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TOPOLOGICAL HOCHSCHILD HOMOLOGY OF K/p AS A $K_p^\wedge$ MODULE

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Abstract

For commutative ring spectra R, one can construct a Thom spectrum for spaces over BGL_1R. This specialises to the classical Thom spectra for spherical fibrations in the case of the sphere spectrum. The construction is useful in detecting A_∞-structures: a loop space (up to homotopy) over BGL_1R yields an A_∞-ring structure on the Thom spectrum. The topological Hochschild homology of these A_∞-ring spectra may be expressed as Thom spectra.

This paper uses the identification of topological Hochschild homology of Thom spectra to make computations. Specifically, we take R to be the p-adic K-theory spectrum and consider a certain map from S^1 to BGL_1R, so that the Thom spectrum is equivalent to the mod p K-theory spectrum. We make computations at odd primes.

1. Introduction

The goal of this paper is to use generalised Thom spectra to calculate the topological Hochschild homology of K/p in the category of modules over $K_p^\wedge$.

Let R be a ring spectrum and GL_1R its space of units. It is the H-space of homotopy automorphisms of R as an R-module. An R-twisting of a space X is a continuous map ζ from X to BGL_1R. Associated to ζ, one can define the Thom spectrum of ζ, X^ζ (see [**2**]). This notion specialises for $R = S^0$ to the Thom spectrum of a spherical fibration. The homotopy groups of X^ζ is the group of twisted R homology classes with respect to the twisting ζ.

Suppose that R is an E_∞-ring spectrum. Then its space of units is an infinite loop space. Given a map $f\colon BG \to B^2GL_1R$, let $\zeta \simeq \Omega f\colon G \to BGL_1R$. Then the Thom spectrum G^ζ admits an A_∞ R-algebra structure.

1.1. K/p as a module over $K_p^\wedge$

Suppose that $R = K_p^\wedge$, the spectrum of p-adic K-theory. Let G be the group S^1. A twisting on S^1 is a map $\zeta\colon S^1 \to BGL_1K_p^\wedge$. This is classified by the group

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$\pi_1(BGL_1K_p^\wedge) \cong \pi_0(GL_1K_p^\wedge) \cong Z_p^\times$. If we choose $\zeta = 1-p \in Z_p^\times$, then the Thom spectrum $(S^1)^\zeta \simeq K/p$, the mod p K-theory spectrum. Moreover, the twisting ζ can be realised as a loop map, and so, for every way of writing $\zeta \simeq \Omega f$ we get an A_∞-ring structure on K/p as an $K_p^\wedge$-module.

1.2. Topological Hochschild homology of Thom spectra

Given a map f from X to B^2GL_1R, let $G \simeq \Omega X$ and $\zeta \simeq \Omega f\colon G \simeq \Omega X \to BGL_1R$. In this case, the Thom spectrum G^ζ has an A_∞-ring structure. We write $\eta^* f$ for the composite

$$\begin{array}{ccccc} LX & \longrightarrow & LB^2GL_1R & \xrightarrow{\cong} & B^2GL_1R \times BGL_1R \\ & & & & \Big\downarrow{\scriptstyle \eta\times id} \\ & & & & BGL_1R \times BGL_1R \longrightarrow BGL_1R, \end{array}$$

where $\eta\colon \Sigma R \to R$ is induced from $S^1 \xrightarrow{\eta} S^0$ via $S^1 \wedge R \to S^0 \wedge R \simeq R$. In the above situation, $THH^R(G^\zeta) \simeq LX^{\eta^* f}$. The case $R = S^0$ was proved in [**5**]. The same argument applies for general R [**3**].

Using this identification of THH as a Thom spectrum, we compute the topological Hochschild homology of K/p. For odd primes p,

$$\pi_*(THH^{K_p^\wedge}(K/p)) = \begin{cases} (Z/(p^\infty))^i & \text{if } * = 2k, \\ 0 & \text{if } * = 2k+1, \end{cases}$$

where i is an integer between 1 and $p-1$ depending on the choice of f with $\zeta \simeq \Omega f$.

Similar results were obtained before by Angeltveit in [**1**]. He used the Bökstedt spectral sequence (see [**6**, Chapter IX]).

We can also form mod p K-theory as a Thom spectrum by starting with $X = S^3$, $R = K_p^\wedge$ and $\zeta = p \in \pi_3(BGL_1K_p^\wedge) = \pi_2(GL_1K_p^\wedge) = Z_p$. Again, this ζ can be realised as a loop map and we can compute THH of these A_∞-ring structures in an analogous way. This gives the same results.

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2. The Thom spectrum

The notion of a generalised Thom spectrum used here is discussed in detail in [**2**]. The construction resembles a twisted version of the group ring. Given an extension of a group G by the units in a field k,

$$(\tau)\colon 1 \to k^* \to E \to G \to 1$$

the algebra $k^\tau[G] = \mathbb{Z}[E] \otimes_{\mathbb{Z}[k^*]} k$ is a twisted group ring. If the extension τ is trivial, one gets the group ring $k[G]$. Imitating this definition of a twisted group ring for spectra leads to the construction of the Thom spectrum. One replaces the field k by an E_∞-ring spectrum R, and the units k^* by the space of units GL_1R acting on R.

2.1. The space of units and the Thom spectrum

The space of units of a ring spectrum is a generalisation of the group of units of a commutative ring, the set of invertible elements under multiplication. It is defined to be the components of $\Omega^\infty R$ that lie over the units in $\pi_0(R)$. Following [**2**], we make the definition:

Definition 2.1. Let R be an E_∞-ring spectrum. Its space of units GL_1R is defined to be the homotopy pullback

$$\begin{array}{ccc} GL_1R & \longrightarrow & \Omega^\infty(R) \\ \downarrow & & \downarrow \\ \pi_0(R)^\times & \longrightarrow & \pi_0(R). \end{array}$$

It follows from the definition that the homotopy classes of maps from a space X to GL_1R are given by

$$[X, GL_1R] = R^0(X)^\times$$

the units of the cohomology ring $R^0(X) = [X, \Omega^\infty R]$.

From the pullback diagram one can read off the homotopy groups of GL_1R:

$$\pi_n(GL_1R) = \begin{cases} \pi_n(R) & \text{if } n > 0, \\ \pi_0(R)^\times & \text{if } n = 0. \end{cases}$$

We note that GL_1R is an H-space for any ring spectrum R. If R is E_∞, then GL_1R is an infinite loop space: there is a connective spectrum gl_1R with 0^{th}-space is GL_1R (Theorem 3.2 in [**2**]).

We can view $\Omega^\infty R$ as the space of endomorphisms $End_R(R,R)$, in the topological category of R-modules, and $GL_1R = Aut_R(R,R) \subset End_R(R,R)$ as the subset of weak equivalences. Therefore, the units GL_1R is the space of homotopy automorphisms of R in the category of R-modules. In this way, the infinite loop space GL_1R acts on the spectrum R by weak equivalences, and R is a module over the E_∞ ring spectrum $\Sigma^\infty GL_1R_+$.

Definition 2.2. Given a map $\zeta\colon X \to BGL_1R$, let P be the GL_1R bundle classified by ζ described as the pullback

$$\begin{array}{ccc} P & \longrightarrow & EGL_1(R) \\ \downarrow & & \downarrow \\ X & \xrightarrow{\zeta} & BGL_1(R) \end{array}$$

and define the associated Thom spectrum to be

$$X^\zeta = \Sigma^\infty P_+ \wedge^L{}_{\Sigma^\infty GL_1(R)_+} R.$$

In the above $\wedge^L$ denotes the derived smash product in the category of modules over the E_∞-ring spectrum $\Sigma^\infty GL_1R_+$ as in [**6**]. We note from Section 7 of [**2**], that the Thom spectrum functor commutes with homotopy colimits, and from Section 8.6 of [**2**] that it generalises the classical Thom spectrum of a spherical fibration.

The Thom spectrum of the map $* \to BGL_1R$ is weakly equivalent to R, since the universal bundle associated to the inclusion of a point in BGL_1R is isomorphic to GL_1R and $\Sigma^\infty GL_1R_+ \wedge^L_{\Sigma^\infty GL_1R_+} R \simeq R$.

Similarly, the Thom spectrum of a map $X \to BGL_1R$ which is null homotopic is weakly equivalent to $R \wedge X_+$. Indeed, the universal bundle associated to the constant map is $X \times GL_1R$. Then the Thom spectrum is $\Sigma^\infty(X \times GL_1R)_+ \wedge^L_{\Sigma^\infty GL_1R_+} R \simeq (\Sigma^\infty X_+ \wedge \Sigma^\infty GL_1R_+) \wedge^L_{\Sigma^\infty GL_1R_+} R \simeq R \wedge X_+$.

Suppose that the space $X \simeq \Sigma Y$, the reduced suspension on Y. Then, a map $X \xrightarrow{\zeta} BGL_1R$ is described by a map $Y \xrightarrow{\widehat{\zeta}} GL_1R$, via $[X, BGL_1R] \cong [\Sigma Y, BGL_1R] \cong [Y, GL_1R]$. Such a $\widehat{\zeta}$ is a unit in $R^0(Y)$ which induces $u_\zeta \colon R \wedge Y_+ \to R$.

Proposition 2.3. *Suppose that ζ is a map from $X \simeq \Sigma Y$ to BGL_1R. Then, the Thom spectrum X^ζ is equivalent to the homotopy colimit of $(R \leftarrow R \wedge Y_+ \to R)$ where one of the maps is the projection p_Y and the other is u_ζ.*

Proof. The space X is the homotopy colimit of $* \leftarrow Y \to *$, and this gives a homotopy pushout square of Thom spectra

$$\begin{array}{ccc} Y^\zeta & \longrightarrow & *^\zeta \\ \downarrow & & \downarrow \\ *^\zeta & \longrightarrow & (\Sigma Y)^\zeta. \end{array}$$

The Thom spectrum $*^\zeta$ is weakly equivalent to R and $Y^\zeta \simeq R \wedge Y_+$, so the homotopy pushout can be written as

$$\begin{array}{ccc} R \wedge Y_+ & \longrightarrow & R \\ \downarrow & & \downarrow \\ R & \longrightarrow & (\Sigma Y)^\zeta. \end{array}$$

From this, one obtains a Mayer Vietoris sequence for calculating the homotopy groups

$$\ldots \to \pi_*(R \wedge Y_+) \to \pi_*(R) \oplus \pi_*(R) \to \pi_*((\Sigma Y)^\zeta) \ldots .$$

To compute the maps in this sequence, one must examine the GL_1R-bundle over $X \simeq \Sigma Y$. This restricts to trivial bundles over the two copies of the cone of Y inside X and on their intersection Y, the bundles are identified via the map $\widehat{\zeta} \colon Y \to GL_1R$.

In the long exact sequence, there are two maps $R_*(Y_+) \to \pi_*(R)$. One of these maps is given by the map from Y to a point (p_Y) and the other is the map u_ζ defined in the preceding paragraph. □

Remark 2.4. The proposition describes the homotopy groups of the Thom spectrum as twisted R-homology groups. An R-twisting on a space X can be defined as a 1-cocycle in the sheaf (of groupoids) – $\{$*units in* $R^0(X)\}$. The groupoid of units in R^0 is classified by the units GL_1R, and therefore, 1-cocycles on X are equivalent to $[X, BGL_1R]$. Therefore, a twisting is given by a continuous map ζ from X to BGL_1R.

