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REMARKS ABOUT Δ-COMPLEXES AND APPLICATIONS

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Abstract

We first consider some generalities regarding Δ-complexes, in particular, we give a brief history of the category of Δ-complexes, and its relation to the category of semi-simplicial complexes introduced in 1950 by Eilenberg and Zilber. A natural construction of Δ-complexes arising from a group action on a simplicial complex is next considered. Finally, an application of this construction to obtain an elementary explicit computation of the cohomology ring $H^*(\mathbb{R}P^n;\mathbb{Z}_2)$, based on a Δ-complex structure, is given.

1. Introduction

The idea and terminology of Δ-complexes seems to originate in about 2002 in the book of Hatcher [**11**], indeed it is introduced in Section 2.1, and can now be found in a few undergraduate texts such as [**2**]. Note that the version of Δ-complexes given in the first edition of [**11**] was somewhat incomplete, and we will be following the improved version given in the subsequent editions. An earlier version of an equivalent idea, due to Eilenberg and Zilber [**7**] in 1950, is mentioned by Hatcher. In the Eilenberg–Zilber paper these objects are called "semi-simplicial complexes," which can be confusing since this term has later been used in a different context. To avoid confusion with subsequent usage we shall call the "semi-simplicial complexes" introduced in [**7**] "EZ-complexes" (face maps only). The "complete semi-simplicial complex" introduced in [**7**] will be called "simplicial sets" (face maps and degeneracy maps), in agreement with the terminology of texts such as [**10, 17, 22**].

In the second section of this note we give a brief historical sketch of this subject and the related idea of geometric realization. In the third section we recall (cf. [**11**]) the definition of the category of Δ-complexes, denoted by **Δ-Comp**. The equivalence of this category with the category of EZ-complexes, as suggested in Section 2.1 of [**11**], is demonstrated. A corollary of this is that the homology functor on **Δ-Comp** is indeed a homology theory satisfying the Eilenberg-Steenrod axioms, and related constructions such as cup product and Steenrod squares (in cohomology theory), and local coefficients, can be carried out on the category **Δ-Comp**.

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The fourth section considers a construction of Δ-complexes that frequently occurs in practice, namely via the action of a finite group on a Δ-complex, in particular, on a simplicial complex. The final section then gives an application of this construction to obtaining a Δ-complex structure for $\mathbb{R}P^n$ and using this to give an elementary explicit computation of the cohomology ring $H^*(\mathbb{R}P^n; \mathbb{Z}_2)$.

2. A brief history

The ideas behind the categories we are studying originated in the 1940s with Eilenberg's work on singular homology theory, and were first given formal expression in the 1950 paper [**7**] of Eilenberg and Zilber. We remind the reader that the terms "semi-simplicial complex, complete semi-simplicial complex" in [**7**] correspond respectively to "EZ-complex, simplicial set" in our terminology. They are respectively denoted by **EZ-Comp**, **Simp**. The simplicial sets have received the lion's share of the attention since 1950, probably because of their advantage in dealing with products of spaces and their suitability for abstract homotopy theory. An excellent brief history of the simplicial sets can be found in Weibel's memoir [**23**], and there are also books dealing with this subject such as [**17**] by May, [**10**] by Goerss and Jardine, as well as chapters in other books such as [**22**] by Selick and various seminar or lecture notes such as [**4**] by Curtis.

We therefore concentrate here on the much less extensive history of the EZ-complexes. The singular complex of a topological space gives a functor from spaces to EZ-complexes (and to simplicial sets). Already in 1950 Giever [**8**] defined an adjoint to this functor, which we now call a geometric realization. In 1951 Hu [**13**] gave a more general version of this geometric realization, which would, in fact, work for any EZ-complex and is the same as Giever's for the singular complex of a space. The geometric realization of a simplicial set was given in 1957 by Milnor [**19**], who emphasizes right at the start of his paper that "This construction will be different from that of Giever and Hu in that the degeneracy operations of K are used." As the titles of the papers by Giever and Hu indicate, their primary aims were in different directions and the geometric realizations/EZ-complexes were used as a convenient tool.

The next work that deals with EZ-complexes is that of Rourke and Sanderson [**21**] in 1971. In this paper EZ-complexes are called Δ-sets, while simplicial sets are called semi-simplicial complexes and abbreviated to css-sets. In fact this paper proves that there is an equivalence of these two categories, and then goes on to discussing the Kan condition and developing homotopy theory. Curiously, although Lemma 1.2 in this paper quotes a result of Eilenberg–Zilber, [**7**] is not mentioned in the bibliography.

It seems that the next appearance of EZ-complexes is not until Hatcher's book in 2002 [**11**], together with the idea of Δ-complexes, as already mentioned in the Introduction. Two recent papers dealing with EZ-complexes are [**9, 18**], where they are called semi-simplicial sets.

3. Generalities about Δ-complexes

The objective of this section is to recall the category of Δ-complexes, in the sense of Hatcher [**11**], the category of EZ-complexes following Eilenberg–Zilber [**7**], and to show that these categories are equivalent (cf. MacLane [**16**, Ch. IV, Theorem 4.1], for the notion of equivalence of categories). In the case of Δ-complexes we give a categorical interpretation as the colimit of a suitable diagram. In the case of EZ-complexes we also give a categorical interpretation as a contravariant functor (cf. Propositions 3.12 and 3.14). Both these interpretations will be convenient for proving the main theorem of this section, Theorem 3.21.

First we fix some notations and terminology. Much of this is standard (cf. [**16**]), however conventions vary. So we fix them here. Throughout this section a *diagram* D in a category $\mathbb{X}$ is itself a category that consists of, for some set I, an I-indexed set

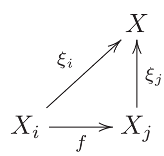
$$\mathrm{Obj}(D) = \{X_i \ : \ i \in I,\ X_i \in \mathrm{Obj}(\mathbb{X})\}$$

together with, for each $i, j \in I$, a (possibly empty) subset of morphisms

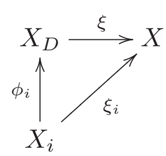
$$D(i,j) \subseteq \mathbb{X}(X_i, X_j) = \hom_{\mathbb{X}}(X_i, X_j)$$

such that, for any $i, j, k \in I$, any $f \in D(i,j)$, and any $g \in D(j,k)$, the composite $g \circ f \in D(i,k)$, and $1_{X_i} \in D(i,i)$. Given this formalization a diagram is said to *commute* if for all $i, j \in I$ the set $D(i,j)$ contains at most one element.

With the diagram D as above, a *cocone* from the base D to the vertex X (or *cocone* from D into X) is a pair $(X, \{(X_i, \xi_i)\})$ where $X \in \mathrm{Obj}(\mathbb{X})$ and $\{(X_i, \xi_i)\} = \{(X_i, \xi_i) \ : \ i \in I\}$ is an I-tuple of pairs (X_i, ξ_i) with each $X_i \in \mathrm{Obj}(\mathbb{X})$ and each $\xi_i \colon X_i \to X$ a morphism in $\hom_{\mathbb{X}}(X_i, X)$ such that, for all $f \in D(i,j)$, the diagram



commutes. A *colimit* of a diagram D is then a cocone $(X_D, \{(X_i, \phi_i)\})$ such that, for any other cocone $(X, \{(X_i, \xi_i)\})$, there is a unique morphism $\xi \colon X_D \to X$ such that, for all $i \in I$ the diagram



commutes. One also writes $X_D = \operatorname{colim} D$. Since colimits are unique up to unique isomorphism (cf. [**16**, Ch. III, §4]) we will often refer to *the* colimit of a diagram D. The dual notion of a *cone* is used in [**16**] to define limits.

Definition 3.1. (i) For any $n, m \in \mathbb{N}$ let $R = \{v_0 < \cdots < v_n\}$ be an ordered set of $n+1$ elements of $\mathbb{R}^m$ such that the set $\{v_i - v_0 \ : \ 1 \leqslant i \leqslant n\}$ is linearly independent. The resulting convex-hull

$$\Delta^n_R := \left\{ \sum_{i=0}^{n} t_i v_i \ : \ t_0, \ldots, t_n \geqslant 0 \text{ and } \sum_{i=0}^{n} t_i = 1 \right\}$$

is called the *ordered n-simplex* generated by R in $\mathbb{R}^m$, and will also be written $\langle v_0, \ldots, v_n \rangle$. We call v_i the *i-th vertex* of Δ^n_R. Given ordered n-simplexes Δ^n_R and

Δ^n_S there is a canonical affine homeomorphism

$$\Delta^n_R \to \Delta^n_S \quad \text{given by} \quad \sum_{i=0}^n t_i u_i \longmapsto \sum_{i=0}^n t_i v_i.$$

The standard n-simplex Δ^n in $\mathbb{R}^{n+1}$ is obtained by taking $v_i = e_{i+1}$ for $0 \leqslant i \leqslant n$.

(ii) Given an n-simplex Δ^n_R and a natural number $k \leqslant n$, a *k-face* of Δ^n_R is a k-simplex generated by $k+1$ vertices of Δ^n_R with the induced ordering. Notice that there are $\binom{n+1}{k+1}$ k-faces of Δ^n_R. If K is a k-face of Δ^n_R then the *k-face inclusion of Δ^k into Δ^n_R associated with K* is the composition ι^n_K of the canonical affine homeomorphism from Δ^k to K followed by subset inclusion

$$\Delta^k \xrightarrow{\cong} K \hookrightarrow \Delta^n_R.$$

In particular, if K is an $(n-1)$-face of the standard n-simplex Δ^n, let k be the index of the unique vertex not contained in K. One denotes the corresponding face inclusion by

$$\iota^n_k \colon \Delta^{n-1} \to \Delta^n.$$

(iii) Given an n-simplex Δ^n, let $\partial\Delta^n$, called the *boundary of* Δ^n, denote the union of the $(n-1)$-faces of Δ^n. Let $\mathring{\Delta}^n := \Delta^n \setminus \partial\Delta^n$ and call it the *interior* of Δ^n. Notice that $\partial\Delta^0 = \emptyset$ and $\mathring{\Delta}^0 = \Delta^0$. This may seem odd, but it simplifies notation later.

