

RATIONAL $O(2)$ -EQUIVARIANT SPECTRA

DAVID BARNES

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Abstract

The category of rational $O(2)$ -equivariant cohomology theories has an algebraic model $\mathcal{A}(O(2))$, as established by work of Greenlees. That is, there is an equivalence of categories between the homotopy category of rational $O(2)$ -equivariant spectra and the derived category of the abelian model $D\mathcal{A}(O(2))$. In this paper we lift this equivalence of homotopy categories to the level of Quillen equivalences of model categories. This Quillen equivalence is also compatible with the Adams short exact sequence of the algebraic model.

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1. Introduction

Equivariant cohomology theories are a fundamental tool for studying spaces with a G -action. To study these cohomology theories, it is helpful to understand the G -spectra that represent them. The homotopy category of G -spectra is particularly complicated. It contains all the information of the stable homotopy category, as well as equivariant information such as the Burnside ring of G and the group cohomology of G . A standard and fruitful method to make this category easier to study is to work rationally. Since the rational stable homotopy category is equivalent to the category of graded rational vector spaces, we have removed most of the topological complexity. However, much of the interesting behaviour that comes from the group is preserved and made tractable.

The category of rational G -spectra has been classified (via Quillen equivalences) in terms of a simple algebraic category for a number of groups. The case of finite groups has been covered by the author [Bar09a] and Kędziorek [Kęd15]. The circle group $\mathbb{T} = SO(2)$ has recently been completed by the author, Greenlees, Kędziorek and Shipley in [BGKS15]. The case of a torus group is considered in Greenlees and Shipley [GS].

In this paper we focus on the group $O(2)$, which is the simplest non-commutative non-finite compact Lie group. The paper [Gre98b] gives an abelian model $\mathcal{A}(O(2))$ for the homotopy category of rational $O(2)$ -spectra. By considering objects with a differential in $\mathcal{A}(O(2))$ we can construct a model category $d\mathcal{A}(O(2))$ and a Quillen equivalence between $d\mathcal{A}(O(2))$ and the model category of rational $O(2)$ -spectra. We call $d\mathcal{A}(O(2))$ the **algebraic model for rational $O(2)$ -spectra**. This Quillen equivalence gives us a triangulated equivalence of the homotopy categories and also tells us that all further homotopical structures (such as homotopy limits or Toda brackets) are preserved by this equivalence.

The algebraic model is explicit and manageable so that constructing objects or maps is straightforward. Furthermore, there is an Adams exact sequence relating maps in the homotopy category of rational $O(2)$ -spectra to the algebraic model, see Theorems 3.8 and 5.5. The algebraic model splits into the product of two simpler categories. The first, called the **toral part** $d\mathcal{A}(\mathcal{C})$, comes from (the homotopy category of) the algebraic model for \mathbb{T} along with a skewed action of $W = O(2)/\mathbb{T}$, see Definition 3.3. Our work will also clarify an imprecision in [Gre98b] regarding the behaviour of W on the toral part, see Remark 3.6. The second, called the **dihedral part** $dg\mathcal{A}(\mathcal{D})$, behaves much more like the case of a finite group (or an exceptional subgroup as in [Kęd15]), see Definition 5.2. The main theorem can be phrased as follows, where we denote rational $O(2)$ -equivariant (orthogonal) spectra by $O(2)\mathrm{Sp}_{\mathbb{Q}}^{\circ}$.

Theorem 1.1. *There is a zig-zag of Quillen equivalences between*

$$O(2) \operatorname{Sp}_{\mathbb{Q}}^{\circ} \quad \text{and} \quad d\mathcal{A}(\mathbb{C}) \times dg\mathcal{A}(\mathbb{D}).$$

This paper represents the prototype for other extensions of a torus by a finite group. Let G be an extension of \mathbb{T}^r by a finite group, then there is a notion of ‘toral’ G -spectra extending our notion of toral $O(2)$ -spectra. A study of the homotopy category of such spectra appears in [Gre15] and that paper gives an abelian model for the homotopy category. A similar method to Section 4 and an extension of the results of [Bar16] should provide a classification of toral G -spectra in terms of the algebraic model built from the abelian model. The key fact is that a map of toral G -spectra is a weak equivalence if and only if it forgets to a weak equivalence of \mathbb{T}^r -spectra.

1.1. Organisation

The model category of rational $O(2)$ -spectra is recalled in Section 2. Theorem 2.5 splits the model category into two parts: the model category of ‘toral spectra’ and the model category of ‘dihedral spectra’, see Definition 2.6. This is the model category version of the fact that the homotopy category splits into two pieces.

The model category of toral spectra only has homotopical information coming from the finite cyclic groups and \mathbb{T} and hence behaves very much like the model category of rational \mathbb{T} -spectra. In Section 3 we introduce the algebraic model for toral spectra. The classification of toral spectra in terms of the algebraic model is given in Section 4. The method is an extension of the method of [BGKS15]. The extra difficulty is accounting for the action of the Weyl group $W = W_{O(2)}\mathbb{T} = O(2)/\mathbb{T}$. The essential point is that the \mathbb{T} -fixed points of a toral $O(2)$ -spectrum have the structure of a spectrum with W -action.

The model category of dihedral spectra contains the homotopical information of rational $O(2)$ -spectra generated by the dihedral subgroups and $O(2)$. In Section 5 we introduce the algebraic model for dihedral spectra and give the classification. This part of the work is simpler than the toral case as all the homotopical information is concentrated in degree zero, see Lemma 5.17.

2. Rational $O(2)$ -spectra

In this section we introduce a model category for rational $O(2)$ -spectra and show that it splits into the product of two localisations. We also give some information on the group $O(2)$ and introduce some basic results on $O(2)$ -spectra.

2.1. The group $O(2)$ and model structures

Let D_{2n}^h denote the dihedral subgroup of order $2n$ containing h , where h is an element of $O(2) \setminus SO(2)$. The closed subgroups of $O(2)$ are $O(2)$, $SO(2)$, the finite cyclic groups C_n ($n \geq 1$) and the finite dihedral groups D_{2n}^h for varying h .

For H a closed subgroup of $O(2)$ we let $N_{O(2)}(H)$ denote the normaliser of H in $O(2)$: So $N_{O(2)}(H)$ is the largest subgroup of $O(2)$ in which H is normal. The Weyl group of H in $O(2)$ is

$$W_{O(2)}(H) := N_{O(2)}(H)/H.$$

The Weyl group of $O(2)$ is the trivial group. The Weyl group of $\mathbb{T} = SO(2)$ is the group of order two, which we call W

$$W := O(2)/SO(2) = W_{O(2)}(SO(2)).$$

The normaliser of D_{2n}^h in $O(2)$ is D_{4n}^h , thus the Weyl group of D_{2n}^h is isomorphic to W . The finite cyclic groups are normal, hence the Weyl group of C_n is $O(2)/C_n \cong O(2)$.

Following [LMSM86, Chapter V, Section 2], define $\mathcal{FO}(2)$ to be the set of those subgroups of $O(2)$ with finite index in their normaliser equipped with the Hausdorff topology. This is an $O(2)$ -space via the conjugation action of $O(2)$ on its subgroups. Let $C(\mathcal{FO}(2)/O(2), \mathbb{Q})$ be the ring of continuous maps from $\mathcal{FO}(2)/O(2)$ to \mathbb{Q} considered as a discrete space. By work of tom Dieck, see [LMSM86, Lemma 2.10], there is an isomorphism of rings

$$A(O(2)) \otimes \mathbb{Q} := [\mathbb{S}, \mathbb{S}]^{O(2)} \otimes \mathbb{Q} \xrightarrow{\cong} C(\mathcal{FO}(2)/O(2), \mathbb{Q}),$$

which sends $f \otimes q$ to $(H) \mapsto q \deg(\Phi^H f)$ (the degree of the H -fixed points of the map $f: \mathbb{S} \rightarrow \mathbb{S}$). We draw $\mathcal{FO}(2)/O(2)$ below as Figure 1. We will sometimes write D_{2n} for (D_{2n}^h) the conjugacy class of D_{2n}^h . The point $O(2)$ is a limit point of this space.

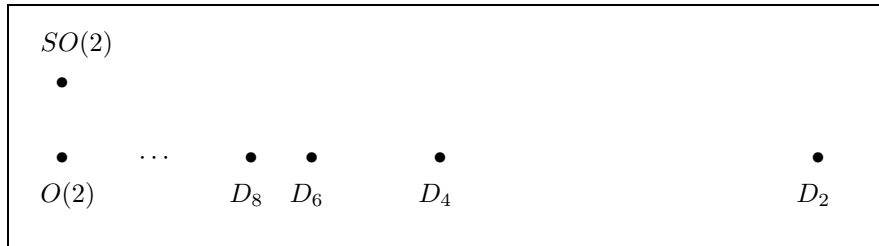


Figure 1: $\mathcal{FO}(2)/O(2)$.

Definition 2.1. Define \mathcal{C} to be the set consisting of the finite cyclic groups and \mathbb{T} . This set is a family in the sense that it is closed under conjugation and taking subgroups. Let \mathcal{D} be the complement of \mathcal{C} in the set of all (closed) subgroups of $O(2)$.

Definition 2.2. We define idempotents of $C(\mathcal{FO}(2)/O(2), \mathbb{Q})$ as follows: $e_{\mathcal{C}}$ is the characteristic function of \mathbb{T} , $e_{\mathcal{D}} := e_{\mathcal{C}} - 1$ and e_n is the characteristic function of D_{2n} for each $n \geq 1$. We also let $f_n = e_{\mathcal{D}} - \sum_{k=1}^{n-1} e_k$.

Our base category of spectra Sp° is the category of orthogonal spectra, equipped with the stable model structure. Let $O(2)\mathrm{Sp}^{\circ}$ be the model category of $O(2)$ -equivariant orthogonal spectra (defined over a complete $O(2)$ -universe \mathcal{U}). This is a proper, cellular, stable model structure where weak equivalences are those maps f such that $\pi_*^H(f)$ is an isomorphism for all closed subgroups H of $O(2)$. See [MM02] for details.

Similarly, we have $\mathbb{T}\mathrm{Sp}^{\circ}$, the model category of \mathbb{T} -equivariant orthogonal spectra. We will index this category of spectra over the universe $i^*\mathcal{U}$ (so our \mathbb{T} -equivariant spectra are indexed on \mathbb{T} -representations of the form i^*V , for V an $O(2)$ -representation).

This non-standard choice of universe is justified by [MM02, Section V.2 and Remark 1.10]. In particular, the homotopy category is the usual \mathbb{T} -equivariant stable homotopy category. One advantage of this convention is that the forgetful functor from $O(2)$ -spectra to \mathbb{T} -spectra is given by $(i^*X)(i^*V) := i^*(X(V))$, that is, the space $X(V)$ with $O(2)$ -action forgotten to a \mathbb{T} -action, for X an $O(2)$ -spectrum.

Following [Bar09b, Section 5] and using [MM02, Theorem IV.6.3], we can create a new model structure on $O(2)\mathrm{Sp}^\circledast$ by localising at a rational sphere spectrum $S^0\mathbb{Q}$. The spectrum $S^0\mathbb{Q}$ can be built as a non-equivariant spectrum and inflated to an $O(2)$ -spectrum or built directly in $O(2)$ -spectra. We call the weak equivalences of the localised model structure **rational equivalences**: those maps f such that $\pi_*^H(f) \otimes \mathbb{Q}$ is an isomorphism for all closed subgroups H of $O(2)$. We call this the **rational model structures**. Analogous model structures exist for \mathbb{T} -spectra and non-equivariant spectra.

Definition 2.3. Let $O(2)\mathrm{Sp}_\mathbb{Q}^\circledast$ be the category of $O(2)$ -equivariant orthogonal spectra equipped with the rational model structure. This category of spectra is indexed on the complete $O(2)$ -universe \mathcal{U} .

Let $\mathbb{T}\mathrm{Sp}_\mathbb{Q}^\circledast$ be the category of \mathbb{T} -equivariant orthogonal spectra equipped with the rational model structure. This category of spectra is indexed on the universe $i^*\mathcal{U}$.

Let $\mathrm{Sp}_\mathbb{Q}^\circledast$ be the category of orthogonal spectra equipped with the rational model structure. This category of spectra is indexed on the universe \mathbb{R}^∞ .

We will also have cause to use a category of naive $W = O(2)/\mathbb{T}$ -equivariant spectra.

Definition 2.4. Let $\mathrm{Sp}_\mathbb{Q}^\circledast[W]$ denote the category of W -objects and W -maps in Sp^\circledast , indexed on the universe \mathbb{R}^∞ . This category is equipped with the ‘free’ model structure lifted from $\mathrm{Sp}_\mathbb{Q}^\circledast$, using the functor $W_+ \wedge (-)$. Hence a map is a weak equivalence or fibration if it is so in $\mathrm{Sp}_\mathbb{Q}^\circledast$ when the W -action is forgotten.

