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DESCENT OF ALGEBRAIC CYCLES

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Abstract

We characterize universally generalizing morphisms which satisfy descent of algebraic cycles as those which are surjective with generically reduced fibres. In doing so, we introduce a naive pullback of cycles for arbitrary morphisms between noetherian schemes, which generalizes the classical pullback for flat morphisms, and then prove basic properties of this naive pullback.

1. Introduction

This paper discusses general descent properties of algebraic cycles. The basic question can be stated as follows: Let X, Y be noetherian schemes with groups of algebraic cycles $\mathcal{Z}^*(X), \mathcal{Z}^*(Y)$ and let $f\colon X \to Y$ be a surjective morphism. Which conditions on f guarantee that the *descent sequence*

$$0 \longrightarrow \mathcal{Z}^*(Y) \xrightarrow{f^*} \mathcal{Z}^*(X) \overset{\mathrm{pr}_1^* - \mathrm{pr}_2^*}{\longrightarrow} \mathcal{Z}^*(X \times_Y X) \tag{1}$$

is exact?

Inspired by known descent theory, it seems natural to assume f is faithfully flat, but this turns out to be insufficient. The obstruction is given by the possible non-reducedness of the fibres of f and can be made precise by introducing a (super)natural number $g_Y(f)$ which is defined as follows. For $y \in Y$ define the natural number

$$g_y(f) := \gcd\{\mathrm{length}(\mathcal{O}_{f^{-1}(\overline{\{y\}}),x}) \mid x \in f^{-1}(\overline{\{y\}}) \text{ generic } \},$$

which measures the non-reducedness of the subscheme $f^{-1}(\overline{\{y\}})$ at its generic points (which in general need not lie over y). Then set

$$g_Y(f) := \mathrm{lcm}\{g_y(f) \mid y \in Y \},$$

where the lcm is taken in the sense of supernatural numbers. The number $g_Y(f)$ therefore takes account of non-reducedness phenomena for all the fibres of f. By introducing a naive pullback of cycles for arbitrary morphisms (cf. Definition 2.3), flatness can be replaced by the weaker notion of a universally generalizing morphism. A morphism $f\colon X \to Y$ of schemes is called generalizing if for every $x \in X$ the induced

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morphism

$$f\colon \mathrm{Spec}(\mathcal{O}_{X,x}) \to \mathrm{Spec}(\mathcal{O}_{Y,f(x)})$$

is surjective. We call f universally generalizing if it stays generalizing after arbitrary base change (cf. [**EGAI**, Définition (3.9.2)]). Typical examples of universally generalizing morphisms are flat morphisms. We obtain the following answer to our question about descent of cycles:

Theorem 1.1. *Assume $f\colon X \to Y$ is a surjective universally generalizing morphism of noetherian schemes such that $X \times_Y X$ is again noetherian. Then the sequence* (1) *is exact up to torsion cohomology at $\mathcal{Z}^*(X)$ and this cohomology vanishes if and only if $g_Y(f) = 1$.*

In particular, descent of cycles holds rationally for arbitrary surjective universally generalizing morphisms $f\colon X \to Y$ between noetherian schemes such that $X \times_Y X$ is noetherian. If, moreover, all fibres of f are (generically) reduced, then in particular $g_Y(f) = 1$ and descent of cycles along f holds true *integrally*. Examples of such morphisms are of course surjective smooth morphisms.

The general question of descent of cycles, although appearing naturally in the discussion of cycle sheaves as in [**SV00**, Chapter 4.2], [**MVW06**, Lemma 3.2], does not seem to have been addressed in the common literature on algebraic cycles, e.g., [**Ful98**]. However, the statement that descent of algebraic cycles holds rationally or along étale morphisms should be known for a long time and the author does not claim originality for this, but was also not able to find precise references. It is worth mentioning that [**SV00**, Chapter 4.2] and [**MVW06**, Lemma 3.2] treat descent of relative cycles along morphisms of the base while our Theorem 1.1 deals with arbitrary absolute cycles and morphisms.

We start this paper by recalling the construction of cycles associated to subschemes and define a naive pullback of cycles. The construction of the naive pullback, as we present it here, exhibits basic problems. For example, the assignment $X \mapsto \mathcal{Z}^*(X)$ is not functorial for *all* scheme morphisms, only for *flat* ones (cf. Example 2.8). After discussing the push-forward of cycles along closed immersions, we will prove our main theorem Theorem 4.8, which then implies Theorem 1.1. As we try to be very general we also discuss the rather trivial case of descent along a universally bijective morphism (cf. Proposition 4.7).

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2. Cycles of subschemes and naive pullback of cycles

Let X be a scheme. We denote by

$$\mathcal{Z}^*(X) := \bigoplus_{x \in X} \mathbb{Z}x$$

the free abelian group generated by the set underlying the topological space of the scheme X and call elements in $\mathcal{Z}^*(X)$ cycles on X. If the local ring $\mathcal{O}_{X,x}$ has finite Krull dimension for every $x \in X$, e.g., if X is locally noetherian, the group $\mathcal{Z}^*(X)$ is naturally graded by setting

$$\mathcal{Z}^r(X) := \bigoplus_{\substack{x \in X, \\ \operatorname{codim}(\overline{\{x\}}, X) = r}} \mathbb{Z}x.$$

We want to attach cycles to closed subschemes. Defining more generally cycles attached to coherent modules turns out to be more flexible. Before giving the definition we recall that the support $\operatorname{Supp}(\mathcal{F})$ of a coherent sheaf $\mathcal{F}$ on a noetherian scheme X is the closed subscheme of X defined by the annihilator

$$\operatorname{Ann}_{\mathcal{O}_X}(\mathcal{F}) := \operatorname{Ker}(\mathcal{O}_X \to \underline{\operatorname{Hom}}(\mathcal{F}, \mathcal{F}))$$

of $\mathcal{F}$.

Definition 2.1. Let X be a noetherian scheme and let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module on X. We define

$$\operatorname{cycl}(\mathcal{F}) := \sum_{\substack{x \in \operatorname{Supp}(\mathcal{F}) \\ \text{generic point}}} \operatorname{length}_{\mathcal{O}_{X,x}}(\mathcal{F}_x) \cdot x \in \mathcal{Z}^*(X).$$

If $Z \subseteq X$ is a closed subscheme, then we set

$$\operatorname{cycl}(Z) := \operatorname{cycl}(\mathcal{O}_Z) = \sum_{\substack{x \in Z \\ \text{generic point}}} \operatorname{length}(\mathcal{O}_{Z,x}) \cdot x.$$

Assuming that X is noetherian is needed at two places. First that Z has only finitely many generic points and secondly to assure that the lengths at these generic points are finite. In general, these zero-dimensional local rings need not be artinian. We see that the theory of cycles (in our definition above) is from the start restricted to *noetherian* schemes.

Lemma 2.2. *Let X be a noetherian scheme and let*

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$

be a short exact sequence of coherent $\mathcal{O}_X$-modules such that Supp($\mathcal{F}$) = Supp($\mathcal{H}$) as subsets of X. Then

$$\mathit{cycl}(\mathcal{G}) = \mathit{cycl}(\mathcal{F}) + \mathit{cycl}(\mathcal{H}).$$

Proof. As $\operatorname{Supp}(\mathcal{G}) = \operatorname{Supp}(\mathcal{F}) \cup \operatorname{Supp}(\mathcal{H})$ we can conclude

$$\operatorname{Supp}(\mathcal{G}) = \operatorname{Supp}(\mathcal{F}) = \operatorname{Supp}(\mathcal{H}),$$

and hence the points where one of the sheaves $\mathcal{F}, \mathcal{G}, \mathcal{H}$ is of finite length are precisely the generic points of $\operatorname{Supp}(\mathcal{G})$. As lengths are additive in short exact sequences the lemma follows. □

Definition 2.3. Let $f \colon X \to Y$ be a morphism of noetherian schemes. We define the naive pullback

$$f^{*,\mathrm{naive}}\colon \mathcal{Z}^*(Y) \to \mathcal{Z}^*(X)$$

by linear extension of the map

$$y \in Y \mapsto \mathrm{cycl}(f^{-1}(\overline{\{y\}})) \in \mathcal{Z}^*(X),$$

where $f^{-1}(\overline{\{y\}})$ denotes the scheme-theoretic pullback of the closed reduced subscheme $\overline{\{y\}} \subseteq Y$.