This next definition is taken from Section 2.1 of [**11**] with the only modifications being a slight categorification of the language. See the same section of [**11**] for several examples (and non-examples). By a "map" we always mean a continuous function.

Definition 3.2. A *Δ-complex* is a pair $\mathcal{X} = (X, \Sigma)$ consisting of a topological space X and a collection of maps

$$\Sigma = \{\sigma_\alpha \colon \Delta^{n_\alpha} \to X \ : \ \alpha \in \mathcal{A},\ n_\alpha \geqslant 0\}$$

satisfying the following conditions.

A1. For each $\sigma_\alpha \in \Sigma$ the composite $\mathring{\Delta}^{n_\alpha} \hookrightarrow \Delta^{n_\alpha} \xrightarrow{\sigma_\alpha} X$ is injective. Call this composite $\mathring{\sigma}_\alpha$. Definition 3.1 (iii) implies that whenever $n_\alpha = 0$, we get that $\mathring{\sigma}_\alpha = \sigma_\alpha$.

A2. Each $x \in X$ is in the image of exactly one such $\mathring{\sigma}_\alpha$.

A3. For each $\sigma_\alpha \colon \Delta^{n_\alpha} \to X$ in Σ, each $k < n_\alpha$, and every face inclusion $\Delta^k \hookrightarrow \Delta^{n_\alpha}$, the composite $\Delta^k \hookrightarrow \Delta^{n_\alpha} \xrightarrow{\sigma_\alpha} X$ is one of the maps in Σ.

A4. A set $A \subseteq X$ is open if and only if $\sigma_\alpha^{-1}(A)$ is open in Δ^{n_α} for each $\sigma_\alpha \in \Sigma$.

We write $\Sigma_n = \{\sigma_\alpha : n_\alpha = n\} \subseteq \Sigma$, and call each $\sigma_\alpha \in \Sigma_n$ an *n-simplex* of $\mathcal{X}$.

Definition 3.3. A *morphism* $\mathcal{X} = (X, \Sigma) \to \mathcal{Y} = (Y, \Sigma')$ of Δ-complexes is a map $f \colon X \to Y$ such that, for each $\sigma \in \Sigma$ we get that the composite

$$\Delta^{n_\alpha} \xrightarrow{\sigma_\alpha} X \xrightarrow{f} Y$$

is an element of Σ'. By a slight abuse of notation, we also write $f \colon \mathcal{X} \to \mathcal{Y}$.

Example 3.4. If X is a Δ-complex with associated maps Σ, then the identity map $1_X\colon X \to X$ clearly satisfies $1_X \circ \sigma \in \Sigma$ for all $\sigma \in \Sigma$, hence 1_X is a morphism of Δ-complexes.

Proposition 3.5. *There is a category,* **Δ-Comp***, whose*

- *objects are* Δ*-complexes,*
- *morphisms are morphisms of* Δ*-complexes,*
- *identity morphisms are the identity maps of Example 3.4,*
- *composition is given by the usual composition of functions.*

Proof. This is routine to check. □

Example 3.6. Any finite simplicial complex (see Definition 4.2) is a Δ-complex, cf. [**2**, Section 6.1]. To obtain a Δ-complex structure one may use a partial order $\preceq$ for the vertices such that the vertices of any simplex are simply ordered under $\preceq$, for example, any total order of the vertices of the simplicial complex will suffice. Note that different partial orders may give rise to different Δ-complex structures. In particular, we may regard Δ^n as a Δ-complex (with $e_1 < \ldots < e_{n+1}$), which we denote simply as Δ^n (or (Δ^n, Σ_n)), and any face inclusion $\Delta^k \overset{\iota}{\hookrightarrow} \Delta^n$ is a morphism of Δ-complexes.

The next lemma identifies a Δ-complex as a suitable colimit, and will be useful towards proving the main Theorem 3.21. A similar result, for simplicial sets, can be found in Section 1.2 of [**10**].

Lemma 3.7. *Let* $\mathcal{X} = (X, \Sigma)$ *be a* Δ*-complex, and let* $D = D_{\mathcal{X}}$ *be the diagram in the category* **Δ-Comp** *with one copy of* Δ^{n_α} *for each* $\sigma_\alpha \in \Sigma$*, denoted* $(\Delta^{n_\alpha}, \sigma_\alpha)$*, and a face inclusion* $\iota\colon \Delta^{n_\alpha} \to \Delta^{n_\beta}$ *if* σ_α *factors as*

$$\sigma_\beta \circ \iota = \sigma_\alpha.$$

Then $\mathcal{X}$ *is the colimit of* D*.*

Proof. Let $\iota\colon \Delta^{n_\alpha} \to \Delta^{n_\beta}$ be a morphism of D, then by construction we know that

$$\sigma_\beta \circ \iota = \sigma_\alpha.$$

Thus (X, Σ) is a cocone for D. Let (Y, Σ') be another cocone for D, where $\Sigma' = \{\phi_\alpha\colon \sigma_\alpha \in \Sigma\}$, $\phi_\alpha\colon \Delta^{n_\alpha} \to Y$. We need to show that there is a unique map $\phi\colon X \to Y$ such that $\phi\sigma_\alpha = \phi_\alpha$ for all $\sigma_\alpha \in \Sigma$.

By A2, for each point $x \in X$ there is a unique $\sigma_{\alpha(x)} \in \Sigma$ and a unique $p_x \in \mathring{\Delta}^{n_{\alpha(x)}}$ such that

$$x = \mathring{\sigma}_{\alpha(x)}(p_x).$$

Define $\phi\colon X \to Y$ by

$$\phi(x) := \phi_{\alpha(x)}(p_x).$$

We now show ϕ is a morphism of cocones for D. Indeed, for any $p \in \Delta^{n_\alpha}$, if $p \in \mathring{\Delta}^{n_\alpha}$ then $(\phi\sigma_\alpha)(p) = \phi_\alpha(p)$ from the above definition of ϕ, taking $\alpha(x) = \alpha$ and $x = \sigma_\alpha(p)$.

Otherwise let Δ^{n_β} be the face of Δ^{n_α} such that $p \in \mathring{\Delta}^{n_\beta}$ and let

$$\iota\colon \Delta^{n_\beta} \to \Delta^{n_\alpha}$$

be the associated face inclusion. By hypothesis $p = \iota(p')$ for some $p' \in \Delta^{n_\beta}$, and compute

$$\begin{aligned}(\phi\sigma_\alpha)(p) &= \phi(\sigma_\alpha(p)) \\ &= \phi_\beta(p') && \text{(by our choice of } \beta \text{ above and our definition of } \phi) \\ &= (\phi_\alpha\iota)(p') && \text{(since } \phi \text{ is a cocone)} \\ &= \phi_\alpha(\iota(p')) \\ &= \phi_\alpha(p).\end{aligned}$$

Hence $\phi \circ \sigma_\alpha = \phi_\alpha$ for each $\sigma_\alpha \in \Sigma$.

Next we show that ϕ is continuous. Let $A \subseteq Y$ be open and let $B := \phi^{-1}(A)$. By definition of the topology on X we have that B is open if and only if $\sigma_\alpha^{-1}(B)$ is open for all $\sigma_\alpha \in \Sigma$, but

$$\sigma_\alpha^{-1}(B) = \sigma_\alpha^{-1}\left(\phi^{-1}(A)\right) = (\phi\sigma_\alpha)^{-1}(A) = \phi_\alpha^{-1}(A)$$

and $\phi_\alpha^{-1}(A)$ is open since ϕ_α is continuous by assumption. We conclude that B is open, and hence that ϕ is continuous.

Finally, to see that ϕ is unique, let $\xi\colon X \to Y$ likewise satisfy $\xi \circ \sigma_\alpha = \phi_\alpha$ for each $\sigma_\alpha \in \Sigma$. Now for each $x \in X$ there is a $\sigma_\beta \in \Sigma$ such that $x \in \operatorname{Im} \sigma_\beta$, say $x = \sigma_\beta(p)$. In particular, we get that

$$\phi(x) = \phi(\sigma_\beta(p)) = (\phi\sigma_\beta)(p) = \phi_\beta(p) = (\xi\sigma_\beta)(p) = \xi(\sigma_\beta(p)) = \xi(x).$$

Since x was arbitrary we conclude that $\phi = \xi$. Thus ϕ is the unique map with this property.

It follows that $\operatorname{colim} D$ exists in the category **Δ-Comp** and equals to $\mathcal{X}$. □

Remark 3.8. If $\mathcal{X}$ and $\mathcal{Y}$ are Δ-complexes, then a morphism $f\colon \mathcal{X} \to \mathcal{Y}$ gives rise to a functor $F\colon D_\mathcal{X} \to D_\mathcal{Y}$, where $F(\Delta^{n_\alpha}, \sigma_\alpha) = (\Delta^{n_\alpha}, f \circ \sigma_\alpha)$, such that

$$f = \operatorname{colim} F\colon \mathcal{X} = \operatorname{colim} D_\mathcal{X} \to \operatorname{colim} D_\mathcal{Y} = \mathcal{Y}.$$

Conversely, any functor $F\colon D_\mathcal{X} \to D_\mathcal{Y}$ arises in this way from a morphism $f\colon \mathcal{X} \to \mathcal{Y}$ of Δ-complexes where, for $x \in X$, say $\mathring{\sigma}_\alpha(\xi) = x \in \operatorname{Im} \mathring{\sigma}_\alpha$ (uniquely) and $F(\Delta^{n_\alpha}, \sigma_\alpha) = (\Delta^{n_\alpha}, \tau_\alpha)$, we set $f(x) = \tau_\alpha(\xi) \in Y$.