2.2. Splitting rational $O(2)$ -spectra

We know by [Gre98b] and [Bar09b, Section 6] that the homotopy theory of rational $O(2)$ -spectra splits into two pieces. Using the idempotents of Definition 2.2 we define $e_e\mathbb{S}$ as the homotopy colimit (mapping telescope) of

$$\mathbb{S} \xrightarrow{e_e} \mathbb{S} \xrightarrow{e_e} \mathbb{S} \xrightarrow{e_e} \dots$$

and we require that this spectrum be cofibrant (either by choice of construction or by replacing it with a cofibrant replacement). Similarly, we have $(1 - e_e)\mathbb{S} \simeq e_{\mathcal{D}}\mathbb{S}$.

We can then Bousfield localise the model category of rational $O(2)$ -spectra at these objects to obtain $L_{e_e\mathbb{S}}O(2)\mathrm{Sp}_\mathbb{Q}^\circledast$ and $L_{e_{\mathcal{D}}\mathbb{S}}O(2)\mathrm{Sp}_\mathbb{Q}^\circledast$ using [MM02, Section IV.6]. The weak equivalences of $L_{e_e\mathbb{S}}O(2)\mathrm{Sp}_\mathbb{Q}^\circledast$ are those maps f such that $e_e\mathbb{S} \wedge f$ is a rational equivalence, and similarly so for $L_{e_{\mathcal{D}}\mathbb{S}}O(2)\mathrm{Sp}_\mathbb{Q}^\circledast$. These are cofibrantly generated, proper, simplicial stable model categories. The result below is [Bar09b, Corollary 6.3].

Theorem 2.5. *The adjoint pair of the diagonal functor Δ and the product functor Π induces a symmetric monoidal Quillen equivalence:*

$$\Delta: O(2)\mathrm{Sp}_\mathbb{Q}^\circledast \rightleftarrows L_{e_e\mathbb{S}}O(2)\mathrm{Sp}_\mathbb{Q}^\circledast \times L_{e_{\mathcal{D}}\mathbb{S}}O(2)\mathrm{Sp}_\mathbb{Q}^\circledast: \Pi.$$

We can identify these localised homotopy categories more clearly. Note that for any X and Y in $O(2)\mathrm{Sp}_{\mathbb{Q}}^{\circ}$, the abelian group $[X, Y]_{\mathbb{Q}}^{O(2)}$ is a module over the ring $[\mathbb{S}, \mathbb{S}]_{\mathbb{Q}}^{O(2)}$ via the smash product. Hence we have the following isomorphisms of sets of maps in the homotopy category:

$$\begin{aligned} [X, Y]_{\mathbb{Q}}^{O(2)} &\cong e_{\mathcal{C}}[X, Y]_{\mathbb{Q}}^{O(2)} \times e_{\mathcal{D}}[X, Y]_{\mathbb{Q}}^{O(2)}, \\ e_{\mathcal{C}}[X, Y]_{\mathbb{Q}}^{O(2)} &\cong \mathrm{Ho} L_{e_{\mathcal{C}}\mathbb{S}}O(2)\mathrm{Sp}_{\mathbb{Q}}^{\circ}(X, Y), \\ e_{\mathcal{D}}[X, Y]_{\mathbb{Q}}^{O(2)} &\cong \mathrm{Ho} L_{e_{\mathcal{D}}\mathbb{S}}O(2)\mathrm{Sp}_{\mathbb{Q}}^{\circ}(X, Y). \end{aligned}$$

We can improve our description of $L_{e_{\mathcal{C}}\mathbb{S}}O(2)\mathrm{Sp}_{\mathbb{Q}}^{\circ}$ by describing $e_{\mathcal{C}}\mathbb{S}$ in terms of a suspension spectrum. Let EC denote the universal $O(2)$ -space corresponding to the family \mathcal{C} , so EC^H is non-equivariantly contractible for each $H \in \mathcal{C}$ and is the empty set for $H \notin \mathcal{C}$. This is an $O(2)$ -CW-complex and is built from cells of the form $O(2)/K_+$ for $K \in \mathcal{C}$. Define \tilde{EC} via the cofibre sequence of $O(2)$ -spaces,

$$EC_+ \rightarrow S^0 \rightarrow \tilde{EC}.$$

By considering geometric fixed points, it is easy to check that the composite map $EC_+ \rightarrow S \rightarrow e_{\mathcal{C}}\mathbb{S}$ of $O(2)$ -spectra is a weak equivalence. It follows that the weak equivalences of $L_{e_{\mathcal{C}}\mathbb{S}}O(2)\mathrm{Sp}_{\mathbb{Q}}^{\circ}$ are those maps f such that $EC_+ \wedge f$ is a rational equivalence. By [MM02, Proposition IV.6.7] it follows that the weak equivalences are those maps f such that i^*f is a weak equivalence of rational \mathbb{T} -spectra.

We will find it convenient to use a slightly different model structure on $O(2)\mathrm{Sp}_{\mathbb{Q}}^{\circ}$ to model $\mathrm{Ho} L_{e_{\mathcal{C}}\mathbb{S}}O(2)\mathrm{Sp}_{\mathbb{Q}}^{\circ}$. We take the following construction from [MM02, Theorem IV.6.5]. The (non-rational) stable model structure on $O(2)$ -spectra has sets of generating cofibrations and acyclic cofibrations obtained by applying the shifted suspension functors F_V to spaces of the form $O(2)/H_+ \wedge A$ for $H \leq O(2)$, V a representation of $O(2)$ and A either a sphere or a disc. If we restrict ourselves to only those with $H \leq \mathbb{T}$, we obtain a new model structure on $O(2)$ -spectra that is stable and cellular. In particular, a map f is a weak equivalence or fibration in this new model structure if and only if i^*f is a weak equivalence or fibration of \mathbb{T} -spectra. We rationalise this model structure as above and denote it $\mathcal{CO}(2)\mathrm{Sp}_{\mathbb{Q}}^{\circ}$. From the descriptions of the weak equivalences it follows immediately that the identity functor from $\mathcal{CO}(2)\mathrm{Sp}_{\mathbb{Q}}^{\circ}$ to $L_{e_{\mathcal{C}}\mathbb{S}}O(2)\mathrm{Sp}_{\mathbb{Q}}^{\circ}$ is the left adjoint of a Quillen equivalence.

Definition 2.6. We call $\mathcal{CO}(2)\mathrm{Sp}_{\mathbb{Q}}^{\circ}$ the model category of **toral $O(2)$ -spectra**. We call $L_{e_{\mathcal{D}}\mathbb{S}}O(2)\mathrm{Sp}_{\mathbb{Q}}^{\circ}$ the model category of **dihedral $O(2)$ -spectra**.

We rephrase the splitting result.

Corollary 2.7. *The model category of rational $O(2)$ -spectra is Quillen equivalent to*

$$\mathcal{CO}(2)\mathrm{Sp}_{\mathbb{Q}}^{\circ} \times L_{e_{\mathcal{D}}\mathbb{S}}O(2)\mathrm{Sp}_{\mathbb{Q}}^{\circ}.$$

In Section 4.2 we will make much use of the \mathbb{T} -fixed points functor, so we introduce that functor and discuss how it acts on the model category $\mathcal{CO}(2)\mathrm{Sp}_{\mathbb{Q}}^{\circ}$. The functor $(-)^{\mathbb{T}}$ of [MM02, Section V.3] first restricts an $O(2)$ -spectrum indexed on a complete $O(2)$ -universe \mathcal{U} to \mathbb{R}^{∞} , then applies the space-level fixed point functor levelwise.

Lemma 2.8. *The (categorical) \mathbb{T} -fixed points functor induces a Quillen pair*

$$\varepsilon^* : \mathrm{Sp}_{\mathbb{Q}}^{\mathcal{O}}[W] \rightleftarrows \mathcal{CO}(2) \mathrm{Sp}_{\mathbb{Q}}^{\mathcal{O}} : (-)^{\mathbb{T}}.$$

Proof. We first consider the adjunction before rationalising. On the left a map is a fibration if and only if it forgets to a fibration of non-equivariant spectra. A map on the right is a fibration if and only if it forgets to a fibration of \mathbb{T} -spectra. The functor $(-)^{\mathbb{T}}$ is a right Quillen functor from \mathbb{T} -spectra to spectra by [MM02, Proposition 3.4] hence we have a Quillen pair before localisation.

The adjunction extends to the rationalised categories as the rational sphere spectrum $S^0\mathbb{Q}$ in each category is given by applying the appropriate inflation functor from non-equivariant spectra. \square

The forgetful and fixed points functors interact well, as the commutative diagram below shows. Analogues of this diagram will appear throughout Section 4.

Proposition 2.9. *There is a diagram of Quillen functors as below, in which both the square of fixed point and forgetful functors commute, as does the square of inflation and forgetful functors.*

$$\begin{array}{ccc} \mathrm{Sp}_{\mathbb{Q}}^{\mathcal{O}}[W] & \begin{array}{c} \xrightarrow{\varepsilon^*} \\ \xleftarrow{(-)^{\mathbb{T}}} \end{array} & \mathcal{CO}(2) \mathrm{Sp}_{\mathbb{Q}}^{\mathcal{O}} \\ \downarrow i^* & & \downarrow i^* \\ \mathrm{Sp}_{\mathbb{Q}}^{\mathcal{O}} & \begin{array}{c} \xrightarrow{\varepsilon^*} \\ \xleftarrow{(-)^{\mathbb{T}}} \end{array} & \mathbb{T} \mathrm{Sp}_{\mathbb{Q}}^{\mathcal{O}} \end{array}$$

Then both functors i^ preserve weak equivalences, cofibrations and fibrations. Furthermore if i^*f is a weak equivalence (or fibration), then f is a weak equivalence (or fibration).*

Proof. The two commutativity statements follow directly from the definitions. We have already discussed weak equivalences and fibrations for both functors i^* in the non-rationalised case. These extend to the rational versions as $S^0\mathbb{Q}$ can be constructed in $\mathrm{Sp}^{\mathcal{O}}$ and then inflated to an equivariant spectrum in any of the other three categories.

For cofibrations, consider a generating cofibration of $\mathcal{CO}(2) \mathrm{Sp}_{\mathbb{Q}}^{\mathcal{O}}$,

$$F_V (O(2)/H_+ \wedge S_+^{n-1}) \longrightarrow F_V (O(2)/H_+ \wedge D_+^n),$$

for H a subgroup of \mathbb{T} and V a representation of $O(2)$. Applying i^* to this gives

$$F_{i^*V} ((\mathbb{T}/H_+ \vee j^*\mathbb{T}/H_+) \wedge S_+^{n-1}) \longrightarrow F_{i^*V} ((\mathbb{T}/H_+ \vee j^*\mathbb{T}/H_+) \wedge D_+^n),$$

where $j^*\mathbb{T}/H_+$ denotes the space \mathbb{T}/H but with the inverse action of \mathbb{T} (so $t \in \mathbb{T}$ acts by t^{-1}). Since this map is a cofibration of \mathbb{T} -spectra, the statement follows. A similar argument holds in the case of $\mathrm{Sp}_{\mathbb{Q}}^{\mathcal{O}}[W]$. \square

3. The toral model

In this section we define $d\mathcal{A}(\mathcal{C})$, the algebraic model for toral spectra and explain how it relates to $d\mathcal{A}(\mathbb{T})$, the algebraic model for rational \mathbb{T} -spectra.

3.1. The model $\mathcal{A}(\mathbb{T})$

The algebraic category for the homotopy category of rational \mathbb{T} -spectra is established in [Gre99]. We adapt that category to the toral case and explain how to relate it to the \mathbb{T} -case. Our starting point is the category of chain complexes with an action of W .

Definition 3.1. The category $\text{Ch}(\mathbb{Q}[W])$ is the category of rational chain complexes that have an action of the group of order two. This is a monoidal category with tensor product given by tensoring over \mathbb{Q} and using the diagonal W -action. The unit of this product is \mathbb{Q} in degree zero with trivial W -action.

There is a proper cofibrantly generated model structure on this category by [Hov99, Proposition 4.2.13]. The fibrations are the surjections and the weak equivalences are the homology isomorphisms. The cofibrations are dimensionwise split injections with cofibrant cokernel. Let S^{n-1} be the chain complex with \mathbb{Q} in degree $n-1 \in \mathbb{Z}$ and zeroes elsewhere and the D^n be the chain complex with \mathbb{Q} in degrees n and $n-1$ (with the identity as the differential between these degrees) and zeroes elsewhere. The generating cofibrations are given by the inclusion maps $S^{n-1} \otimes \mathbb{Q}[W] \rightarrow D^n \otimes \mathbb{Q}[W]$ and the acyclic cofibrations are given by $0 \rightarrow D^n \otimes \mathbb{Q}[W]$ for $n \in \mathbb{Z}$.

Since \mathbb{Q} is a retract of $\mathbb{Q}[W]$ we see that $S^{n-1} \otimes \mathbb{Q} \rightarrow D^n \otimes \mathbb{Q}$ is also a cofibration of $\text{Ch}(\mathbb{Q}[W])$. Hence the cofibrant objects do not have to be W -free.

The forgetful functor from $\text{Ch}(\mathbb{Q}[W])$ to $\text{Ch}(\mathbb{Q})$ is the right adjoint of a strong monoidal Quillen pair. The left adjoint sends a chain complex X to $X \oplus X$ with W acting as the exchange of factors map.