We will see that this definition lacks some properties if f is not flat. First for $y \in Y$ the generic points of $f^{-1}(\overline{\{y\}})$ need not lie over y as examples of closed immersions show. This problem can be solved by requiring that f is *generalizing* (cf. [**EGAI**, Définition (3.9.2)]).

Definition 2.4. A morphism $f\colon X \to Y$ of schemes is called generalizing if for all $x \in X$ the induced morphism

$$f\colon \mathrm{Spec}(\mathcal{O}_{X,x}) \to \mathrm{Spec}(\mathcal{O}_{Y,f(x)})$$

is surjective. Moreover, f is called universally generalizing, if every base change of f is generalizing.

If $f\colon X \to Y$ is generalizing, then for all $y \in Y$ the generic points of $f^{-1}(\overline{\{y\}})$ lie over y. This property turns out to be of fundamental importance for our question. Examples of generalizing morphisms are flat morphisms or open morphisms. Conversely, generalizing morphisms which are locally of finite presentation are open. Another source of examples for universally open morphisms, usually not flat, can be given as follows.

Example 2.5. Assume that $f\colon X \to Y$ is a finite morphism, which is the normalization of a geometrically unibranch integral noetherian scheme Y inside a finite field extension at the generic point of Y. It is known that f satisfies the going-down theorem, i.e., f is generalizing. As f is of finite type and Y noetherian, f is an open morphism. Finally, [**EGAIV**, Corollaire (14.4.3)] shows that f is universally open as Y is geometrically unibranch.

The definition of $f^{*,\mathrm{naive}}$ involves the calculations of lengths, hence a purely topological condition like generalizing cannot be sufficient in the following proposition.

Proposition 2.6. *Let $f\colon X \to Y$ be a flat morphism of noetherian schemes and let $Z \subseteq Y$ be a closed subscheme. Then*

$$f^{*,naive}(cycl(Z)) = cycl(f^{-1}(Z)).$$

Proof. This is proven in [**Ful98**, Lemma 1.7.1]. □

The necessity of flatness in Proposition 2.6 has consequences for the functoriality of $f \mapsto f^{*,\mathrm{naive}}$ and is the main problem encountered with the naive pullback of cycles in the absence of flatness.

Proposition 2.7. *Let $f\colon X \to Y$ and $g\colon Y \to Z$ be morphisms of noetherian schemes. Assume that f is flat. Then*

$$(g \circ f)^{*,naive} = f^{*,naive} \circ g^{*,naive}.$$

Proof. Let $z \in Z$ be a point. Then

$$(g \circ f)^{*,\mathrm{naive}}(z) = \mathrm{cycl}(f^{-1}g^{-1}(\overline{\{z\}})),$$

while

$$f^{*,\mathrm{naive}} \circ g^{*,\mathrm{naive}}(z) = f^{*,\mathrm{naive}}(\mathrm{cycl}(g^{-1}(\overline{\{z\}}))).$$

Both cycles agree by Proposition 2.6 as f is flat. □

As a first example that flatness of f is really needed in Proposition 2.7 one can take $g\colon Y \to Z = \mathrm{Spec}(k)$ a (noetherian) scheme over a field and $f\colon X = Y_{\mathrm{red}} \to Y$ the natural closed immersion. Then in general

$$\begin{aligned}(g \circ f)^{*,\mathrm{naive}}(\mathrm{cycl}(Z)) &= \mathrm{cycl}(Y_{\mathrm{red}}) \\ &\neq f^{*,\mathrm{naive}}(\mathrm{cycl}(Y)) \\ &= f^{*,\mathrm{naive}}(g^{*,\mathrm{naive}}(\mathrm{cycl}(Z))).\end{aligned}$$

An example with smooth schemes is the following.

Example 2.8. Consider now $Y = \mathrm{Spec}(k[t, x]/(x^n - t))$ with $n \geqslant 2$, $X = \mathrm{Spec}(k[t]/(t))$, $Z = \mathrm{Spec}(k[t])$ and as morphisms

$$\begin{aligned}&f\colon X \to Y,\ t \mapsto (t, 0),\\ &g\colon Y \to Z,\ (t, x) \mapsto t.\end{aligned}$$

Define $\mathcal{Z} := \mathrm{cycl}(k[t]/(t)) \in \mathcal{Z}^*(Z)$ which is a cycle on Z. Then

$$(g \circ f)^{*,\mathrm{naive}}(\mathcal{Z}) = \mathrm{cycl}(k[t]/(t)) \in \mathcal{Z}^*(X),$$

while

$$f^{*,\mathrm{naive}}(g^{*,\mathrm{naive}}(\mathcal{Z})) = n \cdot \mathrm{cycl}(k[t]/(t)).$$

Proposition 2.7 is even wrong if g is flat and f a universal homeomorphism, therefore in particular generalizing. We give an example.

Example 2.9. Let $X = \mathrm{Spec}(k[t])$ be the affine line over a field k and consider $Y = \mathrm{Spec}(k[t^2, t^3])$, which is a curve with a cusp at the ideal (t^2, t^3) and normalization X. We take $Z = \mathrm{Spec}(k[t^2])$ and $f\colon X \to Y$, $g\colon Y \to Z$ the morphisms given by the inclusions

$$k[t^2] \subseteq k[t^2, t^3] \subseteq k[t].$$

Then g is flat and f a universal homeomorphism. Let

$$\mathcal{Z} := \mathrm{cycl}(k[t^2]/(t^2)) \in \mathcal{Z}^*(Z).$$

Then

$$g^{*,\mathrm{naive}}(\mathcal{Z}) = \mathrm{cycl}(k[t^2, t^3]/(t^2)) = 2 \cdot \mathrm{cycl}(k[t^2, t^3]/(t^2, t^3)),$$

hence

$$f^{*,\mathrm{naive}}(g^{*,\mathrm{naive}}(\mathcal{Z})) = 4 \cdot \mathrm{cycl}(k[t]/(t)).$$

On the other hand,

$$(g \circ f)^{*,\mathrm{naive}}(\mathcal{Z}) = \mathrm{cycl}(k[t]/(t^2)) = 2 \cdot \mathrm{cycl}(k[t]/(t)).$$

This example can also be modified to show that the naive pullback does not preserve rational equivalence, even for universal homeomorphisms. In fact, the morphism

$f\colon X \to Y$ extends uniquely to the canonical compactifications X', Y' of X, Y giving a morphism $f\colon X' \to Y'$. If we denote by ∞ the boundary of $X \subseteq X'$, resp. $Y \subseteq Y'$, then $f(\infty) = \infty$. The zero divisor of $t^2 \in k[t^2, t^3]$ is

$$\mathcal{Z} := 2 \cdot \mathrm{cycl}(k[t^2, t^3]/(t^2, t^3)) - 2 \cdot \infty,$$

which has a pullback

$$f^{*,\mathrm{naive}}(\mathcal{Z}) = 4 \cdot \mathrm{cycl}(k[t]/(t)) - 2 \cdot \infty$$

not rationally equivalent to zero on $X' \cong \mathbb{P}^1_k$.

Proposition 2.7 shows that the assignment $f \mapsto f^{*,\mathrm{naive}}$ is functorial for *flat* maps between noetherian schemes. For a flat map f we therefore abbreviate $f^{*,\mathrm{naive}}$ by f^* as this is the pull back of cycles usually encountered (for example, in [**Ful98**]). However, in Proposition 3.5 and Lemma 3.6 we investigate situations guaranteeing functoriality for the naive pullback.

Proposition 2.10. *Let $f\colon X \to Y$ be a surjective morphism of noetherian schemes. Then*

$$f^{*,naive}\colon \mathcal{Z}^*(Y) \to \mathcal{Z}^*(X)$$

is injective. If in addition f is generalizing, then conversely injectivity of $f^{*,naive}$ implies surjectivity of f.