Now we define the category of EZ-complexes, **EZ-Comp**. This definition is from [**7**] with minor changes in terminology and notation. In particular, we use the term "EZ-complex" in place of "semi-simplicial complex", and we use an $\mathbb{N}$-indexed collection of sets K_n in place of a set K together with a function $K \to \mathbb{N}$ (here $0 \in \mathbb{N}$).

Definition 3.9 (Eilenberg–Zilber [**7**])**.** An EZ-complex, K, is an $\mathbb{N}$-indexed collection of sets K_n together with a collection of functions

$$\partial_i^n\colon K_n \to K_{n-1}$$

for all $0 < n \in \mathbb{N}$ and all integers $0 \leqslant i \leqslant n$. These functions must satisfy the identity

$$\partial_i^{n-1}\partial_j^n = \partial_{j-1}^{n-1}\partial_i^n$$

for all $n \in \mathbb{N}$ and all $i < j$. Call the elements of K_n the *n-simplexes* of K. For any

n-simplex σ call the $(n-1)$-simplex $\partial_i^n(\sigma)$ the ith face of σ.

Note that for the above Definition 3.9, May [**17**] uses the notation ∂_i, Selick [**22**] uses d_i and $\sigma^{(i)}$ is used in [**7**]. Our aim now is to define a category $\widetilde{\Delta}$ such that any EZ-complex may be described as a contravariant functor from $\widetilde{\Delta}$ to **Set**. The same idea occurs in [**21**], and for simplicial sets can be found in [**17**, p. 4], [**22**, p. 91] and partly in [**7**, §8].

Definition 3.10. Let $\widetilde{\Delta}$ be the category whose

- objects are the ordered sets $[n] = \{0 < 1 < 2 < \cdots < n\}$ for each $n \in \mathbb{N}$,
- morphisms are the injective order-preserving functions,
- identities are the identity functions,
- composition is the usual composition of functions.

For every $n \in \mathbb{N}$ and every integer $0 \leqslant i \leqslant n+1$ define $\delta_i^n\colon [n] \to [n+1]$ by

$$\delta_i^n(k) = \begin{cases} k, & k < i, \\ k+1, & k \geqslant i. \end{cases}$$

These functions are all clearly order-preserving and injective, hence they are morphisms of $\widetilde{\Delta}$.

Lemma 3.11. *For any $n \in \mathbb{N}$ the equation*

$$\delta_j^{n+1} \circ \delta_i^n = \delta_i^{n+1} \circ \delta_{j-1}^n$$

holds for every $0 \leqslant i < j \leqslant n+2$.

The proof is a routine computation. The next result follows immediately from contravariance and Lemma 3.11.

Proposition 3.12. *Given a contravariant functor $L\colon \widetilde{\Delta} \to$ **Set**, the sets $K_n := L([n])$ together with the morphisms $\partial_j^n := L(\delta_j^{n-1})$ for all $n \in \mathbb{N}$ and all $0 \leqslant j \leqslant n$ form an EZ-complex $K := K_L$.*

Lemma 3.13. *Every non-identity morphism $\mu\colon [n] \to [n+m]$ in $\widetilde{\Delta}$ can be uniquely written as a composite*

$$\mu = \delta_{k_{m-1}}^{n+m-1} \circ \cdots \circ \delta_{k_1}^{n+1} \circ \delta_{k_0}^n \quad \textit{with} \quad k_0 < k_1 < \cdots < k_{m-1}.$$

Proof. This can be found in (8.3) of Section 8 of [**7**] (ignoring the degeneracy maps). □

Proposition 3.14. *Given any EZ-complex K we get a contravariant functor $L = L_K\colon \widetilde{\Delta} \to$ **Set** given by $L([n]) := K_n$ for all $n \in \mathbb{N}$ and, for any $\mu\colon [n] \to [n+m]$ write*

$$\mu = \delta_{k_{m-1}}^{n+m-1} \circ \cdots \circ \delta_{k_1}^{n+1} \circ \delta_{k_0}^n$$

and define

$$L(\mu) = \partial_{k_0}^{n-1} \circ \partial_{k_1}^n \circ \cdots \circ \partial_{k_{m-1}}^{n+m-2}.$$

Also set $L(1_{[n]}) = 1_{[n]}$.

Proof. The proof follows by remarking that all morphisms in $\widetilde{\Delta}$ are of the form $\mu\colon [n] \to [n+m]$ for some $n, m \in \mathbb{N}$. For $\nu\colon [n+m] \to [n+m+l]$, the equation $L(\nu \circ \mu) = L(\mu) \circ L(\nu)$ follows from Lemma 3.13. □

Given Propositions 3.12 and 3.14 we can identify an EZ-complex with a contravariant functor from $\widetilde{\Delta}$ to **Set** and vice-versa. A morphism of EZ-complexes is then a natural transformation between the associated contravariant functors. Denote the category of EZ-complexes by **EZ-Comp**.

For the next lemma recall that $\iota_k^n\colon \Delta^{n-1} \to \Delta^n$ is the face inclusion associated to the unique $(n-1)$-face of Δ^n which does not contain the kth vertex of Δ^n. Note that $\iota_k^n(e_i) = e_{\delta_k^{n-1}(i)}$ on the vertices $\{e_1, \ldots, e_n\}$ of Δ^{n-1}. So ι_k^n is "formally" the same as δ_k^{n-1}.

Lemma 3.15. *For any $n \in \mathbb{N}$ and any $1 \leqslant i < j \leqslant n+1$ the following identity holds:*

$$\iota_j^{n+1} \circ \iota_i^n = \iota_i^{n+1} \circ \iota_{j-1}^n.$$

Proof. This is routine to check. Formally (as remarked above) this is same as Lemma 3.11 except for the indexing. □

Definition 3.16. Given a Δ-complex, $\mathcal{X}$, let

$$\mathfrak{K}(\mathcal{X})_n = \{\sigma_\alpha \in \Sigma \ : \ \mathrm{Dom}(\sigma_\alpha) = \Delta^n\} = \Sigma_n$$

for each $0 < n \in \mathbb{N}$ and

$$\partial_i^n(\sigma_\alpha) = \sigma_\alpha \circ \iota_i^n\colon \Delta^{n-1} \to X$$

is the 'pre-compose by the inclusion of the ith face' function for all $0 < n \in \mathbb{N}$ and all $1 \leqslant i \leqslant n$.

We remark that one can think of $\mathfrak{K}(\mathcal{X})$ as a subset of $D_{\mathcal{X}}$.

Proposition 3.17. *Let $\mathfrak{K}(\mathcal{X})$ be the collections of sets $\mathfrak{K}(\mathcal{X})_n$ together with the collection of maps ∂_i^n as defined in Definition 3.16. Then $\mathfrak{K}(\mathcal{X})$ is an EZ-complex.*

Proof. Notice that whenever $1 \leqslant i < j \leqslant n+1$, for any $n \in \mathbb{N}$,

$$\begin{aligned}
\partial_i^{n-1}\partial_j^n(\sigma_\alpha) &= \partial_i^{n-1}(\sigma_\alpha \circ \iota_j^n) \\
&= \sigma_\alpha \circ \iota_j^n \circ \iota_i^{n-1} \\
&= \sigma_\alpha \circ \iota_i^n \circ \iota_{j-1}^{n-1} && \text{by Lemma 3.15} \\
&= \partial_{j-1}^{n-1}\partial_i^n(\sigma_\alpha),
\end{aligned}$$

hence $\mathfrak{K}(\mathcal{X})$ is an EZ-complex as claimed. □

Definition 3.18. Given a morphism $f\colon \mathcal{X} \to \mathcal{Y}$ of Δ-complexes, define

$$\mathfrak{K}(f)\colon \mathfrak{K}(\mathcal{X}) \to \mathfrak{K}(\mathcal{Y})$$

to be the pointwise post-composition function

$$\mathfrak{K}(\mathcal{X})_n \ni \sigma_\alpha \longmapsto f \circ \sigma_\alpha.$$

That this is well-defined is exactly the definition of a morphism of Δ-complexes.

Proposition 3.19. *The assignment of Definition 3.18 is a (covariant) functor*

$$\mathfrak{K}\colon\ \mathbf{\Delta\text{-Comp}} \to \mathbf{EZ\text{-Comp}}\,.$$

Proof. Given a morphism $f\colon \mathcal{X} \to \mathcal{Y}$ in **Δ-Comp** by definition we have that $\mathfrak{K}(f)$ is a morphism in **EZ-Comp** from $\mathfrak{K}(\mathcal{X})$ to $\mathfrak{K}(\mathcal{Y})$, thus domains and codomains are respected as required.

Further, since $\mathfrak{K}(1_X)_n$ is the 'post-compose with the identity' function it is clear that $\mathfrak{K}(1_{\mathcal{X}})$ is the identity morphism.

Finally, given morphisms $f\colon \mathcal{X} \to \mathcal{Y}$ and $g\colon \mathcal{Y} \to \mathcal{Z}$ in **Δ-Comp** we compute

$$\mathfrak{K}(g \circ f)(\sigma_\alpha) = (g \circ f) \circ \sigma_\alpha = g \circ (f \circ \sigma_\alpha) = \mathfrak{K}(g)(f \circ \sigma_\alpha) = \mathfrak{K}(g) \circ \mathfrak{K}(f)(\sigma_\alpha). \quad \square$$

Definition 3.20 (MacLane [**16**]). Let $\mathcal{C}$ and $\mathcal{D}$ be two categories and let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor. Then F induces a function

$$F_{X,Y}\colon \hom_{\mathcal{C}}(X,Y) \to \hom_{\mathcal{D}}(F(X),F(Y))$$

for every pair of objects X and Y in $\mathcal{C}$. The functor F is said to be

- *fully faithful* if $F_{X,Y}$ is bijective for each X and Y in $\mathcal{C}$.
- *essentially surjective* if each object d in $\mathcal{D}$ is isomorphic to an object of the form Fc, for c in $\mathcal{C}$.

