

A LOOPING–DELOOPING ADJUNCTION FOR TOPOLOGICAL SPACES

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Abstract

Every principal G -bundle over X is classified up to equivalence by a homotopy class $X \rightarrow BG$, where BG is the classifying space of G . On the other hand, for every nice topological space X Milnor constructed a strict model of its loop space ΩX , that is a group. Moreover, the morphisms of topological groups $\tilde{\Omega}X \rightarrow G$ generate all the G -bundles over X up to equivalence.

In this paper, we show that the relation between Milnor's loop space and the classifying space functor is, in a precise sense, an adjoint pair between based spaces and topological groups in a homotopical context.

This proof leads to a classification of principal bundles over a fixed space, that is dual to the classification of bundles with a fixed group. Such a result clarifies the deep relation that exists between the theory of bundles, the classifying space construction and the loop space, which are very important in topological K -theory, group cohomology, and homotopy theory.

1. Introduction

Given a topological group G and a CW-complex X , a classical result ([1, 14], or [13] for the case of fibrations) states that every G -bundle E over X is classified up to equivalence by the homotopy class of a map $f: X \rightarrow BG$, so that $f^*EG \simeq E$. Here, the universal bundle EG is a contractible G -bundle over the classifying space BG . An explicit description of EG can be found in [6]. The bijection

$${}_X\mathcal{B}un_G/\simeq \cong \mathcal{T}op(X, BG)/\simeq \quad (1)$$

suggests that the classifying space construction $B: \mathcal{G}p(\mathcal{T}op) \rightarrow \mathcal{T}op$ could behave like a right adjoint in a homotopical context. Moreover, B is in fact a functor, and the bijection (1) is natural in X and G .

For a topological group G , the correspondence (1) witnesses the fact that BG classifies principal G -bundles. Dualizing the picture, it is natural to ask whether there exists an analogous classification for principal bundles over a fixed base space X . The answer was partially given by Milnor in [5]. For any based topological space

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X with a good cellular decomposition, he constructed a principal bundle $\tilde{P}X$ over X whose structure group is denoted $\tilde{\Omega}X$. Such a bundle generates all the other principal bundles over X up to equivalence, giving rise to a surjection of the form

$$\mathcal{G}p(\mathcal{T}op)(\tilde{\Omega}X, G) \twoheadrightarrow {}_X\mathcal{B}un_G/\simeq. \quad (2)$$

This assignment is dual in a precise way to the assignment (1).

The bundle $\tilde{P}X$ is a strictification of the path space of X , so that concatenation becomes associative. The corresponding loop space $\tilde{\Omega}X$ is a group, and the fibration $\tilde{P}X \rightarrow X$ is a principal bundle. Following the intuition that $\tilde{\Omega}X$ is a loop space, the classifying map $\tilde{\Omega}X \rightarrow G$ for a G -bundle over X is the connecting map that appears in the (dual) Nomura–Puppe sequence.

In this paper, we refine the bijection (1), showing that every G -bundle over X is classified by a *based* map $X \rightarrow BG$, and we compute the kernel of the surjection (2), i.e., we determine when two morphisms $\tilde{\Omega}X \rightarrow G$ produce equivalent bundles. This leads to the notion of *algebraic equivalence*, denoted \equiv , of morphisms of topological groups. Although there is no natural extension of $\tilde{\Omega}$ to a functor, we propose a definition of $\tilde{\Omega}$ on arrows. The assignment is *pseudofunctorial* (in the sense of Proposition 4.14), according to the relation of algebraic equivalence, while the bijection obtained by (2) is natural with respect to X and G . This is summed up in the Main Theorem, whose precise statement appears as Theorem 4.16.

Main Theorem. *For every nice connected based space X and nice topological group G , there are natural bijections*

$$\mathcal{G}p(\mathcal{T}op)(\tilde{\Omega}X, G)/\equiv \cong {}_X\mathcal{B}un_G/\simeq \cong \mathcal{T}op_*(X, BG)/\simeq.$$

In this sense, Milnor’s loop space $\tilde{\Omega}$ and the classifying space functor B form an adjoint pair up to homotopy.

The result describes an intrinsic duality that relates loop spaces, classifying spaces and principal bundles, and gives a complete classification of principal bundles over a fixed space.