Proof. We note that under the assumption that f is generalizing the morphism $f^{*,\mathrm{naive}}$ is the direct sum over $y \in Y$ of the maps

$$f^{*,\mathrm{naive}}_{|\mathbb{Z}y}\colon \mathbb{Z}y \to \bigoplus_{x \in f^{-1}(y)} \mathbb{Z}x,$$

because for $y \in Y$ the generic points of the preimage $f^{-1}(\overline{\{y\}})$ all lie over y. In particular, $f^{*,\mathrm{naive}}$ is injective if and only if each $f^{*,\mathrm{naive}}_{|\mathbb{Z}y}$ is injective. But $f^{*,\mathrm{naive}}_{\mathbb{Z}y}$ is injective if and only if $f^{-1}(y) \neq \emptyset$. Hence all statements are, in fact, obvious if f is generalizing. This being said, we now turn to the general case and assume that f is only surjective, but not necessarily generalizing. Let

$$\mathcal{Z} := \sum_{i=1}^{n} m_i y_i \in \mathcal{Z}^*(Y)$$

be a cycle with $f^{*,\mathrm{naive}}(\mathcal{Z}) = 0$. We may assume that the y_i are pairwise distinct and $m_i \neq 0$ for every i. Moreover, we can assume that

$$\mathrm{codim}(\overline{\{y_1\}}, Y) \leqslant \ldots \leqslant \mathrm{codim}(\overline{\{y_n\}}, Y).$$

Let $x_1 \in f^{-1}(y_1)$ be a generic point of the fibre $f^{-1}(y_1)$ (which is non-empty because f is surjective). Because

$$-m_1 f^*(y_1) = \sum_{i=2}^{n} m_i f^*(y_i),$$

there exists some $y_j, j \geqslant 2$, such that x_1 is a generic point of $f^{-1}(\overline{\{y_j\}})$. In particular, $y_1 = f(x_1)$ is a specialization of y_j and as $\mathrm{codim}(\overline{\{y_1\}}, Y) \leqslant \mathrm{codim}(\overline{\{y_j\}}, Y)$ we get $y_j = y_1$, a contradiction. □

We give an example, that the injectivity of $f^{*,\text{naive}}$ does not necessarily imply that f is surjective.

Example 2.11. Let $Y := \operatorname{Spec}(R)$ be the spectrum of a discrete valuation ring R with residue field $k := R/\mathfrak{m}$ and denote by η and $s = \operatorname{Spec}(k)$ its generic and special point respectively. Consider the schemes $X_1 := \operatorname{Spec}(R/\mathfrak{m}^n)$ and $X_2 := s$ and let $x_i \in X_i$ be their unique points. Let $f\colon X = X_1 \coprod X_2 \to Y$ denote the natural morphism. Then

$$f^{*,\text{naive}}(\eta) = nx_1 + x_2,$$
$$f^{*,\text{naive}}(s) = x_1 + x_2.$$

In particular, for $n \geqslant 2$ both cycles are linearly independent. Hence,

$$f^{*,\text{naive}}\colon \mathcal{Z}^*(Y) \to \mathcal{Z}^*(X)$$

is injective although f is not surjective.

In general, the naive pullback does not preserve the natural grading of $\mathcal{Z}^*(X)$ given by codimension. With the notation of Example 2.11, one can take the natural morphism $s \coprod Y \to Y$ as an example. More serious examples include blow-ups, for example, that of points on surfaces. However, in the case of a generalizing morphism this problem disappears as will be recalled in Proposition 2.12.

Proposition 2.12. *Let $f\colon X \to Y$ be a generalizing morphism of noetherian schemes and $y \in Y$. Then for every generic point $x \in f^{-1}(y)$*

$$\mathit{codim}(\overline{\{y\}}, Y) = \mathit{codim}(\overline{\{x\}}, X).$$

In particular, the homomorphism

$$f^{*,\mathit{naive}}\colon \mathcal{Z}^*(Y) \to \mathcal{Z}^*(X)$$

respects the grading by codimension.

Proof. Let $A := \mathcal{O}_{Y,y}$ resp. $B := \mathcal{O}_{X,x}$ be the local rings at $y \in Y$ resp. $x \in X$ and let $\mathfrak{m} := \mathfrak{m}_{Y,y}$ be the maximal ideals in A. As $x \in f^{-1}(y)$ is a generic point, the ring

$$B/\mathfrak{m}B$$

is artinian. More generally, the ring B/IB is artinian for every $\mathfrak{m}$-primary ideal $I \subseteq A$. In other words, for every ideal of definition $I \subseteq A$, that is every $\mathfrak{m}$-primary ideal, the ideal IB is an ideal of definition for B. As the Krull dimension of any local noetherian ring can be computed as the minimal number of generators for ideals of definition, we can conclude

$$\dim(B) \leqslant \dim(A).$$

The morphism $f\colon \operatorname{Spec}(B) \to \operatorname{Spec}(A)$ is generalizing, hence every chain

$$y_n \rightsquigarrow y_{n-1} \rightsquigarrow \ldots \rightsquigarrow y_0$$

of specializations in $\operatorname{Spec}(A)$ can be lifted to a chain

$$x_n \rightsquigarrow x_{n-1} \rightsquigarrow \ldots \rightsquigarrow x_0$$

of specializations in $\operatorname{Spec}(B)$ with $f(x_i) = y_i$. In particular, $\dim(B) \geqslant \dim(A)$ and therefore $\dim(B) = \dim(A)$. By definition, $\dim(B) = \operatorname{codim}(\overline{\{x\}}, X)$ resp. $\dim(A) = \operatorname{codim}(\overline{\{y\}}, Y)$ and the proposition is proven. $\square$

In some easy cases functoriality can be checked directly.

Lemma 2.13. *Let Y be a noetherian scheme and let $f\colon X \to Z$ be a morphism of noetherian schemes over Y. Let $g\colon Y' \to Y$ be an open immersion or the Zariski localization $Y' = Spec(\mathcal{O}_{Y,y})$ at a point $y \in Y$. Then the diagram*

$$\begin{array}{ccc} \mathcal{Z}^*(Z) & \xrightarrow{f^{*,naive}} & \mathcal{Z}^*(X) \\ \downarrow{\scriptstyle p^*} & & \downarrow{\scriptstyle q^*} \\ \mathcal{Z}^*(Z') & \xrightarrow{f'^{*,naive}} & \mathcal{Z}^*(X') \end{array}$$

commutes, where

$$f'\colon X' := X \times_Y Y' \to Z' := Z \times_Y Y'$$

denotes the base change of f to Y' and $p\colon X' \to X$ resp. $q\colon Z' \to Z$ the natural projections.

Proof. Let $z \in Z$ be a point. If $z \notin Z'$, then the claim is immediate as Z' is stable under generalizations. We assume $z \in Z'$ and denote by $\overline{\{z\}}_Z$ the closure of z in Z. Then $\overline{\{z\}}_Z \cap Z' = \overline{\{z\}}_{Z'}$ is the (reduced) closure of z in Z' and therefore the diagram

$$\begin{array}{ccc} \overline{\{z\}}_{Z'} & \longrightarrow & \overline{\{z\}}_Z \\ \downarrow & & \downarrow \\ Z' & \xrightarrow{p} & Z \end{array}$$

is cartesian. Base change along $f\colon X \to Z$ yields the cartesian diagram

$$\begin{array}{ccc} f'^{-1}(\overline{\{z\}}_{Z'}) & \longrightarrow & f^{-1}(\overline{\{z\}}_Z) \\ \downarrow & & \downarrow \\ X' & \xrightarrow{q} & X \end{array}$$

and as $q\colon X' \to X$ is flat, we can conclude by Proposition 2.6

$$\begin{aligned} q^*(f^{*,\mathrm{naive}}(z)) &= q^*(\mathrm{cycl}(f^{-1}(\overline{\{z\}}_Z))) \\ &= \mathrm{cycl}(q^{-1}(f^{-1}(\overline{\{z\}}_Z))) \\ &= \mathrm{cycl}(f'^{-1}(p^{-1}(\overline{\{z\}}_Z))) \\ &= \mathrm{cycl}(f'^{-1}(\overline{\{z\}}_{Z'})) \\ &= f'^{*,\mathrm{naive}}(z) \\ &= f'^{*,\mathrm{naive}}(p^*(z)). \end{aligned}$$