Theorem 3.21. *The functor* $\mathfrak{K}$ *is an equivalence of categories between* **Δ-Comp** *and* **EZ-Comp**.

Proof. Recall (see MacLane [**16**, Theorem IV.4.1 (iii)]) that two categories $\mathbb{X}$ and $\mathbb{Y}$ are equivalent if and only if there is a functor $F\colon \mathbb{X} \to \mathbb{Y}$ such that F is fully faithful and essentially surjective. Thus we need to show that $\mathfrak{K}$ is fully faithful and essentially surjective.

To show that $\mathfrak{K}$ is faithful let $f, g\colon X \to Y$ be distinct morphisms in **Δ-Comp**. We want to show that $\mathfrak{K}(f) \neq \mathfrak{K}(g)$ in **EZ-Comp**. Since $f \neq g$ there is a point $x \in X$ such that $f(x) \neq g(x)$ by the definition of a function. Since X is a Δ-complex we get that there is an α such that x is an element of the image of $\sigma_\alpha\colon \Delta^{n_\alpha} \to X$, hence $f \circ \sigma_\alpha \neq g \circ \sigma_\alpha$. Then

$$\mathfrak{K}(f)(\sigma_\alpha) = f \circ \sigma_\alpha \neq g \circ \sigma_\alpha = \mathfrak{K}(g)(\sigma_\alpha)$$

so $\mathfrak{K}(f) \neq \mathfrak{K}(g)$.

To show that $\mathfrak{K}$ is full, let $\tilde{f}\colon\ \mathfrak{K}(\mathcal{X}) \to \mathfrak{K}(\mathcal{Y})$ be a morphism in **EZ-Comp**, where $\mathcal{X} = (X, \Sigma)$ and $\mathcal{Y} = (Y, \Sigma')$. Recall from Lemma 3.7 that $\mathcal{X} = \operatorname{colim} D_{\mathcal{X}}$ and $\mathcal{Y} = \operatorname{colim} D_{\mathcal{Y}}$. Thus, to define a morphism from $\mathcal{X}$ to $\mathcal{Y}$, it suffices by Remark 3.8 to specify a functor $F\colon D_{\mathcal{X}} \to D_{\mathcal{Y}}$. Define

$$F(\Delta^{n_\alpha}, \sigma_\alpha) = (\Delta^{n_\alpha}, \tilde{f}(\sigma_\alpha))$$

and for any face inclusion $\iota\colon \Delta^{n_\alpha} \to \Delta^{n_\beta}$ define

$$F(\iota) = \tilde{f}(\sigma_\beta) \circ \iota\colon \Delta^{n_\alpha} \to Y.$$

For $\iota_j^{n_\beta}\colon \Delta^{n_\beta - 1} \to \Delta^{n_\beta}$, we have

$$\begin{aligned} F(\iota_j^{n_\beta}) = \tilde{f}(\sigma_\beta) \circ \iota_j &= \partial_j(\tilde{f}(\sigma_\beta)) && \text{Definition 3.16} \\ &= \tilde{f}(\partial_j(\sigma_\beta)) && \text{since } \tilde{f} \text{ is a morphism in } \mathbf{EZ\text{-}Comp} \\ &= \tilde{f}(\sigma_\beta \circ \iota_j) \\ &= \tilde{f}(\sigma_\alpha). \end{aligned}$$

Since any face inclusion $\iota\colon \Delta^m \hookrightarrow \Delta^n$ is a composition $\iota = \iota_{k_{n-m}}^{n-1} \circ \cdots \circ \iota_{k_1}^m$, the above equality and the associative law show $\tilde{f}(\sigma_\beta) \circ \iota = \tilde{f}(\sigma_\alpha)$ (where $m = n_\alpha$ and $n = n_\beta$), hence the collection $\tilde{f}(\sigma_\beta)$ forms a cocone from the base D into Y. The universal property of the colimit then gives a morphism $f = \operatorname{colim} F\colon X \to Y$ such that $f \circ \sigma_\alpha = \tilde{f}(\sigma_\alpha)$ for all $\sigma_\alpha \in \Sigma$. But

$$\mathfrak{K}(f)_n(\sigma_\alpha) = f \circ \sigma_\alpha = \tilde{f}(\sigma_\alpha)$$

by definition, hence $\mathfrak{K}(f) = \tilde{f}$. Since $\tilde{f}$ was arbitrary we get that $\mathfrak{K}$ is full.

To show that $\mathfrak{K}$ is essentially surjective pick any EZ-complex $K\colon \widetilde{\Delta} \to \mathbf{Set}$. We will construct a Δ-complex $\mathcal{X}$ such that $\mathfrak{K}(\mathcal{X}) \cong K$. This is motivated by the similar construction in [**11**, p. 104]. First we define a diagram in **Δ-Comp** in the following way. Define the objects of D to be, for each $n \in \mathbb{N}$ and each $\alpha \in K_n$, a copy of Δ^{n_α} indexed by α. For each $n \in \mathbb{N}$, each integer $m \geqslant 1$, each $\alpha \in K_{n+m}$, and every morphism $\gamma\colon [n] \to [n+m]$ in $\widetilde{\Delta}$ write

$$\gamma = \delta_{k_{m-1}}^{n+m-1} \circ \cdots \circ \delta_{k_1}^{n+1} \circ \delta_{k_0}^{n}$$

and set

$$D(K(\gamma)(\alpha), \alpha) := \{\iota_{k_{m-1}}^{n+m} \circ \cdots \circ \iota_{k_0}^{n+1}\},$$

where $K(\gamma)\colon K_{n+m} \to K_n$. These sets are composition-closed with respect to the index by functoriality. Construct

$$X = \big(\bigsqcup \Delta^{n_\alpha}\big)/\sim,$$

where $\sim$ is the same set of identifications mentioned in Hatcher [**11**, p. 104]. It is routine to check that

$$\big(X, \{\phi_\alpha\colon \Delta^{n_\alpha} \to X \text{ and } \alpha \in \bigcup_{n\in\mathbb{N}} K_n\}\big)$$

is a Δ-complex, that is, it satisfies the conditions A1, …, A4 of Definition 3.2.

Further $\mathfrak{K}(\mathcal{X}) \cong K$ since each $\mathfrak{K}(\mathcal{X})_n$ consists of exactly one face inclusion for each element of K_n, and since, for each $n \in \mathbb{N}$ and each $k \leqslant n$ we have, in $\mathfrak{K}(\mathcal{X})$, that

$$\partial_k^n(\phi_\alpha) = \phi_\alpha \iota_k^n = \phi_{\partial_k^n(\alpha)}$$

by the construction of the diagram D. Thus the morphism $K \to \mathfrak{K}(\mathcal{X})$ defined by sending α to ϕ_α is both well-defined and a pointwise isomorphism. Natural transformations which are pointwise isomorphisms are natural isomorphisms, hence $K \cong \mathfrak{K}(\mathcal{X})$ in **EZ-Comp**. We conclude that $\mathfrak{K}$ is essentially surjective. □

The next corollary deals with the chain complex functors from either of these two categories to **Comp**, the category of chain complexes of abelian groups.

Corollary 3.22. *The functor* $\mathfrak{K}$ *commutes with the chain complex functors on* **Δ-Comp** *and* **EZ-Comp**, *indeed one has the commutative diagram*

$$\begin{array}{ccc} \mathbf{\Delta\text{-}Comp} & \xrightarrow{\mathfrak{K}} & \mathbf{EZ\text{-}Comp} \\ \big\downarrow{\scriptstyle c_\Delta} & & \big\downarrow{\scriptstyle c} \\ \mathbf{Comp} & = & \mathbf{Comp}. \end{array} \tag{1}$$

Proof. For any Δ-complex $\mathcal{X} = (X, \Sigma)$, we find that both $C_n(\mathcal{X})$ and $C_n(\mathfrak{K}(\mathcal{X}))$ are equal to the free abelian group on $\Sigma_n = \mathfrak{K}(\mathcal{X})_n$. Since $\partial_i^n(\sigma_\alpha) = \sigma_\alpha \circ \iota_i^n$, as in Definition 3.16, the boundary maps are also identical. □

From Corollary 3.22 and [**21**, Theorem 1.7, Proposition 2.1] we have the following.

Corollary 3.23. *The categories* **Δ-Comp**, **EZ-Comp** *and* **Simp** *are equivalent.*

Remark 3.24. As proved in [**11**], the natural inclusion of the chains of a Δ-complex X into the singular chains $S(X)$ induces an isomorphism in homology. Therefore the homology of Δ-complexes is a homology theory on **Δ-Comp** satisfying the Eilenberg–Steenrod axioms.

4. Construction of Δ-complexes

We commence this section by proving a basic result, Theorem 4.1, which says that a G-action on a Δ-complex X by Δ-morphisms induces a Δ-complex structure on $|X|/G$. This is followed by a few definitions and remarks pertaining to the case where X is a simplicial complex. Then, after giving some further results, we compare our Definition 4.3 with the corresponding definitions of Illman [**14**] using an example. Throughout the section, it is assumed that G and X are finite.