It is routine to check that this category is a symmetric monoidal model category that satisfies the monoid axiom. We construct a commutative monoid in $\text{Ch}(\mathbb{Q}[W])$.

Definition 3.2. Let $\mathcal{O}_{\mathcal{F}}$ be the graded ring of operations $\prod_{n \geq 1} \mathbb{Q}[c_n]$ with c_n of degree -2 . This ring has trivial differential.

The group W acts on this graded ring (via ring homomorphisms), it is defined by $wc_n = -c_n$. We thus have a map of graded rings (without W -action)

$$w: \mathcal{O}_{\mathcal{F}} \rightarrow \mathcal{O}_{\mathcal{F}}$$

and a change of rings functor w^* from $\mathcal{O}_{\mathcal{F}}$ -mod to itself. For a module N , w^*N is the same underlying set, but now each c_n acts as $-c_n$.

We use the notation

$$\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} = \text{colim}_{n \geq 1} \mathcal{O}_{\mathcal{F}}[c_1^{-1}, \dots, c_n^{-1}].$$

It is easy to see that $\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}}$ is also a graded ring with W -action. As a vector space, $(\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}})_{2k}$ is $\prod_{a \geq 1} \mathbb{Q}$ for $k \leq 0$ and is $\oplus_{a \geq 1} \mathbb{Q}$ for $k > 0$. For any $\mathcal{O}_{\mathcal{F}}$ -module N , we define $\mathcal{E}^{-1}N$ to be $\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes_{\mathcal{O}_{\mathcal{F}}} N$. The tensor product has the diagonal action of W .

Definition 3.3. An object $A = (\beta: N \rightarrow \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes U)$ of $d\mathcal{A}(\mathbb{C})$ consists of the following data:

- an $\mathcal{O}_{\mathcal{F}}$ -module N in the category $\text{Ch}(\mathbb{Q}[W])$,
- an object U of $\text{Ch}(\mathbb{Q}[W])$,
- a map β of $\mathcal{O}_{\mathcal{F}}$ -modules in the category $\text{Ch}(\mathbb{Q}[W])$,
- with the requirement that $\mathcal{E}^{-1}\beta$ is an isomorphism.

Let $B = (\beta': N' \rightarrow \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes U')$ be another object of $d\mathcal{A}(\mathcal{C})$. A map $(\theta, \phi): A \rightarrow B$ in this category consists of the following data:

- a map $\theta: N \rightarrow N'$ of $\mathcal{O}_{\mathcal{F}}$ -modules in the category $\text{Ch}(\mathbb{Q}[W])$,
- a map $\phi: U \rightarrow U'$ in the category $\text{Ch}(\mathbb{Q}[W])$,
- with the requirement that the obvious square involving the structure maps commutes.

We call $d\mathcal{A}(\mathcal{C})$ the **algebraic model for toral spectra**. The subcategory of objects with zero differentials in all places is called $\mathcal{A}(\mathcal{C})$, the **abelian model for toral spectra**.

We let $S^0 = (\mathcal{O}_{\mathcal{F}} \rightarrow \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes \mathbb{Q})$ where \mathbb{Q} has trivial W -action. This is the unit of a monoidal product on $d\mathcal{A}(\mathcal{C})$ (although we will make no direct use of that in this paper).

The abelian and algebraic models for rational \mathbb{T} -equivariant spectra have similar descriptions.

Definition 3.4. The category $d\mathcal{A}(\mathbb{T})$ is defined as in Definition 3.3 but using $\text{Ch}(\mathbb{Q})$ instead of $\text{Ch}(\mathbb{Q}[W])$. We call this the **algebraic model for \mathbb{T} -spectra**.

The full subcategory of $d\mathcal{A}(\mathbb{T})$ consisting of objects with zero differentials is called $\mathcal{A}(\mathbb{T})$, the **abelian model for \mathbb{T} -spectra**.

There is an obvious forgetful functor relating $d\mathcal{A}(\mathbb{T})$ and $d\mathcal{A}(\mathcal{C})$. The results of Section 4 will show that this forgetful functor is the algebraic version of the forgetful functor from $\mathcal{C}O(2)\text{Sp}_{\mathbb{Q}}^0$ to $\mathbb{T}\text{Sp}_{\mathbb{Q}}^0$.

Lemma 3.5. *There is an adjoint pair relating $d\mathcal{A}(\mathbb{T})$ and $d\mathcal{A}(\mathcal{C})$. The left adjoint \mathbb{D} takes*

$$\beta: N \rightarrow \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes U,$$

in \mathcal{A} to the following composite:

$$N \oplus w^*N \xrightarrow{\beta \oplus w^*\beta} (\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes U) \oplus (w^*\mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes U) \xrightarrow{\text{Id} \oplus w} \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes U \oplus \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes U.$$

The W -action then simply swaps the two summands. The right adjoint i^ is the forgetful functor from $d\mathcal{A}(\mathcal{C})$ to $d\mathcal{A}(\mathbb{T})$.*

Remark 3.6. In [Gre98b, Corollary 3.2] the algebraic model for the toral part is described as $\mathcal{A}(\mathbb{T})$ with a W -action. This is not sufficiently precise, the more correct statement is that the algebraic model for the toral part is $\mathcal{A}(\mathcal{C})$ as in Definition 3.3. See Lemma 4.19 for the calculation which shows how the given action of W on $\mathcal{O}_{\mathcal{F}}$ is obtained from topological data.

The problem with the proof of the corollary is as follows, let Z be a \mathbb{T} -spectrum (or a \mathbb{T} -space) with a \mathbb{T} -equivariant map of order two $z: Z \rightarrow Z$. Choose a reflection $\hat{w} \in O(2)$ and let $R_{\hat{w}}: O(2)_+ \rightarrow O(2)_+$ be right multiplication by \hat{w} . Then the map

$$R_{\hat{w}} \wedge z: O(2)_+ \wedge_{\mathbb{T}} Z \longrightarrow O(2)_+ \wedge_{\mathbb{T}} Z$$

is not well defined due to the equalisation of the \mathbb{T} -actions. This is easily seen at the space level, where $R_{\hat{w}} \wedge z([\sigma, x]) = [\sigma\hat{w}, z(x)]$.

Instead, let $j: \mathbb{T} \rightarrow \mathbb{T}$ be the inversion map and j^* be the change of groups functor. If z is a map from Z to j^*Z , such that $j^*z \circ z$ is the identity, then the map $R_{\hat{w}} \wedge z$ is well defined. Obviously, if Y is an $O(2)$ -spectrum then i^*Y has such a map, given by the action of \hat{w} . With this interpretation of W -actions, [Gre98b, Proposition 3.1 and Corollary 3.2] are correct. This idea of skewed actions on a category is considered in more detail in [Bar08b, Chapters 7 and 8].

3.2. Model structures on $d\mathcal{A}(\mathbb{C})$

We use the dualisable model structure on $d\mathcal{A}(\mathbb{T})$, see [Bar16, Theorem 6.6]. This is a proper monoidal model category that satisfies the monoid axiom and whose weak equivalences are the homology isomorphisms. We can lift this model structure to the toral case using the lifting lemma [Hir03, Theorem 11.3.2] and the adjoint pair (\mathbb{D}, i^*) .

Theorem 3.7. *There is a model structure on $d\mathcal{A}(\mathbb{C})$ where the weak equivalences are the homology isomorphisms and the fibrations are those maps which forget to fibrations in the dualisable model structure on $d\mathcal{A}(\mathbb{T})$. This model structure is proper, cofibrantly generated, monoidal and satisfies the monoid axiom. The generating cofibrations and acyclic cofibrations are given by applying \mathbb{D} to the generating sets for the dualisable model structure on $d\mathcal{A}(\mathbb{T})$.*

Note that the ring map $w: \mathcal{O}_{\mathcal{F}} \rightarrow \mathcal{O}_{\mathcal{F}}$ induces a map $w: S^0 \rightarrow w^*S^0$. It follows that S^0 (with W -action) is a retract of $\mathbb{D}S^0$ and hence is cofibrant in $d\mathcal{A}(\mathbb{C})$.

In [Gre99], Greenlees constructs a functor $\pi_*^{\mathcal{A}}$ from the homotopy category of rational \mathbb{T} -spectra to $\mathcal{A}(\mathbb{T})$. For a rational \mathbb{T} -spectrum X , let $\pi_*^{\mathcal{A}}(X)$ be the following object of \mathcal{A} . For details of the spectra $DE\mathcal{F}_+$ and $\tilde{E}\mathcal{F}$ see Definition 4.3. The spectrum $\Phi^{\mathbb{T}}X$ is the geometric \mathbb{T} -fixed points of X :

$$\pi_*^{\mathcal{A}}(X) = (\pi_*^{\mathbb{T}}(X \wedge DE\mathcal{F}_+) \longrightarrow \pi_*^{\mathbb{T}}(X \wedge DE\mathcal{F}_+ \wedge \tilde{E}\mathcal{F}) \cong \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes \pi_*(\Phi^{\mathbb{T}}X)).$$

Since $DE\mathcal{F}_+$ and $\tilde{E}\mathcal{F}$ can be constructed as $O(2)$ -spectra, we can extend $\pi_*^{\mathcal{A}}$ to the toral case by keeping track of the $W = O(2)/\mathbb{T}$ action on $\pi_*^{\mathbb{T}}(X)$ for X an $O(2)$ -spectrum. This functor fits into an Adams short exact sequence for the toral part, the proof follows the same pattern as [Gre99, Theorem 5.6.6].

Theorem 3.8. *For X and Y rational $O(2)$ -spectra, there is an Adams short exact sequence as below, where $[-, -]_*^{\mathbb{C}}$ denotes maps in the homotopy category of toral spectra.*

$$0 \rightarrow \text{Ext}_{\mathcal{A}(\mathbb{C})}(\pi_*^{\mathcal{A}}(\Sigma X), \pi_*^{\mathcal{A}}(Y)) \rightarrow [X, Y]_*^{\mathbb{C}} \rightarrow \text{Hom}_{\mathcal{A}(\mathbb{C})}(\pi_*^{\mathcal{A}}(X), \pi_*^{\mathcal{A}}(Y)) \rightarrow 0$$

4. Toral spectra

In this section we show that the model category of toral $O(2)$ -spectra, $\mathcal{CO}(2) \text{Sp}_{\mathbb{Q}}^0$, is Quillen equivalent to the category $d\mathcal{A}(\mathbb{C})$. The method is an extension of [BGKS15].

The first step is to separate that part of toral $O(2)$ -equivariant homotopy theory that is determined by the finite cyclic subgroups from that determined by \mathbb{T} . Proposition 4.8 gives this separation, see [BGKS15, Proposition 3.2.5] for the \mathbb{T} -equivariant analogue and further explanation of the underlying idea.

The next step is to take \mathbb{T} -fixed points, so that we are now working with non-equivariant spectra. This removal of equivariance is achieved in Corollary 4.15, which is a generalisation of [BGKS15, Corollary 3.3.6]. The major difference is that the \mathbb{T} -fixed points of an $O(2)$ -spectrum define a spectrum with a W -action, see Lemma 2.8.

The third step is to replace categories based on rational spectra (with an action of W) with categories based on chain complexes (with an action of W) using [Shi07], see Theorem 4.17. The remaining steps are analogues of [BGKS15, Section 4], where we complete our series of Quillen equivalences with the model $d\mathcal{A}(\mathbb{C})$ by removing the localisations and cellularisations in our constructions. See Propositions 4.20 and Theorem 4.21.

4.1. Isotropy separation

We briefly recap the notion of a diagram of model categories and the category of generalised diagrams. We let \mathcal{P} denote the pullback category $\bullet \rightarrow \bullet \leftarrow \bullet$.

Definition 4.1. A \mathcal{P} -diagram of model categories R^\bullet is a pair of Quillen pairs

$$\begin{array}{ccc} L: \mathcal{A} & \rightleftarrows & \mathcal{B}: R, \\ F: \mathcal{C} & \rightleftarrows & \mathcal{B}: G, \end{array}$$

with L and F the left adjoints. We will usually draw this as the diagram

$$\mathcal{A} \begin{array}{c} \xrightarrow{L} \\ \xleftarrow{R} \end{array} \mathcal{B} \begin{array}{c} \xleftarrow{F} \\ \xrightarrow{G} \end{array} \mathcal{C}.$$

We can then define the category of generalised diagrams in R^\bullet . This is sometimes also called the category of sections.

Definition 4.2. Given a \mathcal{P} -diagram of model categories R^\bullet as above, we can define a new category, $R^\bullet\text{-mod}$. The objects are pairs of morphisms in \mathcal{B} : $\alpha: La \rightarrow b$ and $\gamma: Fc \rightarrow b$. We write such an object as $(a, \alpha, b, \gamma, c)$. A morphism in $R^\bullet\text{-mod}$ from $(a, \alpha, b, \gamma, c)$ to $(a', \alpha', b', \gamma', c')$ is a triple of maps $x: a \rightarrow a'$ in \mathcal{A} , $y: b \rightarrow b'$ in \mathcal{B} , $z: c \rightarrow c'$ in \mathcal{C} such that we have a commuting diagram in \mathcal{B} :

$$\begin{array}{ccccc} La & \xrightarrow{\alpha} & b & \xleftarrow{\gamma} & Fc \\ \downarrow Lx & & \downarrow y & & \downarrow Fz \\ La' & \xrightarrow{\alpha'} & b' & \xleftarrow{\gamma'} & Fc' \end{array}$$

If each category in the diagram R^\bullet is proper and cellular, then the category $R^\bullet\text{-mod}$ has a proper and cellular model structure with weak equivalences and cofibrations defined objectwise by [GS14b, Proposition 3.3].