Here we denoted by z both the cycles $z \in \mathcal{Z}^*(Z)$ and $p^*(z) = z \in \mathcal{Z}^*(Z')$. □

As a special case of Lemma 2.13 we note that if $f\colon X \to Y$ is a morphism of noetherian schemes such that $X \times_Y X$ is again noetherian and $g\colon Y' \to Y$ as in Lemma 2.13, then the diagram

$$\begin{array}{ccccc}
\mathcal{Z}^*(Y) & \xrightarrow{f^{*,\mathrm{naive}}} & \mathcal{Z}^*(X) & \xrightarrow{\mathrm{pr}_1^{*,\mathrm{naive}} - \mathrm{pr}_2^{*,\mathrm{naive}}} & \mathcal{Z}^*(X \times_Y X) \\
\downarrow{\scriptstyle g^*} & & \downarrow{\scriptstyle g'^*} & & \downarrow{\scriptstyle (g' \times g')^*} \\
\mathcal{Z}^*(Y') & \xrightarrow{f'^{*,\mathrm{naive}}} & \mathcal{Z}^*(X') & \xrightarrow{\mathrm{pr'}_1^{*,\mathrm{naive}} - \mathrm{pr'}_2^{*,\mathrm{naive}}} & \mathcal{Z}^*(X' \times_{Y'} X')
\end{array}$$

with obvious notations commutes.

In Proposition 3.5 we will partly generalize Lemma 2.13.

3. Push forward of cycles for closed immersions

We will need some limited covariant functoriality of cycle groups.

Definition 3.1. Let Y be a noetherian scheme and let $f\colon X \to Y$ be a closed immersion. We define (following, e.g., [**Ful98**]) the push-forward along f by

$$f_*\colon \mathcal{Z}^*(X) \to \mathcal{Z}^*(Y), \sum m_x x \mapsto \sum m_x f(x).$$

Proposition 3.2. *Let Y be a noetherian scheme and let $f\colon X \to Y$ be a closed immersion. Then for all closed subschemes $Z \subseteq X$ we have the equality*

$$f_*(cycl(Z)) = cycl(f(Z)).$$

Proof. The morphism $f\colon Z \to f(Z)$ is an isomorphism, hence generic points are preserved by f. As for such a generic point $z \in Z$ the lengths

$$\mathrm{length}(\mathcal{O}_{Z,z}) = \mathrm{length}(\mathcal{O}_{f(Z),f(z)})$$

agree, the claim follows. □

Proposition 3.3. *Let $f\colon X \to Y$ be a morphism of noetherian schemes. Then for all closed immersions $i\colon Z \to Y$ the diagram*

$$\begin{array}{ccc}
\mathcal{Z}^*(Z) & \xrightarrow{i_*} & \mathcal{Z}^*(Y) \\
\downarrow{\scriptstyle f'^{*,naive}} & & \downarrow{\scriptstyle f^{*,naive}} \\
\mathcal{Z}^*(f^{-1}(Z)) & \xrightarrow{i'_*} & \mathcal{Z}^*(X)
\end{array}$$

is commutative, where $i'\colon f^{-1}(Z) \to X$ resp. $f'\colon f^{-1}(Z) \to Z$ denote the base change of i resp. f.

Proof. Let $z \in Z$ be a point. Then

$$f^{*,\mathrm{naive}}(i_*(z)) = \mathrm{cycl}(f^{-1}(\overline{\{i(z)\}})) = \mathrm{cycl}(i'(f'^{-1}(\overline{\{z\}}))).$$

By Proposition 3.2, this equals

$$i'_*(\mathrm{cycl}(f'^{-1}(\overline{\{z\}}))) = i'_*(f'^{*,\mathrm{naive}}(z))$$

and the proof is finished. □

Recall that a morphism $f\colon X \to Y$ is called weakly immersive if f is a homeomorphism onto its image and for each $x \in X$ the induced morphism $f\colon \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is surjective.

Lemma 3.4. *Let $f\colon X \to Y$ be a generalizing morphism of schemes and assume that $g\colon Y' \to Y$ is weakly immersive. Then the base change*

$$f'\colon X' := X \times_Y Y' \to Y'$$

is again generalizing.

Proof. Let $x' \in X'$ be a point over $y' = f'(x')$ and $x = g'(x')$, $g'\colon X' \to X$. Let $y := f(x) = g(y')$. If $I := \operatorname{Ker}(\mathcal{O}_{Y,y} \to \mathcal{O}_{Y',y'})$ denotes the kernel, then

$$\mathcal{O}_{X',x'} \cong \mathcal{O}_{X,x}/I\mathcal{O}_{X,x}$$

and hence the diagram

$$\begin{array}{ccc} \operatorname{Spec}(\mathcal{O}_{X',x'}) & \longrightarrow & \operatorname{Spec}(\mathcal{O}_{Y',y'}) \\ \downarrow & & \downarrow \\ \operatorname{Spec}(\mathcal{O}_{X,x}) & \longrightarrow & \operatorname{Spec}(\mathcal{O}_{Y,y}) \end{array}$$

is cartesian. In particular, the upper arrow is surjective as f is generalizing, showing that f' is generalizing, too. □

The next result will not be used in the sequel but seems interesting in its own right. Namely, we will partly generalize Proposition 2.7 to generalizing morphisms, which have generically reduced fibres.

Proposition 3.5. *Let $f\colon X \to Y$, $g\colon Y \to Z$ be morphisms of noetherian schemes. Assume that f is generalizing and that g is generalizing with each fibre generically reduced. Then*

$$f^{*,naive} \circ g^{*,naive} = (g \circ f)^{*,naive}.$$

Proof. We have to show $f^{*,\mathrm{naive}} \circ g^{*,\mathrm{naive}}(\mathcal{Z}) = (g \circ f)^{*,\mathrm{naive}}(\mathcal{Z})$ for every cycle $\mathcal{Z} \in \mathcal{Z}^*(Z)$. By Lemma 3.4 we may assume, using Proposition 3.3, that Z is integral and $\mathcal{Z} = \eta$ for the generic point $\eta \in Z$. By Lemma 2.13 (and Lemma 3.4) we may further assume $Z = \operatorname{Spec}(k(\eta))$ as all generic points in Y and X lie over η by assumption. We have to check an equality of cycles whose components all lie over generic points of Y. Hence, we may replace Y by the disjoint union of the spectra of the local rings at the generic points of Y, arriving at a situation, where Y is the spectrum of a finite product of fields as Y is generically reduced. We can conclude that

$$g^*(\eta) = \operatorname{cycl}(Y) = \sum_{y \in Y} y$$

and hence

$$f^{*,\mathrm{naive}}(g^{*,\mathrm{naive}}(\eta)) = f^{*,\mathrm{naive}}(\operatorname{cycl}(Y)) = \operatorname{cycl}(X) = (g \circ f)^{*,\mathrm{naive}}(\eta)$$

as $X = \coprod_{y \in Y} f^{-1}(y)$ with each $f^{-1}(y)$ open. Therefore, we are finished with the proof. □

We can record another situation, where the naive pullback of cycles is well-behaved.