The condition of niceness essentially requires a countable CW-decomposition that is compatible with the base point or with the group structure. This restriction is mostly imposed by the construction \tilde{P} . In any case the conditions are relaxed enough to include interesting examples (e.g., all countable discrete groups, \mathbb{R} , and S^1 as topological groups, and connected countable CW-complexes as based spaces).

Many of the ideas in this paper were inspired by [2], where Farjoun and Hess developed the theory of *twisted homotopical categories*. This is a formal framework based on a *looping–delooping* adjunction

$$\Omega: \text{coMon} \begin{array}{c} \xrightarrow{\quad} \\ \xleftarrow{\quad} \end{array} \text{Mon} : B$$

between suitable categories of monoids and comonoids, where a map $\Omega X \rightarrow G$ (or its adjoint $X \rightarrow BG$) describes a *bundle*. A G -bundle E over X is a twisted version of the tensor product between a comonoid X and a monoid G , and always comes with a diagram of the form

$$G \longrightarrow E \longrightarrow X,$$

that describes the projection on the base space and the inclusion of a distinguished

fibre. Examples are given by twisted tensor products in chain complexes and twisted cartesian products in simplicial sets.

1.1. Related work

It is worth comparing the main result of this paper with the work of Lurie. Applying arguments of higher category theory and topos theory, he showed [4, Lemma 7.2.2.11] that there is an adjunction of quasi-categories between the ∞ -topos of convenient pointed topological spaces and the ∞ -topos of convenient topological groups. His formal framework allows a formulation of the result in term of a strict adjunction in the environment of quasi-categories.

Most likely realizing the π_0 -statement in spaces would give a result close to ours, though Lurie’s equivalence cannot be proved by using the argument presented in this article. In fact, we already have trouble trying to see the loop functor as a simplicial map. We could try to assign to each pointed space (respectively pointed continuous map, respectively homotopy) a topological group (respectively a continuous homomorphism, respectively an equivalence of continuous homomorphisms). The problem is that the way we produce these correspondences often involves either a choice or the solution of a lifting problem. This issue cannot be overcome with the language of enriched categories either. The argument presented below uses more elementary tools, and it makes explicit how the looping–delooping adjunction is directly related to the theory of principal bundles.

A second related article is [3] by Lashof. He assigned to each equivalence class of G -bundles over X a continuous groupoid morphism $\Omega X \rightarrow G$ up to *conjugacy equivalence*, in a bijective way. Lashof’s groupoid structure defined on the loop space does not seem to be directly comparable to Milnor’s group structure on $\tilde{\Omega}$. Also, the groupoid map is not obtained as a restriction of a map between the total objects. However the feeling is that the final classification presented in our paper is similar to his: in both cases a loop object is thought of as a classifier.

1.2. Future directions

We describe briefly here applications of the result in this article that we will elaborate in [11].

It is possible to define a category \mathcal{Bun} of bundles that includes fully faithfully pointed spaces and topological groups as degenerate cases. Given a nice abelian topological group A , e.g., $A := \mathbb{Z}, \mathbb{Z}/n, S^1$, we can define an equivalence relation \simeq_+ on $\mathcal{Bun}(E, A)$ for any bundle E . In particular, for any sufficiently nice topological group G and pointed space X the quotients are

$$\mathcal{Bun}(G, A)/\simeq_+ = \mathcal{Gp}(\mathcal{Top})(G, A)/\cong \quad \text{and} \quad \mathcal{Bun}(X, A)/\simeq_+ = \mathcal{Bun}(X, A)/\simeq.$$

Moreover, the multiplication of A induces on $\mathcal{Bun}(E, A)/\simeq_+$ the structure of an abelian group. Now, given any G -bundle E over X , we can write a sequence in \mathcal{Bun} using the two classifying maps of E :

$$\tilde{\Omega}B \longrightarrow G \longrightarrow E \longrightarrow X \longrightarrow BG.$$

By applying $\mathcal{B}un(-, A)/\simeq_+$, we obtain a sequence of abelian groups

$$\begin{aligned} \mathcal{T}op_\star(BG, A)/\simeq &\longrightarrow \mathcal{T}op_\star(X, A)/\simeq \longrightarrow \mathcal{B}un(E, A)/\simeq_+ \longrightarrow \mathcal{G}p(\mathcal{T}op)(G, A)/\equiv \\ &\longrightarrow \mathcal{G}p(\mathcal{T}op)(\hat{\Omega}X, A)/\equiv \end{aligned}$$

which is proven to be exact.