We can separate the homotopical information of toral $O(2)$ -spectra into three parts. The first part takes care of the homotopical information coming from the finite cyclic subgroups. The second part deals with the homotopical information coming from \mathbb{T} . The third part is a comparison term. We have already removed the behaviour of the dihedral groups in Theorem 2.5. The first step is to carefully construct a commutative ring spectrum with some special properties.

Definition 4.3. Let \mathcal{F} be the collection of finite cyclic subgroups of $O(2)$. There is a universal space for this family called $E\mathcal{F}$ where $E\mathcal{F}^H$ is non-equivariantly contractible for each finite cyclic subgroup H and $E\mathcal{F}^K = \emptyset$ for all other subgroups K . This is an $O(2)$ -CW-complex and is built from cells of the form $O(2)/K_+$ for $K \in \mathcal{F}$. We define $\tilde{E}\mathcal{F}$ via the cofibre sequence of $O(2)$ -spaces,

$$E\mathcal{F}_+ \rightarrow S^0 \rightarrow \tilde{E}\mathcal{F}.$$

We define $DE\mathcal{F}_+$ to be $F(E\mathcal{F}_+, N^\#S)$, where $N^\#$ is the lax monoidal right adjoint described in [EKMM97, Theorem IV.3.9] from $O(2)$ -equivariant EKMM S -modules to $O(2)\mathrm{Sp}^\circ$.

Lemma 4.4. *The spectrum $DE\mathcal{F}_+$ is a commutative ring spectrum that is fibrant in the category of toral $O(2)$ -equivariant orthogonal spectra. This forgets to the commutative ring \mathbb{T} -spectrum $DE\mathcal{F}_+$ constructed in [BGKS15, Definition 3.2.2].*

Proof. The ring spectrum constructed in [BGKS15] is made using the same process as above, but starting in \mathbb{T} -equivariant S -modules. Hence the statement about $i^*DE\mathcal{F}_+$ follows immediately. The spectrum $DE\mathcal{F}_+$ is fibrant in $O(2)\mathrm{Sp}^\circ$, since $N^\#S$ is fibrant. Hence $DE\mathcal{F}_+$ is fibrant in $\mathcal{C}O(2)\mathrm{Sp}^\circ$ (recall the identity functor $\mathcal{C}O(2)\mathrm{Sp}^\circ \rightarrow O(2)\mathrm{Sp}^\circ$ is a left Quillen functor). \square

Note that we do not require $DE\mathcal{F}_+$ to be fibrant in $\mathcal{C}O(2)\mathrm{Sp}_\mathbb{Q}^\circ$: we do not need it to be rational, only that its \mathbb{T} -fixed points are weakly equivalent to its derived \mathbb{T} -fixed points. That is, we need

$$\pi_*(DE\mathcal{F}_+^\mathbb{T}) \cong \pi_*^\mathbb{T}(DE\mathcal{F}_+),$$

which holds as $DE\mathcal{F}_+$ is fibrant in $O(2)\mathrm{Sp}^\circ$.

From this ring spectrum we can make three model categories. In the following, whenever we have a ring object A in a model category \mathcal{M} , we will equip A -mod with the lifted model structure, where fibrations and weak equivalences are defined by forgetting to \mathcal{M} .

- $DE\mathcal{F}_+$ -mod, the category of $DE\mathcal{F}_+$ -modules in $\mathcal{C}O(2)\mathrm{Sp}_\mathbb{Q}^\circ$. This model category captures the information coming from the finite cyclic groups.
- $L_{\tilde{E}\mathcal{F}}\mathcal{C}O(2)\mathrm{Sp}_\mathbb{Q}^\circ$, the model category $\mathcal{C}O(2)\mathrm{Sp}_\mathbb{Q}^\circ$, localised at the homology theory $\tilde{E}\mathcal{F}$. This model category captures the information coming from \mathbb{T} .
- $L_{\tilde{E}\mathcal{F} \wedge DE\mathcal{F}_+} DE\mathcal{F}_+$ -mod the category of $DE\mathcal{F}_+$ -modules in $\mathcal{C}O(2)\mathrm{Sp}_\mathbb{Q}^\circ$, localised at the homology theory $\tilde{E}\mathcal{F} \wedge DE\mathcal{F}_+$. This model category captures the interaction of the first two.

Now we can give our diagram of model categories that separates the behaviour of the finite cyclic groups from the rest.

Definition 4.5. We define S^\bullet to be the following diagram of model categories:

$$DE\mathcal{F}_+ \text{-mod} \begin{array}{c} \xrightarrow{\mathrm{Id}} \\ \xleftarrow{\mathrm{Id}} \end{array} L_{\tilde{E}\mathcal{F} \wedge DE\mathcal{F}_+} DE\mathcal{F}_+ \text{-mod} \begin{array}{c} \xleftarrow{DE\mathcal{F}_+ \wedge -} \\ \xrightarrow{U} \end{array} L_{\tilde{E}\mathcal{F}} \mathcal{C}O(2)\mathrm{Sp}_\mathbb{Q}^\circ.$$

We thus have a cellular model category S^\bullet -mod, that is both proper and stable.

Given any $O(2)$ -spectrum X , we have an S^\bullet -module

$$S^\bullet \wedge X := (DE\mathcal{F}_+ \wedge X, \text{Id}, DE\mathcal{F}_+ \wedge X, \text{Id}, X).$$

The functor $S^\bullet \wedge -$ has a right adjoint called pb , which is constructed just after [BGKS15, Definition 3.2.3]. This right adjoint sends an object $(A, \alpha, B, \gamma, C)$ to the pullback of the diagram

$$A \xrightarrow{\alpha} B \xleftarrow{\gamma} C \wedge DE\mathcal{F}_+ \xleftarrow{C \wedge \text{unit}} C.$$

Similarly to [BGKS15, Proposition 3.2.4], we have a Quillen pair

$$S^\bullet \wedge - : \mathcal{CO}(2)\text{Sp}_{\mathbb{Q}}^{\circ} \rightleftarrows S^\bullet\text{-mod} : \text{pb}.$$

Now we want to relate this to the \mathbb{T} -equivariant separation. The diagram S^\bullet is a diagram of model categories of $O(2)$ -spectra (with extra structure), so at each vertex we can make an analogous category built from \mathbb{T} -equivariant spectra.

Definition 4.6. Let $i^*DE\mathcal{F}_+$ -mod denote modules over $i^*DE\mathcal{F}_+$ in $\mathbb{T}\text{Sp}_{\mathbb{Q}}^{\circ}$. There is a diagram of model categories i^*S^\bullet , made from the three model categories:

$$i^*DE\mathcal{F}_+\text{-mod}, \quad L_{i^*\tilde{E}\mathcal{F} \wedge i^*DE\mathcal{F}_+} i^*DE\mathcal{F}_+\text{-mod} \quad \text{and} \quad L_{i^*\tilde{E}\mathcal{F}} \mathbb{T}\text{Sp}_{\mathbb{Q}}^{\circ}.$$

The above diagram of model categories is precisely the diagram of [BGKS15, Definition 3.2.3] and we have a square of Quillen functors

$$\begin{array}{ccc} \mathcal{CO}(2)\text{Sp}_{\mathbb{Q}}^{\circ} & \xrightleftharpoons[\text{pb}]{S^\bullet \wedge -} & S^\bullet\text{-mod} \\ \downarrow i^* & & \downarrow i^* \\ \mathbb{T}\text{Sp}_{\mathbb{Q}}^{\circ} & \xrightleftharpoons[\text{pb}]{i^*S^\bullet \wedge -} & i^*S^\bullet\text{-mod} \end{array}$$

Lemma 4.7. *The forgetful functors i^* commute with both the horizontal left adjoints and the horizontal right adjoints. They also preserve fibrations and cofibrations. A map (x, y, z) in $S^\bullet\text{-mod}$ is a weak equivalence if and only if $i^*(x, y, z) = (i^*x, i^*y, i^*z)$ is a weak equivalence in $i^*S^\bullet\text{-mod}$.*

Proof. That i^* commutes with the horizontal functors is immediate from the definitions. The left hand i^* preserves fibrations and cofibrations by Proposition 2.9.

The right hand i^* preserves and detects weak equivalences in $S^\bullet\text{-mod}$ as weak equivalences are detected objectwise and $i^* : \mathcal{CO}(2)\text{Sp}_{\mathbb{Q}}^{\circ} \rightarrow \mathbb{T}\text{Sp}_{\mathbb{Q}}^{\circ}$ preserves and detects weak equivalences. Cofibrations are also defined objectwise, so i^* preserves cofibrations as it does so on each component model category of S^\bullet .

On each component model category of the right hand side, i^* preserves fibrations. Since the fibrations in $S^\bullet\text{-mod}$ are defined in terms of certain pullbacks (which are preserved by i^*) it follows that the right hand i^* also preserves fibrations. \square

To turn the horizontal adjunctions of this square into Quillen equivalences, we apply the Cellularization Principle of [GS13, Proposition 2.7]. This result gives conditions under which a Quillen adjunction becomes a Quillen equivalence after cellularising (right Bousfield localising) both sides of the adjunction.

Let e_{C_n} denote the idempotent in the rationalised Burnside ring for C_n corresponding to C_n and let \mathbb{S}_{C_n} denote the C_n -equivariant sphere spectrum. The generators of $\mathcal{C}O(2) \operatorname{Sp}_{\mathbb{Q}}^{\circ}$ are the spectra $O(2)_+ \wedge_{C_n} e_{C_n} \mathbb{S}_{C_n}$ for $n \geq 1$ and the cofibrant replacement of the $O(2)$ -equivariant sphere in $\mathcal{C}O(2) \operatorname{Sp}_{\mathbb{Q}}^{\circ}$: $EC_+ \wedge \mathbb{S}$. Let K_{top} be the set of images of these objects under the functor $S^{\bullet} \wedge -$, and all (integer) suspensions and desuspensions thereof. The elements of this set will be called **cells** and we will cellularise (right Bousfield localise) S^{\bullet} -mod at this set.

To apply the cellularisation principle, we need to know that the cells K_{top} are homotopically compact (also known as small or compact) in the sense of [SS03, Definition 2.1.2]. The arguments of [BGKS15, Section 3.2] apply verbatim, so we leave the details to that reference.

Proposition 4.8. *There is a Quillen equivalence*

$$S^{\bullet} \wedge - : \mathcal{C}O(2) \operatorname{Sp}_{\mathbb{Q}}^{\circ} \rightleftarrows K_{\text{top}}\text{-cell-}S^{\bullet}\text{-mod} : \text{pb}.$$

Proof. This result follows from the same proof as for the \mathbb{T} -case. By [GS13, Proposition 2.7] it suffices to show that the derived unit is a weak equivalence on the generators of $O(2) \operatorname{Sp}_{\mathbb{Q}}^{\circ}$. The pullback of a fibrant replacement of $S^{\bullet} \wedge X$ is given by the homotopy pullback of the following diagram of $O(2)$ -spectra:

$$DEF_+ \wedge X \wedge S^0 \mathbb{Q} \longrightarrow DEF_+ \wedge \tilde{E}\mathcal{F} \wedge X \wedge S^0 \mathbb{Q} \longleftarrow \tilde{E}\mathcal{F} \wedge X \wedge S^0 \mathbb{Q}.$$

Since we are in a stable model category, the above is weakly equivalent to $X \wedge S^0 \mathbb{Q}$ smashed with the homotopy pullback of $DEF_+ \longrightarrow DEF_+ \wedge \tilde{E}\mathcal{F} \longleftarrow \tilde{E}\mathcal{F}$ which is \mathbb{S} . The derived unit map is induced by the unit map $X \rightarrow DEF_+ \wedge X$ and $S^0 \rightarrow \tilde{E}\mathcal{F}$ and hence is a rational equivalence. \square

We now wish to compare with the $\mathbb{T} \operatorname{Sp}_{\mathbb{Q}}^{\circ}$ version. As above, the generators of $\mathbb{T} \operatorname{Sp}_{\mathbb{Q}}^{\circ}$ are the \mathbb{T} -equivariant sphere spectrum \mathbb{S} and the spectra $\mathbb{T}_+ \wedge_{C_n} e_{C_n} \mathbb{S}_{C_n}$ for $n \geq 1$. Let $i^* K_{\text{top}}$ be the set of images of these objects under the functor $i^* S^{\bullet} \wedge -$, and all (integer) suspensions and desuspensions thereof.