Lemma 3.6. *Consider a cartesian diagram*

$$\begin{array}{ccc} X' & \xrightarrow{p} & Y' \\ \downarrow q & & \downarrow g \\ X & \xrightarrow{f} & Y \end{array}$$

of morphisms of noetherian schemes with all morphisms generalizing. Then

$$p^{*,naive} \circ g^{*,naive} = (g \circ p)^{*,naive} = (f \circ q)^{*,naive} = q^{*,naive} \circ f^{*,naive}.$$

Proof. Take $y \in Y$. As f, g, p, q are generalizing, we can, by Lemma 3.4, Proposition 3.3 and Lemma 2.13, assume that $Y = \mathrm{Spec}(k(y))$ is a point. Then f, g and hence p, q are flat and the claim follows from Proposition 2.7. □

4. Descent of algebraic cycles

In this section, let $f\colon X \to Y$ be a surjective morphism of noetherian schemes. We assume that $X \times_Y X$ is again noetherian[1] and denote by

$$\mathrm{pr}_i\colon X \times_Y X \to X,\ i = 1, 2,$$

the two projections.

In the absence of flatness the following lemma is not obvious.

Lemma 4.1. *If f is universally generalizing, the sequence of homomorphisms*

$$0 \longrightarrow \mathcal{Z}^*(Y) \xrightarrow{f^{*,naive}} \mathcal{Z}^*(X) \xrightarrow{p_1 - p_2} \mathcal{Z}^*(X \times_Y X) \qquad (2)$$

is a complex, where $p_i := pr_i^{*,naive}$ for $i = 1, 2$.

Proof. This is a consequence of Lemma 3.6. □

We give an example that (2) need not be a complex in general.

Example 4.2. Let $f\colon X \to Y, \eta, x_1$ and x_2 be as in Example 2.11. We compute

$$\mathrm{pr}_i^{*,\mathrm{naive}}(f^{*,\mathrm{naive}}(\eta)) = \mathrm{pr}_i^{*,\mathrm{naive}}(nx_1 + x_2),$$

for $i = 1, 2$. By definition,

$$\mathrm{pr}_1^{*,\mathrm{naive}}(x_1) = (x_1, x_1) + (x_1, x_2),$$
$$\mathrm{pr}_1^{*,\mathrm{naive}}(x_2) = (x_2, x_1) + (x_2, x_2),$$

while

$$\mathrm{pr}_2^{*,\mathrm{naive}}(x_1) = (x_1, x_1) + (x_2, x_1),$$
$$\mathrm{pr}_2^{*,\mathrm{naive}}(x_2) = (x_1, x_2) + (x_2, x_2).$$

We get

$$\mathrm{pr}_1^{*,\mathrm{naive}}(nx_1 + x_2) = n(x_1, x_1) + n(x_1, x_2) + (x_2, x_1) + (x_2, x_2)$$

[1] Which excludes, for example, proétale covers like $\mathrm{Spec}(k^{\mathrm{sep}}) \to \mathrm{Spec}(k)$ for (most) fields k.

and

$$\mathrm{pr}_2^{*,\mathrm{naive}}(nx_1 + x_2) = n(x_1, x_1) + n(x_2, x_1) + (x_1, x_2) + (x_2, x_2),$$

which are different cycles if $n \geqslant 2$.

This example looks puzzling if compared to the situation for *subschemes*. The transitivity of fibre products implies that for $Z \subseteq Y$ a closed subscheme the pullbacks $\mathrm{pr}_1^{-1}(f^{-1}(Z))$ and $\mathrm{pr}_2^{-1}(f^{-1}(Z))$ are always equal. The reason that this is wrong in general for *cycles* is that in the absence of flatness the schematic pullback pr_i^{-1} "kills lengths" in $f^{-1}(Z)$ while the cycle pullback $\mathrm{pr}_i^{*,\mathrm{naive}}$ does not.

We give a rather trivial condition on f which guarantees that (2) is a complex.

Lemma 4.3. *Assume that f is universally bijective. Then*

$$0 \longrightarrow \mathcal{Z}^*(Y) \overset{f^{*,naive}}{\longrightarrow} \mathcal{Z}^*(X) \overset{p_1 - p_2}{\longrightarrow} \mathcal{Z}^*(X \times_Y X)$$

is a complex, where $p_i := pr_i^{*,naive}$ for $i = 1, 2$. In fact, $p_1 = p_2$.

Proof. We proof directly that $p_1 = p_2$. First, we remark, that pr_1 and pr_2 are, in fact, *homeomorphisms* as the diagonal $X \to X \times_Y X$ is a continuous inverse for both. Let $x \in X$. We calculate

$$\begin{aligned} \mathrm{pr}_1^{*,\mathrm{naive}}(x) &= \mathrm{length}(k(x) \otimes_{\mathcal{O}_{Y,f(x)}} \mathcal{O}_{X,x})(x,x) \\ &= \mathrm{length}(\mathcal{O}_{X,x} \otimes_{\mathcal{O}_{Y,f(x)}} k(x))(x,x) \\ &= \mathrm{pr}_2^{*,\mathrm{naive}}(x). \end{aligned}$$ □

From now on we assume that (2) is a complex and call elements in

$$\mathcal{Z}^*_{\mathrm{desc}}(f) := \mathrm{Ker}(\mathrm{pr}_1^{*,\mathrm{naive}} - \mathrm{pr}_2^{*,\mathrm{naive}})$$

"cycles with descent datum" and elements in

$$\mathcal{Z}^*_{\mathrm{eff.desc}}(f) := \mathrm{Im}(f^{*,\mathrm{naive}})$$

"cycles with effective descent datum".[2] We denote by

$$H_f := \mathcal{Z}^*_{\mathrm{desc}}(f)/\mathcal{Z}^*_{\mathrm{eff.desc}}(f)$$

the cohomology at $\mathcal{Z}^*(X)$ of the complex (2), in other words the group of cycles with descent datum modulo the cycles with effective ones.

We come to an obstruction for the vanishing of H_f. For $y \in Y$ let

$$g_y(f) := \gcd\{\ \mathrm{length}(\mathcal{O}_{f^{-1}(\overline{\{y\}}),x}) \mid x \text{ generic point in } \ f^{-1}(\overline{\{y\}})\}$$

and

$$g_y^{\mathrm{res}}(f) := \gcd\{\ \mathrm{length}(\mathcal{O}_{f^{-1}(y),x}) \mid x \text{ generic point in } f^{-1}(y)\}.$$

We remark that $g_y(f)$ divides $g_y^{\mathrm{res}}(f)$ but both numbers can be different in general. However, they agree if all generic points of $f^{-1}(\overline{\{y\}})$ lie over y. We let

$$g_Y(f) := \mathrm{lcm}\{\ g_y(f) \mid y \in Y\}$$

[2] This should not be confused with "effective" cycles with descent datum, i.e., those cycles with descent datum whose coefficients are non-negative.

be the least common multiple of the $g_y(f)$, which we understand as a supernatural number, i.e., a formal expression

$$\prod_{p \text{ prime}} p^{i_p},$$

with $i_p \in \mathbb{N}_0 \cup \{\infty\}$. Similarly we set

$$g_Y^{\mathrm{res}}(f) := \mathrm{lcm}\{\, g_y^{\mathrm{res}}(f) \mid y \in Y\}.$$

Again $g_Y(f)$ divides $g_Y^{\mathrm{res}}(f)$ (as supernatural numbers), but in general they are different. If f is generalizing, then both agree.

Lemma 4.4. *Let $f\colon X \to Y$ be a morphism of noetherian schemes and $y \in Y$ a point, such that every generic point of $f^{-1}(\overline{\{y\}})$ lies over y. Then the subgroup*

$$\frac{1}{g_y(f)}\mathbb{Z}f^*(y) \subseteq \mathcal{Z}^*(X)$$

is saturated, i.e., $\frac{1}{g_y(f)}\mathbb{Z}f^*(y) = \mathbb{Q}f^*(y) \cap \mathcal{Z}^*(X)$.