Theorem 4.1. *Let* $\mathcal{X} = (X, \Sigma)$ *be a Δ-complex and* G *act on* $\mathcal{X}$ *by Δ-morphisms. Then the orbit space* X/G *has a Δ-complex structure* $(X/G, \bar{\Sigma})$ *and the canonical identification map* $\pi\colon X \twoheadrightarrow X/G$ *induces a morphism* $(X, \Sigma) \twoheadrightarrow (X/G, \bar{\Sigma})$ *of Δ-complexes.*

Proof. We adhere to the notations of Section 3. Since G acts on $\mathcal{X}$ by Δ-morphisms, if $\sigma\colon \Delta^n \to X$ is an n-simplex then the composition $g\sigma$ given by $\Delta^{n_\alpha} \xrightarrow{\sigma} X \xrightarrow{g} X$ is an n-simplex in $\mathcal{X}$. That is, if $\sigma \in \Sigma_n$ then $g\sigma \in \Sigma_n$. Conversely, if $g\sigma = \tau \in \Sigma_n$ then $\sigma = g^{-1}\tau \in \Sigma_n$ for all $g \in G$. Thus X^n is a G-invariant subset of X, where X^n is the union of the images of all $\sigma \in \Sigma_m$, $m \leqslant n$, i.e., the n-skeleton of X. Let $\sigma \in \Sigma_n$. Then

$$\pi^{-1}(\pi(\sigma(\mathring{\Delta}^n))) = \bigcup \{g\ \sigma(\mathring{\Delta}^n) : g \in G\}. \tag{2}$$

Suppose $g\sigma(\mathring{\Delta}^n) \cap \sigma(\mathring{\Delta}^n) \neq \emptyset$ and x belongs to this intersection. So x is in the image of $g\mathring{\sigma}$ and $\mathring{\sigma}$. Then by condition A2 of Definition 3.2, $g\sigma(\Delta^n) = \sigma(\Delta^n)$. Since the G-action on $\mathcal{X}$ preserves the ordering of each simplex, $g\sigma(x) = \sigma(x)$ for all $x \in \Delta^n$.

Summarizing the above, either $g\sigma(\overset{\circ}{\Delta}{}^n) \cap \sigma(\overset{\circ}{\Delta}{}^n) = \emptyset$ or $g|_{\sigma(\Delta^n)} = id_{\sigma(\Delta^n)}$. From this observation and (2) it follows that

$$\pi|_{\sigma(\overset{\circ}{\Delta}{}^n)} : \sigma(\overset{\circ}{\Delta}{}^n) \to \pi(\sigma(\overset{\circ}{\Delta}{}^n))$$

is a homeomorphism. On the other hand, any G-action on $\mathcal{X}$ by Δ-morphisms induces an equivalence relation on Σ where as usual $\sigma \sim \tau$ if and only if $\tau = g\sigma$ for some $g \in G$. Let $A_n = \{\sigma_{n1}, \ldots, \sigma_{nk_n}\}$ be a set of representatives for the equivalence classes of $\sigma \in \Sigma_n$.

So σ_{ni}'s are distinct and

$$X/G = \bigcup_{n=0}^{\dim(X)} \bigcup_{\sigma \in A_n} \pi(\sigma(\Delta^n)).$$

It is now straightforward that the collection of maps

$$\beta_{ni} = \pi \circ \sigma_{ni} \colon \Delta^n \to X/G,$$

where $0 \leqslant n \leqslant \dim(K)$ and $1 \leqslant i \leqslant k_n$, satisfies the conditions of Definition 3.2 of a Δ-complex. Therefore there is a Δ-complex structure on X/G with vertices $V(X)/G$ and clearly the identification map is a morphism of Δ-complexes. $\square$

Definition 4.2. 1. A (*geometric*) *simplicial complex* K is a finite collection of simplexes in some $\mathbb{R}^N$ satisfying: (i) if σ_1 is a face of σ_2 and $\sigma_2 \in K$ then $\sigma_1 \in K$, (ii) if $\sigma_1, \sigma_2 \in K$ then $\sigma_1 \cap \sigma_2$ is a face of both σ_1 and σ_2. The zero dimensional cells of K are called the vertices $V(K)$ of K and the empty set $\emptyset$ is a (-1)-simplex by convention. The dimension of K is the maximum of the dimensions of the simplexes in K.

2. For any simplicial complex K and any subset $L \subseteq K$, one defines its polyhedron as
$$|L| = \bigcup_{\sigma \in L} \sigma \subset \mathbb{R}^N.$$

Note that $|L|$ is a compact subset of $\mathbb{R}^N$. Since we wish to work in the category **Δ-Comp**, we shall usually assume that the vertices of any simplex of K are simply ordered so that if σ_1 is a face of σ_2 then the ordering of the vertices of σ_1 agrees with the ordering coming from σ_2, and in this case call K an *ordered simplicial complex*. For instance, Example 3.6 gives a method to obtain such an ordering from a partial ordering on $V(K)$, but note that orderings that are not of this type also exist (cf. Example 4.12). Next we introduce the idea of a G-action on a (finite) simplicial complex.

Definition 4.3. Let G be a finite group and K be a finite simplicial complex. A G-action on K is called simplicial if

$$\sigma = \langle v_0, \ldots, v_r \rangle \in K \quad \text{then} \quad g\sigma = g(\sigma) = \langle gv_0, \ldots, gv_r \rangle \in K.$$

In addition, if K is an ordered simplicial complex and G preserves the ordering of each simplex of K, we call K an *ordered simplicial G-complex*.

Example 4.4. Let $v_0 = -1$, $v_1 = 1$ and $\langle v_0, v_1\rangle = [-1, 1]$. Then $K = \{\langle v_0\rangle, \langle v_1\rangle, \langle v_0, v_1\rangle\}$ is a simplicial complex. The order two flip $(x \mapsto -x)$ on $\langle v_0, v_1\rangle$ is a simplicial action of $\mathbb{Z}_2$ on K but not an ordered simplicial action.

Remark 4.5. In [**14**, §1] the definitions of both a simplicial G-complex, and an equivariant simplicial G-complex, are given. At the end of this section (Example 4.12) we compare these definitions with Definition 4.3. Note that Definition 4.3 induces a G-action on the polyhedron $|K|$ via

$$\sum_{i=0}^{n} t_i v_i \longmapsto \sum_{i=0}^{n} t_i g(v_i)$$

and we denote both the actions by the same notation. This action is clearly a simplicial homeomorphism for each $g \in G$.

Our primary concern in this section is quotients of ordered simplicial complexes, and, in particular, we get the following as a special case of Theorem 4.1.

Proposition 4.6. *Let K be an ordered simplicial G-complex. Then the orbit space $|K|/G$ is a Δ-complex with vertices $V(K)/G$. Furthermore, the canonical identification map $\pi\colon |K| \twoheadrightarrow |K|/G$ is a morphism of Δ-complexes.*

Proposition 4.7. *Let K be a simplicial complex and τ a free simplicial involution on K, i.e., a free $\mathbb{Z}_2$-action on K. Then there exists an ordering of the vertices of K such that K is an ordered simplicial $\mathbb{Z}_2$-complex.*

Proof. Let V be the set of vertices of K. One can write $V = A \sqcup \tau A$, where A is a subset of V containing exactly half of the vertices of V. Let $v_1 < v_2 < \cdots < v_k$ be an order on A. Define an order on V by

$$v_1 < \tau v_1 < v_2 < \tau v_2 < \cdots < v_k < \tau v_k.$$

This induces a Δ-complex structure on K, which we also denote by K. Since K is a simplicial complex and τ is a free involution, $\langle v_i, \tau v_i\rangle$ is not an edge in K. Let $\sigma = \langle u_1, \ldots, u_\ell\rangle \in K$. Then u_i is either v_{j_i} or τv_{j_i} for $i = 1, \ldots, \ell$ and $j_i < j_{i+1}$ for $i = 1, \ldots, \ell - 1$. So τu_i is respectively either τv_{j_i} or v_{j_i}. Then

$$\tau\sigma = \langle \tau u_1, \ldots, \tau u_\ell\rangle \quad \text{and} \quad \tau u_1 < \cdots < \tau u_\ell.$$

Therefore τ preserves the order on each simplex of K. This proves the proposition. □

Propositions 4.6 and 4.7 imply the following.

Corollary 4.8. *For any free involution τ on a finite simplicial complex K, the space $|K|/\mathbb{Z}_2$ has a Δ-complex structure.*

Next we define a simplicial G-complex and an equivariant simplicial G-complex following Illman [**14**], and we compare these definitions with our Definition 4.3 by means of Proposition 4.11 and Example 4.12.

Definition 4.9 (Illman [**14**])**.** Let K be a simplicial complex and G act on $|K|$. Then K is called a simplicial G-complex, if the map $g\colon K \to K$ is a simplicial homeomorphism for all $g \in G$. Moreover, K is called an equivariant simplicial G-complex if the following also holds:

1. if $\langle v_0, \ldots, v_k \rangle \in K$ and $\langle g_0 v_0, \ldots, g_k v_k \rangle \in K$ for some $g_0, \ldots, g_k \in G$, then there exists $g \in G$ such that $g_i v_i = g v_i$ for $i = 0, \ldots, k$. Here $v_0, \ldots, v_k$ need not be distinct.
2. any simplex $\langle v_0, \ldots, v_k \rangle \in K$ can be ordered such that the inclusions $G_{v_k} \subseteq \cdots \subseteq G_{v_0}$ hold for the isotropy groups.

In the next remark observations 2 and 3 are due to Illman, and given in [**14**] without proof.

Remark 4.10. 1. An equivariant simplicial G-complex is trivially a simplicial G-complex. Also, by the final sentence of Remark 4.5, an ordered simplicial G-complex is a simplicial G-complex.

2. The identification space of an equivariant simplicial G-complex by the action of G is a simplicial complex.
3. For any simplex σ, if $x \in \sigma$ and $gx \in \sigma$ then $gx = x$, and if $x \in \mathring{\sigma}$ and $gx = x$ then $gy = y$ for all $y \in \sigma$.