For $X = \bigcup U_i$ a 1-cocycle defines units over $U_i \cap U_j$ satisfying a cocycle condition on further intersections. A twisted R homology class is an element in each $R_*(U_i)$, two of which are identified using the values of the 1-cocycle on the intersections. The abelian group of these classes is defined to be the twisted R-homology of X with respect to the twisting ζ. This is isomorphic to the homotopy groups of the Thom spectrum X^ζ. The proposition above verifies this in the case $X = \Sigma Y$, where X is the union of two contractible open sets.

2.2. Computations of some Thom spectra

Proposition 2.5. *Suppose that* $\zeta\colon S^1 \to BGL_1K_p^\wedge$ *represents*

$$1 - p \in \pi_1(BGL_1(K_p^\wedge)) = \pi_0(GL_1(K_p^\wedge)) = Z_p^\times.$$

Then, $(S^1)^\zeta \simeq K/p$.

Proof. By Proposition 2.3 with $Y = S^0$, the Thom spectrum is a homotopy pushout

$$\begin{array}{ccc} K_p^\wedge \vee K_p^\wedge & \longrightarrow & K_p^\wedge \\ \downarrow & & \downarrow \\ K_p^\wedge & \longrightarrow & (S^1)^\zeta. \end{array}$$

Therefore, there is a cofibre sequence

$$K_p^\wedge \vee K_p^\wedge \to K_p^\wedge \vee K_p^\wedge \to (S^1)^\zeta.$$

Proposition 2.3 also identifies the left map in the sequence in suitable coordinates, to be given by the matrix

$$\begin{pmatrix} 1 & 1 \\ 1 & 1-p \end{pmatrix}.$$

Therefore, the cofibre sequence can be rewritten as

$$K_p^\wedge \xrightarrow{p} K_p^\wedge \longrightarrow (S^1)^\zeta,$$

so that $(S^1)^\zeta \simeq K_p^\wedge/p \simeq K/p$. □

Remark 2.6. Consider the map $\zeta\colon S^1 \to BGL_1((S^0)_p^\wedge)$ given by $(1-p)$ as in the previous proposition. Then, $(S^1)^\zeta \simeq (S^0)_p^\wedge/p \simeq M_p$ is the mod p Moore spectrum. In fact, for any $\zeta\colon S^1 \to BGL_1R$, $(S^1)^\zeta \simeq$ *cofibre*$(1-\zeta\colon R \to R)$. This follows from the argument above.

Proposition 2.7. *Let* $\zeta\colon S^3 \to BGL_1K_p^\wedge$ *represent the element* p *of*

$$[S^3, BGL_1(K_p^\wedge)] = \pi_3(BGL_1(K_p^\wedge)) = \pi_2(GL_1(K_p^\wedge)) = \pi_2(K_p^\wedge) \cong Z_p.$$

Then $(S^3)^\zeta \simeq K/p$.

Proof. The space S^3 is homotopy equivalent to the suspension of S^2. Proposition 2.3

implies the homotopy pushout

$$\begin{array}{ccc} K_p^\wedge \wedge S^2_+ & \longrightarrow & K_p^\wedge \\ \downarrow & & \downarrow \\ K_p^\wedge & \longrightarrow & (S^3)^\zeta \end{array}$$

and the associated Mayer Vietoris cofibre sequence

$$K_p^\wedge \wedge (S^2) \vee K_p^\wedge \to K_p^\wedge \vee K_p^\wedge \to (S^3)^\zeta.$$

In suitable coordinates, the map in the Mayer Vietoris sequence is given by the matrix

$$\begin{pmatrix} 1 & 0 \\ 1 & p \end{pmatrix}$$

and the sequence can be rewritten as

$$\Sigma^2 K_p^\wedge \xrightarrow{p} K_p^\wedge \longrightarrow (S^3)^\zeta.$$

By Bott periodicity $\Sigma^2 K_p^\wedge \simeq K_p^\wedge$ so that $(S^3)^\zeta \simeq K_p^\wedge/p$, as claimed. □

2.3. Ring structures

Suppose R is an E_∞-ring spectrum so that GL_1R is an infinite loop space. Given $f\colon X \to B^2GL_1R$, and $\zeta\colon G \simeq \Omega X \xrightarrow{\Omega f} BGL_1R$, the Thom spectrum G^ζ has an A_∞-ring structure. This follows from [**3**] where it is proved that the Thom spectrum functor is symmetric monoidal (Proposition 4.10) and loop maps rectify to monoids over an appropriate model of BGL_1R (Appendix A). This raises the question when a map

$$\zeta\colon G \to BGL_1R$$

from a monoid G is homotopy equivalent to a loop map, i.e., $\zeta \simeq \Omega f$ for

$$f\colon BG \to B^2GL_1R.$$

We have the standard maps

$$\Sigma G \xrightarrow{\sigma} BG, \qquad \Sigma GL_1R \xrightarrow{\sigma} BGL_1R,$$

so the question is if

$$\sigma \circ \Sigma\zeta\colon \Sigma G \to B^2GL_1R$$

extends over BG:

$$\begin{array}{ccc} \Sigma G & \xrightarrow{\Sigma\zeta} & \Sigma BGL_1(R) \\ \downarrow{\scriptstyle\sigma} & & \downarrow{\scriptstyle\sigma} \\ BG & \overset{f}{\dashrightarrow} & B^2GL_1(R). \end{array}$$

Proposition 2.8. *Let $G = S^1$, $R = K_p^\wedge$ and $\zeta = 1 - p$ as in Proposition 2.5, then $(S^1)^\zeta \simeq K/p$ has an A_∞-ring structure.*

Proof. The classifying space of S^1 is CP^∞ so, in this case, the diagram above is

$$\begin{array}{ccc} S^2 & \xrightarrow{\Sigma 1 - p} & \Sigma BGL_1(K_p^\wedge) \\ \downarrow{\scriptstyle\sigma} & & \downarrow{\scriptstyle\sigma} \\ CP^\infty & \overset{f}{\dashrightarrow} & B^2GL_1(K_p^\wedge). \end{array}$$

The space CP^∞ has a CW structure made of even dimensional cells so that all the cells are attached along odd dimensional spheres. The spectrum $K_p^\wedge$ has non trivial homotopy groups only in even dimensions and hence, so does $B^2GL_1K_p^\wedge$. Thus, all the obstructions to extending the map $\Sigma 1 - p$ must vanish, which implies that there is an A_∞-ring structure on the Thom spectrum K/p. □

Proposition 2.9. *Suppose that $G = S^3$, $R = K_p^\wedge$, and $\zeta = p$ as in Proposition 2.7, then the Thom spectrum has an A_∞-ring structure.*

Proof. The classifying space of S^3 is the infinite quarternionic projective space HP^∞, and $\Sigma S^3 = S^4 \to BS^3 = HP^\infty$ is obtained by attaching even cells along maps of odd dimensional spheres. Therefore the extension problem can always be solved. □

3. Topological Hochschild homology of Thom spectra

In the last section, we observed that the Thom spectrum of a loop map carries an induced A_∞ structure. In this setting, there is a convenient description of the topological Hochschild homology as a Thom spectrum along the ideas of [**5, 3**] and [**11**]. In the following G will be a group, X a space, and G homotopy equivalent to ΩX as A_∞-spaces. R will be an E_∞ ring spectrum.

The Thom spectrum of a map $G \to BGL_1R$ is a twisted R-module generated by G. If this is a loop map, the construction is that of a twisted group ring. Recall that the Hochschild Homology of group rings over a field is given by

$$HH_*(k[G]) \cong k \otimes H_*(G, G),$$

where G acts on itself by conjugation. This is the homology of the Borel construction $G_{hG} \simeq EG \times_G G \simeq LBG$, the free loop space of BG, and so, $HH_*(k[G]) \cong k \otimes H_*(LBG)$. The analogous statement for topological Hochschild homology is the classical result of Bökstedt and Waldhausen:

$$THH(\Sigma^\infty \Omega X_+) \simeq \Sigma^\infty LX_+.$$

In the category of R-modules, the theorem is $THH^R(R \wedge \Omega X_+) \simeq R \wedge LX_+$, computing the topological Hochschild homology of the Thom spectrum of the constant map. More generally, let $f\colon X \to BGL_1R$ and $\zeta \simeq \Omega f\colon G \to BGL_1R$, the Thom spectrum has an A_∞-ring structure, and the topological Hochschild homology is the Thom spectrum of a map from LX to BGL_1R.

In the second part of the section, we apply the theorem for $R = K_p^\wedge$ and $G = S^1$, in the computation of the previous section. This implies that the Thom spectrum is homotopy equivalent to the cofibre of a certain map $K_p^\wedge \wedge CP_+^\infty \to K_p^\wedge \wedge CP_+^\infty$.

3.1. Identifying topological Hochschild homology as a Thom spectrum

Recall that the free loop space LY fits into a fibration

$$\Omega Y \to LY \to Y.$$

If Y is an H-space, then the fibration splits as $LY \simeq Y \times \Omega Y$. This is an equivalence of H-spaces if Y is homotopy commutative.

Let f be a map from X to B^2GL_1R and $\eta\colon B^2GL_1R \to \Omega B^2GL_1R$ be induced from the Hopf map by

$$\begin{aligned} B^2GL_1R \simeq \mathit{Maps}(S^2, B^4GL_1R) \xrightarrow{\eta^*} & \mathit{Maps}(S^3, B^4GL_1R) \\ \simeq & \mathit{Maps}(S^1, \Omega^2 B^4GL_1R) \\ \simeq & \mathit{Maps}(S^1, B^2GL_1R) \\ \simeq & \Omega B^2GL_1R. \end{aligned}$$

Let $L^\eta f$ be the map from LX to BGL_1R defined by the diagram

$$\begin{array}{ccccc} LX & \xrightarrow{Lf} & LB^2GL_1(R) & \longrightarrow & B^2GL_1(R) \times \Omega B^2GL_1(R) \\ & & & & \downarrow{\scriptstyle \eta \times id} \\ & \searrow^{L^\eta f} & & & \Omega B^2GL_1(R) \\ & & & & \downarrow{\scriptstyle \simeq} \\ & & & & BGL_1(R). \end{array}$$

The map $\eta \times id\colon B^2GL_1R \times \Omega B^2GL_1R \to \Omega B^2GL_1R$ is the product of the maps η and id using the H-space structure of ΩB^2GL_1R. Without proof, we state:

Theorem 3.1. *There is a homotopy equivalence*

$$THH^R(G^\zeta) \simeq (LX)^{L^\eta f}.$$

This was proved in the case of the sphere spectrum in [**5, 11**]. The argument for general E_∞-ring spectra R is given in [**3**, Theorem 1.7].