Proof. As every cycle $\mathcal{Z} \in \mathcal{Z}^*(X) \cap \mathbb{Q}f^*(y)$ is a $\mathbb{Z}$-linear combination of the generic points in $f^{-1}(y)$ the statement follows immediately from the following observation. For $v := (n_1, \dots, n_r) \in \mathbb{Z}^r - \{0\}$ with $d := \gcd\{n_1, \dots, n_r\}$ the group $\mathbb{Z}^r \cap \mathbb{Q}v$ is generated by $\frac{1}{d}v$. □

Proposition 4.5. *Let $f\colon X \to Y$ be as before, i.e., f is surjective such that $X \times_Y X$ is noetherian and (2) is a complex. Then the following hold.*

1. *The subgroup $\mathcal{Z}^*_{desc}(f) \subseteq \mathcal{Z}^*(X)$ is saturated.*
2. *If the subgroup $\mathcal{Z}^*_{eff.desc}(f) \subseteq \mathcal{Z}^*(X)$ is saturated, then $g_Y(f) = 1$.*
3. *If $g_Y^{res}(f) = 1$, then $\mathcal{Z}^*_{eff.desc}(f) \subseteq \mathcal{Z}^*(X)$ is saturated.*

In particular, if f is generalizing, then $\mathcal{Z}^*_{eff.desc}(f)$ is saturated if and only if $g_Y(f) = 1$.

Proof. The group $\mathcal{Z}^*_{\mathrm{desc}}(f)$ is the kernel of a homomorphism of torsion-free abelian groups and hence saturated. If $y \in Y$ is a point, then

$$\mathcal{Z} := \frac{1}{g_y(f)}\mathrm{cycl}(f^{-1}(\overline{\{y\}}))$$

is a cycle (with integral coefficients) and $g_y(f)\mathcal{Z} \in \mathcal{Z}^*_{\mathrm{eff.desc}}(f)$. As f is surjective, $f^{*,\mathrm{naive}}$ is injective by Proposition 2.10 and hence if $\mathcal{Z}$ is a cycle with an effective descent datum, then necessarily $\mathcal{Z} = f^{*,\mathrm{naive}}(\frac{1}{g_y(f)}y)$ and thus, $g_y(f) = 1$. This shows that $g_Y(f) = 1$ if $\mathcal{Z}^*_{\mathrm{eff.desc}}(f)$ happens to be saturated.

Conversely, assume that $g_Y^{\mathrm{res}}(f) = 1$ and let $\mathcal{Z} \in \mathcal{Z}^*(X)$ be a cycle such that $m\mathcal{Z} = f^*(\mathcal{Y})$ for some integer $m \neq 0$ and some (unique) cycle

$$\mathcal{Y} = \sum_{i=1}^{n} m_i y_i \in \mathcal{Z}^*(Y).$$

We assume that $m_i \neq 0$ for all i, that the y_i are pairwise different and that,

$$\mathrm{codim}(\overline{\{y_1\}}) \leqslant \ldots \leqslant \mathrm{codim}(\overline{\{y_n\}}).$$

We argue by induction on the number

$$s := \#\{\ f(x_i) \mid \mathcal{Z} = \sum m_i x_i \text{ with } m_i \neq 0\}.$$

If $s = 1$, then $n = 1$ and Lemma 4.4 can be applied to conclude $\mathcal{Z} \in \mathbb{Z}f^{*,\mathrm{naive}}(y_1)$. In the general case, let $x_1, \dots, x_r$ be the generic points in $f^{-1}(\overline{\{y_1\}})$ and assume that $x_1, \dots, x_d$ lie over y_1 while $x_{d+1}, \dots, x_r$ do not ($r = d$ is allowed). We define

$$U := X - \bigcup_{i=d+1}^{r} \overline{\{x_i\}}$$

and

$$j \colon U \to X$$

as the natural open immersion of U in X. Furthermore, let

$$g \colon Y_1 := \mathrm{Spec}(\mathcal{O}_{Y,y_1}) \to Y$$

be the Zariski localization at y_1. The diagram

$$\begin{array}{ccc}
\mathcal{Z}^*(U \times_Y Y_1) & \xleftarrow{(\mathrm{Id}_U \times g)^*} & \mathcal{Z}^*(U) \\
\uparrow {\scriptstyle (j \times \mathrm{Id}_{Y_1})^*} & & \uparrow {\scriptstyle j^*} \\
\mathcal{Z}^*(X \times_Y Y_1) & \xleftarrow{(\mathrm{Id}_X \times g)^*} & \mathcal{Z}^*(X) \\
\uparrow {\scriptstyle (f \times \mathrm{Id}_{Y_1})^{*,\mathrm{naive}}} & & \uparrow {\scriptstyle f^{*,\mathrm{naive}}} \\
\mathcal{Z}^*(Y_1) & \xleftarrow{g^*} & \mathcal{Z}^*(Y)
\end{array}$$

is commutative by Lemma 2.13. We conclude that

$$(\mathrm{Id}_U \times g)^* j^*(m\mathcal{Z}) = m_1 (f \circ j \times \mathrm{Id}_{Y_1})^{*,\mathrm{naive}}(y_1),$$

as y_1 is not a specialization of one of the $y_2, \dots, y_n$ and therefore $g^*(\mathcal{Y}) = m_1 y_1$. By the case $n = 1$, applied to $(\mathrm{Id}_U \times g)^* j^*(\mathcal{Z})$ and $f \circ j \times \mathrm{Id}_{Y_1} \colon U \times_Y Y_1 \to Y_1$, it follows that m divides m_1. By construction, the induction hypothesis may be applied to

$$\mathcal{Z}' := \mathcal{Z} - \frac{m_1}{m} f^*(y_1),$$

as no point occurring in $\mathcal{Z}'$ lies over y_1. Using induction we obtain that $\mathcal{Z}' \in \mathcal{Z}^*_{\mathrm{eff.desc}}(f)$ and hence $\mathcal{Z} \in \mathcal{Z}^*_{\mathrm{eff.desc}}(f)$. This finishes the proof. □

We give an example to show that in general $g_Y(f) \neq g_Y^{\mathrm{res}}(f)$ and that $g_Y^{\mathrm{res}}(f) \neq 1$ is possible even if $\mathcal{Z}^*_{\mathrm{eff.desc}}(f)$ is saturated.

Example 4.6. Let $Y = \mathrm{Spec}(R)$ be the spectrum of a discrete valuation ring R with generic point $\eta = \mathrm{Spec}(K)$ and special point $s = \mathrm{Spec}(k)$. Let $\pi \in R$ be a uniformizer and define

$$X_1 := \mathrm{Spec}(R[t]/(\pi^n t^n, t^m)),$$

with $n, m \geqslant 1$. Then as a topological space $X_1 = \{\eta_1, s_1\}$ with η_1 specialising to s_1.

We define

$$X := X_1 \coprod \{s\}$$

and take $f\colon X \to Y$ as the natural morphism. Then

$$f^{*,\text{naive}}(\eta) = e\eta_1 + s$$

and

$$f^{*,\text{naive}}(s) = ms_1 + s,$$

with $e := \min\{n, m\}$, hence $\mathcal{Z}^*_{\text{eff.desc}}(f)$ is generated by

$$e\eta_1 + s, e\eta_1 - ms_1.$$

In particular,

$$\mathcal{Z}^*(X)/\mathcal{Z}^*_{\text{eff.desc}}(f) \cong \mathbb{Z}\eta_1 \oplus \mathbb{Z}s_1/(e\eta_1 - ms_1) \cong \mathbb{Z} \oplus \mathbb{Z}/d,$$

with $d = \gcd\{e, m\}$. It is torsionfree, i.e., $\mathcal{Z}^*_{\text{eff.desc}}(f)$ is saturated, if and only if $d = 1$. But

$$g^{\text{res}}_\eta(f) = \text{length}(K[t]/(\pi^n t^n, t^m)) = \min\{n, m\} = e \neq 1,$$

for general n, m, while $g_Y(f) = 1$ for every n, m.

Example 4.6 shows that in the statement of Proposition 4.5 the number $g_Y(f)$ cannot be replaced by $g^{\text{res}}_Y(f)$. It also shows that descent of algebraic cycles does not hold in general, as then $\mathcal{Z}^*_{\text{eff.desc}}(f)$ must be separated.