Proposition 4.11. *Let K be an equivariant simplicial G-complex. Then there exists an ordering for the simplexes of K with respect to which K is an ordered simplicial G-complex.*

Proof. By Remark 4.10 (2) K/G is a simplicial complex. Writing the vertices of K as v and of K/G as $[v]$, we can then order the vertices $[v] \in K/G$ using a suitable partial order $\preceq$ as in Example 3.6. Lifting this to a partial order $\leqslant$ for the vertices of K, i.e. $v_1 \leqslant v_2$ if and only if $[v]_1 \preceq [v_2]$, then $\leqslant$ has the property that the vertices of any simplex of K are simply ordered, and it is preserved under the G-action. □

We next give an example of an ordered simplicial G-complex which is not an equivariant simplicial G-complex. So, the category of ordered simplicial G-complexes properly contains the category of equivariant simplicial G-complexes.

Example 4.12. In this example the arrows on the edges imply the ordering of simplexes. For example, the 2-simplex $v_0 v_1 v_2$ in Figure 1 is ordered as $\langle v_1, v_2, v_0 \rangle$.

Consider the 2-dimensional simplicial complex K corresponding to Figure 1. The

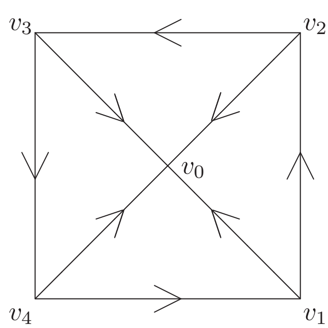


Figure 1: An ordered simplicial G-complex.

cyclic group $\mathbb{Z}_4 \cong G = \langle \alpha \mid \alpha^4 = 1 \rangle$ acts on K as determined by

$$\alpha v_0 = v_0, \;\; \alpha v_4 = v_1, \;\text{ and } \alpha v_i = v_{i+1} \text{ for } i = 1, 2, 3.$$

Then v_0 is fixed by G. Notice that the G-action is order preserving. So K is an ordered simplicial G-complex. Thus by Proposition 4.6, $|K|/G$ is a Δ-complex with vertices $V(K)/G$.

Observe that the G-action on K does not satisfy Remark 4.10 (2) or Remark 4.10 (3), so this is not an equivariant simplicial G-complex.

In the above we discussed group actions which preserve the ordering of each simplex. More generally we may ask the following.

Question: Let K be simplicial G-complex. Do there exist any other (possibly more general) sufficient conditions which ensure that $|K|/G$ has the structure of a Δ-complex?

5. Application to real projective space

A fairly simple way to triangulate $\mathbb{R}P^n$ is to start with the standard triangulation of S^n as the boundary of an n-simplex. The antipodal map τ is then not a simpicial map, however, if one passes to the first barycentric subdivision the antipodal map is now simplicial with $\sigma \cap \tau\sigma = \emptyset$ for all simplexes, hence a triangulation of $\mathbb{R}P^n = S^n/\mathbb{Z}_2$ is induced with $2^{n+1} - 1$ vertices, see Example 3.20 in [**5**]. Also, Basak and the second author constructed some triangulation of $\mathbb{RP}^n$ with $2^n + n + 1$ vertices, see Theorem 3.24 in [**1**]. Using these triangulations to compute the cohomology ring of $\mathbb{R}P^n$ still seems very complicated (the authors succeeded for $n \leqslant 3$).

However, $\mathbb{R}P^n$ admits a much simpler structure as a Δ-complex with only $n+1$ vertices, already mentioned as an exercise in [**11**]. After setting up some notation, we shall show how this Δ-complex can be used to give an elementary and explicit computation of the cohomology ring. A similar computation is discussed in [**12**] using an alternative technique, block-dissections.

Following Coxeter [**3**], the regular polytope $\beta_n \subset \mathbb{R}^{n+1}$ with the $2(n+1)$ vertices $\pm e_i$, $1 \leqslant i \leqslant n+1$, is called the n-dimensional cross polytope (or orthoplex), and is homeomorphic to S^n. For $n = 1, 2$ it is respectively the familiar square, octahedron. It will be useful to label the vertices as $v_i = e_i$, $w_i = -e_i$. Of course β_n is a simplicial complex, with vertex set

$$V = \{v_i, w_i : 1 \leqslant i \leqslant n+1\} = \{\lambda v_1, \dots, \lambda v_{n+1} : \lambda = \pm 1\}.$$

Any subset of V that contains no pair v_i, w_i will form the vertices of a simplex of the simplicial complex β_n. Equivalently, an r-simplex of β_n can be written as $\sigma = \langle \lambda_0 v_{i_0}, \dots, \lambda_r v_{i_r} \rangle$, where $1 \leqslant i_0 < i_1 < \ldots < i_r \leqslant n+1$, $\lambda_i = \pm 1$ and $\langle x_0, \dots x_r \rangle$ denotes the convex hull of points $x_0, \dots, x_r \in \mathbb{R}^{n+1}$. From this description we see that the number of r-simplexes equals $2^{r+1} \cdot \binom{n+1}{r+1}$.

Now let τ be the simplicial map of β_n to itself induced by defining $\tau(\lambda v_i) = -\lambda v_i$ on the vertices of β_n. Clearly τ is a simplicial involution and gives rise to a free simplicial action of the group $C_2 = \{e, \tau\}$ on β_n. Also note $|\beta_n| \cong S^n$ with $\tau(x) = -x$

for any $x \in |\beta_n|$, hence

$$|\beta_n|/C_2 \cong \mathbb{R}P^n.$$

Each r-simplex σ of $\mathbb{R}P^n$ can be written as

$$\sigma = \langle x_{i_0}, x_{i_1}, \ldots, x_{i_r} \rangle = \langle -x_{i_0}, -x_{i_1}, \ldots, -x_{i_r} \rangle,$$

where $1 \leqslant i_0 < i_1 < \ldots < i_r \leqslant n+1$ and $x_{i_j} \in \{v_{i_j}, w_{i_j}\}$, $0 \leqslant j \leqslant r$ (as before $v_{i_j} = e_{i_j}$, $w_{i_j} = -e_{i_j}$). This notation is compatible with the simple order $v_1 < w_1 < v_2 < \cdots < v_{n+1} < w_{n+1}$ of the vertices of β_n so gives both β_n and $\mathbb{R}P^n$ a Δ-complex structure (which would also arise by applying Proposition 4.6 or Corollary 4.8).

We now turn our attention to describing the chain complex $\mathcal{C}$ arising from this Δ-complex, and then using it to give an elementary computation of the cohomology ring $H^*(\mathbb{R}P^n; \mathbb{Z}_2)$. Of course this is well known to be the truncated polynomial algebra $\mathbb{Z}_2[u]/(u^{n+1} = 0)$ and can be found in many texts, e.g. [**11**], but using more sophisticated machinery. Our goal here is to utilize the above Δ-complex structure for $\mathbb{R}P^n$ to compute its homology and cohomology (ring) as well as to exhibit explicit generators for these, in terms of the simplexes of $\mathbb{R}P^n$.

In particular, the following notation for the 2^n n-simplexes of $\mathbb{R}P^n$ will be convenient. From the above discussion each such simplex can be uniquely written $\sigma = \langle x_1, \ldots, x_{n+1} \rangle$ where $x_i \in \{v_i, w_i\}$ and (for uniqueness) $x_{n+1} = v_{n+1}$. Equivalently,

$$\sigma = \langle \lambda_1 v_1, \ldots, \lambda_n v_n, v_{n+1} \rangle, \quad \lambda_i = \pm 1.$$

Then these 2^n simplexes can be written σ_A, one for each $A \subseteq \underline{n} = \{1, 2, \ldots, n\}$, where $A = \{i : \lambda_i = -1\}$. For example, $\sigma_\emptyset = \langle v_1, v_2, \ldots, v_{n+1} \rangle$. The coefficients of each chain complex will be $\mathbb{Z}$ unless otherwise indicated, and $|A|$ will denote the cardinality of A. We remark that there is a slight abuse of notation here since we are using the same notation for the simplex σ in S^n and its image in $\mathbb{R}P^n$, but there should be no confusion since we only deal with $\mathbb{R}P^n$.

We first consider the chain complex $C_*(\mathbb{R}P^n, \mathbb{R}P^{n-1})$ and its differential

$$\tilde{d}\colon C_j(\mathbb{R}P^n, \mathbb{R}P^{n-1}) \to C_{j-1}(\mathbb{R}P^n, \mathbb{R}P^{n-1}),$$

in particular, for $j = n$.

Lemma 5.1. (a) *$H_n(\mathbb{R}P^n, \mathbb{R}P^{n-1}) \approx \mathbb{Z}$, with generator* $s_n := \sum_{A \subseteq \underline{n}} (-1)^{|A|} \sigma_A$.

(b) *For $0 \leqslant i \leqslant n-1$, $H_i(\mathbb{R}P^n, \mathbb{R}P^{n-1}) = 0$.*

Proof. (a) Note that $C_n(\mathbb{R}P^n, \mathbb{R}P^{n-1}) = C_n(\mathbb{R}P^n)$ since $C_n(\mathbb{R}P^{n-1}) = 0$, and also that $C_{n+1}(\mathbb{R}P^n, \mathbb{R}P^{n-1}) = 0$. It follows that

$$H_n(\mathbb{R}P^n, \mathbb{R}P^{n-1}) = \mathrm{Ker}(\tilde{d_n}\colon C_n(\mathbb{R}P^n) \to C_{n-1}(\mathbb{R}P^n, \mathbb{R}P^{n-1})).$$

Since each $(n-1)$-simplex of $\mathbb{R}P^n \setminus \mathbb{R}P^{n-1}$ occurs in the boundary of precisely two σ_A's, any n-cycle $x \in \mathrm{Ker}(\tilde{d})$ must be an integral multiple of $\sum_{A \subseteq \underline{n}} \lambda_A \cdot \sigma_A$ for some $\lambda_A = \pm 1$. Without loss of generality fix $\lambda_\emptyset = +1$, then all the other signs λ_A are uniquely determined by the cycle condition $\tilde{d}(x) = 0$. It then only remains to check that $\tilde{d}(s_n) = 0$ since this will imply that $x = m \cdot s_n$ for some $m \in \mathbb{Z}$ and hence gives the conclusion of the lemma. To complete the proof then, consider any $n-1$ simplex

ω of $\mathbb{R}P^n \setminus \mathbb{R}P^{n-1}$. It must have the form σ_A with a single vertex v_j missing, where $1 \leqslant j \leqslant n$. Then it occurs in the boundary of precisely two σ_A, one for which $j \notin A$ and one for which $j \in A$. The cardinalities of these two subsets A differ by 1, so the form of s_n guarantees that ω occurs with opposite signs in $\tilde{d}(s_n)$ and thus $\tilde{d}(s_n) = 0$.