3.2. The example of $G = S^1$ and $R = K_p^\wedge$

By Proposition 2.8, we have the commutative diagram

$$\begin{array}{ccc} S^2 & \xrightarrow{\Sigma 1 - p} & \Sigma BGL_1(K_p^\wedge) \\ \downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle \sigma} \\ CP^\infty & \overset{f}{\dashrightarrow} & B^2GL_1(K_p^\wedge) \end{array}$$

and write $THH^{K_p^\wedge}(K/p, f)$ for the topological Hochschild homology corresponding to this A_∞-ring structure.

Proposition 3.2.

$$THH^{K_p^\wedge}(K/p, f) \simeq (LCP^\infty)^{\widehat{f}},$$

where $\widehat{f}$ *is the composite*

$$LCP^\infty \xrightarrow{Lf} LB^2GL_1K_p^\wedge \simeq B^2GL_1K_p^\wedge \times BGL_1K_p^\wedge \xrightarrow{p_2} BGL_1K_p^\wedge.$$

Proof. By Theorem 3.1, $THH^{K_p^\wedge}(K/p, f) \simeq (LCP^\infty)^{L^\eta f}$. Since $\pi_1(K_p^\wedge) = 0$, $\eta = 0$ in this case. Hence, the proposition. □

The focus of the rest of the paper will be the calculation of $\pi_*((LCP^\infty)^{\widehat{f}}) \cong THH^{K_p^\wedge}(K/p, f)$. First of all we note that:

Proposition 3.3. *There is a long exact sequence*

$$K_p^\wedge{}_* CP^\infty \to K_p^\wedge{}_* CP^\infty \to \pi_* THH^{K_p^\wedge}(K/p, f) \to K_p^\wedge{}_{*-1} CP^\infty \dots .$$

Proof. Note that CP^∞ is an infinite loop space, and hence homotopy commutative, which implies that $LCP^\infty \simeq \Omega CP^\infty \times CP^\infty \simeq S^1 \times CP^\infty$. The space S^1 is a union of two contractible open sets whose intersection is S^0, so, there is a homotopy pushout

$$\begin{array}{ccc} CP^\infty \sqcup CP^\infty & \longrightarrow & CP^\infty \\ \downarrow & & \downarrow \\ CP^\infty & \longrightarrow & LCP^\infty \end{array} \tag{3.1}$$

and hence, a homotopy pushout square of Thom spectra

$$\begin{array}{ccc} (CP^\infty \sqcup CP^\infty)^{\widehat{f}} & \longrightarrow & (CP^\infty)^{\widehat{f}} \\ \downarrow & & \downarrow \\ (CP^\infty)^{\widehat{f}} & \longrightarrow & (LCP^\infty)^{\widehat{f}} \end{array}$$

The two maps $CP^\infty \to LCP^\infty$ in (3.1) are the inclusion of constant loops, so, the two compositions $CP^\infty \to LCP^\infty \to LB^2GL_1K_p^\wedge \to BGL_1K_p^\wedge$ are nullhomotopic and the Thom spectra are $\simeq K_p^\wedge \wedge CP_+^\infty$. The map from $CP^\infty \sqcup CP^\infty$ to $BGL_1K_p^\wedge$ factors through $CP^\infty \to BGL_1K_p^\wedge$ so, the Thom spectrum $(CP^\infty \sqcup CP^\infty)^f \simeq K_p^\wedge \wedge CP_+^\infty \vee K_p^\wedge \wedge CP_+^\infty$. Therefore, the pushout can be written as

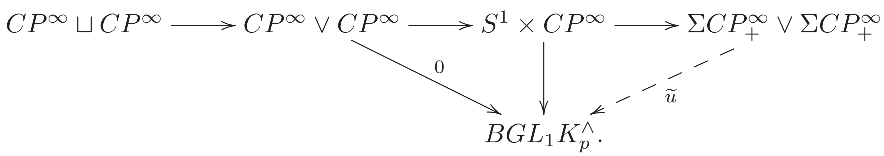
$$\begin{array}{ccc} K_p^\wedge \wedge CP_+^\infty \vee K_p^\wedge \wedge CP_+^\infty & \longrightarrow & K_p^\wedge \wedge CP_+^\infty \\ \downarrow & & \downarrow \\ K_p^\wedge \wedge CP_+^\infty & \longrightarrow & (LCP^\infty)^{Lf}. \end{array}$$

This gives a Mayer Vietoris sequence on homotopy groups

$$\cdots \to K_p^\wedge{}_*(CP^\infty) \oplus K_p^\wedge{}_*(CP^\infty) \to K_p^\wedge{}_*(CP^\infty) \oplus K_p^\wedge{}_*(CP^\infty) \to \pi_*((LCP^\infty)^{Lf}) \cdots$$

To simplify, one needs to understand the left hand map i.e., how $K_p^\wedge \wedge CP_+^\infty \vee K_p^\infty \wedge CP_+^\infty$ maps to the two different copies of $K_p^\wedge \wedge CP_+^\infty$ in the pushout square. For that one needs to examine the structure of $P^{\widehat{f}}$, the $GL_1K_p^{\wedge}$-bundle over $S^1 \times CP^{\infty}$ classified by $\widehat{f}$.

Following the pushout square (3.1), we see that $P^{\widehat{f}}$ is obtained by identifying two trivial bundles over CP^{∞} after restricting over $CP^{\infty} \sqcup CP^{\infty}$, via a map $u\colon CP^{\infty} \sqcup CP^{\infty} \to GL_1K_p^{\wedge}$. The adjoint of u is the map $\widetilde{u}$ in the diagram:

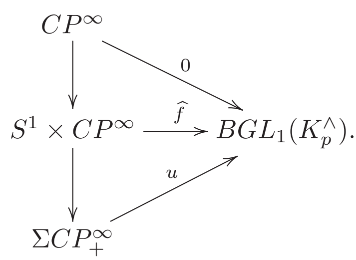


The top row is the cofibre sequence associated to the pushout (3.1). Since the map $S^1 \times CP^{\infty} \to BGL_1K_p^{\wedge}$ is nullhomotopic on $CP^{\infty} \vee CP^{\infty}$, it factors through $\Sigma CP^{\infty}_{+} \vee \Sigma CP^{\infty}_{+}$ as $\widetilde{u}$.

The map u gives two units u_1, u_2 in the ${K_p^{\wedge}}^0(CP^{\infty})$. In the Mayer Vietoris sequence for the Thom spectrum, these describe the map $K_p^{\wedge} \wedge CP^{\infty}_{+} \vee K_p^{\wedge} \wedge CP^{\infty}_{+} \to K_p^{\wedge} \wedge CP^{\infty}_{+} \vee K_p^{\wedge} \wedge CP^{\infty}_{+}$ as the matrix

$$\begin{pmatrix} 1 & u_2 \\ u_1 & 1 \end{pmatrix}.$$

In fact, u_1 and u_2 are equal because each summand in ΣCP^{∞}_{+} of $\Sigma CP^{\infty}_{+} \vee \Sigma CP^{\infty}_{+}$ is the cofibre of the map $CP^{\infty} \to LCP^{\infty} = S^1 \times CP^{\infty}$ given by the inclusion of the constant loops and both can be defined by the same diagram



In terms of u, we can rewrite the Mayer Vietoris sequence as the long exact sequence

$$\cdots \longrightarrow {K_p^{\wedge}}_*(CP^{\infty}) \xrightarrow{u-1} {K_p^{\wedge}}_*(CP^{\infty}) \longrightarrow \pi_*((LCP^{\infty})^{Lf}) \longrightarrow \cdots \qquad (\alpha)$$

which proves the required statement. $\square$

To calculate $\pi_*(THH^{K_p^{\wedge}}(K/p, f))$, it remains to understand the map u. This is done as follows:

Proposition 3.4. *The adjoint of the map $u\colon \Sigma CP^{\infty}_{+} \to BGL_1R$, is homotopy equivalent to the composite $\Sigma^2 CP^{\infty}_{+} \xrightarrow{\mu} CP^{\infty} \xrightarrow{f} B^2GL_1K_p^{\wedge}$, where μ is the composition $\Sigma^2 CP^{\infty}_{+} \simeq S^2 \wedge CP^{\infty}_{+} \xrightarrow{\sigma \wedge id} CP^{\infty} \wedge CP^{\infty}_{+} \to CP^{\infty}$.*

Proof. The following diagram commutes:

$$\begin{array}{ccccc} S^1\wedge(S^1\times CP^\infty) & \xrightarrow{\simeq} & S^1\wedge LCP^\infty & \xrightarrow{S^1\wedge Lf} & S^1\wedge LB^2GL_1(K_p^\wedge)\\ && \downarrow{\scriptstyle ev} && \downarrow{\scriptstyle ev}\\ && CP^\infty & \xrightarrow{f} & B^2GL_1(K_p^\wedge). \end{array}$$

Consider the inclusion of the based loops $BGL_1K_p^\wedge \hookrightarrow LB^2GL_1K_p^\wedge$. Under the composite

$$S^1 \times BGL_1K_p^\wedge \to S^1 \times LB^2GL_1K_p^\wedge \overset{ev}{\to} B^2GL_1K_p^\wedge,$$

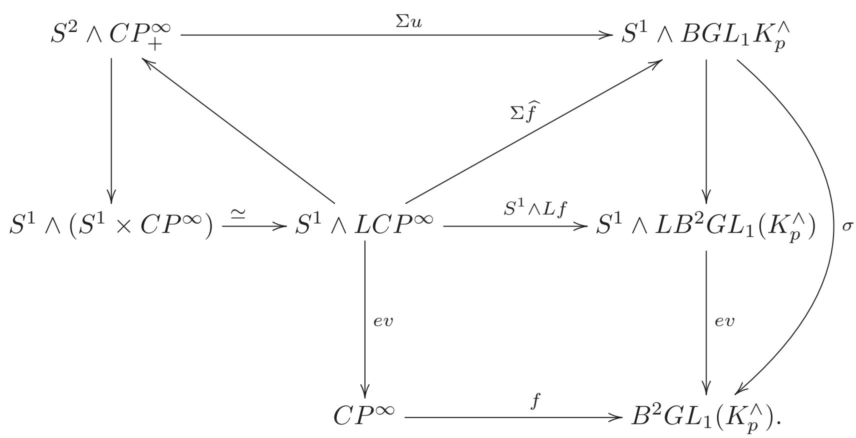
the copies $S^1 \times *$ and $* \times BGL_1K_p^\wedge$ map trivially. Thus, it factors through $S^1 \wedge BGL_1K_p^\wedge$ as $\Sigma BGL_1K_p^\wedge \overset{\sigma}{\to} B^2GL_1K_p^\wedge$. We are trying to figure out the map

$$S^1 \times LCP^\infty \to S^1 \times LB^2GL_1K_p^\wedge \to S^1 \times BGL_1K_p^\wedge \to B^2GL_1K_p^\wedge.$$

Then, this factors through

$$S^1 \wedge LCP^\infty \to S^1 \wedge LB^2GL_1K_p^\wedge \to S^1 \wedge BGL_1K_p^\wedge \overset{\sigma}{\to} B^2GL_1K_p^\wedge.$$

Also $LCP^\infty \to BGL_1K_p^\wedge$ factors through $S^1 \wedge CP^\infty_+$ as u. Putting all the remarks together, we have a commutative diagram



The left hand vertical map from $S^2 \wedge CP^\infty_+$ to $S^1 \wedge (S^1 \times CP^\infty)$ is the inclusion of a factor in the splitting of the suspension of $S^1 \wedge (S^1 \times CP^\infty) \simeq (S^2 \wedge CP^\infty_+) \vee (S^1 \wedge CP^\infty)$. It follows that $\widetilde{u} \simeq \sigma \circ \Sigma u \simeq f \circ g$, where

$$g\colon S^2 \times \Sigma CP^\infty_+ \to S^1 \wedge (S^1 \times CP^\infty) \simeq S^1 \wedge LCP^\infty \overset{ev}{\to} CP^\infty$$

and the composition $g \simeq \mu$. □

4. The structure of $GL_1(K_p^\wedge)$

In this section, we prove a splitting of $GL_1K_p^\wedge$ using the logarithm $l_p\colon gl_1K_p^\wedge \to K_p^\wedge$ defined by Rezk (see [**10**]). Throughout this section, we assume that p is an odd prime.

