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POSTNIKOV DECOMPOSITION AND THE GROUP OF SELF-EQUIVALENCES OF A RATIONALIZED SPACE

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Abstract

Let X be a simply connected rational CW complex of finite type. Write $X^{[n]}$ for the nth Postnikov section of X. Let $\mathcal{E}(X^{[n+1]})$ denote the group of homotopy self-equivalences of $X^{[n+1]}$. We use Sullivan models in rational homotopy theory to construct two short exact sequences:

$$\mathrm{Hom}\big(\pi_{n+1}(X);H^{n+1}(X^{[n]})\big)\rightarrowtail\mathcal{E}(X^{[n+1]})\twoheadrightarrow D_n^{n+1},$$

$$\mathrm{Hom}\big(\pi_{n+1}(X);H^{n+1}(X^{[n]})\big)\rightarrowtail\mathcal{E}_\sharp(X^{[n+1]})\twoheadrightarrow G_n^{n+1},$$

where D_n^{n+1} is a subgroup of $\mathrm{aut}(\mathrm{Hom}(\pi_q(X);\mathbb{Q}))\times\mathcal{E}(X^{[n]})$ which is defined in terms of the Whitehead exact sequence of X and where G_n^{n+1} is a certain subgroup of $\mathcal{E}_\sharp(X^{[n]})$. Here $\mathcal{E}_\sharp(X^{[n]})$ is the subgroup of those elements inducing the identity on the homotopy groups. Moreover, we give an alternative proof of the Costoya–Viruel theorem [**9**]: Every finite group occurs as $\mathcal{E}(X)$ where X is rational.

1. Introduction

Let $\mathcal{E}(X)$ denote the group of self homotopy equivalences of a simply connected CW-complex X and let $\mathcal{E}_\sharp(X)$ denote the subgroup represented by self-equivalences that induce the identity map on $\pi_*(X)$. The subgroup $\mathcal{E}_\sharp(X)$ is not in general trivial, for instance in [**3**] it is shown that $\mathcal{E}_\sharp(S^2\times S^n)\cong\pi_{n+2}(S^2)\oplus\mathbb{Z}_2$ where $n\geqslant 3$.

The study of the groups $\mathcal{E}(X)$ and $\mathcal{E}_\sharp(X)$ by means of a cellular decomposition of X is a difficult problem with a long history. See Rutter [**11**, Chapter 11] for a survey.

When X is a simply connected rational CW complex of finite type, i.e., $\pi_n(X)$ is a vector space of finite dimension for every $n\geqslant 1$, the group $\mathcal{E}(X)$ has emerged as a recent object of interest, for instance the realization problem, namely which group occurs as $\mathcal{E}(X)$? Arkowitz–Lupton [**2**] gave the first examples of finite groups occurring as $\mathcal{E}(X)$. Further examples were given by the author in [**6, 7**]. Costoya–Viruel [**9**] then proved the remarkable result that every finite group G occurs as $G=\mathcal{E}(X)$ for some elliptic rational space X. All these works have been accomplished

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using Sullivan models in rational homotopy theory. A similar approach of studying the group $\mathcal{E}(X)$ based on Anick's DG Lie algebra models [**1**] over a certain subring of $\mathbb{Q}$ has been developed by Benkhalifa–Smith [**8**].

The aim of this paper is to study the effect of rational cell-attachment on the group of self-equivalences using the Postnikov tower. More precisely let X be a simply connected rational CW complex of finite type. Write $X^{[n]}$ for the nth Postnikov section of X [**4**, page 237]. We consider the situation in which

$$X^{[n]} \cup_\alpha \Big(\bigcup_{i\in I} e_i^q\Big), \qquad \text{where } I \text{ is finite and } q > n, \tag{1}$$

is the space obtained by attaching rational cells to $X^{[n]}$ by a map $\alpha\colon \underset{i\in I}{\vee} S^q_{\mathbb{Q}} \to X^{[n]}$ where $S^q_{\mathbb{Q}}$ is the rational sphere of dimension q [**10**, page 102]. Let Z^q be the pullback of the map $X^{[n]} \to K(\pi_q(X); q+1)$ over the path fibration:

$$\begin{array}{ccc} Z^q & \longrightarrow & PK(\pi_q(X); q+1) \\ \downarrow & & \downarrow \\ X^{[n]} & \longrightarrow & K(\pi_q(X); q+1). \end{array}$$

Recall that if $q = n+1$, then the space Z^q coincides with $(n+1)$th Postnikov section $X^{[n+1]}$ of X.

We prove:

Theorem 1.1. *There exist two short exact sequences of groups:*

$$\mathrm{Hom}\big(\pi_q(X); H^{n+1}(X^{[n]})\big) \rightarrowtail \mathcal{E}(Z^q) \twoheadrightarrow D^q_n,$$

$$\mathrm{Hom}\big(\pi_q(X); H^{n+1}(X^{[n]})\big) \rightarrowtail \mathcal{E}_\sharp(Z^q) \twoheadrightarrow G^q_n,$$

where D^q_n (respect. G^q_n) is a certain subgroup of $\mathrm{aut}(\mathrm{Hom}(\pi_q(X); \mathbb{Q})) \times \mathcal{E}(X^{[n]})$ *(respect. of $\mathcal{E}_\sharp(X^{[n]})$) (see (14) and (15) for the definitions).*

Consequently, we prove the following first result concerning the question of (in)finiteness of the group $\mathcal{E}(X^{[n]})$.

Corollary 1.2. *Let X be a simply connected rational CW complex of finite type. Assume that $\pi_i(X) \otimes \mathbb{Q} \neq 0$ for $i = n-1, n, n+1$.*

If $\mathcal{E}(X^{[n]})$ is finite, then $\mathcal{E}(X^{[n+1]})$ and $\mathcal{E}(X^{[n-1]})$ are infinite.

Next we prove the following result showing the relationship between the finiteness of the group $\mathcal{E}(Z^{n+2})$ and the Postnikov invariant $[k^n]$ of the space X [**4**, page 237].

Corollary 1.3. *Let X be a simply connected rational CW complex of finite type. Assume that $\mathcal{E}(Z^{n+2})$ is finite, then the space $X^{[n+1]}$ has the homotopy type of $X^{[n]} \times K(\pi_{n+1}(X), n+1)$.*

In addition we give a second proof of Costoya–Viruel in [**9**] that *every finite group occurs as* $\mathcal{E}(X)$ for some elliptic rational space X.

We establish these results in an algebraic context using the notion of Sullivan models in rational homotopy theory [**12**]. Recall that if X is a simply connected rational CW complex of finite type, then there exists a free commutative cochain algebra $(\Lambda V, \partial)$ called the Sullivan model of X, unique up to isomorphism, which determines completely the homotopy type of the space X. Moreover, the Sullivan model recovers homotopy data via the identifications:

$$\operatorname{Hom}\big(\pi_*(X), \mathbb{Q}\big) \cong V^*, \quad H^*(X;\mathbb{Q}) \cong H^*(\Lambda V, \partial) \quad \text{and} \quad \mathcal{E}(X) \cong \frac{\operatorname{aut}(\Lambda V, \partial)}{\simeq},$$

where $\frac{\operatorname{aut}(\Lambda V,\partial)}{\simeq}$ is the group of homotopy cochain self-equivalences of $(\Lambda V, \partial)$ modulo the relation of homotopy between free commutative cochain algebras (see [**12**]). We write

$$\mathcal{E}(\Lambda V) = \operatorname{aut}(\Lambda V, \partial)/\simeq,$$

for this group. Similarly, we have $\mathcal{E}_\sharp(X) \cong \operatorname{aut}_\sharp(\Lambda V, \partial)/\simeq$. We denote the latter group by $\mathcal{E}_\sharp(\Lambda V)$. Here $\operatorname{aut}_\sharp(\Lambda V, \partial)$ is the group of homotopy cochain self-equivalences inducing the identity automorphism on V^*.

The exact sequences in Theorem 1.1 are then the translation of the exact sequences given, in (11), for $\mathcal{E}(\Lambda V)$ and $\mathcal{E}_\sharp(\Lambda V)$. We end this work by giving an example showing that there exists a free commutative cochain algebra $(\Lambda V, \partial)$, which is not of finite type, such that $\mathcal{E}(\Lambda V) \cong \mathbb{Z}$.

2. Homotopy self-equivalences of cochain morphisms

2.1. Notation and fundamental results

All vector spaces, algebras, tensor products, etc. are defined over $\mathbb{Q}$ and this ground field will be in general suppressed from the notation.