Lemma 4.9. *The functors i^* below are right Quillen functors that commute with both the horizontal left adjoints and the horizontal right adjoints. Furthermore the functors i^* preserve and detect all weak equivalences.*

$$\begin{array}{ccc} \mathcal{C}O(2) \operatorname{Sp}_{\mathbb{Q}}^{\circ} & \begin{array}{c} \xrightarrow{S^{\bullet} \wedge -} \\ \xleftarrow{\text{pb}} \end{array} & K_{\text{top}}\text{-cell-}S^{\bullet}\text{-mod} \\ \downarrow i^* & & \downarrow i^* \\ \mathbb{T} \operatorname{Sp}_{\mathbb{Q}}^{\circ} & \begin{array}{c} \xrightarrow{i^* S^{\bullet} \wedge -} \\ \xleftarrow{\text{pb}} \end{array} & i^* K_{\text{top}}\text{-cell-}i^* S^{\bullet}\text{-mod} \end{array}$$

Proof. The commutativity follows immediately from the definitions. Let $\mathbb{R} \text{pb}$ denote the right derived functor of pb. Then $f: X \rightarrow Y$ in $K_{\text{top}}\text{-cell-}S^{\bullet}\text{-mod}$ is a weak equivalence if and only if $\mathbb{R} \text{pb } f$ is a weak equivalence of $\mathcal{C}O(2) \operatorname{Sp}_{\mathbb{Q}}^{\circ}$. This holds if and only if $i^* \mathbb{R} \text{pb } f = \mathbb{R} \text{pb } i^* f$ is a weak equivalence in $\mathbb{T} \operatorname{Sp}_{\mathbb{Q}}^{\circ}$ by Proposition 2.9. Finally, that holds if and only if $i^* f$ is a weak equivalence in $i^* K_{\text{top}}\text{-cell-}i^* S^{\bullet}\text{-mod}$, as the lower adjunction is a Quillen equivalence. Thus the right hand i^* is a right

Quillen functor after cellularisation as the fibrations are unchanged and it preserves weak equivalences. \square

Thus we have separated the homotopical information of $S_{\mathbb{C}}$ -mod into a diagram of three model categories. The advantage of doing so is that we may now remove the equivariance from the model category whilst keeping the correct homotopy category.

4.2. Removing equivariance

We start with the general pattern used in this section. Let A be a (commutative) ring spectrum in $O(2)$ -equivariant spectra. Then $A^{\mathbb{T}}$ is a (commutative) ring object in $\mathrm{Sp}^{\mathbb{O}}[W]$ and there is a map of commutative rings $a: \varepsilon^* A^{\mathbb{T}} \rightarrow A$. Using [GS14a, Section 4] we get a Quillen adjunction

$$a_{\#}: A \wedge_{\varepsilon^* A^{\mathbb{T}}} \varepsilon^*(-): A^{\mathbb{T}}\text{-mod} \rightleftarrows A\text{-mod}: (-)^{\mathbb{T}}.$$

That reference has a number of examples where this kind of adjunction is a Quillen equivalence. We want to use this type of adjunction to remove \mathbb{T} -equivariance from S^{\bullet} -mod. We do so by considering each of the three component model categories in turn. In the first we consider $A = DE\mathcal{F}_+$.

Lemma 4.10. *The adjunction below is a Quillen equivalence.*

$$a_{\#}: DE\mathcal{F}_+^{\mathbb{T}}\text{-mod} \rightleftarrows DE\mathcal{F}_+\text{-mod}: (-)^{\mathbb{T}}$$

Proof. The forgetful functors, to $\mathrm{Sp}_{\mathbb{Q}}^{\mathbb{O}}[W]$ on the left and $\mathbb{T}\mathrm{Sp}_{\mathbb{Q}}^{\mathbb{O}}$ on the right, commute with both $a_{\#}$ and $(-)^{\mathbb{T}}$, just as in Proposition 2.9. Furthermore these forgetful functors preserve fibrant objects and cofibrant objects and preserve and detect weak equivalences. We also know that the analogous adjunction at the level of \mathbb{T} -equivariant spectra is a Quillen equivalence by [BGKS15, Proposition 3.3.1].

Let $X \in DE\mathcal{F}_+^{\mathbb{T}}\text{-mod}$ be cofibrant and $Y \in DE\mathcal{F}_+\text{-mod}$ be fibrant. The map $f: a_{\#}X \rightarrow Y$ is a weak equivalence if and only if the adjoint map $\widehat{f}: X \rightarrow Y^{\mathbb{T}}$ is a weak equivalence, as this holds for i^*f and $i^*(\widehat{f}) = i^*f$. \square

The next step is to repeat the above with an additional Bousfield localisation. The model category $L_{\widetilde{E}\mathcal{F} \wedge DE\mathcal{F}_+} DE\mathcal{F}_+\text{-mod}$ can be described as the localisation of the model category $DE\mathcal{F}_+\text{-mod}$ at the set of maps Σ^*f : the set of all suspensions and desuspensions of $f: DE\mathcal{F}_+ \rightarrow DE\mathcal{F}_+ \wedge \widetilde{E}\mathcal{F}$. Let $(\Sigma^*f)^{\mathbb{T}}$: be the set of maps obtained by applying the derived right adjoint to the maps Σ^*f . By [Hir03, Theorem 3.3.20, part 1b] we obtain a Quillen equivalence.

Lemma 4.11. *The adjunction below is a Quillen equivalence.*

$$a_{\#}: L_{(\Sigma^*f)^{\mathbb{T}}} DE\mathcal{F}_+^{\mathbb{T}}\text{-mod} \rightleftarrows L_{\Sigma^*f} DE\mathcal{F}_+\text{-mod}: (-)^{\mathbb{T}}$$

The final version is to use $A = S^0$, in which case the left adjoint $a_{\#}$ is simply $\varepsilon^*(-)$.

Lemma 4.12. *The adjunction below is a symmetric monoidal Quillen equivalence.*

$$\varepsilon_*: \mathrm{Sp}_{\mathbb{Q}}^{\mathbb{O}}[W] \rightleftarrows L_{\widetilde{E}\mathcal{F}} \mathcal{CO}(2) \mathrm{Sp}_{\mathbb{Q}}^{\mathbb{O}}: (-)^{\mathbb{T}}$$

Proof. This follows by the same arguments as for [BGKS15, Proposition 3.3.3], namely that the derived right adjoint behaves as the geometric \mathbb{T} -fixed point functor, the left hand side is generated by W_+ and the right hand side is generated by $O(2)/\mathbb{T}_+$. \square

We can extend the functor $(-)^{\mathbb{T}}$ to the level of diagrams of model categories.

Definition 4.13. We define S_{top}^\bullet to be the diagram of model categories below. Here, $DEF_+^{\mathbb{T}}\text{-mod}$ means modules over $DEF_+^{\mathbb{T}}$ in the model category $\text{Sp}_{\mathbb{Q}}^{\circlearrowleft}[W]$.

$$DEF_+^{\mathbb{T}}\text{-mod} \begin{array}{c} \xrightarrow{\text{Id}} \\ \xleftarrow{\text{Id}} \end{array} L_{(\Sigma^* f)^{\mathbb{T}}} DEF_+^{\mathbb{T}}\text{-mod} \begin{array}{c} \xleftarrow{DEF_+ \wedge -} \\ \xrightarrow{} \end{array} \text{Sp}_{\mathbb{Q}}^{\circlearrowleft}[W]$$

The unmarked functor is simply the forgetful functor.

Because of the way we have constructed S_{top}^\bullet it follows that $(a_{\#}, (-)^{\mathbb{T}})$ gives a map of diagrams of model categories from S_{top}^\bullet to S^\bullet . Since each of the components is a Quillen equivalence, we immediately obtain the following.

Theorem 4.14. *There is a Quillen equivalence:*

$$S_{\text{top}}^\bullet\text{-mod} \begin{array}{c} \xrightarrow{a_{\#}} \\ \xleftarrow{(-)^{\mathbb{T}}} \end{array} S^\bullet\text{-mod}.$$

We define $K_{\text{top}}^{\mathbb{T}}$, the set of cells for $S_{\text{top}}^\bullet\text{-mod}$, to be the set of objects given by applying the derived functor of $(-)^{\mathbb{T}}$ to K_{top} . By [GS13, Corollary 2.8] we see that the Quillen equivalence above is preserved by cellularisation.

Corollary 4.15. *The adjunction below is a Quillen equivalence.*

$$K_{\text{top}}^{\mathbb{T}}\text{-cell-} S_{\text{top}}^\bullet\text{-mod} \begin{array}{c} \xrightarrow{a_{\#}} \\ \xleftarrow{(-)^{\mathbb{T}}} \end{array} K_{\text{top}}\text{-cell-} S^\bullet\text{-mod}$$

The forgetful functor i^* relates the toral spectra version and the \mathbb{T} -equivariant version of this result [BGKS15]. Recall that we have the diagram of model categories based on \mathbb{T} -spectra which we call i^*S^\bullet , see Definition 4.6. Just as we constructed S_{top}^\bullet , we can also make a diagram of model categories $i^*S_{\text{top}}^\bullet$ that is the ‘fixed points’ of i^*S^\bullet . We then have the following result which is similar in nature to Proposition 2.9.

Corollary 4.16. *There is a diagram of Quillen functors as below. We note that i^* commutes with both $a_{\#}$ and $(-)^{\mathbb{T}}$ up to natural isomorphism.*

$$\begin{array}{ccc} S_{\text{top}}^\bullet\text{-mod} & \begin{array}{c} \xrightarrow{a_{\#}} \\ \xleftarrow{(-)^{\mathbb{T}}} \end{array} & S^\bullet\text{-mod} \\ \downarrow i^* & & \downarrow i^* \\ i^*S_{\text{top}}^\bullet\text{-mod} & \begin{array}{c} \xrightarrow{a_{\#}} \\ \xleftarrow{(-)^{\mathbb{T}}} \end{array} & i^*S^\bullet\text{-mod} \end{array}$$

4.3. Moving to algebra

We want to replace the model category $K_{\text{top}}^{\mathbb{T}}\text{-cell-} S_{\text{top}}^\bullet\text{-mod}$ by a Quillen equivalent $\text{Ch}(\mathbb{Q}[W])$ -model category. The method is the same as in [BGKS15, Section 3.4], we

briefly describe the process. We forget structure to get a diagram of model categories based on symmetric spectra with W -action (rather than orthogonal spectra). Then we ‘smash’ with $\mathbb{H}\mathbb{Q}$ to get a diagram of model categories based on $\mathbb{H}\mathbb{Q}$ -modules in symmetric spectra with W -action. Then one can apply [Kęd15, Lemma 5.7] (an extension of [Shi07]) to get a diagram of model categories based on $\text{Ch}(\mathbb{Q}[W])$. We leave the fine details to the references and simply state the consequences.

Theorem 4.17. *There exists a commutative ring S_t in $\text{Ch}(\mathbb{Q}[W])$ and a set of maps of S_t -modules A such that*

- *there is an isomorphism $H_*(S_t) \cong \pi_*^\mathbb{T}(DE\mathcal{F}_+)$,*
- *for any $a \in A$, there is a canonical $g \in (\Sigma^*f)^\mathbb{T}$ with $H_*(a) \cong \pi_*(g)$,*
- *there is a zig-zag of Quillen equivalences between $S_{\text{top}}^\bullet\text{-mod}$ and the diagram of model categories $S_t^\bullet\text{-mod}$*

$$S_t\text{-mod} \begin{array}{c} \xrightarrow{\text{Id}} \\ \xleftarrow{\text{Id}} \end{array} L_A S_t\text{-mod} \begin{array}{c} \xleftarrow{S_t \wedge -} \\ \xrightarrow{U} \end{array} \text{Ch}(\mathbb{Q}[W]) .$$

As in [BGKS15, Section 3.4], the zig-zag between $S_{\text{top}}^\bullet\text{-mod}$ and $S_t^\bullet\text{-mod}$ actually consists of objectwise Quillen equivalences. In particular, there is a zig-zag of Quillen equivalences between $S_t\text{-mod}$ and $DE\mathcal{F}_+^\mathbb{T}\text{-mod}$ and A is the set of images of the maps in $(\Sigma^*f)^\mathbb{T}$ under the derived zig-zag.

Since cellularisation is compatible with Quillen equivalences [GS13, Corollary 2.8], there is a set of cells K_t in $S_t\text{-mod}$ which gives the following result. These cells K_t are the images of the objects in K_{top} under the derived zig-zag of Quillen equivalences.

Corollary 4.18. *There is a zig-zag of Quillen equivalences between the model category $K_{\text{top}}^\mathbb{T}\text{-cell-}R_{\text{top}}^\bullet\text{-mod}$ and the model category $K_t\text{-cell-}S_t^\bullet\text{-mod}$.*

For each $\tau \in K_t$ there is a canonical $\sigma \in K_{\text{top}}$ with $H_(\tau) \cong \pi_*(\sigma)$.*

This process is compatible with the forgetful functor, in the sense that at each stage of the zig-zag, there is a commutative square of functors similar to that of Proposition 2.9. In particular, there is a zig-zag of Quillen equivalences between the model categories

$$i^* K_{\text{top}}^\mathbb{T}\text{-cell-}i^* S_{\text{top}}^\bullet\text{-mod} \quad \text{and} \quad i^* K_t\text{-cell-}i^* S_t^\bullet\text{-mod}$$

and the forgetful functor

$$i^* : K_t\text{-cell-}S_t^\bullet\text{-mod} \longrightarrow i^* K_t\text{-cell-}i^* S_t^\bullet\text{-mod}$$

preserves and detects weak equivalences and fibrations.