Proposition 4.1 (Rezk, [**10**]). *Let R be an E_∞ ring spectrum. Then there is a logarithmic cohomology operation, $l_{p,n}$, from $gl_1(R)$ to $L_{K(n)}(R)$ for every n, and prime p. If R is $K(n)$-local, this is a map from $gl_1(R)$ to R. When $n = 1$, $l_p \colon gl_1 R \to R$ is given by the formula:*

$$l_p(x) = -\frac{1}{p}\log\left(\frac{\psi(x)}{x^p}\right).$$

[Recall that a θ-algebra structure is described by operations ψ and θ (ψ is a ring homomorphism) such that $\psi(x) = x^p + p\theta(x)$.]

Proposition 4.2. *Suppose that $R = K_p^\wedge$. The operation $l_p \colon gl_1 K_p^\wedge \to K_p^\wedge$ factors through $ku_p^\wedge$, the connective cover of $K_p^\wedge$. On homotopy groups, the map is an isomorphism on π_n for $n > 2$. At $n = 2$, it is 0. And for $n = 0$, this is the map*

$$Z_p^\times \cong Z/(p-1) \times Z_p \stackrel{p_2}{\to} Z_p.$$

Proof. The spectrum $K_p^\wedge$ is $K(1)$-local, and the operation ψ is the Adams operation ψ_p. Since $gl_1 K_p^\wedge$ is connective, the map l_p factors through $ku_p^\wedge$. Recall, that the homotopy groups of $gl_1 K_p^\wedge$ are given by

$$\pi_n(gl_1 K_p^\wedge) = \begin{cases} ({K_p^\wedge}^0(S^n))^\times = \pi_n(K_p^\wedge) & \text{if } n > 0, \\ ({K_p^\wedge}^0(S^0))^\times = \pi_0(K_p^\wedge)^\times & \text{if } n = 0. \end{cases}$$

Since $\pi_n K_p^\wedge$ is nonzero only for even n, it suffices to restrict our attention to even dimensional spheres. The K-theory of S^{2n} is generated by ϵ where for the map $p \colon (S^2)^n \to S^{2n}$ which quotients out the lower cells, ϵ splits as the product

$$p^*(\epsilon) = \prod (1 - L_i),$$

where L_i is the canonical line bundle over the i^{th} copy of $S^2 = CP^1$. We have

$$\pi_{2n}(gl_1(K_p^\wedge)) = \widetilde{gl_1(K_p^\wedge)}^0(S^{2n}) = (\widetilde{K_p^\wedge}^0(S^{2n}))^\times = 1 + \epsilon\pi_{2n}(K_p^\wedge).$$

To calculate l_p on $\pi_{2n} gl_1 K_p^\wedge$, one needs to compute $l_p(1 + k\epsilon)$ for $1 + k\epsilon$ in $gl_1 {K_p^\wedge}^0(S^{2n}) = \pi_0(gl_1({K_p^\wedge}^{S^{2n}}))$. To accomplish this, we need to calculate $\psi_p(\epsilon)$. We do this by calculating $\psi_p(p^*(\epsilon))$ and using that p^* induces an injection in K-theory. Since the Adams operation ψ_p raises line bundles to the p^{th} power,

$$\begin{aligned} \psi_p(L_i) &= L_i^p \\ \implies \psi_p(1 - L_i) &= 1 - L_i^p \\ &= 1 - (1 - (1 - L_i))^p. \end{aligned}$$

The element $1 - L_i$ lies in the K-theory of S^2, so it squares to 0. Therefore,

$$\begin{aligned} \psi_p(1 - L_i) &= 1 - (1 - p(1 - L_i)) \\ &= p(1 - L_i) \\ \implies \psi_p(\epsilon) &= p^n \epsilon \\ \implies \psi_p(1 + \epsilon) &= 1 + p^n \epsilon. \end{aligned}$$

Hence,

$$\begin{aligned} l_p(1+k\epsilon) &= -\frac{1}{p}\log\left(\frac{\psi(1+k\epsilon)}{(1+k\epsilon)^p}\right) \\ &= -\frac{1}{p}\log\left(\frac{1+p^n k\epsilon}{(1+k\epsilon)^p}\right) \\ &\equiv -\frac{1}{p}\log(1+(p^n-p)k\epsilon) \pmod p, \end{aligned}$$

which becomes multiplication by $1-p^{n-1} \pmod p$ if $n>0$. Since the homotopy group $\pi_{2n}(gl_1 K_p^\wedge) = Z_p$ for $n>0$, this is an isomorphism for $n>1$. For $n=1$, this map is 0. For $n=0$, the map $l_p\colon Z_p^\times \cong \mu_{p-1}\times Z_p \to Z_p$ is given by

$$-\frac{1}{p}\log(x^{1-p}).$$

This map has kernel ν_{p-1}, the group of $(p-1)^{st}$ roots of unity, as it takes p-adic integers of the form $1+pk$ to

$$\begin{aligned} l_p(1+pk) &= -\frac{1}{p}\log((1+pk)^{1-p}) \\ &= -\frac{1}{p}\log(1+p(1-p)k) \\ &= -(1-p)k + O(p) \\ &\equiv -k \pmod p. \end{aligned}$$

Therefore, the map l_p on $Z_p^\times = \nu_{p-1}\times Z_p$, has kernel ν_{p-1} and is an isomorphism onto Z_p. □

Recall that the spectrum $ku_p^\wedge$ splits into Adams summands,

$$ku_p^\wedge \simeq B \vee \Sigma^2 B \ldots \Sigma^{2p-4} B,$$

where B is the p-adic Adams summand($\pi_*(B) = Z_p[v_1]$). Using this, we identify the image of the logarithmic cohomology operation. We construct $K_p(\widehat{2})$ from the spectrum $ku_p^\wedge$ by killing the 2^{nd} homotopy group:

Definition 4.3. Let B_2 be the 2-connective cover of B. Define

$$K_p(\widehat{2}) = B \vee \Sigma^2 B_2 \ldots \vee \Sigma^{2p-4} B.$$

Proposition 4.4. *There is a split cofibre sequence*

$$H\nu_{p-1} \vee \Sigma^2 HZ_p \to gl_1(K_p^\wedge) \to K_p(\widehat{2}).$$

Proof. From the definition above, note that $gl_1 K_p^\wedge \xrightarrow{l_1} ku_p^\wedge \to K_p(\widehat{2})$ is surjective on homotopy groups. The fibre F has homotopy only in dimensions 0 and 2. The Postnikov tower of F then is a cofibre sequence

$$\Sigma^2 HZ_p \to F \to H\nu_{p-1} \to \Sigma^3 HZ_p.$$

Since the group $H^3(H\nu_{p-1}; Z_p) = 0$, the sequence splits and one obtains

$$F \simeq H\nu_{p-1} \vee \Sigma^2 HZ_p.$$

Therefore, there is a cofibre sequence

$$H\nu_{p-1} \vee \Sigma^2 HZ_p \to gl_1(K_p^\wedge) \to K_p(\widehat{2}).$$

The next term in this sequence is

$$\Sigma(H\nu_{p-1} \vee \Sigma^2 HZ_p) \simeq \Sigma H\nu_{p-1} \vee \Sigma^3 HZ_p$$

and the next map is $K_p(\widehat{2}) \to \Sigma H\nu_{p-1} \vee \Sigma^3 HZ_p$. Since the spaces in the Adams summands are retracts of $bu_p^\wedge$, their homology concentrated in even dimensions. Therefore,

$$[\Sigma^{2k} B, \Sigma H\nu_{p-1} \vee \Sigma^3 HZ_p] \cong H^1(B; \nu_{p-1}) \oplus H^3(B; Z_p) \cong 0.$$

Since the spectrum B_2 is 3-connected,

$$[\Sigma^2 B_2, \Sigma H\nu_{p-1} \vee \Sigma^3 HZ_p] \cong H^{-1}(B_2; \nu_{p-1}) \oplus H^1(B_2; Z_p) \cong 0$$
$$\implies [K_p(\widehat{2}), H^1(B; \nu_{p-1}) \oplus H^3(B; Z_p)] = 0.$$

Hence, the cofibre sequence splits and

$$gl_1(K_p^\wedge) \simeq K_p(\widehat{2}) \vee H\nu_{p-1} \vee \Sigma^2 HZ_p$$

completing the proof. □

We will use this decomposition later to calculate homotopy classes of extensions. For that, we also have to understand how the splitting looks like when we map a space X to $GL_1(K_p^\wedge)$. Recall, $[X, GL_1(K_p^\wedge)] = {K_p^\wedge}^0(X)^\times$. The map l_p gives the way to map this to $[X, K_p(\widehat{2})]$. The map ${K_p^\wedge}^0(X)^\times \to H^0(X; \nu_{p-1})$ is the composite

$$X \to GL_1(K_p^\wedge) \to \pi_0 GL_1(K_p^\wedge) \cong Z_p^\times \cong \nu_{p-1} \times Z_p \to \nu_{p-1} \simeq K(\nu_{p-1}, 0).$$

The third factor is $\Sigma^2 HZ_p$, and we have to understand the map from $H^2(X; Z_p)$ to ${K_p^\wedge}^0(X)^\times$. Now, $H^2(X; Z_p) = [X, K(Z_p, 2)] = [X, {CP^\infty}_p^\wedge]$. The space CP^∞ classifies line bundles which are invertible elements in K-theory.