A commutative cochain algebra is a (positive) graded differential vector space equipped with a graded linear map ∂^* (called the differential) ($\partial^n \colon A^n \to A^{n+1}$, such that $\partial^2 = 0$) together with linear maps $A^* \otimes A^* \to A^*$, denoted by $(a \otimes b \to a.b)$ and $i \colon \mathbb{Q} \to A^*$ which satisfy the associativity, commutativity (in the graded sense), and unit conditions and the following relations:

$$A^i.A^j \subset A^{i+j} \quad \text{and} \quad d(a.b) = d(a).b + (-1)^{|a|}a.d(b) \quad , \quad a, b \in A.$$

A commutative cochain algebra (A^*, d) is said to be 1-connected if $H^0(A) = \mathbb{Q}$ and $H^1(A) = 0$, and of finite type if each vector space $H^n(A)$ is finite dimensional.

Definition 2.1. Let V be a (positive) graded vector space and let $T(V)$ be the free tensor algebra over V. Put $T(V) = \mathbb{Q} \oplus T^{\geqslant 1}(V)$ and define ΛV, called the free commutative algebra, to be $T(V)$ divided by the two-sided ideal generated by elements of the form $(a.b).c - a.(b.c)$ and $a.b - (-1)^{|a||b|}b.a$ where $a, b, c \in T(V)$.

The free commutative algebra ΛV can also be described as follows: $\Lambda V = E(V^{odd}) \otimes S(V^{even})$, where $E(V^{odd})$ is the exterior algebra on the oddly graded part of V and $S(V^{even})$ is the symmetric algebra on the evenly graded part of V. Moreover, if ∂ is a differential on ΛV, then $(\Lambda V, \partial)$ is called a free commutative cochain algebra (fcca for short).

Let us recall the following results (see [**10, 12**]) which are fundamental in rational homotopy theory.

Theorem 2.2. *For every 1-connected commutative cochain algebra* (A^*, d)*, there exists a quasi-isomorphism* $(\Lambda V, \partial) \to (A^*, d)$ *such that* ∂ *is decomposable, i.e.,* $\operatorname{Im}\partial \subset \Lambda^+ V.\Lambda^+ V$*. The fcca* $(\Lambda V, \partial)$ *is called a minimal Sullivan model of* (A^*, d)*, it is unique up to isomorphism. Moreover, two 1-connected commutative cochain algebras are quasi-isomorphic if and only if their minimal Sullivan models are isomorphic.*

Theorem 2.3. *Let* X *be a simply connected CW-complex having rational homology of finite type. Let* $A_{PL}(X)$ *be the simplicial cochain algebra associated with* X *(see [**10, 12**]). The minimal Sullivan model* $(\Lambda V, \partial)$ *of* $A_{PL}(X)$ *is called the minimal Sullivan model of* X*. Recall that* $H^*(\Lambda V, \partial) \cong H^*(X, \mathbb{Q})$*, as graded algebras, and* $V^n \cong \operatorname{Hom}_{\mathbb{Z}}(\pi_n(X); \mathbb{Q})$ *for every* $n \geqslant 2$*. In addition there exists a bijection between the set of rational homotopy types of simply connected CW-complexes having rational homology of finite type and the set of isomorphism classes of fccas.*

2.2. Notion of homotopy for free commutative cochain algebras

Definition 2.4. Let $(\Lambda(V), \partial)$ be a 1-connected fcca. Define the vector spaces $\overline{V}$ and $\widehat{V}$ by $(\overline{V})^n = V^{n+1}$ and $(\widehat{V})^n = V^n$. We then define the fcca $(\Lambda(V, \overline{V}, \widehat{V}), D)$ with the differential D is given by

$$D(v) = \partial(v), \qquad D(\widehat{v}) = 0, \qquad D(\overline{v}) = \widehat{v}.$$

We define a derivation S of degree -1 of the fcca $(\Lambda(V, \overline{V}, \widehat{V}), D)$ by putting $S(v) = \overline{v}$, $S(\overline{v}) = S(\widehat{v}) = 0$.

A homotopy between two cochain morphisms $\alpha, \alpha' \colon (\Lambda(V), \partial) \to (\Lambda(V), \partial)$ is a cochain morphism

$$F \colon (\Lambda(V, \overline{V}, \widehat{V}), D) \to (\Lambda(V), \partial),$$

such as $F(v) = \alpha(v)$ and $F \circ e^\theta(v) = \alpha'(v)$, where

$$e^\theta(v) = v + \widehat{v} + \sum_{n\geqslant 1} \frac{1}{n!}(S \circ \partial)^n(v),\ v \in V \quad \text{and} \quad \theta = D \circ S + S \circ D.$$

Thereafter we will need the following lemma:

Lemma 2.5. *Let* $q > n$ *and let* $V = V^q \oplus V^{\leqslant n}$ *and* $\alpha, \alpha' \colon (\Lambda(V), \partial) \to (\Lambda(V), \partial)$ *be two cochain morphisms satisfying:*

$$\alpha(v) = v + z, \quad \alpha(v) = v + z' \ \text{ on } V^q \quad \text{and} \quad \alpha = \alpha' = \mathrm{id} \ \text{ on } V^{\leqslant n}.$$

Assume that $z - z' = \partial(u)$*, where* $u \in \Lambda(V)$*. Then* α *and* α' *are homotopic.*

Proof. Define F by setting

$$\begin{array}{llll} F(v) = v + y, & F(\widehat{v}) = z' - z & \text{and} \quad F(\overline{v}) = z & \text{for} \quad v \in V^q, \\ F(v) = v, & F(\widehat{v}) = 0 & \text{and} \quad F(\overline{v}) = 0 & \text{for} \quad v \in V^{\leqslant n} \end{array}$$

then F is the needed homotopy. ∎

2.3. The graded linear map b^*

Definition 2.6. Let $(\Lambda(V^q \oplus V^{\leqslant n}), \partial)$ be a 1-connected fcca where $q > n$. We define the linear map $b^q \colon V^q \to H^{q+1}(\Lambda V^{\leqslant n})$ by setting

$$b^q(v) = [\partial(v)], \qquad v \in V^q. \tag{2}$$

Here $[\partial(v)]$ denotes the cohomology class of $\partial(v) \in (\Lambda V^{\leqslant n})^{q+1}$.

For every 1-connected cdga $(\Lambda(V^q \oplus V^{\leqslant n}), \partial)$, the linear map b^q is natural. Namely, if $[\alpha] \in \mathcal{E}(\Lambda(V^q \oplus V^{\leqslant n}))$, then the following diagram commutes:

$$\begin{array}{ccc} V^q & \xrightarrow{\quad \widetilde{\alpha}^q \quad} & V^q \\ \big\downarrow{\scriptstyle b^q} & & \big\downarrow{\scriptstyle b^q} \\ H^{q+1}(\Lambda V^{\leqslant n}) & \xrightarrow{H^{q+1}(\alpha^{\leqslant n})} & H^{q+1}(\Lambda V^{\leqslant n}), \end{array} \tag{3}$$

where $\widetilde{\alpha} \colon V^* \to V^*$ is the graded homomorphism induced by α on the indecomposables and where $\alpha^{\leqslant n} \colon (\Lambda V^{\leqslant n}, \partial) \to (\Lambda V^{\leqslant n}, \partial)$ is the restriction of α.

2.4. The groups $\mathcal{D}_n^q$

Definition 2.7. Given a 1-connected fcca $(\Lambda(V^q \oplus V^{\leqslant n}), \partial)$ where $q \geqslant n$, let $\mathcal{D}_n^q$ be the subset of $\operatorname{aut}(V^q) \times \mathcal{E}(\Lambda V^{\leqslant n})$ consisting of the couples $(\xi, [\alpha^{\leqslant n}])$ making the following diagram commute:

$$\begin{array}{ccc} V^q & \xrightarrow{\quad \xi \quad} & V^q \\ \big\downarrow{\scriptstyle b^q} & & \big\downarrow{\scriptstyle b^q} \\ H^{q+1}(\Lambda V^{\leqslant n}) & \xrightarrow{H^{q+1}(\alpha^{\leqslant n})} & H^{q+1}(\Lambda V^{\leqslant n}). \end{array} \tag{4}$$

Clearly, $\mathcal{D}_n^q$ is a subgroup of $\operatorname{aut}(V^q) \times \mathcal{E}(\Lambda V^{\leqslant n})$.