We proceed with the question under which conditions on f the group H_f is zero. Examples with $H_f \neq 0$, additionally to Example 4.6, are the following. Firstly, local artinian schemes X over a field $Y = \text{Spec}(k)$ (such that $X \times_Y X$ is noetherian). Secondly, set

$$Y = \text{Spec}(k[t]), X = \text{Spec}(k[t, x]/(x^n - t))$$

and

$$f\colon X \to Y, (t, x) \mapsto t$$

the natural projection. The morphism f is faithfully flat with $n \mid g_Y(f)$ as the fibre over $t = 0$ is

$$\text{Spec}(k[x]/(x^n)).$$

The cycle $\mathcal{Z} := \text{cycl}(\text{Spec}(k[t, x]/(t, x)))$ is a cycle with descent datum, but

$$m\mathcal{Z} = f^*(\text{cycl}(\text{Spec}(k[t]/(t))))$$

has an effective descent datum for $m \in \mathbb{Z}$ only if $n \mid m$.

The reason that descent can fail for generalizing morphisms is basically that two subschemes can have the same cycle without being equal, hence $\mathcal{Z}^*_{\text{desc}}(f)$ is in some sense "too large". Concretely, in the example $Y = \text{Spec}(k)$ and $X = \text{Spec}(k[t]/(t^2))$ consider the subscheme $X_{\text{red}} \subseteq X$. Then $\text{pr}_1^{-1}(X_{\text{red}}) \neq \text{pr}_2^{-1}(X_{\text{red}})$, but both schemes have length 2 with same support. In particular, their cycles agree.

As another numerical invariant we define $\pi_Y^{\mathrm{res}}(f)$ as the product[3] of the $g_y^{\mathrm{res}}(f)$ over all $y \in Y$. We arrive at our first condition on f guaranteeing descent of cycles (for rather trivial reasons).

Proposition 4.7. *Assume that f is universally bijective and that Y is of finite Krull dimension. Then H_f is a torsion group annihilated by $\pi_Y^{res}(f)$.*[4]

Proof. By Lemma 4.3, we conclude that $\mathcal{Z}^*(X) = \mathcal{Z}^*_{\mathrm{desc}}(f)$. Let $x \in X$ be point. We prove $g_Y^{\mathrm{res}}(f)x \in \mathrm{Im}(f^{*,\mathrm{naive}})$ by induction on $d := \dim \overline{\{f(x)\}}$. More precisely, our induction hypothesis states that x is annihilated by

$$\prod_{y \in \overline{\{f(x)\}}} g_y(f).$$

If $d = 0$, then $f^{*,\mathrm{naive}}(f(x)) = g^{\mathrm{res}}_{f(x)}(f)x$ and the claim is established. For general $x \in X$ we can write by induction

$$f^{*,\mathrm{naive}}(f(x)) = g^{\mathrm{res}}_{f(x)}(f)x + \mathcal{Z},$$

for some $\mathcal{Z} \in \mathcal{Z}^*(X)$ with $n\mathcal{Z} \in \mathrm{Im}(f^{*,\mathrm{naive}})$ and n dividing

$$\prod_{x \neq y \in \overline{\{f(x)\}}} g_y(f),$$

because x is the only point lying over $f(x)$. In particular, $ng^{\mathrm{res}}_{f(x)}(f)x \in \mathrm{Im}(f^{*,\mathrm{naive}})$ and the proof is finished. □

Now we come to our main theorem giving a general condition on f such that descent of algebraic cycles holds for f. By Proposition 2.10, Theorem 4.8 implies Theorem 1.1 from the introduction.

Theorem 4.8. *Assume that $f\colon X \to Y$ is a universally generalizing morphism between noetherian schemes such that $X \times_Y X$ is again noetherian. Then the following hold.*

1. *The group H_f is torsion, which is annihilated by $g_Y(f)$.*
2. *The descent sequence*

$$0 \longrightarrow \mathcal{Z}^*(Y) \xrightarrow{f^{*,naive}} \mathcal{Z}^*(X) \xrightarrow{pr_1^{*,naive} - pr_2^{*,naive}} \mathcal{Z}^*(X \times_Y X)$$

is exact if and only if f is surjective and $g_Y(f) = 1$.

Proof. The second claim follows directly from the first and from Proposition 2.10. To prove the first claim assume that

$$\mathcal{Z} = \sum_{i=1}^{n} m_i x_i \in \mathcal{Z}^*(X)$$

is a cycle with $\mathrm{pr}_1^*(\mathcal{Z}) = \mathrm{pr}_2^*(\mathcal{Z})$. As $f, \mathrm{pr}_1, \mathrm{pr}_2$ are generalizing we may assume by

[3] In the sense of supernatural numbers.

[4] By which we mean that every element in H_f is annihilated by some natural number $\neq 0$ dividing $\pi_Y^{\mathrm{res}}(f)$.

Proposition 2.12 that $\mathcal{Z} \in \mathcal{Z}^r(X)$ for some r, hence

$$\operatorname{codim}(\overline{\{x_i\}}, X) = r$$

for every $i = 1, \ldots, n$. Even further, as the decompositions

$$\mathcal{Z}^*(X) = \bigoplus_{y \in Y} \bigoplus_{x \in f^{-1}(y)} \mathbb{Z}x$$

and

$$\mathcal{Z}^*(X \times_Y X) = \bigoplus_{y \in Y} \bigoplus_{z \in \operatorname{pr}_1^{-1} f^{-1}(y)} \mathbb{Z}z$$

are preserved under pr_1^* and pr_2^*, we may further assume that $x_1, \ldots, x_n$ lie in the fibre $f^{-1}(y)$ for some $y \in Y$. Let $Y' = \overline{\{y\}}$ be the (reduced) closure of y and denote by $i\colon Y' \to Y$ the natural closed immersion. Let $i'\colon X' \to X$ be the base change of i along f. Then, by Proposition 3.3, the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{Z}^*(Y') & \longrightarrow & \mathcal{Z}^*(X') & \longrightarrow & \mathcal{Z}^*(X' \times_{Y'} X') \\ & & \downarrow i_* & & \downarrow i'_* & & \downarrow (i' \times i')_* \\ 0 & \longrightarrow & \mathcal{Z}^*(Y) & \longrightarrow & \mathcal{Z}^*(X) & \longrightarrow & \mathcal{Z}^*(X \times_Y X) \end{array}$$

of descent sequences with all vertical arrows injective commutes. As $\mathcal{Z}$ is the image of an element with descent datum under i'_*, we can replace Y by Y' and X by X' and assume that $y \in Y$ is generic. Next we want to reduce to the case that $Y = \operatorname{Spec}(k(y))$ is a field. Let $j\colon Y' := \operatorname{Spec}(k(y)) \to Y$ be the inclusion and j' its base change to X. Then the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{Z}^*(Y) & \longrightarrow & \mathcal{Z}^*(X) & \longrightarrow & \mathcal{Z}^*(X \times_Y X) \\ & & \downarrow j^* & & \downarrow j'^* & & \downarrow (j' \times j')^* \\ 0 & \longrightarrow & \mathcal{Z}^*(Y') & \longrightarrow & \mathcal{Z}^*(X') & \longrightarrow & \mathcal{Z}^*(X' \times_{Y'} X') \end{array}$$

commutes by Lemma 2.13. If $j'^*(\mathcal{Z})$ is the image of some $\mathcal{Y} = my \in \mathcal{Z}^*(Y')$, then $\mathcal{Z} = f^{*,\mathrm{naive}}(my)$ as the generic fibre of f agrees with X' and f is generalizing. In other words, we may assume that

$$Y = \operatorname{Spec}(k),\ k := k(y),$$

is a point. We next show that (if Y is a point) the generic points of X are precisely the x_i (recall $\mathcal{Z} = \sum m_i x_i$). Let $\eta \in X$ be a generic point of X. By the assumption $\operatorname{pr}_1^*(\mathcal{Z}) = \operatorname{pr}_2^*(\mathcal{Z})$ it follows that

$$\coprod_{j=1}^{n} x_j \times_Y X = \coprod_{j=1}^{n} X \times_Y x_j.$$

As Y is a point, for every j there exist a point $z \in X \times_Y x_j$ with $\operatorname{pr}_1(z) = \eta$ showing that

$$\eta \in \operatorname{pr}_1(\coprod_{j=1}^{n} X \times_Y x_j) = \operatorname{pr}_1(\coprod_{j=1}^{n} x_j \times_Y X) = \{x_1, \ldots, x_n\}.$$