Proposition 4.5. *The map $H^2(X; Z_p) \to {K_p^\wedge}^0(X)^\times$ is given by $f \in [X, {CP^\infty}_p^\wedge] \mapsto L^f$ where L^f is the line bundle classified by f.*

Proof. The formula in the statement of the proposition defines a map of infinite loop spaces $CP_p^{\infty\wedge} \to GL_1 K_p^\wedge$, and hence, a map of spectra $\Sigma^2 HZ_p \to gl_1 K_p^\wedge$. Composing it with l_p, we get

$$\begin{aligned} l_p(L^f) &= -\frac{1}{p}\log\left(\frac{\psi_p(L^f)}{(L^f)^p}\right) \\ &= -\frac{1}{p}\log\left(\frac{(L^f)^p}{(L^f)^p}\right) \\ &= -\frac{1}{p}\log(1) \\ &= 0. \end{aligned}$$

The computation above shows that the composition $\Sigma^2 HZ_p \to gl_1(K_p^\wedge) \to K_p(\widehat{2})$ equals 0. Therefore, it factors through $\nu_{p-1} \times \Sigma^2 HZ_p$ in the diagram:

$$\begin{array}{ccccc} & & \Sigma^2 HZ_p & & \\ & \swarrow & \downarrow & & \\ H\nu_{p-1} \times \Sigma^2 HZ_p & \longrightarrow & gl_1(K_p^\wedge) & \longrightarrow & K_p(\widehat{2}). \end{array}$$

To complete this proof, we need to show that the map $\Sigma^2 HZ_p \to H\nu_{p-1} \vee \Sigma^2 HZ_p \to \Sigma^2 HZ_p$ is an equivalence. The only non zero homotopy group of $\Sigma^2 HZ_p$ is π_2, so it suffices to check that the map $[S^2, CP^\infty] \to H^2(S^2; Z_p)$ as described by the statement is an isomorphism. The left group is isomorphic to Z_p, via $k \mapsto L^k$, $L =$ the tangent bundle of S^2. The right group is $H^2(S^2; Z_p) \cong Z_p$ inside $K_p^\wedge(S^2)^\times$ as elements $1 + k\epsilon$, $\epsilon = 1 - L$. The map between the two is $L^k \mapsto (1-\epsilon)^k = 1 - k\epsilon$ because $\epsilon^2 = 0$, and is evidently an isomorphism. □

5. Calculation of *THH*

In this section, we complete the computation of *THH* for odd primes p. We first parameterise the homotopy classes of extensions f

$$\begin{array}{ccc} S^2 & \xrightarrow{\Sigma 1-p} & \Sigma BGL_1(K_p^\wedge) \\ \downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle \sigma} \\ CP^\infty & \xrightarrow{f} & B^2GL_1(K_p^\wedge) \end{array}$$

using the results of the previous section.

Recall that

$$\begin{aligned} GL_1(K_p^\wedge) &= \nu_{p-1} \times K(Z_p, 2) \times \Omega^\infty K_p(\widehat{2}) \\ \implies B^2GL_1(K_p^\wedge) &= B^2\nu_{p-1} \times K(Z_p, 4) \times \Omega^\infty \Sigma^2 K_p(\widehat{2}). \end{aligned}$$

The condition on the map f is that its restriction to S^2 is $1-p$. The homotopy classes of maps from S^2 to $B^2GL_1(K_p^\wedge)$ is split into three factors:

1. $[S^2, B^2\nu_{p-1}] = H^2(S^2; \nu_{p-1}) \cong \nu_{p-1}$,
2. $[S^2, K(Z_p, 4)] = H^4(S^2; Z_p) = 0$,
3. $[S^2, \Omega^\infty \Sigma^2 K_p(\widehat{2})] = [S^2, \Omega^\infty(\Sigma^2 B \vee \Sigma^4 B_2 \vee \Sigma^6 B \ldots \vee \Sigma^{2p-4} B] = [S^2, \Omega^\infty \Sigma^2 B]$ $= B^2(S^2) \cong Z_p$.

In the splitting

$$[S^2, B^2GL_1(K_p^\wedge)] = \nu_{p-1} \oplus B^2(S^2) \oplus H^4(S^2; Z_p) = \nu_{p-1} \oplus Z_p \oplus 0,$$

$1-p$ is in the factor Z_p, where it equals $l_p(1-p) = \alpha_p$ and

$$\begin{aligned} \alpha_p &= -\frac{1}{p}\log((1-p)^{1-p}) \\ &\equiv -\frac{1}{p}\log(1-(1-p)p) \\ &\equiv -1 \pmod{p}. \end{aligned}$$

5.1. Calculation at the prime 3

Let us begin the calculation at the prime 3. The cofibre sequence for $gl_1 K_3^\wedge$ is

$$HZ/2 \vee \Sigma^2 HZ_3 \to gl_1(K_3^\wedge) \to K_3(\widehat{2})$$

and

$$K_3(\widehat{2}) = B \vee \Sigma^2 B_2.$$

Therefore,

$$GL_1 K_p^\wedge = Z/2 \times K(Z_p, 2) \times \Omega^\infty B \times \Omega^\infty B_2.$$

We will study the extension to CP^∞ of the map $1-p$, to the four factors $Z/2$, $K(Z_3, 2)$, $\Omega^\infty B$, $\Omega^\infty B_2$ one by one. Let us start with the factor B. The Adams summands are the eigenspaces of the action of the $(p-1)^{st}$ roots of unity by Adams operations. The spectrum B is fixed by all the Adams operations. The projection from $K_p^{\wedge *}(X)$ to $B^*(X)$ is given by

$$\pi = \frac{1}{p-1}(1 + \psi_\zeta + \psi_{\zeta^2} + \cdots + \psi_{\zeta^{p-2}}),$$

where $\zeta \in \nu_{p-1} \subset Z_p^\times$.

For the prime 3, we can take $\zeta = -1$ and then the projection operator is

$$\pi = \frac{1 + \psi_{-1}}{2}.$$

Let us start by working out an example.

Example 5.1. Consider the element $\beta L \in K_3^{\wedge 2}(CP^\infty)$ where β is the Bott element. Applying the projection, we get

$$\pi(\beta L) = \frac{\beta(L - L^{-1})}{2}.$$

Restricting to S^2, using $L = 1 - \epsilon$ and $\epsilon^2 = 0$, we obtain

$$\begin{aligned} \frac{\beta((1-\epsilon) - (1-\epsilon)^{-1})}{2} &= \frac{\beta((1-\epsilon) - (1+\epsilon))}{2} \\ &= -\beta\epsilon \\ &= -1. \end{aligned}$$

In order for it to be an extension of the kind required, this restriction must be α_3, so we multiply by $-\alpha_3$. This defines

$$f = -\alpha_3 \frac{\beta(L - L^{-1})}{2}.$$

Recall that, $THH^{K_3^\wedge}(K/3, f)$ is the cofibre of

$$K_3^\wedge \wedge CP^\infty \xrightarrow{u-1} K_3^\wedge \wedge CP^\infty \qquad (\beta)$$

where $u \in K_3^{\wedge 0}(CP^\infty)^\times = [CP^\infty_+, GL_1(K_3^\wedge)]$ is the adjoint of

$$\begin{array}{ccc} S^2 \wedge CP^\infty_+ & & \\ \downarrow{\scriptstyle \mu} & \searrow^{\widetilde{u}} & \\ CP^\infty & \xrightarrow{f} & B^2GL_1(K_3^\wedge). \end{array}$$

The group structure of CP^∞ classifies tensor product of line bundles so, $\mu^* L = L \otimes L$. This implies

$$\mu^*(f) = -\alpha_3 \frac{\beta(L \otimes L - L^{-1} \otimes L^{-1})}{2}.$$

The K-theory of S^2 is generated by $\epsilon = 1 - L$ with $\epsilon^2 = 0$. We can rewrite the equation using the generator

$$\begin{aligned} \mu^*(f) &= -\alpha_3 \frac{\beta((1-\epsilon) \otimes L - (1+\epsilon) \otimes L^{-1})}{2} \\ &= -\alpha_3 \frac{\beta\epsilon \otimes (L + L^{-1})}{2}. \end{aligned}$$

Using the suspension isomorphism (given by $\beta\epsilon = 1$) we get

$$\mu^*(f) = -\alpha_3 \frac{L + L^{-1}}{2}.$$

To get u we need to invert the logarithmic cohomology operation. Suppose that $u = h(x) \in K_3^{\wedge 0}(CP^\infty)^\times$. Then, we have to solve

$$\begin{aligned} -\frac{1}{3} \log\left(\frac{\psi_3(h(x))}{h(x)^3}\right) &= -\alpha_3 \frac{L + L^{-1}}{2} \\ \Longrightarrow \quad \frac{\psi_3(h(x))}{h(x)^3} &= \exp\left(3\alpha_3 \frac{L + L^{-1}}{2}\right). \end{aligned} \tag{5.1}$$

Note that $\psi_3(x) = 1 - (1-x)^3$ and hence,

$$\frac{h(1-(1-x)^3)}{h(x)^3} = \exp\left(3\alpha_3 \frac{L + L^{-1}}{2}\right).$$

Let us look at the equation $\pmod{(3^2, x^3)}$. The right side of the equation can be written in terms of x using $L = 1 - x$, and then, $L^{-1} = 1 + x + x^2 \pmod{(3^2, x^3)}$. Therefore, the right side simplifies to

$$\begin{aligned} \exp\left(3\alpha_3 \frac{L + L^{-1}}{2}\right) &= \exp\left(3\alpha_3 \frac{2 + x^2}{2}\right) \\ &= 1 + 3\alpha_3 + 3\frac{\alpha_3 x^2}{2}. \end{aligned}$$

Now we will simplify the left side of (5.1). Suppose that $h(x) = a + bx + cx^2$. In order to solve the equation, we have to invert l_3. We know that l_3 has a kernel $Z/2 \vee K(Z_3, 2)$, so the equation can be solved once we know the restriction to these.

In the part of $HZ/2$, $\sigma\colon S^2 \to CP^\infty$ induces an isomorphism in $H^2(-; Z/2)$. Therefore, the extension is 0 here. The map $K_3^{\wedge 0}(CP^\infty)^\times \to H^0(CP^\infty; Z/2)$ sends $a \mapsto a$

(mod 3) (identifying $Z/2$ with the group of units in $\mathbb{F}_3$). Therefore, since $\mu^*(0) = 0$, we get the equation

$$a \equiv 1 \pmod 3.$$

In the factor $K(Z_3, 2)$, there is no restriction on f. Assume that it is trivial, so $\mu^*(0) = 0$. This maps into $GL_1(K_3^\wedge)$ by taking a line bundle over CP^∞ to the corresponding unit in K-theory. If we look at $k \in Z_p = H^2(CP^\infty; Z_3) = [CP^\infty, K(Z_3, 2)]$, this is the line bundle $L^k = (1-x)^k = 1 - kx + \frac{k(k-1)}{2}x^2 \pmod{x^3}$. This is the only factor that gives a non zero coefficient of x so, we get that $b = 0$.

Therefore, $h(x) = a + cx^2 \pmod{(3^2, x^3)}$ and $a \equiv 1 \pmod 3$. The left side of (5.1) is

$$\begin{aligned}\frac{\psi(h(1-(1-x)^3))}{h(x)^3} &\equiv \frac{h(3x - 3x^2 + x^3)}{h(x)^3} \\ &\equiv \frac{a}{a^3 + 3ca^2x^2} \\ &\equiv a^{-2}\left(1 - 3\frac{c}{a}x^2\right) \pmod{(3^2, x^3)}.\end{aligned}$$

Working $\pmod{(3^2, x^3)}$, we have

$$\begin{aligned} & a^{-2}\left(1 - 3\frac{c}{a}x^2\right) = 1 + 3\alpha_3 + 3\frac{\alpha_3 x^2}{2} \\ \implies & a \equiv 1 + 3\alpha_3 \pmod{3^2} \text{ and } c \equiv \alpha_3 \pmod 3.\end{aligned}$$

Thus $a = 1 + 3(\mathit{unit})$ and c is a unit (since α_3 is a unit). Therefore, $u - 1$ looks like $3(\mathit{unit}) + x^2(\mathit{unit})$. We can choose a different parameterisation for K-theory of CP^∞ to assume that $u - 1 = 3 + x^2$.