Proposition 2.8. *The map* $\mathrm{g} \colon \mathcal{E}(\Lambda(V^q \oplus V^{\leqslant n})) \to \mathcal{D}^q$ *given by*

$$\mathrm{g}([\alpha]) = (\widetilde{\alpha}^q, [\alpha^{\leqslant n}])$$

is a surjective homomorphism of groups.

Proof. First it is well-known [**10**, Proposition 12.8] that if two cochain morphisms are homotopic, then they induce the same graded linear maps on the indecomposables, i.e., $\widetilde{\alpha} = \widetilde{\alpha'}$, moreover, $\alpha^{\leqslant n}, \alpha'^{\leqslant n}$ are homotopic and by using the diagram (3) we deduce that the map g is well-defined.

Next let $(\xi, [\alpha^{\leqslant n}]) \in \mathcal{D}_n^q$. Recall that, in the diagram (4), we have:

$$\begin{aligned} H^{q+1}(\alpha^{\leqslant n}) \circ b^q(v) &= \alpha^{\leqslant n} \circ \partial(v) + \operatorname{Im} \partial^{\leqslant n}, \\ b^q \circ \xi^q(v) &= \partial \circ \xi^q(v) + \operatorname{Im} \partial^{\leqslant n}, \end{aligned} \tag{5}$$

where $\partial^{\leqslant n} \colon (\Lambda V^{\leqslant n})^q \to (\Lambda V^{\leqslant n})^{q+1}$.

Since by Definition 2.7 this diagram commutes, the element $(\alpha^{\leqslant n} \circ \partial - \partial \circ \xi^q)(v) \in \operatorname{Im} \partial^{\leqslant n}$. As a consequence there exists $u_v \in (\Lambda V^{\leqslant n})^q$ such that

$$(\alpha^{\leqslant n} \circ \partial - \partial \circ \xi^q)(v) = \partial^{\leqslant n}(u_v). \tag{6}$$

Thus we define $\alpha\colon (\Lambda(V^q \oplus V^{\leqslant n}), \partial) \to (\Lambda(V^q \oplus V^{\leqslant n}), \partial)$ by setting

$$\alpha(v) = \xi^q(v) + u_v, \qquad \text{and} \qquad \alpha = \alpha^{\leqslant n} \text{ on } V^{\leqslant n}.$$

As $\partial(v) \in (\Lambda V^{\leqslant n})^q$ then, by (6), we get

$$\partial \circ \alpha(v) = \partial(\xi^q(v)) + \partial^{\leqslant n}(u_v) = \alpha^{\leqslant n} \circ \partial(v) = \alpha \circ \partial(v).$$

So α is a cochain morphism. Now due to the fact that $u_v \in (\Lambda V^{\leqslant n})^q$ and $q > n$, the linear map $\widetilde{\alpha}^q\colon V^q \to V^q$ coincides with ξ^q.

Then it is well-known (see [**10**]) that any cochain morphism between two 1-connected fccas inducing a graded isomorphism on the indecomposables is a homotopy equivalence. Consequently, $[\alpha] \in \mathcal{E}(\Lambda V, \partial)$. Therefore g is surjective.

Finally, the following relations:

$$\begin{aligned} \mathrm{g}([\alpha][\alpha']) = \mathrm{g}([\alpha \circ \alpha']) &= (\widetilde{\alpha \circ \alpha'}^q, [\alpha^{\leqslant n} \circ \alpha'^{\leqslant n}]) \\ &= (\widetilde{\alpha}^q, [\alpha^{\leqslant n}]) \circ (\widetilde{\alpha'}^q, [\alpha'^{\leqslant n}]) = \mathrm{g}([\alpha]) \circ \mathrm{g}([\alpha']) \end{aligned}$$

assure that g is a homomorphism of groups. □

2.5. Characterization of $\ker \mathrm{g}$

Next by definition we have

$$\ker \mathrm{g} = \Big\{ [\alpha] \in \mathcal{E}(\Lambda(V)) \ \mid \ \widetilde{\alpha}^q = id_{V^q}, \quad [\alpha^{\leqslant n}] = [id_{\Lambda(V^{\leqslant n})}] \Big\},$$

therefore for every $[\alpha] \in \ker \mathrm{g}$ we have

$$\begin{aligned} &\alpha(v) = v + z, \qquad z \in \Lambda^q(V^{\leqslant n}), \\ &\alpha^{\leqslant n} \simeq id_{\Lambda(V^{\leqslant n})}. \end{aligned} \tag{7}$$

So define

$$\theta_\alpha\colon V^q \to \Lambda^q(V^{\leqslant n}) \quad \text{by} \quad \theta_\alpha(v) = \alpha(v) - v. \tag{8}$$

Notice that the relations (7) and (8) imply that

$$\theta_{\alpha' \circ \alpha} = \theta_{\alpha'} + \theta_\alpha. \tag{9}$$

Lemma 2.9. *Let $[\alpha] \in \ker \mathrm{g}$. Then there exists $[\beta] \in \ker \mathrm{g}$ satisfying:*

1. *$\theta_\beta(v)$ is a cocycle in $\Lambda^q(V^{\leqslant n})$ for every $v \in V^q$*
2. *$\beta_{\leqslant n} = \alpha_{\leqslant n}$*
3. *$[\beta] = [\alpha]$*

Proof. Since $[\alpha^{\leqslant n}] = [id_{\Lambda(V^{\leqslant n})}]$ there is a homotopy

$$F\colon (\Lambda(V^{\leqslant n}, \overline{V}^{\leqslant n}, \widehat{V}^{\leqslant n}), D) \to (\Lambda(V^{\leqslant n}), \partial),$$

such that $F(v) = v$ and $F \circ e^\theta(v) = \alpha^{\leqslant n}(v)$. Therefore for $v \in V^q$ the element $F\Big(\sum\limits_{n \geqslant 1} \frac{1}{n!}(S \circ \partial)^n(v)\Big)$ is a well-defined element in $\Lambda^q(V^{\leqslant n})$. Thus we define β by setting

$$\beta(v) = \begin{cases} v, & \text{for } v \in V^q; \\ \alpha(v) - F\Big(\sum\limits_{n \geqslant 1} \frac{1}{n!}(S \circ \partial)^n(v)\Big), & \text{for } v \in V^{\leqslant n}. \end{cases}$$

Given $v \in V^q$ we compute

$$\begin{aligned}\partial(\theta_\beta(v)) &= \partial\left(\alpha(v) - F\Big(\sum_{n\geqslant 1}\frac{1}{n!}(S\circ\partial)^n(v)\Big) - v\right)\\ &= \alpha(\partial(v)) - \alpha\circ F\Big(\sum_{n\geqslant 1}\frac{1}{n!}(S\circ\partial)^n(v)\Big) - \partial(v)\\ &= F\circ e^{\theta}(\partial(v)) - F\circ D\Big(\sum_{n\geqslant 1}\frac{1}{n!}(S\circ\partial)^n(v)\Big) - \partial(v)\\ &= F\circ D\circ e^{\theta}(v) - F\circ D\Big(\sum_{n\geqslant 1}\frac{1}{n!}(S\circ\partial)^n(v)\Big) - \partial(v)\\ &= F\circ D(v+\widehat{v}) - \partial(v)\\ &= \partial(v) - \partial(v) = 0.\end{aligned}$$

Thus β satisfies (1). For (2) and (3), we define $G\colon (\Lambda(V,\overline{V},\widehat{V}),D)\to(\Lambda(V),\partial)$ by setting $G=F$ on $(\Lambda(V^{\leqslant n},\overline{V}^{\leqslant n},\widehat{V}^{\leqslant n}),D)$ while, for $v\in V^q$, we set $G(v)=\beta(v)$ and $G(\widehat{v})=G(\overline{v})=0$. First, it is easy to check that G is a cochain morphism. Next, for $v\in V^q$ we have

$$\begin{aligned}G\circ e^{\theta}(v) &= G\left(v+\widehat{v}+\sum_{n\geqslant 1}\frac{1}{n!}(S\circ\partial)^n(v)\right) = G(v) + G\left(\sum_{n\geqslant 1}\frac{1}{n!}(S\circ\partial)^n(v)\right)\\ &= \beta(v) + F\left(\sum_{n\geqslant 1}\frac{1}{n!}(S\circ\partial)^n(v)\right) = \alpha(v).\end{aligned}$$

Therefore $\beta\simeq\alpha$ and the lemma is proved. □

Thus Lemma 2.9 and the relation (8) allow us to define a map $\Phi\colon \ker \mathrm{g}\to \mathrm{Hom}\big(V^q, H^q(\Lambda(V^{\leqslant n}))\big)$ by setting $\Phi([\beta])(v)=\{\theta_\beta(v)\}$ for $v\in V^q$ where $[\beta]$ is chosen as in Lemma 2.9.