4.4. Simplifying the algebra

We can simplify the diagram S_t in two ways. First by removing the localisation at A , secondly by replacing the commutative dga S_t by a simpler commutative dga. The key to both is formality arguments, we use the fact that the homology of S_t or the homology of the maps in A is sufficiently well-structured to determine their homology type. The method is an extension of [BGKS15, Section 4], where we include the action of $W = O(2)/\mathbb{T}$ on $\mathcal{O}_{\mathcal{F}}$. This extension is possible since in the previous sections we have shown that at each our stage forgetful functors restrict to

the categories and objects used in [BGKS15]. Let $\mathcal{O}_{\mathcal{F}}$ be the graded ring $\prod_{n \geq 0} \mathbb{Q}[c_n]$ from Section 3.1. Recall that each c_n has degree -2 .

Lemma 4.19. *There is a zig-zag of quasi-isomorphisms of commutative ring objects in $\text{Ch}(\mathbb{Q}[W])$ between S_t and $\mathcal{O}_{\mathcal{F}}$.*

Proof. We already have isomorphisms of commutative ring objects in $\text{Ch}(\mathbb{Q}[W])$

$$H_*(S_t) \cong \pi_*(DE\mathcal{F}_+^{\mathbb{T}}) \cong \pi_*^{\mathbb{T}}(DE\mathcal{F}_+) = [S^0, DE\mathcal{F}_+]_*^{\mathbb{T}} \cong [E\mathcal{F}_+, S^0]_*^{\mathbb{T}}.$$

By [Gre08, Theorems 7.4 and 7.5], in particular the first line of the proof of Theorem 7.5, it follows that

$$[E\mathcal{F}_+, S^0]_*^{\mathbb{T}} \cong [E\mathcal{F}_+, E\mathcal{F}_+]_*^{\mathbb{T}} \cong \prod_{H \in \mathcal{F}} H^*(B(\mathbb{T}/H)) = \prod_{H \in \mathcal{F}} \mathbb{Q}[c] = \mathcal{O}_{\mathcal{F}}.$$

The $O(2)$ -action on $E\mathcal{F}_+$ induces a W -action on $\pi_*^{\mathbb{T}}(DE\mathcal{F}_+)$ and hence there is a W -action on $H^*(CP^\infty) = H^*(B\mathbb{T}/H) = \mathbb{Q}[c]$. This action sends c^i to $(-1)^i c^i$ as it is induced by the self-map of \mathbb{T} given by $t \mapsto t^{-1}$, which is exactly conjugation by a reflection of $O(2)$. This fact is also noted immediately above Lemma 7.1 of [Gre01]. Thus we know the homology of S_t as a graded ring with W -action. Our next task is to show that S_t is quasi-isomorphic to its homology.

There is a cycle $x_n \in S_t$ which corresponds to the idempotent $e_n \in \mathcal{O}_{\mathcal{F}}$, which projects onto factor n . Since this may not be W -fixed, let y_n be the average of x_n and $w x_n$. We then have a quasi-isomorphism $S_t \rightarrow \prod_{n \geq 1} S_t[y_n^{-1}]$. For each n , pick a representative $a_n \in S_t[y_n^{-1}]$ for the homology class c_n . Now let $b_n = 1/2(a_n - w a_n)$. Then the map sending c_n to b_n gives a W -equivariant quasi-isomorphism $\mathbb{Q}[c_n] \rightarrow S_t[y_n^{-1}]$. Putting these together for each n gives the other half of our zig-zag. \square

The other simplification is formally the same as the argument in [BGKS15, Section 4], so we leave the details to the reference.

Proposition 4.20. *There is a Quillen equivalence between S_t^\bullet -mod, and a diagram of model categories S_a^\bullet -mod:*

$$\mathcal{O}_{\mathcal{F}}\text{-mod} \begin{array}{c} \xleftarrow{\varepsilon^{-1}\mathcal{O}_{\mathcal{F}} \otimes_{\mathcal{O}_{\mathcal{F}}} -} \\ \xrightarrow{\varepsilon^{-1}\mathcal{O}_{\mathcal{F}} \otimes_{\mathcal{O}_{\mathcal{F}}} -} \end{array} \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}}\text{-mod} \begin{array}{c} \xleftarrow{\varepsilon^{-1}\mathcal{O}_{\mathcal{F}} \otimes -} \\ \xrightarrow{\varepsilon^{-1}\mathcal{O}_{\mathcal{F}} \otimes -} \end{array} \text{Ch}(\mathbb{Q}[W]),$$

where both unmarked functors are forgetful functors. Let K_a be the images of the cells K_t in S_a^\bullet -mod. Then we have Quillen equivalences between K_t -cell- S_t^\bullet -mod and K_a -cell- S_a^\bullet -mod.

As in previous cases, application of the forgetful functor from $\text{Ch}(\mathbb{Q}[W]) \rightarrow \text{Ch}(\mathbb{Q})$ reduces us to the case of rational \mathbb{T} -spectra. The forgetful functor

$$i^* : K_a\text{-cell-}S_a^\bullet\text{-mod} \longrightarrow i^*S_a\text{-cell-}i^*S_a^\bullet\text{-mod}$$

preserves and detects weak equivalences and fibrations.

4.5. Comparison with the algebraic model

We now turn to comparing S_a^\bullet -mod to the algebraic model $d\mathcal{A}(\mathcal{C})$ of Section 3. We first introduce an adjoint pair relating S_a^\bullet -mod and $d\mathcal{A}(\mathcal{C})$. An object

$$\beta: M \rightarrow \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes V,$$

of $d\mathcal{A}(\mathbb{C})$ gives an object of S_a^\bullet -mod defined by

$$(M, \mathcal{E}^{-1}\beta, \mathcal{E}^{-1}\mathcal{O}_{\mathcal{F}} \otimes V, \text{Id}, V).$$

This functor, which we call l^* , includes $d\mathcal{A}(\mathbb{C})$ into S_a^\bullet -mod, it has a right adjoint Γ . For more details, see [Bar16, Section 7].

Theorem 4.21. *The pair (l^*, Γ) induces a Quillen equivalence between the model categories $d\mathcal{A}(\mathbb{C})$ and K_a -cell- S_a^\bullet -mod. Furthermore, the square of left adjoints (and the square of right adjoints) of the diagram below commute.*

$$\begin{array}{ccc} d\mathcal{A}(\mathbb{C}) & \begin{array}{c} \xrightarrow{l^*} \\ \xleftarrow{\Gamma} \end{array} & K_a\text{-cell-}S_a^\bullet\text{-mod} \\ \mathbb{D} \updownarrow i^* & & \mathbb{D} \updownarrow i^* \\ d\mathcal{A}(\mathbb{T}) & \begin{array}{c} \xrightarrow{l^*} \\ \xleftarrow{\Gamma} \end{array} & i^*K_a\text{-cell-}i^*S_a^\bullet\text{-mod} \end{array}$$

Proof. Recall \mathbb{D} , the left adjoint to the forgetful functors i^* , see Lemma 3.5. That the left adjoints commute is immediate from the definitions. It follows automatically that the square of right adjoints also commutes.

The lower adjunction is a Quillen equivalence by [BGKS15, Proposition 4.2.4]. The weak equivalences and fibrations for K_a^{dual} -cell- S_a^\bullet -mod and $d\mathcal{A}(\mathbb{C})$ are defined in terms of the functors i^* . It is also routine to check that i^* preserves cofibrant objects. Hence the adjunction between $d\mathcal{A}(\mathbb{C})$ and K_a^{dual} -cell- R_a^\bullet -mod is a Quillen equivalence by the same argument as in Lemma 4.10. \square

We summarise this section with the following result.

Corollary 4.22. *There is a zig-zag of Quillen equivalences between the model category $d\mathcal{A}(\mathbb{C})$ and the model category of toral $O(2)$ -spectra, $\mathcal{C}O(2)\text{Sp}_{\mathbb{Q}}^\circ$. Furthermore, these Quillen equivalences are compatible with the two forgetful functors*

$$i^*: \mathcal{C}O(2)\text{Sp}_{\mathbb{Q}}^\circ \rightarrow \mathbb{T}\text{Sp}_{\mathbb{Q}}^\circ \quad \text{and} \quad i^*: d\mathcal{A}(\mathbb{C}) \rightarrow d\mathcal{A}(\mathbb{T}).$$

Hence the algebraic forgetful functor correctly models the spectrum-level forgetful functor.

Remark 4.23. As with [BGKS15], all of the Quillen equivalences of this section are in fact symmetric monoidal, giving us a classification of ring objects in $\mathcal{C}O(2)\text{Sp}_{\mathbb{Q}}^\circ$ in $d\mathcal{A}(\mathbb{C})$. However, the classification of the dihedral part of $O(2)\text{Sp}_{\mathbb{Q}}^\circ$ is not monoidal, so we do not obtain a monoidal classification of $O(2)\text{Sp}_{\mathbb{Q}}^\circ$ overall.

5. Dihedral spectra

In this section we find a model category based on chain complexes of \mathbb{Q} -modules that is Quillen equivalent to the model category of dihedral spectra $L_{e_{\mathcal{D}}S}O(2)\text{Sp}_{\mathbb{Q}}^\circ$. The material we present is an updated version of the preprint [Bar08a]. An early draft of this paper purported to give a symmetric monoidal Quillen equivalence. That result relied on there being a commutative ring G -spectrum weakly equivalent to $e_{\mathcal{D}}S$. A

correct interpretation of such a result would require careful use of N_∞ -spectra such as in [BH15] (and possible further work in that area). Hence we leave monoidal considerations to future work.

5.1. The dihedral model

The paper [Gre98b] constructs an algebraic model for the homotopy category of dihedral spectra. We call this category $\mathcal{A}(\mathcal{D})$ and write $dg\mathcal{A}(\mathcal{D})$ for the category of differential graded objects in $\mathcal{A}(\mathcal{D})$. In this subsection, we recap the definition of that category and equip it with a model structure. The category $\mathcal{A}(\mathcal{D})$ can also be described as the category of rational $O(2)$ -Mackey functors with support in the dihedral groups, see [Gre98a, Examples C(iii)].

Recall that W is used to denote the group of order two. For R a ring, let $\text{Ch}(R)$ denote the category of chain complexes of R -modules. We use $\mathbb{Q}[W]$ to denote the rational group ring of W . We will often consider \mathbb{Q} -modules as objects of $\mathbb{Q}[W]$ -mod with trivial W -action without comment or decoration.

Definition 5.1. We define a category called $\mathcal{A}(\mathcal{D})$. An object M consists of the following data: a \mathbb{Q} -module M_∞ , a collection of $\mathbb{Q}[W]$ -modules M_k for $k \geq 1$ and a map of $\mathbb{Q}[W]$ -modules $\sigma_M: M_\infty \rightarrow \text{colim}_n \prod_{k \geq n} M_k$.

A map $f: M \rightarrow M'$ in this category consists of a map $f_\infty: M_\infty \rightarrow M'_\infty$ in \mathbb{Q} -mod and maps $f_k: M_k \rightarrow M'_k$ in $\mathbb{Q}[W]$ -mod making the square below commute.

$$\begin{array}{ccc} M_\infty & \xrightarrow{\sigma_M} & \text{colim}_n \prod_{k \geq n} M_k \\ f_\infty \downarrow & & \downarrow \text{colim}_n \prod_{k \geq n} f_k \\ M'_\infty & \xrightarrow{\sigma_{M'}} & \text{colim}_n \prod_{k \geq n} M'_k \end{array}$$

We will also write $\text{tails}(M)$ for $\text{colim}_n \prod_{k \geq n} M_k$.

Definition 5.2. Let $dg\mathcal{A}(\mathcal{D})$ be the category of chain complexes in $\mathcal{A}(\mathcal{D})$. We call this the **algebraic model for dihedral spectra**. We shall also need to use $g\mathcal{A}(\mathcal{D})$, the category of graded objects in $\mathcal{A}(\mathcal{D})$.

We see that an object M of $dg\mathcal{A}(\mathcal{D})$ consists of a rational chain complex M_∞ and a collection $M_k \in \text{Ch}(\mathbb{Q}[W])$ for $k \geq 1$ with a map of chain complexes of $\mathbb{Q}[W]$ -modules $\sigma_M: M_\infty \rightarrow \text{colim}_n \prod_{k \geq n} M_k$. A map f in this category consists of a map $f_\infty \in \text{Ch}(\mathbb{Q})$ and maps $f_k \in \text{Ch}(\mathbb{Q}[W])$ such that the analogous square to the above definition commutes.

We want to show how to construct an object of $\mathcal{A}(\mathcal{D})$ from a rational $O(2)$ -spectrum. We first need to discuss some more idempotents of rationalised Burnside rings.

Recall the idempotents $e_e, e_{\mathcal{D}}$ and e_n for $n \geq 1$ from Definition 2.2. We also have $f_n = e_{\emptyset} - \sum_{k=1}^{n-1} e_k$.