(b) A short non-simplicial proof of (b) follows from Proposition 2.22 of Hatcher [**11**], namely $H_i(\mathbb{R}P^n, \mathbb{R}P^{n-1}) \approx \widetilde{H}_i(\mathbb{R}P^n/\mathbb{R}P^{n-1}) \approx \widetilde{H}_i(S^n) = 0$, $i < n$. We also give a simplicial proof of this fact by means of an explicit contracting homotopy in the chain complex itself, but defer this to the Appendix so that we can continue with the homology calculations. □

Using this lemma, the next step is to compute the connecting homomorphism

$$\partial\colon H_n(\mathbb{R}P^n, \mathbb{R}P^{n-1}) \to H_{n-1}(\mathbb{R}P^{n-1}).$$

Since, as established in Section 3, Δ-complexes give rise to a homology theory, knowing this connecting homomorphism together with Lemma 5.1 will suffice to determine (by induction) the exact homology sequence of the pair $(\mathbb{R}P^n, \mathbb{R}P^{n-1})$, hence, in particular, the (integral) homology of $\mathbb{R}P^n$ with explicit generators. The differential in the chain complex $C_*(\mathbb{R}P^n)$ will be denoted

$$d\colon C_j(\mathbb{R}P^n) \to C_{j-1}(\mathbb{R}P^n)$$

(in contrast with $\tilde{d}$ defined earlier). We may regard s_n as an n-chain in $C_n(\mathbb{R}P^n, \mathbb{R}P^{n-1})$ or in $C_n(\mathbb{R}P^n)$ since, as noted in the proof of Lemma 5.1, these are equal.

Theorem 5.2. (a) $\partial(s_n) = (1 + (-1)^n)s_{n-1}$,

(b)

$$H_i(\mathbb{R}P^n) = \begin{cases} \mathbb{Z} & : \; i = 0 \textit{ and } i = n \textit{ odd}, \\ \mathbb{Z}_2 & : \; 0 < i < n \textit{ and } n \textit{ odd}, \\ 0 & : \; \textit{otherwise}. \end{cases}$$

Proof. (a) Since, as noted above, $j_*\colon C_n(\mathbb{R}P^n) \to C_n(\mathbb{R}P^n, \mathbb{R}P^{n-1})$ is the identity map, we have

$$\partial(s_n) = (i_*)^{-1} d (j_*)^{-1}(s_n) = (i_*)^{-1} d(s_n).$$

As shown in the proof of Lemma 5.1, any simplex containing the vertex v_{n+1} vanishes in $d(s_n)$, so we need only consider the coefficients of simplexes of the form $\langle \lambda_1 v_1, \dots, \lambda_n v_n \rangle$ in $d(s_n)$. Write such a simplex (non-uniquely) as τ_A, $A \subseteq \underline{n}$, noting that $\tau_A = \tau_B$ if and only if $B = \underline{n} \setminus A$. By adjoining the vertex v_{n+1} to τ_A, τ_B we obtain two distinct n-simplexes σ_A, σ_B of $\mathbb{R}P^n$. Now $d(\sigma_A)$ will contain τ_A with coefficient $(-1)^{n+1} \cdot (-1)^{|A|}$ whereas $d(\sigma_B)$ will contain $\tau_B = \tau_A$ with coefficient

$$(-1)^n \cdot (-1)^{|B|} = (-1)^n \cdot (-1)^{n-|A|}.$$

For n odd these two coefficients add to 0, while for n even they add to $2 \cdot (-1)^{|A|}$. Since in the n even case this sum is precisely $i_*(2s_{n-1})$, and 0 in the odd case, (a) is proved.

(b) This is a straightforward induction, starting the induction with the known case $\mathbb{R}P^1 \cong S^1$, and using the exact homology sequence of the pair $(\mathbb{R}P^n, \mathbb{R}P^{n-1})$ together with (a) as well as Lemma 5.1 to complete the inductive step. Note that there are two cases in the inductive step, $(n-1)$ odd to n even or vice-versa. We omit the easily verified details. □

Applying the universal coefficient theorem, or repeating the above proof with $\mathbb{Z}_2$ coefficients, now gives the following corollary at once.

Corollary 5.3. *For all j, $0 \leqslant j \leqslant n$, one has $H_j(\mathbb{R}P^n;\mathbb{Z}_2) \approx \mathbb{Z}_2$ and $H^j(\mathbb{R}P^n;\mathbb{Z}_2) \approx \mathbb{Z}_2$.*

Using this corollary we can now turn to the final goal of this section, computing $H^*(\mathbb{R}P^n;\mathbb{Z}_2)$ as a graded ring with explicit generator. We shall use the notation e.g. $\langle v_1, v_2, v_3\rangle^*$ to denote the dual cochain of the simplex $\langle v_1, v_2, v_3\rangle$ with coefficients $\mathbb{Z}$ or $\mathbb{Z}_2$, i.e. $\langle v_1, v_2, v_3\rangle^*(\sigma) = 1$ if $\sigma = \langle v_1, v_2, v_3\rangle$ and is 0 otherwise, and similarly for any other r-simplex. Since cup products are to be considered, a suitable partial order of the vertices of the Δ-complex must be used, we choose the (simple) order $v_1 < w_1 < v_2 < \ldots < v_{n+1} < w_{n+1}$. As with any connected n-manifold, any n dimensional "co-simplex" $\langle \lambda_1 v_1, \ldots, \lambda_{n+1}v_{n+1}\rangle^*$ is a cocycle that generates $H^n(\mathbb{R}P^n;\mathbb{Z}_2)$.

To complete the main goal of this section, we now exhibit an explicit 1-cocycle $u \in C^1(\mathbb{R}P^n;\mathbb{Z}_2)$ such that its cohomology class $[u] \in H^1(\mathbb{R}P^n;\mathbb{Z}_2)$ satisfies $[u]^n \neq 0$. Indeed, define

$$u := \sum_{1\leqslant i<j\leqslant n+1} \langle v_i, w_j\rangle^*. \tag{3}$$

Theorem 5.4. *The cochain u is a cocycle, i.e. $\delta(u) = 0$, and $[u]^n \neq 0$.*

Proof. To show that $\delta(u) = 0$ it suffices to show $0 = (\delta(u))(\sigma) = u(d\sigma)$ for each 2-simplex $\sigma = \langle \pm v_r, \pm v_s, \pm v_t\rangle,\ r < s < t$. Because $\sigma = \tau(\sigma)$, we can suppose without loss of generality that there is at most one minus sign, thus σ has one of the four forms $\langle v_r, v_s, v_t\rangle, \langle w_r, v_s, v_t\rangle, \langle v_r, w_s, v_t\rangle, \langle v_r, v_s, w_t\rangle$. In the first case

$$d\sigma = \langle v_s, v_t\rangle + \langle v_r, v_t\rangle + \langle v_r, v_s\rangle$$

and each summand $\langle v_i, w_j\rangle^*$ of u evaluates to 0 on each of the three terms in $d\sigma$. In the second case

$$d\sigma = \langle v_s, v_t\rangle + \langle w_r, v_t\rangle + \langle w_r, v_s\rangle = \langle v_s, v_t\rangle + \langle v_r, w_t\rangle + \langle v_r, w_s\rangle$$

so the summands $\langle v_r, w_t\rangle^*, \langle v_r, w_s\rangle^*$ of u each evaluate to 1 on $d\sigma$ while all other summands evaluate to 0, giving $u(d\sigma) = 1 + 1 = 0$. The third and fourth cases are similar to the second.

Next, for the n-fold cup product $[u]^n$, one must show that $u^{\otimes n}$ evaluates to 1 on a generator of $H_n(\mathbb{R}P^n;\mathbb{Z}_2)$. From the above discussion (cf. Lemma 5.1 and the remark just before Theorem 5.2) we can take this generator to be $s_n := \sum\limits_{A\,\subseteq\,\underline{n}} \sigma_A$ (note that we are using $\mathbb{Z}_2$ coefficients). Each σ_A has v_{n+1} as final vertex, so in the sum s_n there will be a single "alternating" σ_A which we call ω, namely $\omega = \langle v_1, w_2, v_3, \ldots, w_n, v_{n+1}\rangle$ for n even or $\omega = \langle w_1, v_2, w_3, \ldots, w_n, v_{n+1}\rangle$ for n odd. Then $s_n = \omega + \varphi$ where φ is the sum of all σ_A with a repetition, i.e. a $v_i v_{i+1}$ or $w_j w_{j+1}$ in σ_A.