When $A \simeq K(Z; n)$ is an Eilenberg-MacLane space, some of the quotients involved coincide with the (group) cohomology of X and G ,

$$\mathcal{G}p(\mathcal{T}op)(G, A)/\equiv = H_{\mathcal{G}p}^{n+1}(G; Z) \quad \text{and} \quad \mathcal{T}op_\star(X, A)/\simeq = H^n(X; Z).$$

In particular we obtain a long exact sequence

$$\cdots \longrightarrow H_{\mathcal{G}p}^n(G; Z) \xrightarrow{\chi} H^n(X; Z) \longrightarrow \mathcal{B}un(E, A)/\simeq_+ \longrightarrow H_{\mathcal{G}p}^{n+1}(G; Z) \longrightarrow \cdots$$

where the connecting map χ is the *characteristic map* of E , which is used in the literature to define *characteristic classes* of bundles [7]. The exactness of this sequence contains a lot of geometric information. For instance, when $Z := \mathbb{Z}/2$, X is a real k -manifold and E is its frame bundle, we recover the classical description of the first Stiefel-Whitney class as obstruction to orientability.

The dual formalism produces the well known homotopy long exact sequence induced by a bundle E .

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2. Principal bundles

In this section we recall the notion of principal bundle and two useful constructions: pullback and pushforward. They allow us to compare bundles with different structure groups or different base spaces.

2.1. Tensor product of modules

Let G be a topological group, and let Mod_G and ${}_G Mod$ denote the categories of right and left G modules. Given M a right G -module and M' a left G -module, the **tensor product** $M \otimes_G M'$ is defined to be the coequalizer in $\mathcal{T}op$ of

$$M \times G \times M' \begin{array}{c} \xrightarrow{\mu \times M'} \\ \xrightarrow{M \times \mu'} \end{array} \rightrightarrows M \times M'.$$

Thus, $M \otimes_G M' = M \times M' / \sim$, where the relation is generated by the pairs $(m \cdot g, m') \sim (m, g \cdot m')$. This construction extends to a bifunctor

$$- \otimes_G -: Mod_G \times_G Mod \longrightarrow \mathcal{T}op,$$

which is associative in the obvious sense.

Every morphism $a: G \rightarrow G'$ of topological groups induces an adjunction

$$a_* := - \otimes_G G': \text{Mod}_G \begin{array}{c} \xrightarrow{\quad} \\ \xleftarrow{\quad} \end{array} \text{Mod}_{G'} : U_a$$

between the **pushforward** a_* and the forgetful functor. For any left G -module M , we denote the unit of this adjunction

$$\eta_a: M \longrightarrow a_*M.$$

2.2. Cotensor product of comodules

Let X be a topological space, and let coMod_X and ${}_X\text{coMod}$ denote the categories of right and left X -comodules. Note that, since we are working in a cartesian category, a right or left comodule M over X is equivalent to a map $\pi: M \rightarrow X$.

Given M a (right) X -comodule and M' a (left) X -comodule, the **cotensor product** $M \times_X M'$ is defined to be the equalizer in $\mathcal{T}op$ of

$$M \times M' \begin{array}{c} \xrightarrow{(M, \pi) \times M'} \\ \xrightarrow{M \times (\pi', M')} \end{array} M \times X \times M'.$$

Thus, $M \times_X M'$ is just the pullback of π and π' . This construction extends to a bifunctor

$$- \times_X -: \text{coMod}_X \times {}_X\text{coMod} \longrightarrow \mathcal{T}op$$

that is associative in the obvious sense.

Every continuous map $f: X' \rightarrow X$ of spaces induces an adjunction

$$U_f: {}_{X'}\text{coMod} \begin{array}{c} \xrightarrow{\quad} \\ \xleftarrow{\quad} \end{array} {}_X\text{coMod} : X' \times_X - =: f^*$$

between the forgetful functor and the **pullback** f^* . For any X -module N , we denote the counit

$$\epsilon_f: f^*N \longrightarrow N.$$

2.3. Mixed modules

Let G be a topological group and X a topological space. A **mixed module** over X and G is a topological space M endowed with right G -module and (left) X -comodule structures such that the action is fibrewise, i.e., $\pi(m \cdot g) = \pi(m)$ for any $g \in G$ and $m \in M$. A **morphism of mixed modules over X and G** is a continuous map that is a G -equivariant morphism of spaces over X . An isomorphism of mixed modules over X and G is called an **equivalence**. We denote by ${}_X\text{Mix}_G$ the category of mixed modules over X and G . The **trivial mixed module** over X and G is $X \times G$.