Now $K^\wedge_{3\,*}(CP^\infty) = K^\wedge_{3\,*}\{\beta_0, \beta_1, \ldots\}$ where β_i is dual to x^i. Therefore,

$$\langle (u-1)(\beta_i), x^j \rangle = \langle \beta_i, x^j(3 + x^2) \rangle = \begin{cases} 3 & \text{if } j = i, \\ 1 & \text{if } j = i - 2, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, the map $u - 1$ on $K^\wedge_{p\,*}(CP^\infty)$ is given by

$$(u-1)(\beta_i) = \begin{cases} 3\beta_i & \text{if } i = 0, 1, \\ 3\beta_i + \beta_{i-2} & \text{if } i > 1. \end{cases}$$

Following the cofibre (β), we understand that $u - 1$ is injective, and its cokernel has two copies of $Z/(3^\infty)$ in even dimensions. Thus,

$$\pi_k(THH^{K_3^\wedge}(K/3), f) = \begin{cases} 0 & \text{if } k \text{ is odd}, \\ Z/(3^\infty) \oplus Z/(3^\infty) & \text{if } k \text{ is even}, \end{cases}$$

completing the calculation in this example.

Now we perform the calculation at the prime 3 for all extensions that are non trivial only on the factor $\Omega^\infty B$ of $GL_1(K_3^\wedge)$. The extension in the example was of this kind. So, we are looking at elements in $B^2(CP^\infty)$ which restrict to α_3 in S^2.

An element in $K_3^{\wedge 2}(CP^\infty)$ is given by $\beta g(x)$. Therefore, an element in $B^2(CP^\infty)$ is

$$\pi(\beta g(x)) = \frac{\beta\left(g(x) - g\left(1 - \frac{1}{1-x}\right)\right)}{2}.$$

Suppose that $g(x) = a' + b'x + c'x^2 \pmod{(3^2, x^3)}$. Restricting to S^2 (using $x = \epsilon$ and $\epsilon^2 = 0$) we get b'. We need to get α_3. Thus, to get an extension we must have $b' = \alpha_3$. This gives us all possible extensions f on the factor B. Let us work as before $\pmod{(3^2, x^3)}$. Then,

$$\begin{aligned} f &= \frac{\beta\left(g(x) - g\left(1 - \frac{1}{1-x}\right)\right)}{2} \\ &= \frac{\beta(a' + b'x + c'x^2 - g(-x - x^2))}{2} \\ &= \frac{\beta(a' + b'x + c'x^2 - (a' - b'x - b'x^2 + c'x^2))}{2} \\ &= \frac{\beta(2b'x + b'x^2)}{2} \\ &= \beta b'x + \frac{\beta b'}{2}x^2. \end{aligned}$$

We have to calculate u using

$$\begin{array}{ccc} S^2 \times CP^\infty & & \\ \downarrow \mu & \overset{u}{\searrow} & \\ CP^\infty & \xrightarrow{f} & B^2GL_1(K_3^\wedge). \end{array}$$

By definition, the multiplication map takes x to the formal group, which for K-theory is the multiplicative group. Therefore,

$$\begin{aligned} & x \mapsto \epsilon \otimes 1 + 1 \otimes x - \epsilon \otimes x \\ \Longrightarrow\ & x^2 \mapsto (\epsilon \otimes 1 + 1 \otimes x - \epsilon \otimes x)^2 \\ & \qquad = 1 \otimes x^2 + 2\epsilon \otimes x - 2\epsilon \otimes x^2. \end{aligned}$$

To get μ^* we must project onto the factor $S^2 \wedge CP^\infty_+$. Thus, we obtain

$$\mu^*(x) = \epsilon \otimes 1 - \epsilon \otimes x\ ,\ \mu^*(x^2) = 2\epsilon \otimes x - 2\epsilon \otimes x^2.$$

Using these formulae and the suspension isomorphism $\beta\epsilon = 1$ we calculate $\mu^*(f)$.

$$\begin{aligned} \mu^*(f) &= \beta b'(\epsilon \otimes 1 - \epsilon \otimes x) + \frac{\beta b'}{2}(2\epsilon \otimes x - 2\epsilon \otimes x^2) \\ &= b'(1 - x) + \frac{b'}{2}(2x - 2x^2) \\ &= b' - b'x^2 \end{aligned}$$

To get u, we have to invert the logarithmic cohomology operation l_3, as in the example.

Suppose that $u = h(x)$. Then, we need to solve

$$l_3(u) = \frac{\psi_3(h(x))}{h(x)^3} = \exp(-3b'(1-x^2)).$$

We have the formula $\psi_3(x) = 1 - (1-x)^3$. Similar to the example, we assume that in our extension the contribution from $HZ/2$ is 1 and HZ_3 is 0. In the same way, this implies that if $h(x) = a + bx + cx^2$,

$$a \equiv 1 \pmod 3, \quad b = 0.$$

Then, the equation becomes

$$\begin{aligned} a^{-2}(1 - 3\frac{c}{a}x^2) &= \exp(-3b'(1-x^2)) \\ &= 1 - 3b' + 3b'x^2. \end{aligned}$$

In the same way, we understand that the unit $u = 1 + 3.unit + x^2.unit$, and so, we obtain the same computation

$$\pi_k(THH^{K_3^\wedge}(K/3), f) = \begin{cases} 0 & \text{if } k \text{ is odd,} \\ Z/(3^\infty) \oplus Z/(3^\infty) & \text{if } k \text{ is even.} \end{cases}$$

Now we want to see what happens if we allow extensions with non trivial contributions from the other 3 factors of $GL_1(K_3^\wedge) = Z/2 \times K(Z_3, 2) \times \Omega^\infty B \times \Omega^\infty B_2$. In the part $Z/2$, the restriction $H^2(CP^\infty; Z/2) \to H^2(S^2; Z/2)$ is an isomorphism. So, this factor always contributes trivially.

For the factor $K(Z_3, 2)$, the group $[S^2, B^2K(Z_3, 2)] = [S^2, K(Z_3, 4)] = H^4(S^2; Z_3) = 0$. Therefore, there is no condition on f here. The group $H^4(CP^\infty; Z_3)$ is generated by x^2 and f is given by ax^2 for some $a \in Z_p$. To compute u, consider:

$$\begin{array}{ccc} S^2 \times CP^\infty & & \\ \downarrow \mu & \searrow u & \\ CP^\infty & \xrightarrow{f} & K(Z_3, 4). \end{array}$$

Note that in this case, $\mu^*(x) = \epsilon \otimes 1 + 1 \otimes x$, which implies

$$\begin{aligned} \mu^*(x^2) &= (\epsilon \otimes 1 + 1 \otimes x)^2 \\ &= 2\epsilon \otimes x + 1 \otimes x^2. \end{aligned}$$

To get u we have to project to $S^2 \wedge CP^\infty_+$ and apply the suspension isomorphism. Then, we get $2ax \in H^2(CP^\infty; Z_3)$. Recall from the previous section that, from this we get the unit by taking L^{2a}, where $L = (1-x)$ is the canonical line bundle. Therefore, the contribution to u from this factor is $(1-x)^{2a}$.

Now if a is divisible by 3 then, we still get that our $u = 1 + 3.unit + x^2.unit$ which results in the same calculation for $\pi_*(THH^{K_3^\wedge}(K/3, f))$. If a is not divisible by 3 then, it is a unit, so that $u = 1 + 3.unit + x.unit$. Therefore, by reparameterising we can write $u - 1 = 3 + x$.

$$\implies \langle (u-1)(\beta_i), x^j \rangle = \langle \beta_i, x^j(3+x) \rangle = \begin{cases} 3 & \text{if } j = i, \\ 1 & \text{if } j = i-1, \\ 0 & \text{otherwise} \end{cases}$$

$$\implies (u-1)(\beta_i) = \begin{cases} 3\beta_i & \text{if } i=0, \\ 3\beta_i + \beta_{i-1} & \text{if } i>0 \end{cases}$$

Therefore, in this case,

$$\pi_k(THH^{K_3^\wedge}(K/3), f) = \begin{cases} 0 & \text{if } k \text{ is odd,} \\ Z/(3^\infty) & \text{if } k \text{ is even.} \end{cases}$$

Now consider the factor $\Sigma^2 B_2$. We know that this is 5-connected. So, if we look at extensions we know that they always restrict to $0 \in K_3^\wedge(CP^2)$. Since we are working $\pmod{x^3}$, this means that these extensions always give 0.

Therefore, we get that, depending on f either $\pi_*(THH^{K_3^\wedge}(K/3, f)) = (Z/(3^\infty))^i$ in even degrees where $i = 1$ or 2 depending on f. This finishes our calculation at the prime 3.

5.2. Calculation at primes $\geqslant 5$

Let us now look at the other odd primes and work $\pmod{(x^p, p^2)}$. Recall that there is a splitting

$$GL_1(K_p^\wedge) = \nu_{p-1} \times K(Z_p, 2) \times \Omega^\infty K_p(\widehat{2}),$$
$$K_p(\widehat{2}) = B \vee \Sigma^2 B_2 \vee \ldots \Sigma^{2p-4} B.$$

We start by working in the factor B of $K_p(\widehat{2})$. The projection operator from ${K_p^\wedge}^*(X)$ to $B^*(X)$ is given by

$$\pi = \frac{1 + \psi_\zeta + \psi_{\zeta^2} + \ldots \psi_{\zeta^{p-2}}}{p-1}.$$

Define κ to be the composite

$${K_p^\wedge}^*(CP^\infty) \xrightarrow{\mu^*} {K_p^\wedge}^*(S^2 \wedge CP_+^\infty) \xrightarrow{\cong} {K_p^\wedge}^{*-2}(CP^\infty).$$

First observe that the following diagram commutes:

$$\begin{array}{ccc} {K_p^\wedge}^2(CP^\infty) & \xrightarrow{\psi_a} & {K_p^\wedge}^2(CP^\infty) \\ \downarrow{\scriptstyle\kappa} & & \downarrow{\scriptstyle\kappa} \\ {K_p^\wedge}^0(CP^\infty) & \xrightarrow{\psi_a} & {K_p^\wedge}^0(CP^\infty). \end{array}$$

This implies all Adams operations hence π, commutes with κ.