Proposition 2.10. *The map* $\Phi\colon \ker \mathrm{g}\to \mathrm{Hom}\big(V^q, H^q(\Lambda(V^{\leqslant n}))\big)$ *is an isomorphism.*

Proof. First we prove that Φ is well-defined. Suppose $[\beta]=[\beta']$ satisfy the conclusion of Lemma 2.9. Since both maps then restrict to the identity on $\Lambda(V^{\leqslant n})$, the homotopy $F\colon(\Lambda(V,\overline{V},\widehat{V}),D)\to(\Lambda(V),\partial)$ between them can be chosen so that

$$F(\widehat{V}^{\leqslant n}) = F(\overline{V}^{\leqslant n}) = 0. \tag{10}$$

Given $v\in V^q$, according to (8) we then have

$$\begin{aligned}\theta_{\beta'}(v)-\theta_\beta(v) &= \beta(v)-\beta'(v) = F\circ e^{\theta}(v) - F(v)\\ &= F(v') + F\Big(\sum\frac{1}{n!}(S\circ\partial)^n(v)\Big)\\ &= F(D(sv)) + F\Big(\sum_{n\geqslant 1}\frac{1}{n!}(S\circ\partial)^n(v)\Big)\\ &= \partial(F(sv)) + F\Big(\sum_{n\geqslant 1}\frac{1}{n!}(S\circ\partial)^n(v)\Big) = \partial(F(sv)).\end{aligned}$$

Thus $\theta_{\beta'}(v)-\theta_\beta(v)$ is a coboundary. Notice that the relation (10) implies that $F\Big(\sum_{n\geqslant 1}\frac{1}{n!}(S\circ\partial)^n(v)\Big)=0$.

For the injectivity, assume that $\Phi([\beta])(v)=\Phi([\beta'])(v)$ in $H^{q+1}(\Lambda(V^{\leqslant n}))$, then $\theta_{\beta'}(v)-\theta_\beta(v)=\beta(v)-\beta'(v)$ is a coboundary and Lemma 2.5 implies that $[\beta]=[\beta']$. For the surjectivity, given a homomorphism $\chi\in\operatorname{Hom}\big(V^q,H^q(\Lambda(V^{\leqslant n}))\big)$, write $\chi(v)=\{\widetilde{\chi(v)}\}$, where $\widetilde{\chi(v)}$ is a cocycle. We define $\beta\colon(\Lambda(V),\partial)\to(\Lambda(V),\partial)$ by

$$\beta(v)=v+\widetilde{\chi(v)}\quad\text{for}\quad v\in V^q\qquad\text{and}\qquad\beta=id\quad\text{on}\quad V^{\leqslant n}.$$

Then β is a cochain morphism with $\Phi([\beta])=\chi$.

Finally, given $\beta,\beta'\in\ker\mathrm{g}$ as in Lemma 2.9. So $\beta(v)=v+\theta_\beta(v)$ and $\beta'(v)=v+\theta_{\beta'}(v)$ for $v\in V^q$. Therefore, by (9) we get

$$\beta'\circ\beta(v)=v+\theta_{\beta'}(v)+\theta_\beta(v)=v+\theta_{\beta'\circ\beta}(v).$$

Consequently, $\Phi([\beta'].[\beta])=\Phi([\beta'\circ\beta])=\theta_{\beta'\circ\beta}=\theta_{\beta'}+\theta_\beta=\Phi([\beta'])+\Phi([\beta])$. Thus Φ is a homomorphism of groups. □

Summarizing, we have proven:

Theorem 2.11. *Let $q>n$ and let $(\Lambda(V^q\oplus V^{\leqslant n}),\partial)$ be a 1-connected fcca. Then there exists a short exact sequence of groups*

$$\operatorname{Hom}\big(V^q,H^q(\Lambda(V^{\leqslant n}))\big)\rightarrowtail\mathcal{E}(\Lambda(V^q\oplus V^{\leqslant n}))\overset{\mathrm{g}}{\twoheadrightarrow}\mathcal{D}^q_n. \tag{11}$$

We now focus on the subgroup $\mathcal{E}_\sharp(\Lambda(V^q\oplus V^{\leqslant n}))$ of $\mathcal{E}(\Lambda(V^q\oplus V^{\leqslant n}))$ of the elements inducing the identity on the graded vector space of indecomposables. Let us define $\mathcal{G}^q_n$ as the subgroup of $\mathcal{E}_\sharp(\Lambda(V^{\leqslant n}))$ of those elements $[\alpha]$ satisfying $H^{q+1}(\alpha)\circ b^q=b^q$ where $b^q\colon V^q\to H^{q+1}(\Lambda(V^{\leqslant n}))$ is as in (2).

Theorem 2.12. *Let $q>n$ and let $(\Lambda(V^q\oplus V^{\leqslant n}),\partial)$ be a 1-connected fcca. Then there exists a short exact sequence of groups*

$$\operatorname{Hom}\big(V^q,H^q(\Lambda(V^{\leqslant n}))\big)\rightarrowtail\mathcal{E}_\sharp(\Lambda(V^q\oplus V^{\leqslant n}))\twoheadrightarrow\mathcal{G}^q_n.$$

Proof. First let $[\alpha]\in\ker\mathrm{g}$. From the relation (7) we deduce that $\widetilde{\alpha}^q=id_{V^q}$ and $\alpha^{\leqslant n}\simeq id_{\Lambda(V^{\leqslant n})}$, therefore $\alpha^{\leqslant n}$ induces the identity on the indecomposables. So $\widetilde{\alpha}=id_V$. It follows that $\ker\mathrm{g}\subseteq\mathcal{E}_\sharp(\Lambda(V^q\oplus V^{\leqslant n}))$.

Next from (11) we obtain the short exact sequence

$$\operatorname{Hom}\big(V^q,H^q(\Lambda(V^{\leqslant n}))\big)\rightarrowtail\mathcal{E}_\sharp(\Lambda(V^q\oplus V^{\leqslant n}))\twoheadrightarrow\mathrm{g}\Big(\mathcal{E}_\sharp(\Lambda(V^q\oplus V^{\leqslant n}))\Big),$$

where

$$\mathrm{g}\Big(\mathcal{E}_\sharp(\Lambda(V^q\oplus V^{\leqslant n}))\Big)=\Big\{\Psi([\alpha])=(\widetilde{\alpha}^q,[\alpha^{\leqslant n}])\ \mid\ [\alpha]\in\mathcal{E}_\sharp(\Lambda(V^q\oplus V^{\leqslant n}))\Big\}.$$

As $[\alpha]\in\mathcal{E}_\sharp(\Lambda(V^q\oplus V^{\leqslant n}))$, the graded automorphism $\widetilde{\alpha}\in\operatorname{aut}(V^q\oplus V^{\leqslant n})$ is the identity which, in turn, implies

$$\mathrm{g}\Big(\mathcal{E}_\sharp(\Lambda(V^q\oplus V^{\leqslant n}))\Big)=\Big\{(\mathrm{id}_{V^q},[\alpha^{\leqslant n}])\ \mid\ [\alpha^{\leqslant n}]\in\mathcal{E}_\sharp(\Lambda(V^{\leqslant n}))\Big\}.$$

As $[\alpha]\in\mathcal{E}_\sharp(\Lambda(V^q\oplus V^{\leqslant n}))$, the pair $(\mathrm{id}_{V^q},[\alpha^{\leqslant n}])$ makes the diagram (3) commute. As a result we can identify $\mathrm{g}\Big(\mathcal{E}_\sharp(\Lambda(V^q\oplus V^{\leqslant n}))\Big)$ with the subgroup $\mathcal{G}^q_n$. □

Corollary 2.13. *Let $q > n$ and let $(\Lambda(V^q \oplus V^{\leqslant n}), \partial)$ be a 1-connected fcca. If $\mathcal{E}_\sharp(\Lambda(V^{\leqslant n}))$ is trivial, then*

$$\operatorname{Hom}\big(V^q, H^q(\Lambda(V^{\leqslant n}))\big) \cong \mathcal{E}_\sharp(\Lambda(V^q \oplus V^{\leqslant n})).$$

Corollary 2.14. *Let $q > 2n+1$ and let $(\Lambda(V^q \oplus V^{\leqslant 2n+1}), \partial)$ be an n-connected fcca. Then*

$$\operatorname{Hom}\big(V^q, H^q(\Lambda(V^{\leqslant 2n+1}))\big) \cong \mathcal{E}_\sharp(\Lambda(V^q \oplus V^{\leqslant 2n+1})).$$

Proof. As $(\Lambda(V^q \oplus V^{\leqslant 2n+1}), \partial)$ is n-connected, then $V^1 = \cdots = V^n = 0$. So, for degree reasons, the group $E_\sharp(\Lambda(V^{\leqslant 2n+1}))$ is trivial and we then apply Corollary 2.13. □

3. Topological applications

All the CW-complexes which we consider in this section are simply connected having rational homology of finite type.