For any continuous map $f: X' \rightarrow X$ and homomorphism of topological groups $a: G \rightarrow G'$, the adjunctions above restrict and corestrict to mixed modules and commute:

$$a_* \circ f^* \cong f^* \circ a_*: {}_X\text{Mix}_G \longrightarrow {}_{X'}\text{Mix}_{G'}.$$

2.4. Principal bundles

Let G be a topological group and X a topological space. A **principal bundle** with **structure group** G and **base space** X , or a **G -bundle over X** , is a mixed

module P over X and G that is **locally trivial**, i.e., there exists an open covering $\{U_i\}_{i \in I}$ of X and a **local trivialization** $\{\psi_i\}_{i \in I}$, where

$$\psi_i: P|_{U_i} := \pi^{-1}(U_i) \longrightarrow U_i \times G$$

is an equivalence of mixed modules over U_i and G . The trivial module $X \times G$ is a bundle. We denote by ${}_X\mathcal{B}un_G$ the full subcategory of ${}_X\mathcal{M}ix_G$ of G -bundles over X . Every morphism of G -bundles over X is in fact an equivalence.

Pullback and pushforward constructions restrict and corestrict to principal bundles, i.e., they define assignments

$$(-)^*(-): \mathcal{T}(X', X) \times {}_X\mathcal{B}un_G \rightarrow {}_{X'}\mathcal{B}un_G \text{ and}$$

$$(-)_*(-): \mathcal{G}p(\mathcal{T}op)(G, G') \times {}_X\mathcal{B}un_G \rightarrow {}_X\mathcal{B}un_{G'}.$$

Using the universal property of pullbacks, it is easy to show that a G -equivariant morphism $P \rightarrow E$ between G -bundles that induces a morphism $f: X \rightarrow Y$ on the base spaces also induces an equivalence of G -bundles over X

$$P \simeq f^*E.$$

Similarly, using the fact that the pushforward is a coequalizer, a morphism $Q \rightarrow E$ of bundles over X that is equivariant according to a map $a: H \rightarrow G$ between the structure groups induces an equivalence of H -bundles over X

$$a_*Q \simeq E.$$

3. Classification of bundles with fixed structure group

For a topological group G , a classification (stated here as Theorem 3.8) of G -bundles over CW-complexes was proven by Dold (in [1] for bundles over paracompact spaces, which include CW-complexes) and Steenrod (in [14] for bundles over normal, locally compact and countably compact spaces): every G -bundle over a CW-complex X is determined up to equivalence by a homotopy class of maps from X into the classifying space BG . The two references share the same ideas.

Exploiting the Serre model structure on spaces [10, Chapter II, Section 3], we improve the classical result by proving a strong universal property of the classifying space of a group in Theorem 3.4. This implies the classification mentioned above. On the other hand, it also allows a variant of the main classification where we take into consideration only continuous maps $X \rightarrow BG$ that are pointed (Theorem 3.10). Such a strong universal property was already proven by Steenrod in the case of bundles over locally finite complexes, as Theorem 19.4 of [14].

We think of the pullback construction as a tool to produce new principal bundles with the same structure group. Given a topological group G , one can fix a suitable G -bundle Q over some space X , and let the map along which we change the base space vary. For a fixed G -bundle Q over Y , this process is encoded by the following assignment:

$$(-)^*Q: \mathcal{T}op(X, Y) \longrightarrow {}_X\mathcal{B}un_G,$$

where $[f: X \rightarrow Y] \mapsto f^*Q$. This map is not surjective in a strict sense, but the goal is to capture all the G -bundles for a fixed group G up to equivalence. Therefore, the

function we are really interested in is

$$(-)^*Q: \mathcal{Top}(X, Y) \longrightarrow {}_X\mathcal{Bun}_G \twoheadrightarrow {}_X\mathcal{Bun}_G/\simeq,$$

or its pointed version

$$(-)^*Q: \mathcal{Top}_*(X, Y) \hookrightarrow \mathcal{Top}(X, Y) \longrightarrow {}_X\mathcal{Bun}_G \twoheadrightarrow {}_X\mathcal{Bun}_G/\simeq,$$

for a fixed bundle Q . There is a part of the kernel that does not depend on Q and Y , which is a consequence of the *Covering Homotopy Theorem* ([1, Theorem 7.8], or [14, Theorem 11.3]).