Write $x = 1 - L$ for the generator in ${K_p^\wedge}^2(CP^\infty)$ and ϵ its restriction to S^2. We have to look for f as in the diagram:

$$\begin{array}{ccc} S^2 & \xrightarrow{\Sigma 1-p} & \Sigma BGL_1(K_p^\wedge) \\ \downarrow{\scriptstyle\sigma} & & \downarrow{\scriptstyle\sigma} \\ CP^\infty & \xrightarrow{f} & B^2GL_1(K_p^\wedge). \end{array}$$

Suppose that f is given by $\pi(\beta g(x))$, where

$$g(x) = a_0 + a_1 x + \cdots + a_{p-1}x^{p-1} \pmod{(x^p, p^2)}.$$

Claim 5.2.

$$\kappa(\beta g(x)) = g'(x)(1-x)$$

Proof. It is enough to check this on the generators x^n. The multiplication takes x to the formal group of K-theory, which is the multiplicative formal group.

$$\mu^*(x) = \epsilon \otimes 1 - \epsilon \otimes x$$

Therefore,

$$\begin{aligned} \mu^*(\beta x^n) &= \beta(\epsilon \otimes 1 + 1 \otimes x - \epsilon \otimes x)^n \\ &= \beta(1 \otimes x^n + n\epsilon \otimes x^{n-1} - n\epsilon \otimes x^n). \end{aligned}$$

κ is obtained by projecting this onto the factor $S^2 \wedge CP^\infty_+$ of the product, and then applying the suspension isomorphism ($\beta\epsilon = 1$). Therefore, we obtain

$$\begin{aligned} \kappa(\beta x^n) &= nx^{n-1} - nx^n \\ &= nx^{n-1}(1-x) \\ &= (x^n)'(1-x), \end{aligned}$$

which proves the claim. □

If we restrict f to S^2, we get

$$\begin{aligned} \pi(\beta g(\epsilon)) &= \pi(\beta(a_0 + a_1\epsilon)) \\ &= \left(\frac{1 + \psi_\zeta + \psi_{\zeta^2} + \dots \psi_{\zeta^{p-2}}}{p-1}\right)(\beta(a_0 + a_1\epsilon)). \end{aligned}$$

The action of the Adams operations on the Bott element and ϵ are given by

$$\psi_a(\beta) = \frac{\beta}{a}\ ,\ \psi_a(\epsilon) = 1 - (1-\epsilon)^a = a\epsilon.$$

Therefore,

$$\begin{aligned} \pi(\beta g(\epsilon)) &= \left(\frac{1 + \psi_\zeta + \psi_{\zeta^2} + \dots \psi_{\zeta^{p-2}}}{p-1}\right)(\beta(a_0 + a_1\epsilon)) \\ &= \frac{\beta(a_0(1 + \zeta^{-1} + \zeta^{-2} + \dots + \zeta^{2-p}) + (p-1)a_1\epsilon)}{p-1} \\ &= \frac{\beta(a_0(1 + \zeta^{-1} + \zeta^{-2} + \dots + \zeta^{2-p}))}{p-1} + a_1\epsilon. \end{aligned}$$

Since ζ is a $(p-1)^{st}$ root of unity, we get

$$\begin{aligned} 1 + \zeta^{-1} + \zeta^{-2} + \dots + \zeta^{2-p} &= \zeta^{p-1} + \zeta^{p-2} + \dots + \zeta \\ &= 0. \end{aligned}$$

This shows that $\pi(\beta g(x))$ restricts to $a_1 \in B^2(S^2)$. Thus, we have that $a_1 = l_p(1-p) = \alpha_p$.

We need to calculate u from the extension $\pi(\beta g(x))$ by solving

$$\begin{aligned} l_p(u) &= \kappa\pi(\beta g(x)) \\ &= \pi\kappa(\beta g(x)) \\ &= \pi(g'(x)(1-x)). \end{aligned}$$

Suppose that $h(x) = g'(x)(1-x) = c_0 + c_1 x + \cdots + c_{p-1}x^{p-1} \pmod{(x^p, p^2)}$. Then

$$\begin{aligned} \pi(h(x)) &= \frac{1 + \psi_\zeta + \psi_{\zeta^2} + \ldots \psi_{\zeta^{p-2}}}{p-1}(h(x)) \\ &= \sum_{i=0}^{p-2} \frac{h(1-(1-x)^{\zeta^i})}{p-1}. \end{aligned}$$

Let us look at the coefficient of x^a in the above equation.

$$\begin{aligned} [\pi(x^n)]_a &= \left[\sum_{i=0}^{p-2} \frac{(1-(1-x)^{\zeta^i})^n}{p-1}\right]_a \\ &= \sum_{i=0}^{p-2} \frac{[(1-(1-x)^{\zeta^i})^n]_a}{p-1} \\ &= \sum_{i=0}^{p-2} \frac{[\sum \binom{n}{l}(-1)^l(1-x)^{l\zeta^i}]_a}{p-1} \\ &= \sum_{i=0}^{p-2} \frac{\sum \binom{n}{l}(-1^l)\binom{l\zeta^i}{a}}{p-1} \end{aligned}$$

Since ζ is a $(p-1)^{st}$ root of unity,

$$(\zeta)^i + (\zeta^2)^i + \cdots + (\zeta^{p-1})^i = \begin{cases} 0 & \text{if } i = 1, 2, \ldots, p-2, \\ p-1 & \text{if } i = 0, p-1. \end{cases}$$

The binomial coefficient $\binom{y}{a}$ is a polynomial in y of degree a with the constant term 0 and the top coefficient $1/a!$. Therefore,

$$\binom{l(\zeta)}{a} + \binom{l(\zeta^2)}{a} + \cdots + \binom{l(\zeta^{p-1})}{a} = \begin{cases} 0 & \text{if } a = 1, 2, \ldots, p-2, \\ \frac{p-1}{(p-1)!} l^{p-1} & \text{if } a = p-1, \\ p-1 & \text{if } a = 0. \end{cases}$$

Therefore, we get

$$[\pi(x^n)]_a = \begin{cases} 0 & \text{if } a = 1, 2, \ldots, p-2, \\ \frac{1}{(p-1)!}\sum \binom{n}{l}(-1)^l l^{p-1} & \text{if } a = p-1, \\ \sum \binom{n}{l}(-1)^l & \text{if } a = 0 \end{cases}$$

$$\Longrightarrow [\pi(x^n)]_0 = \sum \binom{n}{l}(-1)^l = \begin{cases} (1-1)^n = 0 & \text{if } n > 0, \\ 1 & \text{if } n = 0. \end{cases}$$

The other possible non zero coefficient is $[\pi(x^n)]_{p-1}$. If $n = 0$, this must be 0. If $n > 0$

this gives

$$\begin{aligned}[\pi(x^n)]_{p-1} &\equiv \frac{1}{(p-1)!}\sum \binom{n}{l}(-1)^l l^{p-1}\\ &\equiv \frac{1}{(p-1)!}\sum \binom{n}{l}(-1)^l l^{p-1}\\ &\equiv -\sum \binom{n}{l}(-1)^l\\ &\equiv -(1-1)^n+1\\ &\equiv 1 \pmod p.\end{aligned}$$

Summarising the calculation (mod p), we get

$$[\pi(x^n)]_a = \begin{cases} 1 & \text{if } a=0,\ n=0,\\ 1 & \text{if } a=p-1,\ n>0,\\ 0 & \text{otherwise.}\end{cases}$$

Now we are in a position to calculate $\pi(h(x)) \pmod p$

$$\begin{aligned}\pi(h(x)) &= \pi(c_0+c_1x+\cdots+c_{p-1}x^{p-1})\\ &= c_0\pi(1)+c_1\pi(x)+\cdots+c_{p-1}\pi(x^{p-1})\\ &= c_0+c_1x^{p-1}+\cdots+c_{p-1}x^{p-1}\\ &= c_0+bx^{p-1},\end{aligned}$$

where $c_0 = a_1$ and

$$\begin{aligned}b &= c_1+\ldots c_{p-1}\\ &= a_1-2a_2+2a_2-3a_3\ldots-(p-1)a_{p-1}+(p-1)a_{p-1}-pa_p\\ &\equiv a_1 \pmod p.\end{aligned}$$

Thus the equation for $u \pmod p$ reduces to

$$\begin{aligned} l_p(u) &= a_1+bx^{p-1} \pmod p\\ \implies -\frac{1}{p}\log\left(\frac{\psi_p(u(x))}{u^p}\right) &= a_1+bx^{p-1} \pmod p\\ \implies \frac{\psi_p(u(x))}{u^p} &= \exp(-p(a_1+bx^{p-1})) = 1-pa_1+pbx^{p-1} \pmod{p^2}.\end{aligned}$$

We are looking at extensions which are non trivial only on the factor B. This implies $u(x)\in B^0(CP^\infty)$ which implies u is in the image of π. By the calculations above, this implies that $u(x) = d_0+d_1x^{p-1} \pmod{x^p}$. Then

$$\begin{aligned}\frac{\psi_p(u(x))}{u^p} &= \frac{d_0}{{d_0}^p+pd_0^{p-1}d_1x^{p-1}}\\ &= (d_0)^{1-p}\left(1-p\frac{d_1}{d_0}x^{p-1}\right).\end{aligned}$$

Therefore, we obtain

$$\begin{aligned} d_0^{1-p} &= 1 - pa_1 \\ \implies d_0 &= (1-pa_1)^{\frac{1}{1-p}} \\ &= 1 - \frac{p}{1-p}a_1 \\ &= 1 - pa_1 \pmod{p^2} \\ \implies d_1 &= -{d_0}^p b \\ &\equiv -1 \pmod{p}. \end{aligned}$$

Therefore, $d_0 = 1 + p.unit$ and $d_1 = unit$. Thus, $u = 1 + p.unit + unit.x^{p-1}$. We can reparameterise so that $u = 1 + p + x^{p-1}$.

$$\begin{aligned} \langle (u-1)(\beta_i), x^j \rangle &= \langle \beta_i, x^j(p + x^{p-1}) \rangle \\ &= \begin{cases} p & \text{if } j = i, \\ 1 & \text{if } j = i-(p-1), \\ 0 & \text{otherwise} \end{cases} \\ \implies (u-1)(\beta_i) &= \begin{cases} p\beta_i & \text{if } i = 0, \\ p\beta_i + \beta_{i-(p-1)} & \text{if } i > 0 \end{cases} \end{aligned}$$

Inputting this in the long exact sequence (α), we get

$$\pi_k(THH^{K_p^\wedge}(K/p, f)) = \begin{cases} 0 & \text{if } k \text{ is odd,} \\ (Z/(p^\infty))^{p-1} & \text{if } k \text{ is even.} \end{cases}$$

Now let us look at what happens if we allow non trivial extensions in the other factors. Under restriction to S^2, $H\nu_{p-1}^2(S^2) \cong H\nu_{p-1}^2(CP^\infty) = \nu_{p-1}$. The element $1-p$ gives $1 \in \nu_{p-1}$. So, this part always contributes trivially.

The factor $\Sigma^2 B_2$ is $(2p-1)$-connected. So, $[CP^{p-1}, \Sigma^2 B_2] = 0$. Thus $\pmod{x^p}$ this factor is always trivial.