Let $(\Lambda(V), \partial)$ be a 1-connected fcca. Recall that in [**5**] it is shown that with $(\Lambda(V), \partial)$ we can associate the following long exact sequence

$$\cdots \to V^n \xrightarrow{b^n} H^{n+1}(\Lambda(V^{\leqslant n-1})) \to H^{n+1}(\Lambda(V)) \to V^{n+1} \xrightarrow{b^{n+1}} \cdots,$$

called the Whitehead exact sequence of $(\Lambda(V), \partial)$. Recall that b^* is the graded linear map defined in (2).

Now let X be a simply connected rational CW-complex of finite type and let $X^{[n]}$ be the nth Postnikov section of X. For $q > n$, as in (1), let

$$X^{[n]} \cup_\alpha \Big(\bigcup_{i\in I} e_i^q\Big), \qquad \text{where } I \text{ is finite and } q > n,$$

be the space obtained by attaching rational cells to $X^{[n]}$ by a map $\alpha\colon \bigvee_{i\in I} S^q_{\mathbb{Q}} \to X^{[n]}$ where $S^q_{\mathbb{Q}}$ is the rational sphere of dimension q. If $(\Lambda(V), \partial)$ is the Sullivan model of X, then it is well-known that $(\Lambda(V^{\leqslant n}), \partial)$ is the Sullivan model of $X^{[n]}$ while $(\Lambda(V^q \oplus V^{\leqslant n}), \partial)$ is the Sullivan model of the pullback Z^q of the map $X^{[n]} \to K(\pi_q(X); q+1)$, whose homotopy class is the cohomology class in $H^{q+1}(X^{[n]}; \pi_q(X)) = [X^{[n]}; K(\pi_q(X); q+1)]$ given algebraically by

$$b^q \in \operatorname{Hom}\Big(V^q; H^{q+1}(\Lambda(V^{\leqslant n}))\Big) \cong H^{q+1}(X^{[n]}; \pi_q(X)),$$

over the path fibration:

$$\begin{array}{ccc} Z^q & \longrightarrow & PK(\pi_q(X); q+1) \\ \big\downarrow & & \big\downarrow \\ X^{[n]} & \longrightarrow & K(\pi_q(X); q+1). \end{array} \tag{12}$$

Observe that the fibration long exact sequence implies that $\pi_q(X) \cong \pi_q(Z^q)$. Hence by the virtues of the properties of the Sullivan model, the Whitehead exact sequence of $(\Lambda(V^q \oplus V^{\leqslant n}), \partial)$ yields the following exact sequence

$$\cdots \xrightarrow{h^q} \mathrm{Hom}\big(\pi_q(X),\mathbb{Q}\big) \xrightarrow{b^q} H^{q+1}(X^{[n]}) \to H^{q+1}(Z^q) \xrightarrow{h^{q+1}} \mathrm{Hom}\big(\pi_{q+1}(X),\mathbb{Q}\big) \xrightarrow{b^{q+1}} \cdots, \tag{13}$$

where h^* is the dual of the Hurewicz homomorphism.

Let D_n^q be the subgroup of $\mathrm{aut}\big(\mathrm{Hom}(\pi_q(X),\mathbb{Q}) \times \mathcal{E}(X^{[n]})\big)$ of those pairs $(\xi,[f])$ making the following diagram commute:

$$\begin{array}{ccc} \mathrm{Hom}\big(\pi_q(X),\mathbb{Q}\big) & \xrightarrow{\quad\xi\quad} & \mathrm{Hom}\big(\pi_q(X),\mathbb{Q}\big) \\ \downarrow{\scriptstyle b^q} & & \downarrow{\scriptstyle b^q} \\ H^{q+1}(X^{[n]}) & \xrightarrow{H^{q+1}(f)} & H^{q+1}(X^{[n]}) \end{array} \tag{14}$$

and let

$$G_n^q = \Big\{[f] \in \mathcal{E}_\sharp(X^{[n]}) \quad \text{such that} \quad H^{q+1}(f)\circ b^q = b^q\Big\}. \tag{15}$$

Clearly G_n^q is a subgroup of $\mathcal{E}_\sharp(X^{[n]})$. From Theorems 2.11 and 2.12 we deduce the following topological results:

Theorem 3.1. *Let X be a CW-complex. Then for every n and for every $q > n$ there exist two short exact sequence of groups:*

$$\begin{aligned} \mathrm{Hom}\big(\pi_q(X); H^q(X^{[n]})\big) &\rightarrowtail \mathcal{E}(Z^q) \twoheadrightarrow D_n^q, \\ \mathrm{Hom}\big(\pi_q(X); H^q(X^{[n]})\big) &\rightarrowtail \mathcal{E}_\sharp(Z^q) \twoheadrightarrow G_n^q. \end{aligned} \tag{16}$$

Moreover, if $\mathcal{E}_\sharp(X^{[n]})$ is a trivial group, then $\mathrm{Hom}\big(\pi_q(X); H^q(X^{[n]})\big) \cong \mathcal{E}_\sharp(Z^q)$. *Here Z^q is the space given in (12).*

Proof. The two sequences (16) follow from Theorems 2.11 and 2.12. Notice that Sullivan theory implies the following identifications:

$$\mathcal{E}(Z^q) \cong \mathcal{E}(\Lambda(V^q \oplus V^{\leqslant n})), \qquad E_\sharp(Z^q) \cong \mathcal{E}_\sharp(\Lambda(V^q \oplus V^{\leqslant n})),$$
$$D_n^q \cong \mathcal{D}_n^q, \qquad G_n^q \cong \mathcal{G}_n^q.$$

Finally, the last assertion follows by applying Corollary 2.13. □

Remark 3.2. According to Theorem 3.1, if we take $X^{[n]} = K(\pi,n)$, where π is a vector space of finite dimension, then we get

$$\mathcal{E}_\sharp(Z^q) \cong \mathrm{Hom}\big(\pi_q(X); H^q(K(\pi,n))\big).$$

Indeed, we know that the Sullivan model of $K(\pi,n)$ is $(\Lambda(V^n),0)$ where $V^n \cong H^n(K(\pi,n),\mathbb{Q})$. Therefore the group $\mathcal{E}_\sharp(K(\pi,n)) \cong \mathcal{E}_\sharp(\Lambda(V^n))$ is trivial.

Let X be a simply connected rational CW-complex of finite type. As the space Z^{n+1} coincides with $X^{[n+1]}$, Theorem 3.1 implies:

Corollary 3.3. *Let X be a simply connected rational CW-complex of finite type. The following two short sequences are exact:*

$$\mathrm{Hom}\big(\pi_q(X); H^q(X^{[n]})\big) \rightarrowtail \mathcal{E}(X^{[n+1]}) \twoheadrightarrow D_{n-1}^n,$$
$$\mathrm{Hom}\big(\pi_q(X); H^q(X^{[n]})\big) \rightarrowtail \mathcal{E}_\sharp(X^{[n+1]}) \twoheadrightarrow \mathrm{G}_{n-1}^n.$$

Moreover, if $\mathcal{E}_\sharp(X^{[n-1]})$ is finite, then $\mathrm{Hom}\big(\pi_q(X); H^q(X^{[n]})\big) \cong \mathcal{E}_\sharp(X^{[n]})$.

Now let us consider the dual of the Hurewicz homomorphism h^* given in the long exact sequence (13) and let $\mathcal{E}_\sharp^{(q+1)}(X^{[n]})$ denote the subgroup of $\mathcal{E}_\sharp(X^{[n]})$ consisting of the self-homotopy equivalences $[f]$ such that $H^{q+1}(f)\colon H^{q+1}(X^{[n]}) \to H^{q+1}(X^{[n]})$ is the identity.