Proposition 3.1 ([1, Corollary 7.10], [14, Theorem 11.5]). *For any G -bundle Q over a CW-complex Y , its pullbacks via homotopic maps $X \rightarrow Y$ give equivalent G -bundles over X , i.e.,*

$$f \simeq g: X \rightarrow Y \implies f^*Q \simeq g^*Q.$$

So far, this is the best we can deduce without specializing to any particular bundle Q . We will now fix a group G and focus on the *universal bundle* of G , proposed by Milnor in [6]. He considers the join of infinitely many copies of G

$$\tilde{E}G := \star_{n=0}^{+\infty} G,$$

which he endows with a right G -action induced by right multiplication on every copy of G . Motivated by its good properties, the orbit space $\tilde{B}G$ is often called **classifying space** of G . Moreover, $\tilde{B}G$ comes with a natural base point, and can be considered as a pointed space.

Theorem 3.2 ([6, Sections 3,5]). *The constructions \tilde{E} and \tilde{B} are functorial into spaces (or pointed spaces in the case of \tilde{B}). The space $\tilde{E}G$ is contractible, and has the structure of a G -bundle over $\tilde{B}G$. Moreover, if G is a countable CW-group, then $\tilde{B}G$ is a countable CW-complex.*

The conditions for topological groups and for spaces appearing in the last part of the statement will play a role in Section 5, since *Milnor’s loop space* $\tilde{\Omega}$ will be defined only for spaces that admit a good CW-structure.

Definition 3.3 (Nice spaces and nice groups).

- A topological space X is **nice** if it is connected and admits a countable CW-decomposition. We denote by \mathcal{T} the category of nice pointed spaces and continuous maps.
- A pointed topological space X is **nice** if it is connected and admits a countable CW-decomposition such that the base point is a vertex. We denote by \mathcal{T}_* the category of nice pointed spaces and pointed continuous maps.
- A topological group G is **nice** if it admits the structure of a countable CW-group, i.e., a countable CW-decomposition with respect to which the group structure maps are cellular. We denote by \mathcal{G} the category of nice topological groups and topological group homomorphisms.

The following result is new in this generality. In fact, it generalizes Theorem 19.4 of [14], where Steenrod proves it only for locally finite complexes, and it implies Theorem 3.7, which was proven by Dold in [1], as Theorem 7.5.

Theorem 3.4 (Strong universal property of the classifying space). *Let P be a G -bundle over a CW-complex Y . For each $K \subset Y$ such that (X, Y) is a CW-pair and $\rho: P|_K \rightarrow \tilde{E}G$ right G -equivariant, there exists a right G -equivariant morphism*

$$\phi: P \longrightarrow \tilde{E}G$$

that extends ρ to P . In particular, the following diagram commutes, where r and f are the maps induced by ρ and ϕ on orbit spaces.

$$\begin{array}{ccccc}
 & & P|_K & \xrightarrow{\pi|_K} & K \\
 & \swarrow & \downarrow \rho & & \downarrow r \\
 P & \xrightarrow{\pi} & Y & & \\
 \downarrow \phi & & \downarrow f & & \\
 \tilde{E}G & \xrightarrow{\quad} & \tilde{E}G & \xrightarrow{\quad} & \tilde{B}G \\
 \downarrow & & \downarrow & & \downarrow \\
 \tilde{E}G & \xrightarrow{\quad} & \tilde{B}G & &
 \end{array}$$

In the proof we use the following fact.