Next lets look at the factor $\Sigma^2 HZ_p$. Since $[S^2, \Sigma^4 HZ_p] = HZ_p^4(S^2) = 0$, we have no condition on f from this factor. The group $HZ_p^4(CP^\infty)$ is generated by x^2. Suppose that f is given by $ax^2 \in HZ_p^4(CP^\infty)$. To compute the contribution to u, we have the diagram

$$\begin{array}{ccc} S^2 \times CP^\infty & & \\ \downarrow \mu & \searrow u & \\ CP^\infty & \xrightarrow{f} & K(Z_p, 4). \end{array}$$

Under μ, x pulls back to the formal group and thus

$$\begin{aligned} \mu^*(x^2) &= (\epsilon \otimes 1 + 1 \otimes x)^2 \\ &= 2\epsilon \otimes x + 1 \otimes x^2. \end{aligned}$$

Projecting this to the factor $S^2 \wedge CP_+^\infty$, and applying the suspension isomorphism we get $2ax \in HZ_p^2(CP^\infty)$. The map from $HZ_p^2(CP^\infty) \to [CP^\infty, GL_1(K_p^\wedge)] = {K_p^\wedge}^0(CP^\infty)^\times$ is given by $\alpha x \to (1+x)^\alpha$.

Therefore, if a is divisible by p then we still get that $u = 1 + p.unit + x^{p-1}.unit$. This does not change the calculation of $THH^{K_p^\wedge}(K/p, f)$. If a is not divisible by

p, then it is a unit. Then, $u = 1 + p.unit + x.unit$. This can be reparameterised to $u = 1 + p + x$. Then

$$\begin{aligned}\langle (u-1)(\beta_i), x^j\rangle &= \langle \beta_i, x^j(p+x)\rangle \\ &= \begin{cases} p & \text{if } j = i, \\ 1 & \text{if } j = i-1, \\ 0 & \text{otherwise} \end{cases} \\ \implies (u-1)(\beta_i) &= \begin{cases} p\beta_i & \text{if } i = 0, \\ p\beta_i + \beta_{i-1} & \text{if } i > 0. \end{cases}\end{aligned}$$

Therefore, we obtain

$$\pi_k(THH^{K_p^\wedge}(K/p, f)) = \begin{cases} 0 & \text{if } k \text{ is odd,} \\ Z/(p^\infty) & \text{if } k \text{ is even.} \end{cases}$$

The other factors are $\Sigma^{2k}B$ for $k = 2, 3, \ldots,\ p-2$. These correspond to the eigenspaces of the action of the Adams operations where ψ_{ζ^i} acts as ζ^{ki}. The projection operator is given by

$$\pi_k = \frac{1 + \zeta^{-k}\psi_\zeta + \zeta^{-2k}\psi_{\zeta^2} + \cdots + \zeta^{-k(p-2)}\psi_{\zeta^{p-2}}}{p-1}.$$

The group $[S^2, \Omega^\infty\Sigma^2\Sigma^{2k}B] = B^{2k+2}(S^2) = 0$, so, there is no condition on restriction to S^2. Then, we may choose any $\pi_k(\beta h(x))$ for f, and u must satisfy

$$\begin{aligned} l_p(u) &= \kappa(\pi_k(\beta h(x))) \\ &= \pi_k(\kappa(\beta h(x))) \\ &= \pi_k(h'(x)(1-x)). \end{aligned}$$

Now assume $g(x) = h'(x)(1-x) = c_0 + c_1x + \cdots + c_{p-1}x^{p-1}$. Then

$$\begin{aligned}\pi_k(g(x)) &= \frac{1 + \zeta^{-k}\psi_\zeta + \zeta^{-2k}\psi_{\zeta^2} + \cdots + \zeta^{-k(p-2)}\psi_{\zeta^{p-2}}}{p-1}(g(x)) \\ &= \frac{1 + \zeta^{-k}g(1-(1-x)^\zeta) + \zeta^{-2k}g(1-(1-x)^{\zeta^2}) + \cdots \zeta^{-k(p-2)}g(1-(1-x)^{\zeta^{p-2}})}{p-1}.\end{aligned}$$

The following proposition is useful to complete the calculation

Proposition 5.3. *There is a polynomial* $f_k(x) = x^k + a_{k+1}x^{k+1} + \ldots$ *such that,* $Im(\pi_k)$ *has polynomials that are multiples of* $f_k \pmod{x^p}$.

Proof. These polynomials are in the p-adic K-theory of CP^∞. By looking $\pmod{x^p}$, we are restricting to the K-theory of CP^{p-1}. It splits into eigenspaces

$$K_p^{\wedge 0}(CP^{p-1}) = \oplus_{k=0}^{p-2}\Lambda_k,$$

where $\Lambda_k = [CP^{p-1}, \Omega^\infty\Sigma^{2k}B] = B^{2k}(CP^{p-1})$ is the eigenspace on which the Adams operations ψ_ζ act as multiplication by ζ^k. π_k is the projection on to the eigenspace Λ_k. In this decomposition, $\dim(\Lambda_0) = 2$ and $\dim(\Lambda_k) = 1$ for all $k \geqslant 1$. Therefore, $\Lambda_k = span(f_k)$ for some polynomial f_k. To see how the polynomial f_k looks we compute

$\pi_k(x)$ (note that $\pi_k(1) = 0$, so we don't get any information out of it).

$$\begin{aligned}\pi_k(x) &= \frac{1 + \zeta^{-k}\psi_\zeta + \zeta^{-2k}\psi_{\zeta^2} + \cdots + \zeta^{-k(p-2)}\psi_{\zeta^{p-2}}}{p-1}(x) \\ &= \frac{1}{p-1}\sum_{i=0}^{p-2}\zeta^{-ik}(1-(1-x)^{\zeta^i}) \\ &= \frac{1}{p-1}\sum_{i=0}^{p-2}\zeta^{-ik}\sum_{n=1}^{\infty}(-1)^{i-1}\binom{\zeta^i}{n}x^n \\ &= \frac{1}{p-1}\sum_{i=0}^{p-2}\sum_{n=1}^{\infty}(-1)^{i-1}\zeta^{-ik}\binom{\zeta^i}{n}x^n\end{aligned}$$

Let us look at the coefficient of x^n in the above formula. $\binom{y}{n}$ is a polynomial of degree n in y, and therefore, $(-1)^{i-1}y^{-k}\binom{y}{n}$ has terms of degree $-k$ to $-k+n$. So, if we sum the series, it is 0 if $n < k$. Thus, the first possible non zero coefficient of x is in degree k. The coefficient of x^k in $\pi_k(x)$ is given by

$$\begin{aligned}[\pi_k(x)]_k &\equiv \frac{1}{p-1}\sum_{i=0}^{p-2}(-1)^{i-1}\zeta^{-ik}\binom{\zeta^i}{k} \\ &\equiv \frac{1}{p-1}\sum_{i=0}^{p-2}(-1)^{i-1}\frac{1}{k!} \\ &\equiv \frac{1}{(p-1)k!} \\ &\not\equiv 0 \pmod p.\end{aligned}$$

So, this is a unit in Z_p. Therefore, $Im(\pi_k) = Span(f_k)$ where f_k looks like $x^k + O(x^{k+1})$. □

Therefore, $\pi_k(g(x)) = cf_k(x)$ for some constant c. The equation for u is

$$l_p(u) = \pi_k(g(x)) = cf_k(x) \quad \Longrightarrow \quad \frac{\psi_p(u)}{u^p} = \exp(-pcf_k(x)).$$

If c is divisible by p, then $\pmod{p^2}$ the above equation is 0. If c is not divisible by p, then the coefficient of x^k in the right side is p times an unit. We can solve for u as in the cases before. From here, we get a contribution $= unit.x^k$. Therefore, the unit becomes $u = 1 + p.unit + x^k.unit$. As before, we have the long exact sequence (α)

$$K^\wedge_{p\,*}(CP^\infty) \xrightarrow{u-1} K^\wedge_{p\,*}(CP^\infty) \longrightarrow \pi_*(THH^{K_p^\wedge}(K/p, f))$$

and

$$\begin{aligned}\langle (u-1)(\beta_i), x^j\rangle &= \langle \beta_i, x^j(p + x^k)\rangle \\ &= \begin{cases} p & \text{if } j = i, \\ 1 & \text{if } j = i-k, \\ 0 & \text{otherwise}\end{cases} \\ \Longrightarrow\ (u-1)(\beta_i) &= \begin{cases} p\beta_i & \text{if } i = 0, \\ p\beta_i + \beta_{i-k} & \text{if } i > 0.\end{cases}\end{aligned}$$

Therefore, we obtain that

$$\implies \pi_n(THH^{K_p^\wedge}(K/p, f)) = \begin{cases} 0 & \text{if } n \text{ is odd ,} \\ (Z/(p^\infty))^k & \text{if } n \text{ is even.} \end{cases}$$

This ends the calculation for all odd primes. The homotopy groups of $THH^{K_p^\wedge}(K/p)$ are 0 in odd degrees and $(Z/(p^\infty))^k$ in even degrees, where k is a number between 1 and $p-1$ depending on the A_∞ structure on K/p. This result was proved before by Angeltveit [**1**]. He used the Bökstedt spectral sequence to calculate topological Hochschild homology.

Remark 5.4. This is the calculation identifying K/p as the Thom spectrum of S^1. A similar calculation can be carried out for the Thom spectrum of S^3 to get the same results.

References

[**1**] V. Angeltveit, *Topological Hochschild homology and cohomology of A_∞ ring spectra,* Geom. Topol. **12** (2008), 987–1032.

[**2**] M. Ando, A.J. Blumberg, D. Gepner, M.J. Hopkins, C. Rezk, *Units of ring spectra and Thom spectra,* Preprint, 2008.

[**3**] S. Basu, S. Sagave, C. Schlichtkrull, *Generalized Thom spectra and their topological Hochschild homology,* Preprint, 2016.

[**4**] A. Blumberg, *Topological Hochschild homology of Thom spectra which are E_∞ ring spectra,* J. Topol. **3** (2010), 535–560.

[**5**] A. Blumberg, R.L. Cohen, C. Schlichtkrull, *Topological Hochschild homology of Thom spectra and the free loop space,* Geom. Topol. **14** (2010), 1165–1242.

[**6**] A.D. Elmendorf, I. Kriz, M. Mandell, J.P. May, *Rings, modules and algebras in stable homotopy theory.* With an appendix by M. Cole. Mathematical Surveys and Monographs, **47**. American Mathematical Society, Providence, RI, 1997.

[**7**] M.J. Hopkins, *$K(1)$-local E_∞ ring spectra, Topological modular forms*, 287–302, Math. Surveys Monogr., **201**, Amer. Math. Soc., Providence, RI, 2014.

[**8**] L.G. Lewis, J.P. May, M. Steinberger, *Equivariant stable homotopy theory.* With contributions by J. E. McClure. Lecture Notes in Mathematics, **1213**. Springer-Verlag, Berlin, 1986.

[**9**] J.P. May, *E_∞ ring spaces and E_∞ ring spectra,* With contributions by Frank Quinn, Nigel Ray, and Jørgen Tornehave. Lecture Notes in Mathematics, **577**. Springer-Verlag, Berlin-New York, 1977.

[**10**] C. Rezk, *The units of a ring spectrum and a logarithmic cohomology operation,* J. Amer. Math. Soc. **19** (2006), 969–1014.

[**11**] C. Schlichtkrull, *Units of ring spectra and their traces in Algebraic K-theory,* Geom. Topol. **8** (2004), 645–673.

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