Lemma 5.3. *Let D_{2n}^h be a dihedral subgroup of $O(2)$ of order $2n$. Then for each $k \mid n$ the rational Burnside ring of D_{2n}^h has idempotents e_{C_k} and $e_{D_{2k}}$. The collection of idempotents e_{C_k} and $e_{D_{2k}}$ for $k \mid n$ gives a maximal orthogonal decomposition of the identity.*

The inclusion map $D_{2n}^h \rightarrow O(2)$ induces the following map $A(O(2)) \rightarrow A(D_{2n}^h)$:

$$\begin{aligned} e_{\mathcal{C}} &\mapsto \Sigma_{k|n} e_{C_k}, \\ e_{\mathcal{D}} &\mapsto \Sigma_{k|n} e_{D_{2k}}, \\ e_k &\mapsto e_{D_{2k}} && k \mid n, \\ e_k &\mapsto 0 && k \nmid n. \end{aligned}$$

Proof. Consider the induced map $\mathcal{F}D_{2n}^h/D_{2n}^h \rightarrow \mathcal{F}O(2)/O(2)$ (which sends the cyclic groups to $SO(2)$) and use tom Dieck’s isomorphism. \square

The following definition and theorem are taken from [Gre98b]. Note that for any compact Lie group G and closed subgroup H , the action of $N_G H/H$ on G/H induces an action of $N_G H/H$ on $[G/H_+, X]_*^G \cong \pi_*^H(X)$.

Definition 5.4. Let X be an $O(2)$ -spectrum with rational homotopy groups. We let $\underline{\pi}_*^{\mathcal{D}}(X)$ denote the following object of $g\mathcal{A}(\mathcal{D})$. Let $k \geq 1$ and define

$$\underline{\pi}_*^{\mathcal{D}}(X)_k = e_{D_{2k}} \pi_*^{D_{2k}^h}(X) \quad \underline{\pi}_*^{\mathcal{D}}(X)_\infty = \operatorname{colim}_n (f_n \pi_*^{O(2)}(X)).$$

Note that $\underline{\pi}_*^{\mathcal{D}}(X)_k$ is isomorphic to the homotopy groups of the D_{2k} -geometric fixed points of X . Whenever $k \geq n$, there is a map

$$f_n \pi_*^{O(2)}(X) \longrightarrow e_{D_{2k}} \pi_*^{D_{2k}^h}(X)$$

induced from the inclusion $D_{2k}^h \rightarrow O(2)$ and multiplication by $e_{D_{2k}}$. Thus we obtain a map

$$f_n \pi_*^{O(2)}(X) \longrightarrow \prod_{k \geq n} e_{D_{2k}} \pi_*^{D_{2k}^h}(X).$$

Taking colimits over n defines the structure map σ of $\underline{\pi}_*^{\mathcal{D}}(X)$.

Since any fibrant object of $L_{e_{\mathcal{D}}} \mathbb{S}O(2) \operatorname{Sp}_{\mathbb{Q}}^{\mathcal{O}}$ has rational homotopy groups, this construction defines a functor

$$\underline{\pi}_*^{\mathcal{D}} : \operatorname{Ho} \left(L_{e_{\mathcal{D}}} \mathbb{S}O(2) \operatorname{Sp}_{\mathbb{Q}}^{\mathcal{O}} \right) \longrightarrow g\mathcal{A}(\mathcal{D}).$$

Thus one has a map of graded \mathbb{Q} -modules

$$[X, Y]_*^{\mathcal{D}O(2)} \rightarrow \operatorname{Hom}_{g\mathcal{A}(\mathcal{D})}(\underline{\pi}_*^{\mathcal{D}}(X), \underline{\pi}_*^{\mathcal{D}}(Y)).$$

This fits into an Adams short exact sequence as below, see [Gre98b, Corollary 5.5].

Theorem 5.5. *Let X and Y be $O(2)$ -spectra with rational homotopy groups. Then there is a short exact sequence*

$$0 \rightarrow \operatorname{Ext}(\underline{\pi}_*^{\mathcal{D}}(\Sigma X), \underline{\pi}_*^{\mathcal{D}}(Y)) \rightarrow [X, Y]_*^{\mathcal{D}O(2)} \rightarrow \operatorname{Hom}_{g\mathcal{A}(\mathcal{D})}(\underline{\pi}_*^{\mathcal{D}}(X), \underline{\pi}_*^{\mathcal{D}}(Y)) \rightarrow 0.$$

5.2. Adjunctions and model structures

We now introduce a particularly useful construction, $\widehat{\mathfrak{A}}$. We will soon see that this construction is an explicit description of the ‘global sections’ of an object of $dg\mathcal{A}(\mathcal{D})$.

Definition 5.6. Let $N \geq 1$ and take $M \in dg\mathcal{A}(\mathcal{D})$. Then $\widehat{\mathfrak{A}}_N M$ is defined as the following pullback in the category of $\operatorname{Ch}(\mathbb{Q}[W])$ -modules:

$$\begin{array}{ccc}
 \mathfrak{F}_N M & \xrightarrow{\alpha_{M,N}} & \prod_{k \geq N} M_k \\
 \downarrow & & \downarrow \\
 M_\infty & \longrightarrow & \text{tails}(M)
 \end{array}$$

Define $\mathfrak{F}_N^W M$ to be $(\mathfrak{F}_N M)^W$, the W -fixed points of $\mathfrak{F}_N M$.

It follows immediately from the definition above that \mathfrak{F}_N and \mathfrak{F}_N^W are exact functors. Furthermore, there are natural isomorphisms

$$\begin{aligned}
 \mathfrak{F}_N M &\cong \mathfrak{F}_{N+1} M \oplus M_N, \\
 M_\infty &\cong \text{colim}_N \mathfrak{F}_N M, \\
 \text{tails}(M)^W &\cong \text{colim}_n \prod_{k \geq n} (M_k^W).
 \end{aligned}$$

The notation \mathfrak{F}_N is to make the reader think of some combination of a direct product and a direct sum. Indeed if $M_\infty = 0$, then $\mathfrak{F}_N M = \bigoplus_{k \geq N} M_k$. Whereas if $M_\infty = \text{tails}(M)$ and $\sigma = \text{Id}$, then $\mathfrak{F}_N M = \prod_{k \geq N} M_k$.

Our first use of \mathfrak{F}_N is to give a construction of limits in $\mathcal{A}(\mathcal{D})$.

Lemma 5.7. *The category $dg\mathcal{A}(\mathcal{D})$ contains all small limits and colimits.*

Proof. Let M^i be a small diagram of objects of $dg\mathcal{A}(\mathcal{D})$. Define

$$(\text{colim}_i M^i)_\infty = \text{colim}_i (M^i_\infty) \quad \text{and} \quad (\text{colim}_i M^i)_k = \text{colim}_i (M^i_k).$$

The structure map for $\text{colim}_i M^i$ is induced by the composite

$$M^i_\infty \longrightarrow \text{colim}_n \prod_{k \geq n} M^i_k \longrightarrow \text{colim}_n \prod_{k \geq n} \text{colim}_i M^i_k.$$

For limits, we define

$$(\lim_i M^i)_k = \lim_i (M^i_k) \quad \text{and} \quad (\lim_i M^i)_\infty = \text{colim}_N \lim_i (\mathfrak{F}_N^W M^i).$$

The structure map of $\lim_i M^i$ is the composite below, where the middle map is induced by the maps $\alpha_{M^i,N}^W: \mathfrak{F}_N^W M^i \rightarrow \prod_{k \geq N} M^i_k$.

$$(\lim_i M^i)_\infty = \text{colim}_N \lim_i \left(\mathfrak{F}_N^W M^i \right) \longrightarrow \text{colim}_N \lim_i \prod_{k \geq N} M^i_k = \text{tails}(\lim_i M^i)$$

It is routine to check that these constructions give the colimit and limit. □

We will need the following fact to construct the model structure on $\mathcal{A}(\mathcal{D})$.

Lemma 5.8. *The functors \mathfrak{F}_N and \mathfrak{F}_N^W preserve filtered colimits for all $N \geq 1$.*

Proof. One checks injectivity and surjectivity of the canonical map

$$\text{colim}_i \mathfrak{F}_N M^i \longrightarrow \mathfrak{F}_N \text{colim}_i M^i$$

by first dealing with the term at infinity, then dealing with the finite number of terms that are not determined by the term at infinity. □

We introduce a collection of useful adjunctions relating $dg\mathcal{A}(\mathcal{D})$ to rational chain complexes and W -equivariant rational chain complexes.

Definition 5.9. Let A be a rational chain complex, $R \in \text{Ch}(\mathbb{Q}[W])$ and $M \in dg\mathcal{A}(\mathcal{D})$.

Define $i_k R$ to be the object of $dg\mathcal{A}(\mathcal{D})$ with $(i_k R)_\infty = 0$, $(i_k R)_n = 0$ for $n \neq k$ and $(i_k R)_k = R$. Now define p_k by setting $p_k M = M_k \in \text{Ch}(\mathbb{Q}[W])$. Then i_k is both right and left adjoint to p_k :

$$i_k: \text{Ch}(\mathbb{Q}[W]) \rightleftarrows dg\mathcal{A}(\mathcal{D}): p_k \quad p_k: dg\mathcal{A}(\mathcal{D}) \rightleftarrows \text{Ch}(\mathbb{Q}[W]): i_k.$$

Let $p_\infty M = M_\infty \in \text{Ch}(\mathbb{Q})$ and define $(i_\infty A)_\infty = A$ and $(i_\infty A)_k = 0$. Then we have an adjunction

$$p_\infty: dg\mathcal{A}(\mathcal{D}) \rightleftarrows \text{Ch}(\mathbb{Q}): i_\infty.$$

We set cA to be the object of $dg\mathcal{A}(\mathcal{D})$ with $cA_k = A = cA_\infty$ and structure map induced by the diagonal map $A \rightarrow \prod_{k \geq 1} A$. Then we have the ‘constant sheaf’ and ‘global sections’ adjunction

$$c: \text{Ch}(\mathbb{Q}) \rightleftarrows dg\mathcal{A}(\mathcal{D}): \bigoplus_1^W.$$

We put a model structure on $dg\mathcal{A}(\mathcal{D})$. We use the functors i_k and c above to create the generating sets. Let $I_{\mathbb{Q}}$ and $J_{\mathbb{Q}}$ denote the sets of generating cofibrations and acyclic cofibrations for the projective model structure on rational chain complexes, see [Hov99, Section 2.3]. Similarly, we have generating sets $I_{\mathbb{Q}[W]}$ and $J_{\mathbb{Q}[W]}$ for $\text{Ch}(\mathbb{Q}[W])$.

Proposition 5.10. *Define a map f in $dg\mathcal{A}(\mathcal{D})$ to be a weak equivalence or fibration if f_∞ and each f_k is a homology isomorphism or surjection. These classes define a cofibrantly generated and proper model structure on the category $dg\mathcal{A}(\mathcal{D})$.*

The generating cofibrations I are the collections $cI_{\mathbb{Q}}$ and $i_k I_{\mathbb{Q}[W]}$ for $k \geq 1$. The generating acyclic cofibrations J are $cJ_{\mathbb{Q}}$ and $i_k J_{\mathbb{Q}[W]}$ for $k \geq 1$.

Proof. Lemma 5.8 shows that \bigoplus_1^W preserves filtered colimits. The required smallness conditions on the generating sets follows immediately.

The rest of the proof is routine. As an example of the kind of argument we need to make, we prove that the acyclic fibrations are the maps with the right lifting property with respect to I . Let $f: A \rightarrow B$ be such a map. Using the adjunctions of Definition 5.9 it follows that each $f_k: A_k \rightarrow B_k$ is a surjection and a homology isomorphism, as is $\bigoplus_1^W f: \bigoplus_1^W A \rightarrow \bigoplus_1^W B$.

Since $\bigoplus_N^W A \cong \bigoplus_{N+1}^W A \oplus A_N^W$ it follows that each $\bigoplus_N^W f$ is a surjection and a homology isomorphism for each $N \geq 1$. Taking colimits over N we see that f_∞ is a surjection and homology isomorphism.

Left properness is immediate because colimits are defined term-wise. For right properness the only difficulty occurs at infinity, but the same method as in the preceding paragraph suffices, using exactness of \bigoplus_N^W to see that it preserves surjections and homology isomorphisms. \square

Note that with this model structure, the adjunctions (i_k, p_k) and (c, \bigoplus_N^W) are Quillen pairs between rational chain complexes (with a W -action in the first case) and $\mathcal{A}(\mathcal{D})$.

Lemma 5.11. *The collection $i_k\mathbb{Q}[W]$ for $k \geq 1$ and $c\mathbb{Q}$ are a set of homotopically compact, cofibrant and fibrant generators for this category.*

Proof. Every object of $dg\mathcal{A}(\mathcal{D})$ is fibrant and these objects are images of cofibrant objects under left Quillen functors. Homotopy compactness is simple to check for $i_k\mathbb{Q}[W]$. For $c\mathbb{Q}$ it relies on the fact that \bigoplus_1^W commutes with arbitrary coproducts (as they are filtered colimits of finite products).