Corollary 3.4. *Let X be a simply connected rational CW-complex and let Z^q be the space given in (12). Assume that h^q is nil and h^{q+1} is injective. There exist two short exact sequence of groups:*

$$\begin{aligned} &\mathrm{Hom}\big(\pi_q(X); H^q(X^{[n]})\big) \rightarrowtail \mathcal{E}(Z^q) \twoheadrightarrow \mathcal{E}^{(q+1)}(X^{[n]}),\\ &\mathrm{Hom}\big(\pi_q(X); H^q(X^{[n]})\big) \rightarrowtail \mathcal{E}_\sharp(Z^q) \twoheadrightarrow \mathcal{E}_\sharp^{(q+1)}(X^{[n]}). \end{aligned} \tag{17}$$

Proof. First notice that if h^q is nil and h^{q+1} is injective, then according to the long exact sequence (13) the map $b^q\colon \mathrm{Hom}\big(\pi_q(X), \mathbb{Q}\big) \to H^{q+1}(X^{[n]})$ is an isomorphism. Then for every $[f] \in \mathcal{E}(X^n)$ the pair $\big((b^q)^{-1} \circ H^{q+1}(f) \circ b^q, [f]\big)$ makes the diagram (14) commute. Therefore we get a map $\mathcal{E}(X^n) \to D_n^q$ defined by $[f] \mapsto \big((b^q)^{-1} \circ H^{q+1}(f) \circ b^q, [f]\big)$ and it is easy to see that it is an isomorphism of groups.

Likewise by (15) we can say that the group G_n^q coincides with the subgroup $\mathcal{E}_\sharp^{(q+1)}(X^n)$. Thus the sequences (16) imply the sequences (17). □

Corollary 3.5. *Let X be simply connected rational CW-complex of finite order. Assume that $\pi_i(X) \neq 0$ for $i = n-1, n, n+1$. If $\mathcal{E}(X^{[n]})$ is finite, then $\mathcal{E}(X^{[n+1]})$ and $\mathcal{E}(X^{[n-1]})$ are infinite.*

Proof. Working algebraically, we assume given a fcca $(\Lambda(V), \partial)$ with $V^i \neq 0$ for $i = n-1, n, n+1$ and suppose that $\mathcal{E}(\Lambda(V^{\leqslant n+1}))$ is finite. We prove $\mathcal{E}(\Lambda(V^{\leqslant n}))$ is infinite. The result then follows from the properties of the Sullivan model.

Since $\mathcal{E}(\Lambda(V^{\leqslant n+1}))$ is finite, applying Theorem 2.11 gives that $H^{n+1}(\Lambda(V^{\leqslant n-1})) = 0$. This implies the the linear map $b^n\colon V^n \to H^{n+1}(\Lambda(V^{\leqslant n-1})) = 0$ vanishes. Now by taking $\alpha = \mathrm{id}\colon (\Lambda(V^{\leqslant n-1}), \partial) \to (\Lambda(V^{\leqslant n-1}), \partial)$ and $\xi^a \in \mathrm{aut}(V^n), a \in \mathbb{Q}$, such that $\xi^a(v) = av$ for $v \in V^n$, the following diagram commutes obviously:

$$\begin{array}{ccc} V^n & \xrightarrow{\quad \xi^a \quad} & V^n \\ \Big\downarrow {\scriptstyle b^n = 0} & & \Big\downarrow {\scriptstyle b^n = 0} \\ H^{n+1}(\Lambda(V^{\leqslant n-1})) & \xrightarrow{H^{n+1}(\alpha) = \mathrm{id}} & H^{n+1}(\Lambda(V^{\leqslant n-1})). \end{array}$$

Therefore there exist an infinity of pairs $(\xi^a, [\mathrm{id}]) \in \mathcal{D}_{n-1}^n$, so the group $\mathcal{D}_{n-1}^n$ is infinite, it follows that $\mathcal{E}(\Lambda(V^{\leqslant n}))$ is infinite by Theorem 2.11. Finally, if $\mathcal{E}(\Lambda(V^{\leqslant n-1}))$ is finite, then by the above argument the group $\mathcal{E}(\Lambda(V^{\leqslant n}))$ must be infinite. Contradiction. □

Corollary 3.6. *Let X be a simply connected rational CW complex of finite type. Assume that $\mathcal{E}(Z^{n+2})$ is finite, then the space $X^{[n+1]}$ has the homotopy type of $X^{[n]} \times K(\pi_{n+1}(X), n+1)$.*

Proof. As $\mathcal{E}(Z^{n+2})$ is finite, Theorem 3.1 implies that

$$\mathrm{Hom}\big(\pi_q(X); H^q(X^{[n]})\big) = H^{n+2}(X^{[n]}, \pi_{n+1}(X)) = 0.$$

This implies that the Postnikov invariant $[k^n] \in H^{n+2}(X^{[n]}, \pi_{n+1}(X))$ is nil. As a result the space $X^{[n+1]}$ has the homotopy type of $X^{[n]} \times K(\pi_{n+1}(X), n+2)$. □

Remark 3.7. Let X be as in Corollary 3.6. Then if the Postnikov invariant $[k^{n+1}]$ is not nil, then $\mathcal{E}(Z^{n+2})$ is infinite.

3.1. Realization problem

Recall that the realizability problem for groups deals with the following question: Given a group G, is there a space X such that $G = \mathcal{E}(X)$? A complete answer to the realizability problem for finite groups is given by Costoya–Viruel [**9**]. Here, in this section, we give an alternative proof based on the main result of this work.

Let $G = \{g_1, g_2, \dots, g_n\}$ be a group of order n. By Cayley's theorem there is a monomorphism $G \to S_n$ given by $g_s \mapsto \sigma_s \colon g_k \longrightarrow g_s g_k$, $1 \leqslant k \leqslant n$. For $2 \leqslant s \leqslant n$ write $\sigma_s = \begin{pmatrix} 1 & 2 & \cdots & n \\ s & \sigma_s(2) & \cdots & \sigma_s(n) \end{pmatrix}$ and let

$$\sigma_2 = \Big(1\, 2\, \sigma_1(2) \dots, \sigma_2^{\kappa_2}(2)\Big)\Big(i_1 \sigma_1(i_1) \, \dots \, \sigma_2^{\kappa_{i_1}}(i_1)\Big) \dots \Big(i_k \, \sigma_1(i_k) \dots \sigma_2^{\kappa_{i_k}}(i_k)\Big) \quad (18)$$

be the decomposition of σ_2 as a product of cycles. Notice that the monomorphism $G \to S_n$ implies that

$$\sigma_{\sigma_s(i)}(j) = \sigma_s \circ \sigma_i(j), \qquad 1 \leqslant i, j, s \leqslant n. \quad (19)$$

For a group G, we define $\big(\Lambda(x_1, x_2, y_1, y_2, y_3, \{z_j, w_j\}_{1 \leqslant j \leqslant n}), \partial\big)$, where $|x_1| = 8$, $|x_2| = 10$, $|w_j| = 40$, by

$$\partial(x_1) = \partial(x_2) = \partial(w_j) = 0, \quad \partial(y_1) = x_1^3 x_2, \quad \partial(y_2) = x_1^2 x_2^2, \quad \partial(y_3) = x_1 x_2^3,$$

$$\partial(z_1) = w_1^3 + w_1 w_2 x_2^4 + \sum_{\tau=1}^{k} w_1 w_{i_\tau} x_2^4 + u + x_1^{15}, \quad u = y_1 y_2 x_1^4 x_2^2 - y_1 y_3 x_1^5 x_2 + y_2 y_3 x_1^6,$$

$$\partial(z_j) = w_j^3 + w_j w_{\sigma_j(2)} x_2^4 + \sum_{\tau=1}^{k} w_j w_{\sigma_s(i_\tau)} x_2^4 + u + x_1^{15}, \quad 2 \leqslant j \leqslant n. \quad (20)$$

Thus applying Theorem 2.11 to the fcca $\big(\Lambda(x_1, x_2, y_1, y_2, y_3, \{z_j, w_j\}_{1 \leqslant j \leqslant n}), \partial\big)$ we get the following exact sequence

$$H^{119}(\Lambda(x_1, x_2, y_1, y_2, y_3, w_1, \dots, w_n)) \rightarrowtail \mathcal{E}(\Lambda(x_1, x_2, y_1, y_2, y_3, \{z_j, w_j\}_{1 \leqslant j \leqslant n})) \twoheadrightarrow \mathcal{D}_{40}^{119}.$$

Lemma 3.8. $H^{119}(\Lambda(x_1, x_2, y_1, y_2, y_3, w_1, \dots, w_n)) = 0$.

