \chi_T(A) + \chi_T(B) - \chi_T(A\cap B),$$

for subcomplexes A and B of a finite CW-complex X. Recall the example given in our introduction. The finite category C is

$$x \rightrightarrows y \longrightarrow z,$$

with the terminal object z, D_1 is the full subcategory with $\mathrm{ob}(D_1) = \{x, y\}$, and D_2 is the full subcategory with $\mathrm{ob}(D_2) = \{y, z\}$. The Euler characteristic $\chi(C) = \chi(D_1 \cup D_2) = 1$, since C has a terminal object. However,

$$\chi(D_1) + \chi(D_2) - \chi(D_1 \cap D_2) = 0 + 1 - 1 = 0.$$

The inclusion–exclusion principle does not hold in this case. In [**FLS11**], Fiore, Luck, and Sauer used the homotopy colimit of diagram in **Cat**, instead of the genuine colimit or union.

Now, we consider the following situation. Let C be a finite category, and let $\mathcal{D} = \{D_a\}_{a \in A}$ be a cover of C, indexed by a totally ordered finite set A. We denote the full subcategory of Δ_{inj}, consisting of $[n]$ for $0 \leqslant n \leqslant A^\sharp - 1$ by $\Delta_{A^\sharp}$. The reduced Čech nerve $\check{C}^o(\widetilde{\mathcal{D}})$ can be regarded as a functor $\Delta_{A^\sharp}^{\mathrm{op}} \to \mathbf{Cat}$. The category $\Delta_{A^\sharp}^{\mathrm{op}}$ is a finite acyclic category, which never has circuit of morphisms. Moreover, each $\check{C}^o(\widetilde{\mathcal{D}})_n$ is a finite coproduct of finite categories. Leinster provided the product formula for the Euler characteristic of finite categories, with respect to Grothendieck construction in [**Lei08**]. By applying this to $\mathrm{Gr}(\check{C}^o(\widetilde{\mathcal{D}}))$, we obtain the inclusion–exclusion principle for ideal (filter) covers of a finite category.

Theorem 3.3 (Inclusion–exclusion principle)**.** *Let $\mathcal{D} = \{D_a\}_{a \in A}$ be an ideal cover or a filter cover of a finite category C, indexed by a finite set A. If each $D_{a_0 \ldots a_i}$ and C have Euler characteristics, then we have*

$$\chi(C) = \sum_{i=0}^{A^\sharp - 1} \sum_{a_0 < \ldots < a_i} (-1)^i \chi(D_{a_0 \ldots a_i}).$$

Proof. Let $\mathcal{D}$ be an ideal cover of C. The category $\Delta_{A^\sharp}^{\mathrm{op}}$ has a unique weighting $w[n] = (-1)^n$ (see Example 3.4 (d) of [**Lei08**]). By applying Proposition 2.8 of [**Lei08**] to $\mathrm{Gr}(\check{C}^o(\widetilde{\mathcal{D}}))$, we have

$$\chi(\mathrm{Gr}(\check{C}^o(\widetilde{\mathcal{D}}))) = \sum_{i=0}^{A^\sharp - 1} (-1)^i \chi(\check{C}^o(\widetilde{\mathcal{D}})_i) = \sum_{i=0}^{A^\sharp - 1} \sum_{a_0 < \ldots < a_i} (-1)^i \chi(D_{a_0 \ldots a_i}).$$

Then, Proposition 2.4 of [**Lei08**] and our Proposition 2.15 imply the result. If $\mathcal{D}$ is a filter cover of C, then $\mathcal{D}^{\mathrm{op}}$ is an ideal cover of C^{op}. Since the equality $\chi(X) = \chi(X^{\mathrm{op}})$ holds for any finite category X having Euler characteristic, we obtain the desired formula. □

Note that the order on the index set is not essential, since we can choose a total order on any finite set, and Proposition 2.9 also holds in the case of reduced Čech nerves.

Corollary 3.4. *Let both A and B be filters or ideals of a finite category C. If each of A, B, $A \cap B$, and $A \cup B$ has Euler characteristic, then we have*

$$\chi(A \cup B) = \chi(A) + \chi(B) - \chi(A \cap B).$$

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Kohei Tanaka `tanaka@shinshu-u.ac.jp`

Institute of Social Sciences, School of Humanities and Social Sciences, Academic Assembly, Shinshu University, 3-1-1, Matsumoto, Nagano, 390-8621, Japan