Assume that $[\sigma, M]_*^{A(\mathcal{D})} = 0$ for each σ in the collection. It follows that each M_k must be acyclic as

$$0 = [i_k\mathbb{Q}[W], M]_*^{A(\mathcal{D})} \cong [\mathbb{Q}[W], M_k]_*^{\text{Ch}(\mathbb{Q}[W])}.$$

It follows that the canonical map $M \rightarrow i_\infty M_\infty$, which is the identity at infinity and zero elsewhere, is a weak equivalence. Thus we know that the graded groups

$$[c\mathbb{Q}, M]_*^{A(\mathcal{D})} \cong [c\mathbb{Q}, i_\infty M_\infty]_*^{A(\mathcal{D})} \cong [\mathbb{Q}, M_\infty]_*^{\mathbb{Q}} \cong H_*(M_\infty)$$

are zero. Hence M_∞ is acyclic and $M \rightarrow 0$ is a weak equivalence. \square

It is time we make our analogy to sheaves clear. In particular, this explains why the construction of limits in $\mathcal{A}(\mathcal{D})$ are more complicated than colimits: $\mathcal{A}(\mathcal{D})$ is essentially a category of sheaves described in terms of stalks.

Definition 5.12. Let \mathcal{P} be the space $\mathcal{F}O(2)/O(2) \setminus \{SO(2)\}$ (\mathcal{P} for points). Let \mathcal{O} be the constant sheaf of \mathbb{Q} on \mathcal{P} considered as a sheaf of rings. Let $W\mathcal{O}$ -mod denote the category of W -equivariant objects and W -equivariant maps in \mathcal{O} -mod.

To specify an \mathcal{O} -module M one only needs to give the stalks at the points k and ∞ and a map of \mathbb{Q} -modules $M_\infty \rightarrow \text{tails}(M)$. The global sections of M are then given by $\bigoplus_1 M$. Hence any object of $dg\mathcal{A}(\mathcal{D})$ defines an object of $W\mathcal{O}$ -mod. We call this functor inc and see that it is full and faithful. Thus we can view $dg\mathcal{A}(\mathcal{D})$ as a full subcategory of $W\mathcal{O}$ -mod.

The inclusion functor has a right adjoint called ‘fix’. On a W -equivariant \mathcal{O} -module M , $\text{fix}(M)_k = M_k$, $\text{fix}(M)_\infty = M_\infty^W$ and the structure map is $M_\infty^W \rightarrow M_\infty \rightarrow \text{tails}(M)$.

$$\text{inc}: dg\mathcal{A}(\mathcal{D}) \rightleftarrows dgW\mathcal{O}\text{-mod}: \text{fix}$$

In particular, one could also describe the limit of some diagram M^i in $dg\mathcal{A}(\mathcal{D})$ as $\text{fix} \lim_j \text{inc} M^i$, where the limit on the right is taken in the category of $W\mathcal{O}$ -modules.

Remark 5.13. The adjunction $\text{inc}: dg\mathcal{A}(\mathcal{D}) \rightleftarrows W\mathcal{O}\text{-mod}: \text{fix}$ can be used to put a model structure on $W\mathcal{O}$ -mod. Define a map f to be a weak equivalence or fibration if $\text{fix} f$ is. Then we have a new cofibrantly generated model structure on $W\mathcal{O}$ -mod. With this model structure the adjunction (inc, fix) becomes a Quillen equivalence.

5.3. The dihedral comparison

In this subsection we give the proof that $dg\mathcal{A}(\mathcal{D})$ and $L_{e_{\mathcal{D}}}\mathbb{S}O(2) \text{Sp}_{\mathbb{Q}}^{\mathbb{Q}}$ are Quillen equivalent. Since we are not considering monoidal products, we use the tilting theorem of Schwede and Shipley [SS03, Theorem 5.1.1]. Recall that a set of **tiltors** is a set of homotopically compact generators for the homotopy category such that $[T, T']_*$ is concentrated in degree zero for any T, T' in the set. One could use a similar method to that of [Bar09a], but the argument given below is somewhat simpler.

Lemma 5.14. *The model category $L_{e_{\mathcal{D}}\mathbb{S}}O(2) \mathrm{Sp}_{\mathbb{Q}}^{\mathcal{O}}$ has a set of tiltors given by the following countably infinite collection of cofibrant-fibrant objects. Let $\widehat{f}_{\mathcal{D}}$ denote fibrant replacement in $L_{e_{\mathcal{D}}\mathbb{S}}O(2) \mathrm{Sp}_{\mathbb{Q}}^{\mathcal{O}}$ and define*

$$\mathcal{G}_{top} = \{\widehat{f}_{\mathcal{D}}S^0\} \cup \{\widehat{f}_{\mathcal{D}}e_H O(2)/H_+ \mid H \in \mathcal{D} \setminus \{O(2)\}\}.$$

Proof. For any two dihedral subgroups H and K , the set of maps

$$[O(2)/H_+, O(2)/K_+]_*^{\mathcal{D}O(2)}$$

is concentrated in degree zero, where $[-, -]_*^{\mathcal{D}O(2)}$ denotes maps in the homotopy category of $L_{e_{\mathcal{D}}\mathbb{S}}O(2) \mathrm{Sp}_{\mathbb{Q}}^{\mathcal{O}}$. Hence it follows that, in the homotopy category, maps between elements of \mathcal{G}_{top} are concentrated in degree 0.

To show that \mathcal{G}_{top} generates, we must prove that if X is an object of $L_{e_{\mathcal{D}}\mathbb{S}}O(2) \mathrm{Sp}_{\mathbb{Q}}^{\mathcal{O}}$ such that $[\sigma, X]_*^{\mathcal{D}O(2)} = 0$ for all $\sigma \in \mathcal{G}_{top}$, then $X \rightarrow *$ is a π_* -isomorphism. Let $X \in L_{e_{\mathcal{D}}\mathbb{S}}O(2) \mathrm{Sp}_{\mathbb{Q}}^{\mathcal{O}}$ be fibrant, by Theorem [MM02, IV.6.13] $\pi_*^H(X) = 0$ for any $H \in \mathcal{C}$ and we see immediately that

$$\pi_*^{O(2)}(X) = [S^0, X]_*^{O(2)} = [S^0, X]_*^{\mathcal{D}O(2)} = 0.$$

Let H be a finite dihedral group. By [Gre98a, Examples C(i)] there is a natural isomorphism

$$\pi_*^H(X) \cong \bigoplus_{(K) \leq H} (e_K \pi_*^K(X))^{W_H K}.$$

Since we have assumed that $e_K \pi_*^K(X) = [e_K O(2)/K_+, X]_*^{\mathcal{D}O(2)}$ is zero for each finite dihedral K , $\pi_*^H(X) = 0$. Hence our set generates the homotopy category. Homotopy compactness follows from the isomorphisms

$$[e_K O(2)/K_+, X]_*^{\mathcal{D}O(2)} = [e_K O(2)/K_+, X]_*^{O(2)} = e_K \pi_*^K(X),$$

which hold whenever X is fibrant in $L_{e_{\mathcal{D}}\mathbb{S}}O(2) \mathrm{Sp}_{\mathbb{Q}}^{\mathcal{O}}$. □

We identify a ringoid \underline{R} from our algebraic model and give an algebraic version of the tilting result we want for dihedral spectra.

Definition 5.15. Define a set of objects \mathcal{G}_a of $dg\mathcal{A}(\mathcal{D})$

$$\mathcal{G}_a = \{c\mathbb{Q}\} \cup \{i_k \mathbb{Q}[W] \mid k \geq 1\}.$$

Let \underline{R} denote the ringoid given by taking the full subcategory of $\mathcal{A}(\mathcal{D})$ on the object set \mathcal{G}_a , considered as a category enriched over rational vector spaces. A **module over \underline{R}** is a contravariant additive functor from \underline{R} to rational vector spaces.

By Lemma 5.11, \mathcal{G}_a is a set of homotopically compact cofibrant-fibrant generators for the homotopy category of $dg\mathcal{A}(\mathcal{D})$. A standard variation of the tilting theorem (using rational chain complexes instead of symmetric spectra) gives the following result.

Proposition 5.16. *The model category of chain complexes of modules over \underline{R} (with fibrations the objectwise surjections and weak equivalences the objectwise homology isomorphisms) is Quillen equivalent to $dg\mathcal{A}(\mathcal{D})$.*

We now prove that \underline{R} is isomorphic to the endormorphism ringoid of \mathcal{G}_{top} .

Lemma 5.17. *The functor $\underline{\pi}_*^{\mathcal{D}}$ induces an isomorphism of categories (enriched over rational vector spaces) from the full subcategory of $\text{Ho}(L_{e_{\mathcal{D}}\mathbb{S}}O(2)\text{Sp}_{\mathbb{Q}}^{\circ})$ with object set \mathcal{G}_{top} to \underline{R} .*

Proof. This is a series of routine calculations using [Gre98b]. Let H and K be finite dihedral groups with $|H| = 2k$ and $|K| = 2m$ and let $\sigma_H = \widehat{f}_{\mathcal{D}}e_HO(2)/H_+$. Then

$$\underline{\pi}_*^{\mathcal{D}}(S_{\mathcal{D}}) = c\mathbb{Q} \quad \text{and} \quad \underline{\pi}_*^{\mathcal{D}}(\sigma_H) = \underline{\pi}_*^{\mathcal{D}}(e_HO(2)/H_+) = i_k\mathbb{Q}[W].$$

The functor $\underline{\pi}_*^{\mathcal{D}}$ from Definition 5.4 gives maps as below. These maps are isomorphisms by Theorem 5.5 as $i_k\mathbb{Q}[W]$ and $c\mathbb{Q}$ are projective, see the proof of [Gre98b, Remark 4.3]. The equalities on the right hold as the objects of \mathcal{G}_a are cofibrant, fibrant and concentrated in degree zero:

$$\begin{aligned} [S_{\mathcal{D}}, S_{\mathcal{D}}]_*^{\mathcal{D}O(2)} &\xrightarrow{\cong} [c\mathbb{Q}, c\mathbb{Q}]_*^{dg\mathcal{A}(\mathcal{D})} = \mathcal{A}(\mathcal{D})(c\mathbb{Q}, c\mathbb{Q}), \\ [\sigma_H, S_{\mathcal{D}}]_*^{\mathcal{D}O(2)} &\xrightarrow{\cong} [i_k\mathbb{Q}[W], c\mathbb{Q}]_*^{dg\mathcal{A}(\mathcal{D})} = \mathcal{A}(\mathcal{D})(i_k\mathbb{Q}[W], c\mathbb{Q}), \\ [S_{\mathcal{D}}, \sigma_H]_*^{\mathcal{D}O(2)} &\xrightarrow{\cong} [c\mathbb{Q}, i_k\mathbb{Q}[W]]_*^{dg\mathcal{A}(\mathcal{D})} = \mathcal{A}(\mathcal{D})(c\mathbb{Q}, i_k\mathbb{Q}[W]), \\ [\sigma_H, \sigma_H]_*^{\mathcal{D}O(2)} &\xrightarrow{\cong} [i_k\mathbb{Q}[W], i_k\mathbb{Q}[W]]_*^{dg\mathcal{A}(\mathcal{D})} = \mathcal{A}(\mathcal{D})(i_k\mathbb{Q}[W], i_k\mathbb{Q}[W]), \\ [\sigma_K, \sigma_H]_*^{\mathcal{D}O(2)} &\xrightarrow{\cong} [i_m\mathbb{Q}[W], i_k\mathbb{Q}[W]]_*^{dg\mathcal{A}(\mathcal{D})} = \mathcal{A}(\mathcal{D})(i_m\mathbb{Q}[W], i_k\mathbb{Q}[W]). \quad \square \end{aligned}$$

We can now give the classification theorem for dihedral spectra.

Theorem 5.18. *The model categories $L_{e_{\mathcal{D}}\mathbb{S}}O(2)\text{Sp}_{\mathbb{Q}}^{\circ}$ and $dg\mathcal{A}(\mathcal{D})$ are Quillen equivalent. Hence the homotopy categories of $L_{e_{\mathcal{D}}\mathbb{S}}O(2)\text{Sp}_{\mathbb{Q}}^{\circ}$ and $dg\mathcal{A}(\mathcal{D})$ are equivalent.*

Proof. The model category $L_{e_{\mathcal{D}}\mathbb{S}}O(2)\text{Sp}_{\mathbb{Q}}^{\circ}$ is simplicial, cofibrantly generated, proper and stable as these properties are preserved by the localisations we have applied to $O(2)\text{Sp}_{\mathbb{Q}}^{\circ}$.

Lemma 5.17 gives an isomorphism of ringoids, hence [SS03, Theorem 5.1.1] implies that $L_{e_{\mathcal{D}}\mathbb{S}}O(2)\text{Sp}_{\mathbb{Q}}^{\circ}$ is Quillen equivalent to the model category of chain complexes of \underline{R} -modules. The result then follows by Proposition 5.16. \square

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David Barnes d.barnes@qub.ac.uk

Queen's University Belfast, School of Mathematics and Physics, Pure Mathematics Research Centre, University Road, Belfast, BT7 1NN, United Kingdom