We assumed that there are no specializations among the x_i, hence conversely every x_i is generic. Let

$$g\colon X' := \coprod_{i=1}^{n} \operatorname{Spec}(\mathcal{O}_{X,x_i}) \to X$$

be the natural inclusion, which is a flat morphism. By Proposition 2.7, the diagram

$$\begin{array}{ccccccc} 0 \longrightarrow & \mathcal{Z}^*(Y) & \longrightarrow & \mathcal{Z}^*(X) & \longrightarrow & \mathcal{Z}^*(X \times_Y X) \\ & \downarrow \mathrm{Id}_Y & & \downarrow g^* & & \downarrow (g\times g)^* \\ 0 \longrightarrow & \mathcal{Z}^*(Y) & \longrightarrow & \mathcal{Z}^*(X') & \longrightarrow & \mathcal{Z}^*(X' \times_{Y'} X') \end{array}$$

commutes. By construction of X', the morphism g^* induces isomorphisms

$$g^*\colon \mathcal{Z}^*_{\mathrm{eff.desc}}(f) \to \mathcal{Z}^*_{\mathrm{eff.desc}}(g \circ f)$$

and

$$g^*\colon \mathcal{Z}^*_{\mathrm{desc}}(f) \to \mathcal{Z}^*_{\mathrm{desc}}(g \circ f).$$

Hence we may replace X by $X' = \coprod\limits_{i=1}^{n} \operatorname{Spec}(\mathcal{O}_{X,x_i})$. Let $R_i := \mathcal{O}_{X,x_i}$ with residue field k_i and $r_i := \mathrm{length}_{R_i}(R_i)$. We first show that m_i/r_i is independent of i. We compute (using Lemma 2.2)

$$\mathrm{pr}_1^*(\mathcal{Z}) = \sum_{i=1}^{n} m_i \sum_{j=1}^{n} \mathrm{cycl}(k_i \otimes_k R_j) = \sum_{i=1}^{n} m_i \sum_{j=1}^{n} r_j \mathrm{cycl}(k_i \otimes_k k_j),$$

which by assumption equals

$$\mathrm{pr}_2^*(\mathcal{Z}) = \sum_{i=1}^{n} m_i \sum_{j=1}^{n} r_j \mathrm{cycl}(k_j \otimes_k k_i).$$

We can now conclude

$$m_i r_j = m_j r_i$$

for all $i, j = 1, \ldots, n$, as the supports of $\operatorname{Spec}(k_i \otimes_k k_j) \subseteq X \times_Y X$ are disjoint and thus the cycles $\mathrm{cycl}(k_i \otimes_k k_j), i, j = 1, \ldots, n$, are linearly independent. Setting $q := \frac{m_1}{r_1}$ the cycle

$$\mathcal{Z} = \sum_{i=1}^{n} m_i x_i = q \sum_{i=1}^{n} r_i x_i = q f^{*,\mathrm{naive}}(y)$$

lies in $\mathbb{Q} f^{*,\mathrm{naive}}(y) \cap \mathcal{Z}^*(X) = \frac{1}{g_y(f)} \mathbb{Z} f^*(y)$ and hence

$$g_y(f)\mathcal{Z} \in \mathrm{Im}(f^*),$$

which finishes the proof. Indeed, in the general situation the group

$$H_f = \mathcal{Z}^*_{\mathrm{desc}}(f) / \mathcal{Z}^*_{\mathrm{eff.desc}}(f)$$

is the direct sum over $y \in Y$ of the groups

$$H_{y,f} := \Big(\Big(\bigoplus_{x \in f^{-1}(y)} \mathbb{Z}x\Big) \cap \mathcal{Z}^*_{\mathrm{desc}}(f)\Big) / \mathbb{Z} f^*(y),$$

as f is universally generalizing, and we proved that for $y \in Y$ the direct summand $H_{y,f}$ is annihilated by $g_y(f)$. Hence, $H_f = \bigoplus_{y \in Y} H_{y,f}$ is annihilated by

$$g_Y(f) = \operatorname{lcm}\{\, g_y(f) \mid y \in Y\}. \qquad \square$$

Finally, we want to remark that Theorem 4.8 implies easily that various presheaves of cycles are actually sheaves in the étale topology. For example, let X be a scheme, separated and of finite type over a field k. Then the presheaf with transfers $\mathbb{Z}_{\mathrm{tr}}(X)$ represented by X (cf. [**MVW06**, Definition 2.8]) is a sheaf for the étale topology, i.e., for every étale surjection $f\colon Y \to Z$ of (quasi-compact) smooth, separated schemes over k the sequence

$$0 \longrightarrow \mathbb{Z}_{\mathrm{tr}}(X)(Z) \xrightarrow{f^*} \mathbb{Z}_{\mathrm{tr}}(X)(Y) \xrightarrow{\mathrm{pr}_1^* - \mathrm{pr}_2^*} \mathbb{Z}_{\mathrm{tr}}(X)(Y \times_Z Y)$$

is exact. We shortly present how to deduce this from Theorem 4.8.

Corollary 4.9. *Let k be a field and let X be a separated scheme of finite type over k. Then the presheaf with transfers $\mathbb{Z}_{tr}(X)$ is a sheaf for the étale topology. Similarly, for every $r \geqslant 0$ the presheaf with transfers $z_{\mathrm{equi}}(X,r)$ of equidimensional cycles over X (cf. [**MVW06**, Definition 16.1]) is an étale sheaf.*

Proof. We only treat the case of $\mathbb{Z}_{\mathrm{tr}}(X)$ as the case $z_{\mathrm{equi}}(X,r)$ is similar (using that the property of being equidimensional descends along étale morphisms). Let $Y \to Z$ be an étale surjection of smooth, separated schemes over k. By [**SV00**, Lemma 3.3.12] the pullback

$$f^*\colon \mathbb{Z}_{\mathrm{tr}}(X)(Z) \to \mathbb{Z}_{\mathrm{tr}}(X)(Y)$$

of relative cycles is induced by the pullback of absolute cycles

$$(\mathrm{Id}_X \times f)^*\colon \mathcal{Z}^*(X \times_k Z) \to \mathcal{Z}^*(X \times_k Y)$$

along the inclusions $\mathbb{Z}_{\mathrm{tr}}(X)(Z) \subseteq \mathcal{Z}^*(X \times_k Z)$ and $\mathbb{Z}_{\mathrm{tr}}(X)(Y) \subseteq \mathcal{Z}^*(X \times_k Y)$. By Theorem 4.8 descent of algebraic cycles holds for $\mathrm{Id}_X \times f$ and therefore it suffices to show that for every $w \in X \times_k Z$ the subscheme $\overline{\{w\}}$ is finite and surjective over a component of Z if and only if the components of the cycle $f^*(w) \in \mathcal{Z}^*(X \times_k Y)$ are finite and surjective over a component of Y. But this last statement follows as the properties of being finite and dominant over a component descend along quasi-compact faithfully flat morphisms. $\square$

Of course Corollary 4.9 is well-known. For example, the sheaf property is proven in [**MVW06**, Lemma 6.2] for $\mathbb{Z}_{\mathrm{tr}}(X)$ and left as an exercise in the case of $z_{\mathrm{equi}}(X,r)$. The argument in Corollary 4.9 also applies to other presheaves of cycles, for example, to Bloch's cycle complex. More precisely, by the same arguments as in Corollary 4.9 for two (equidimensional) schemes T, X separated and of finite type over a field k the presheaf of complexes

$$U \mapsto z^i(T \times U, \bullet)$$

sending $U \in X_{\acute{e}t}$ in the small étale site of X to Bloch's cycle complex $z^i(T \times U, \bullet)$ (cf. [**MVW06**, Definition 17.1]) is a complex of étale sheaves, a result which again was well-known.

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