Now from (3), it is clear that

$$u^{\otimes n} = \sum_{i_r<j_r} \langle v_{i_1}, w_{j_1}\rangle^* \otimes \langle v_{i_2}, w_{j_2}\rangle^* \otimes \cdots \otimes \langle v_{i_n}, w_{j_n}\rangle^*$$

evaluates to zero on all terms of φ, hence for n even

$$u^{\otimes n}(s_n) = u^{\otimes n}(\omega) = \langle v_1, w_2\rangle^*(\langle v_1, w_2\rangle) \cdot \langle v_2, w_3\rangle^*(\langle w_2, v_3\rangle) \cdots = 1^n = 1,$$

using that $\langle w_2, v_3\rangle = \langle v_2, w_3\rangle$, $\langle w_4, v_5\rangle = \langle v_4, w_5\rangle$, etc. The argument for n odd is essentially the same. Hence $[u]^n \neq 0$. □

Corollary 5.5. *For each j, $0 \leqslant j \leqslant n$, $[u]^j$ generates $H^j(\mathbb{R}P^n; \mathbb{Z}_2)$, which is thus isomorphic to the truncated polynomial algebra $\mathbb{Z}_2[\,[u]\,]/([u]^{n+1} = 0)$.*

6. Appendix

In this appendix we show $H_j(\mathbb{R}P^n, \mathbb{R}P^{n-1}) = 0$, $0 \leqslant j \leqslant n-1$, by means of an explicit contracting homotopy s in the chain complex

$$C_*\colon C_n \xrightarrow{\tilde{d}_n} C_{n-1} \xrightarrow{\tilde{d}_{n-1}} \cdots \xrightarrow{\tilde{d}_2} C_1 \xrightarrow{\tilde{d}_1} C_0 \xrightarrow{\tilde{d}_0} 0,$$

where we write C_j for $C_j(\mathbb{R}P^n, \mathbb{R}P^{n-1})$ in this appendix, and as in Section 5 $\tilde{d}$ for its differential. Recall that this means a sequence of homomorphisms $s_j\colon C_j \to C_{j+1}$, $0 \leqslant j \leqslant n-1$, with $\tilde{d}_{j+1}s_j + s_{j-1}\tilde{d}_j = id_{C_j}$ and implies that $H_j(\mathbb{R}P^n, \mathbb{R}P^{n-1}) = 0$, $0 \leqslant j \leqslant n-1$. Note that $s_{-1} = 0$.

Since $C_*(\mathbb{R}P^n, \mathbb{R}P^{n-1}) = C_*(\mathbb{R}P^n)/C_*(\mathbb{R}P^{n-1})$, any non-zero simplex σ of the chain complex $C_*(\mathbb{R}P^n, \mathbb{R}P^{n-1})$ must contain the vertex v_{n+1}. We can therefore write any r-simplex σ uniquely as

$$\sigma = \langle x_{i_1}, x_{i_2}, \ldots, x_{i_r}, v_{n+1}\rangle \quad \text{where} \quad 1 \leqslant i_1 < i_2 < \ldots < i_r \leqslant n,$$

and henceforth use this notation. Of course, these are the generators of the free abelian group C_*.

We next define a homomorphism

$$\Lambda_p\colon C_{r-p} \to C_r,$$

show it is an anti-chain map (i.e., $\tilde{d}\Lambda_p = (-1)^p\Lambda_p\tilde{d}$), prove one combinatorial lemma, and after that it will be fairly easy to construct the contracting homotopy.

Definition 6.1. Let $\sigma = \langle x_{m_1}, \ldots, x_{m_{r-p}}, v_{n+1}\rangle$ be an $(r-p)$-simplex in C_{r-p}, $p + 1 \leqslant m_1 < \ldots < m_{r-p} \leqslant n$. Set

$$\Lambda_p(\sigma) = \sum_{\lambda_i=\pm 1} (\prod_{i=1}^{p} \lambda_i)\langle \lambda_1 v_1, \ldots, \lambda_p v_p, \sigma\rangle \in C_r.$$

We remark that this sum has 2^p terms, e.g. $\Lambda_2(\sigma) = \langle v_1, v_2, \sigma\rangle - \langle v_1, w_2, \sigma\rangle - \langle w_1, v_2, \sigma\rangle + \langle w_1, w_2, \sigma\rangle$.

Lemma 6.2. *The map Λ_p satisfies $\tilde{d}\Lambda_p = (-1)^p\Lambda_p\tilde{d}$.*

Proof. One has

$$\tilde{d}\Lambda_p(\sigma) = \sum_{\lambda_i=\pm 1} (\prod_{i=1}^{p} \lambda_i)\ \tilde{d}\langle \lambda_1 v_1, \dots, \lambda_p v_p, \sigma\rangle.$$

The first terms of this sum, i.e., those that arise by omitting $\lambda_1 v_1$ (for $\lambda_1 = \pm 1$), gives

$$\sum_{\lambda_2,\dots,\lambda_p=\pm 1} (\prod_{i=2}^{p} \lambda_i)\ (\langle \lambda_2 v_2, \dots, \lambda_p v_p, \sigma\rangle - \langle \lambda_2 v_2, \dots, \lambda_p v_p, \sigma\rangle) = 0.$$

The same happens for the terms omitting $\lambda_2 v_2, \dots$, omitting $\lambda_p v_p$. What remains gives us

$$\tilde{d}\Lambda_p(\sigma) = \sum_{\lambda_i=\pm 1} (\prod_{i=1}^{p} \lambda_i)\ \langle \lambda_1 v_1, \dots, \lambda_p v_p, (-1)^p \tilde{d}\sigma\rangle = (-1)^p \Lambda_p(\tilde{d}\sigma).$$ □

Lemma 6.3. *Let* $\omega = \langle w_1, \dots, w_p, \sigma\rangle$, $p \geqslant 1$. *Then*

$$\langle v_1, w_2, \dots, w_p, \sigma\rangle - \Lambda_1\langle v_2, w_3, \dots, w_p, \sigma\rangle + \Lambda_2\langle v_3, w_4, \dots, w_p, \sigma\rangle - \cdots + (-1)^p \Lambda_p\langle\sigma\rangle = \omega. \quad (4)$$

Proof. The final term $(-1)^p\Lambda_p\langle\sigma\rangle$ clearly has ω as one of its 2^p terms (corresponding to $\lambda_1 = \ldots = \lambda_p = -1$). It is not difficult to see that the other $2^p - 1$ terms in $(-1)p\Lambda_p\langle\sigma\rangle$ cancel the other $1 + 2 + \cdots + 2^p = 2^p - 1$ terms in (4) in pairs. Indeed, for each term other than ω in $(-1)^p\Lambda_p\langle\sigma\rangle$, we have $1 \leqslant m := \max\{j : \lambda_j = -1\} \leqslant p$, and one easily sees that this term will cancel the corresponding term in the $(-1)^{m-1}\Lambda_m$ term in (4), (where Λ_0 is the first term). □

Now to define a contracting homotopy s, we start with $s_0(v_{n+1}) = \langle v_1, v_{n+1}\rangle$, then $\tilde{d}s_0(v_{n+1}) = \langle v_{n+1}\rangle$ as required.

Definition 6.4. For $0 < r < n$, we define $s_r\langle x_{i_1}, \dots, x_{i_r}, v_{n+1}\rangle = s_r(\omega)$ using the following four cases.

1. If $i_1 > 1$, then $s_r(\omega) = \langle v_1, \omega\rangle$,
2. If $i_1 = 1$ and $x_{i_1} = v_1$, then $s_r(\omega) = 0$,
3. If $\omega = \langle w_1, \dots, w_p, x_{i_{p+1}}, \dots, v_{n+1}\rangle = \langle w_1, \dots, w_p, \sigma\rangle$ with $p < i_{p+1}$, then $s_r(\omega) = \Lambda_p(\langle v_{p+1}, \sigma\rangle)$,
4. (remaining cases) If $\omega = \langle w_1, \dots, w_p, v_{p+1}, x_{i_{p+2}}, \dots, v_{n+1}\rangle$, then $s_r(\omega) = 0$.

Proposition 6.5. *The homomorphism s is a contracting homotopy for C_*.*

Proof. We prove that s is a contracting homotopy for the cases 1 to 4 in the given order. In fact, 1, 2 and 4 are routine verifications and we give only the proof of 3 here.

Let $\omega = \langle w_1, w_2, \ldots, w_p, \sigma\rangle$, where $p \geqslant 1$.

$$\begin{aligned}\tilde{d}s_r(\omega) + s_{r-1}\tilde{d}(\omega) &= \tilde{d}\Lambda_p(\langle v_{p+1}, \sigma\rangle) + s_{r-1}(\langle w_2, \ldots, w_p, \sigma\rangle - \langle w_1, w_3, \ldots, w_p, \sigma\rangle \\ &\quad + \langle w_1, w_2, w_4, \ldots, w_p, \sigma\rangle - \cdots + (-1)^p\langle w_1, \ldots, w_p, \tilde{d}\sigma\rangle).\end{aligned}$$

Applying Lemma 6.2, we get

$$\begin{aligned}\tilde{d}s_r(\omega) + s_{r-1}\tilde{d}(\omega) &= (-1)^p\Lambda_p\tilde{d}\langle v_{p+1}, \sigma\rangle + \langle v_1, w_2, \ldots, w_p, \sigma\rangle - \Lambda_1\langle v_2, w_3, \ldots, w_p, \sigma\rangle \\ &\quad + \Lambda_2\langle v_3, w_4, \ldots, w_p, \sigma\rangle - \cdots + (-1)^p\Lambda_p\langle v_{p+1}, \tilde{d}\sigma\rangle \\ &= (-1)^p\Lambda_p\langle\sigma\rangle - (-1)^p\Lambda_p\langle v_{p+1}, \tilde{d}\sigma\rangle + \langle v_1, w_2, \ldots, w_p, \sigma\rangle \\ &\quad - \Lambda_1\langle v_2, w_3, \ldots, w_p, \sigma\rangle + \Lambda_2\langle v_3, w_4, \ldots, w_p, \sigma\rangle - \cdots \\ &\quad + (-1)^p\Lambda_p\langle v_{p+1}, \tilde{d}\sigma\rangle.\end{aligned}$$

Canceling the second and final terms, Lemma 6.3 shows that the remaining terms equal ω. □

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