Proposition 3.5 ([8, Proposition 6.1]). *Let Q be a G -bundle over Y and Z a right G -space. The tensor product $Q \otimes_G Z$ is identified with the orbit space of the right G -action on $Q \times Z$ given by $(q, z) \cdot g := (q \cdot g, z \cdot g)$. There is a bijection*

$$\text{Mod}_G(Q, Z) \cong \Gamma(Y, Q \otimes_G Z)$$

between the set of right G -equivariant maps $Q \rightarrow Z$ and the set of sections of the fibration $Q \otimes_G Z \rightarrow Y$, induced by $pr_1: Q \times Z \rightarrow Q$. The assignment is given by $\phi \mapsto \sigma_\phi: Y \rightarrow Q \otimes_G Y$, where σ_ϕ is induced by $(Q, \phi): Q \rightarrow Q \times Z$. \square

Proof of Theorem 3.4. According to Proposition 3.5, the G -equivariant morphism $\rho: P|_K \rightarrow \tilde{E}G$ determines a section σ_ρ of $P|_K \otimes_G \tilde{E}G \rightarrow K$. This allows us to draw a commutative square.

$$\begin{array}{ccc}
 & P|_K \otimes_G \tilde{E}G & \\
 \sigma_\rho \nearrow & & \searrow \iota \otimes_G \tilde{E}G \\
 K & & P \otimes_G \tilde{E}G \\
 \downarrow i & \xrightarrow{\quad} & \downarrow \sim \\
 Y & \xlongequal{\quad} & Y
 \end{array}$$

The vertical arrow on the left is a Serre cofibration. As for the vertical arrow on the right, it is an acyclic Serre fibration. Indeed, $P \otimes_G \tilde{E}G$ is a fibre bundle over a paracompact space Y , every fibre bundle over a paracompact space is *numerable* [1, Section 7], and every numerable fibre bundle is a Serre fibration [12, Theorem 7.12]. Moreover, the fibre is contractible, and a Serre fibration with a contractible fibre is a weak equivalence.

Thus, using the Serre model structure on spaces [10, Chapter II, Section 3], there exists a lift $\sigma: Y \rightarrow P \otimes_G \tilde{E}G$:

$$\begin{array}{ccc}
 & P|_K \otimes_G EG & \\
 \sigma_\rho \nearrow & & \searrow \iota \otimes_G EG \\
 K & \xrightarrow{\quad} & P \otimes_G EG \\
 \downarrow i & \dashrightarrow \sigma & \downarrow \sim \\
 Y & \xlongequal{\quad} & Y
 \end{array}$$

From the commutativity of the lower triangle we see that the lift σ is a section of $P \otimes_G \tilde{E}G \rightarrow Y$, and therefore corresponds, according to Proposition 3.5, to a right G -equivariant map $\phi: P \rightarrow \tilde{E}G$ that induces some $f: Y \rightarrow BG$.

Moreover, using the commutativity of the upper part of the diagram, we have:

$$(\iota \otimes_G EG) \circ \sigma_\rho = \sigma \circ i = (\iota \otimes_G EG) \circ \sigma_{\phi \circ \iota},$$

where the second equality is induced at the level of orbits by the displayed commutative diagram of right G -equivariant maps

$$\begin{array}{ccc}
 P|_K \xrightarrow{(P|_K, \phi \circ \iota)} P|_K \times Q & & K \xrightarrow{\sigma_{\phi \circ \iota}} P|_K \otimes_G Q \\
 \downarrow \iota & \sim & \downarrow i \\
 P \xrightarrow{(P, \phi)} P \times Q & & Y \xrightarrow{\sigma_\phi} P \otimes_G Q \\
 & & \downarrow \iota \otimes_G Q
 \end{array}$$

Since $\iota \otimes_G EG$ is injective, the sections are equal, that is $\sigma_\rho = \sigma_{\phi \circ \iota}$. Thus, using again Proposition 3.5, we have

$$\rho = \phi \circ \iota \text{ and } r = f \circ i. \quad \square$$

We use the result above to conclude that the process of pulling back the universal bundle is injective, in the sense of the following proposition.

Proposition 3.6. *If the pullbacks of the universal bundle $\tilde{E}G$ via two maps $f, g: X \rightarrow \tilde{B}G$ are equivalent G -bundles over the CW-complex X , then f is homotopic to g , i.e.,*

$$f^* \tilde{E}G \simeq g^* \tilde{E}G \implies f \simeq g: X \rightarrow \tilde{B}G.$$

Proof. Here is essentially the same argument used by Dold (Theorem 7.5) and Steenrod (Theorem 19.3), but presented as an application of Theorem 3.4. Let $\alpha: f^* \tilde{E}G \rightarrow g^* \tilde{E}G$ be an equivalence of G -bundles over X . We define a morphism

$$\varphi: \{0, 1\} \times (f^* \tilde{E}G) := f^* \tilde{E}G \sqcup f^* \tilde{E}G \longrightarrow \tilde{E}G$$

determined by the summands

$$\epsilon_f: f^* \tilde{E}G \longrightarrow EG \text{ and } \epsilon_g \circ \alpha: f^* \tilde{E}G \longrightarrow g^* \tilde{E}G \longrightarrow \tilde{E}G.$$

It is G -equivariant because each summand is, and it induces $f + g: \{0, 1\} \times X =$

$X \sqcup X \rightarrow \tilde{B}G$. The inclusion

$$i \times (f^*P): \{0, 1\} \times (f^*P) \longrightarrow I \times (f^*P)$$

is also right G -equivariant.