Proof. First Λ^{119} is spanned by

$$y_1 x_1^7 x_2^3,\ y_1 x_1^2 x_2^7,\ y_2 x_1^3 x_2^6,\ y_2 x_1^8 x_2^2,\ y_3 x_1^9 x_2,\ y_3 x_1^4 x_2^5,\ y_1 w_j x_1^2 x_2^3,\ y_2 w_j x_1^3 x_2^2,\ y_3 w_j x_1^4 x_2.$$

Next, if

$$\phi = A_1 y_1 x_1^7 x_2^3 + A_2 y_1 x_1^2 x_2^7 + A_3 y_2 x_1^3 x_2^6 + A_4 y_2 x_1^8 x_2^2 + A_5 y_3 x_1^9 x_2 + A_6 y_3 x_1^4 x_2^5$$
$$+ (\sum_{j=1}^{n} A_7^{(j)} y_1 w_j x_1^2 x_2^3) + (\sum_{j=1}^{n} A_8^{(j)} y_2 w_j x_1^3 x_2^2) + (\sum_{j=1}^{n} A_9^{(j)} y_3 w_j x_1^4 x_2),$$

it follows that

$$\partial(\phi) = (A_1 + A_4 + A_5)x_1^{10}x_2^4 + (A_2 + A_3 + A_6)x_1^5x_2^8 + \Big(\sum_{j=1}^{n} A_7^{(j)} + A_8^{(j)} + A_9^{(j)}\Big)w_j x_1^5 x_2^4. \tag{21}$$

So the space of 119-cocycles is spanned by $y_3x_1^9x_2 - y_1x_1^7x_2^3$, $y_2x_1^2x_2^8 - y_1x_1^7x_2^3$, $y_1x_1^2x_2^7 - y_3x_1^4x_2^5$, $y_2x_1^3x_2^6 - y_3x_1^4x_2^5$, $y_3w_jx_1^4x_2 - y_1w_jx_1^2x_2^3$, $y_2w_jx_1^3x_2^2 - y_1w_jx_1^2x_2^3$.

But we have

$$\partial(y_3y_1x_2^6) = y_1x_1^7x_2^3 - y_3x_1^9x_2, \qquad \partial(y_2y_1x_1^5x_2) = y_2x_1^8x_2^2 - y_1x_1^7x_2^3,$$
$$\partial(y_3y_1x_1x_2^4) = y_1x_1^2x_2^7 - y_3x_1^4x_2^5, \qquad \partial(y_2y_3x_1^2x_2^3) = y_2x_1^3x_2^6 - y_3x_1^4x_2^5,$$
$$\partial(y_1y_2w_jx_2) = y_2w_jx_1^3x_2^2 - y_1w_jx_1^2x_2^3, \quad \partial(y_1y_3w_jx_1) = y_3w_jx_1^4x_2 - y_1w_jx_1^2x_2^3$$

and the lemma is proved. □

Let $g_s \in G$. For every $j \leqslant n$, define

$$\xi_s(z_j) = z_{\sigma_s(j)}, \qquad \alpha_s(w_j) = w_{\sigma_s(j)}, \qquad \alpha_s = id \quad \text{on} \quad x_1, x_2, y_1, y_2, y_3.$$

Clearly, $[\alpha_s] \in \mathcal{E}(\Lambda(x_1, x_2, y_1, y_2, y_3, w_1, \dots, w_n))$ and $b^{119} \circ \xi_s = H^{120}(\alpha_s) \circ b^{119}$. So we get a map $\Omega \colon G \longrightarrow \mathcal{D}_{40}^{119}$ defined by $\Omega(g_s) = (\xi_s, [\alpha_s])$.

Proposition 3.9. *Ω is an isomorphism of groups.*

Proof. Let $(\xi, [\alpha]) \in \mathcal{D}_{40}^{119}$. For degree reasons we have

$$\begin{aligned} &\alpha(x_1) = \beta_1x_1, \quad \alpha(x_2) = \beta_2x_2, \quad \alpha(y_1) = p_1y_1, \quad \alpha(y_2) = p_2y_2, \quad \alpha(y_3) = p_3y_3, \\ &\alpha(w_j) = a_{j1}w_1 + \dots + a_{jn}w_n + \gamma_{j1}x_1^5 + \gamma_{j2}x_2^4, \qquad \xi(z_j) = \lambda_{j1}z_1 + \dots + \lambda_{jn}z_n. \end{aligned} \tag{22}$$

As $\alpha(\partial(y_i)) = \partial(\alpha(y_i))$ it follows that

$$p_1 = \beta_1^3\beta_2, \qquad p_2 = \beta_1^2\beta_2^2, \qquad p_3 = c_1 = \beta_1\beta_2^3. \tag{23}$$

Thus

$$\alpha(\partial(z_j)) = \alpha(w_j^3) + \alpha(w_jw_{\sigma_j(2)}x_2^4) + \alpha\Big(\sum_{\tau=1}^{k} w_jw_{\sigma_j(i_\tau)}x_2^4\Big) + \alpha(u) + \alpha(x_1^{15}),$$

$$\partial(\xi(z_j)) = \lambda_{j1}\big(w_1^3 + w_1w_2x_2^4 + \sum_{\tau=1}^{k} w_1w_{\sigma_s(i_\tau)}x_2^4 + u + x_1^{15}\big) + \cdots$$
$$+\lambda_{jn}\big(w_n^3 + w_nw_{\sigma_n(2)}x_2^4 + \sum_{\tau=1}^{k} w_nw_{\sigma_s(i_\tau)}x_2^4 + v^2 + u + x_1^{15}\big).$$

As $(\xi, [\alpha]) \in \mathcal{D}_{40}^{119}$ and due to (5) there exists $\varphi_j \in \Lambda^{119}(x_1, x_2, y_1, y_2, y_3, w_1, \dots, w_n)$ such that

$$\partial(\xi(z_j)) - \alpha(\partial(z_j)) = \partial(\varphi_j), \qquad \forall j \leqslant n. \tag{24}$$

By expanding $\alpha(w_j^3) = \big(a_{j1}w_1 + \dots + a_{jn}w_n + \gamma_{j1}x_1^5 + \gamma_{j2}x_2^4\big)^3$ the monomials $a_{ji}^2a_{js}w_s^2w_t$, $\gamma_{j1}^2a_{js}x_1^{10}w_s, \gamma_{j2}^3x_2^{12}$ where $1 \leqslant s \neq t \leqslant n$ appear. As α is a homotopy equivalence it induces an isomorphism on the indecomposables. So $a_{j1},\dots,a_{jn}$ cannot be all nil. Equating the coefficients in (24) and (21) leads to

$$a_{js}^2 a_{jt}=0,\qquad \gamma_{j1}=\gamma_{j2}=0,\qquad 1\leqslant s\neq t\leqslant n,\quad 1\leqslant j\leqslant n. \tag{25}$$

As a result if $a_{1s}\neq 0$, then $a_{1j}=0$ for every $1\leqslant s\neq j\leqslant n$. It follows that $\alpha(w_1)=a_{1s}w_s$ and

$$\begin{aligned}\alpha(\partial(z_1))&=a_{1s}^3w_s^3+a_{1s}w_s\alpha(w_2x_2^4)+\sum_{\tau=1}^k a_{1s}w_s\alpha(w_{i_\tau}x_2^4)+\alpha(u)+\alpha(x_1^{15}),\\ \partial(\xi(z_1))&=\lambda_{s1}\big(w_s^2+w_sw_{\sigma_s(2)}x_2^4+\sum_{\tau=1}^k w_sw_{\sigma_t(i_\tau)}x_2^4+u+x_1^{15}\big)+\cdots.\end{aligned}\tag{26}$$

Notice the relations (23) imply that $\alpha(u)=\beta_1^9\beta_2^5(u)$ and $\alpha(x_1^{15})=\beta_1^{15}$. Consequently, equating the coefficients in (26) and using (22), (23), for every $\tau\leqslant k$ and for every $2\leqslant j\leqslant n$ we get

$$a_{1s}^3=a_{1s}a_{2\sigma_s(2)}\beta_2^4=a_{1s}a_{i_\tau\sigma_s(i_\tau)}\beta_2^4=\beta_1^9\beta_2^5=\beta_1^{15}=\lambda_{s1},\qquad \lambda_{sj}=0 \tag{27}$$

it follows that $a_{2\sigma_s(2)}\neq 0$ and $a_{i_\tau\sigma_s(i_\tau)}\neq 0$ for every $\tau\leqslant k$. So

$$\alpha(w_2)=a_{2\sigma_s(2)}w_2,\qquad \alpha(w_{i_\tau})=a_{i_\tau\sigma_s(i_\tau)}w_{i_\tau}.$$

Likewise we have

$$\begin{aligned}\alpha(\partial(z_2))&=a_{2\sigma_s(2)}^3w_{\sigma_s(2)}^3+a_{2\sigma_s(2)}w_{\sigma_s(2)}\alpha(w_{\sigma_2(2)}x_2^4)\\ &\qquad+\sum_{\tau=1}^k a_{2\sigma_s(2)}w_{\sigma_s(2)}\alpha(w_{\sigma_2(i_\tau)}x_2^4)+\alpha(u)+\alpha(x_1^{15}),\\ \partial(\xi(z_2))&=\lambda_{\sigma_s(2)2}\big(w_{\sigma_s(2)}^3+w_{\sigma_s(2)}w_{\sigma_{\sigma_s(2)}(2)}x_2^4\\ &\qquad+\sum_{\tau=1}^k w_{\sigma_s(2)}w_{\sigma_{\sigma_s(2)}(i_\tau)}x_2^4+u+x_1^{15}\big)+\dots.\end{aligned}\tag{28}$$

Due to (19) we can write:

$$w_{\sigma_{\sigma_s(2)}(2)}=w_{\sigma_s(\sigma_2(2))},\qquad w_{\sigma_{\sigma_s(2)}(i_\tau)}=w_{\sigma_s(\sigma_2(i_\tau))},\qquad \tau\leqslant k.$$

Taking in consideration (25) and comparing the coefficients in (28) we get

$$\begin{gathered}a_{2\sigma_s(2)}^3=a_{2\sigma_s(2)}a_{\sigma_s(2),\sigma_s(\sigma_2(2))}\beta_2^4=a_{2\sigma_s(2)}a_{\sigma_s(i_\tau),\sigma_s(\sigma_2(i_\tau))}\beta_2^4=\beta_1^9\beta_2^5=\beta_1^{15}=\lambda_{\sigma_s(2)2},\\ \lambda_{\sigma_s(2)j}=0,\qquad 1\leqslant j\neq 2\leqslant n,\quad \forall\tau\leqslant k.\end{gathered}\tag{29}$$

Now set $\sigma_2^p(2)=r_p$. By iterating the above process we get

$$\begin{aligned}a_{r_p\sigma_s(r_p)}^3&=a_{r_p\sigma_s(r_p)}a_{\sigma_s(r_p),\sigma_s(\sigma_2(r_p))}\beta_2^4=a_{r_p\sigma_s(r_p)}a_{\sigma_s(\sigma_2^p(i_\tau)),\sigma_s(\sigma_2^{p+1}(i_\tau))}\beta_2^4\\ &=\beta_1^9\beta_2^5=\beta_1^{15}=\lambda_{\sigma_s(r_p)r_p},\quad \lambda_{\sigma_s(r_p)j}=0,\quad 1\leqslant j\neq r_p\leqslant n.\end{aligned}\tag{30}$$

Now comparing (27), (29) and (30), for all $1\leqslant\tau\leqslant k$ we get

$$\begin{gathered}a_{1s}=a_{2\sigma_s(2)}=a_{r_p\sigma_s(r_p)}=\beta_2=\beta_1=\lambda_{s1}=\lambda_{\sigma_s(2)2}=\lambda_{\sigma_s(r_p)r_p}=1,\\ a_{i_\tau\sigma_s(i_\tau)}=a_{\sigma_s(i_\tau),\sigma_s(\sigma_2(i_\tau))}=a_{\sigma_s(\sigma_2^p(i_\tau)),\sigma_s(\sigma_2^{p+1}(i_\tau))}=\lambda_{\sigma_s(\sigma_2^{p+1}(i_\tau)),\sigma_s(\sigma_2^p(i_\tau))}=1.\end{gathered}$$

Hence, for every $\tau \leqslant k$ and $p \leqslant \kappa_2$,

$$\alpha(w_{\sigma_2^p(2)}) = w_{\sigma_s(\sigma_2^p(i_2))}, \qquad \alpha(w_{\sigma_2^p(i_\tau)}) = w_{\sigma_s(\sigma_2^p(i_\tau))}. \tag{31}$$

Finally, due to (18) we can say that every $1 \leqslant j \leqslant n$ occurs as $\sigma_2^p(2)$ or $\sigma_2^p(i_\tau)$, where $\tau \leqslant k$, for a certain p. Therefore from (31) we deduce that $\alpha(w_j) = \alpha(w_{\sigma_s(j)})$.

Summarizing, we have proven that if $(\xi, [\alpha]) \in \mathcal{D}_{40}^{119}$, then there exists a permutation $\sigma_s = \begin{pmatrix} 1 & 2 & \cdots & n \\ s & \sigma_s(2) & \cdots & \sigma_s(n) \end{pmatrix}$ such that

$$\xi(z_j) = z_{\sigma_s(j)}, \qquad \alpha(w_j) = w_{\sigma(j)}, \qquad \alpha = id, \qquad \text{on} \quad x_1, x_2, y_1, y_2, y_3.$$

This allows us to define $\Omega' \colon \mathcal{D}_{40}^{119} \to G$ by setting $\Omega'(\xi, [\alpha]) = g_s$, where the element g_s corresponds to the permutation σ_s via Cayley theorem. Clearly Ω' is the inverse of Ω and as $g_s g_{s'}$ correspond to the permutation $\sigma_s \circ \sigma_{s'}$ it is easy to see that Ω is an isomorphism. □

Applying Theorem 2.11, Lemma 3.8 and Proposition 3.9 we deduce that

$$\mathcal{E}(\Lambda(x_1, x_2, y_1, y_2, y_3, \{z_j, w_j\}_{1 \leqslant j \leqslant n})) \cong G$$

and by the Sullivan model there exists a simply connected CW-complex X such that $\mathcal{E}(X_{\mathbb{Q}}) \cong G$.

Remark 3.10. The fcca $\big(\Lambda(x_1, x_2, y_1, y_2, y_3, \{z_j, w_j\}_{1 \leqslant j \leqslant n}), \partial\big)$ given in (20), which is not elliptic, is a little modification of the elliptic fcca used by Costoya–Viruel in [**9**] to show their main result.

Using the arguments of the proof of Proposition 3.9 and omitting the details we can prove that:

Example 3.11. If we define $\big(\Lambda(x_1, x_2, y_1, y_2, y_3, \{z_n, w_n\}_{n \in \mathbb{Z}}), \partial\big)$, where $|x_1| = 8$, $|x_2| = 10$, $|w_n| = 40$ for all $n \in \mathbb{Z}$, by

$$\partial(x_1) = \partial(x_2) = \partial(w_n) = 0, \quad \partial(y_1) = x_1^3 x_2, \quad \partial(y_2) = x_1^2 x_2^2, \quad \partial(y_3) = x_1 x_2^3,$$
$$\partial(z_n) = w_n^3 + w_n w_{n+1} x_2^4 + y_1 y_2 x_1^4 x_2^2 - y_1 y_3 x_1^5 x_2 + y_2 y_3 x_1^6 + x_1^{15},$$

then

$$\mathcal{E}\big(\Lambda(x_1, x_2, y_1, y_2, y_3, \{z_n, w_n\}_{n \in \mathbb{Z}})\big) \cong \mathbb{Z}. \tag{32}$$

Indeed, let $m \in \mathbb{Z}$. For every $n \in \mathbb{Z}$, define

$$\xi_m(z_n) = z_{n+m}, \qquad \alpha_m(w_n) = w_{n+m}, \qquad \alpha_m = id \quad \text{on} \quad x_1, x_2, y_1, y_2, y_3,$$

so that we get a homomorphism:

$$\Omega \colon \mathbb{Z} \to \mathcal{D}_{40}^{119}, \qquad \Omega(m) = (\xi_m, [\alpha_m]).$$

Now if $(\xi, [\alpha]) \in \mathcal{D}_{40}^{119}$, then there is a unique $m \in \mathbb{Z}$ such that

$$\xi(z_n) = z_{n+m}, \qquad \alpha(w_n) = w_{n+m}, \qquad \alpha = id \quad \text{on} \quad x_1, x_2, y_1, y_2, y_3.$$

So Ω is an isomorphism. Finally, (32) follows by applying Theorem 2.11 and Lemma 3.8.

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