All the conditions to apply Theorem 3.4 (with $P = I \times (f^*\tilde{E}G)$, $Y = I \times X$, $K = \{0, 1\} \times X$) are satisfied. We thus get a morphism of G -bundles $\phi: I \times (f^*\tilde{E}G) \rightarrow \tilde{E}G$ and a map $F: I \times X \rightarrow \tilde{B}G$. Such an F is the required homotopy. \square

As an immediate consequence of Theorem 3.4, taking K to be empty, and Proposition 3.6, we obtain the following classical statements.

Proposition 3.7 (Universal property of the classifying space). *Let P be a G -bundle over a CW-complex X . There exists a right G -equivariant morphism*

$$\phi: P \longrightarrow \tilde{E}G$$

that induces a map on the base spaces

$$f: X \longrightarrow \tilde{B}G.$$

As a consequence,

$$P \simeq f^*\tilde{E}G. \quad \square$$

Theorem 3.8 (Classification of G -bundles). *Let X be a CW-complex. The pullback of the classifying bundle $\tilde{E}G$ induces a bijection*

$$(-)^*\tilde{E}G: \mathcal{Top}(X, \tilde{B}G)/\simeq \longrightarrow {}_X\mathcal{Bun}_G/\simeq$$

between the homotopy classes of continuous maps $X \rightarrow \tilde{B}G$ and the equivalence classes of G -bundles over X . \square

By taking K to be the base point of X in Proposition 3.4, we can establish based variants of Proposition 3.7 and Theorem 3.8.

Proposition 3.9. *Let P be a G -bundle over a CW-complex X pointed on a vertex. There exists a right G -equivariant morphism*

$$\phi: P \longrightarrow \tilde{E}G$$

that induces a based map at the level of base spaces

$$f: X \longrightarrow \tilde{B}G.$$

As a consequence,

$$P \simeq f^*\tilde{E}G. \quad \square$$

Theorem 3.10 (Pointed classifications of G -bundles). *Let X be a CW-complex based at a vertex. The pullback of the universal bundle $\tilde{E}G$ induces a bijection*

$$(-)^*\tilde{E}G: \mathcal{Top}_*(X, \tilde{B}G)/\simeq \longrightarrow {}_X\mathcal{Bun}_G/\simeq$$

between the (unpointed!) homotopy classes of pointed continuous maps $X \rightarrow \tilde{B}G$ and the equivalence classes of G -bundles over X . \square

The next result is essentially classical.

Proposition 3.11 (Uniqueness of the classifying space). *The classifying space of a topological group G is unique up to homotopy. More precisely, for a CW-complex B pointed on a vertex the following are equivalent.*

- (1) Homotopy type: B has the same homotopy type as $\tilde{B}G$.
- (2) Universal property: there exists a natural bijection

$$\mathcal{Top}(X, B)/\simeq \longrightarrow {}_X\mathcal{Bun}_G/\simeq$$

between the homotopy classes of continuous maps $X \rightarrow B$ and the equivalence classes of G -bundles over X , for every CW-complex X .

- (2') Pointed universal property: there exists a natural bijection

$$\mathcal{Top}_*(X, B)/\simeq \longrightarrow {}_X\mathcal{Bun}_G/\simeq$$

between the (unpointed!) homotopy classes of pointed continuous maps $X \rightarrow B$ and the equivalence classes of G -bundles over X , for every CW-complex X based at a vertex.

- (3) Contractibility: B is a classifying space for G , i.e., there exists a G -bundle E over B that is weakly contractible.

Proof. • [(3) \implies (2')]: See Proposition 3.9.

- [(2') \implies (2)]: By the Yoneda Lemma, a natural bijection as in (2') has to be induced by pulling back a G -bundle E over B . Consider the function

$$(-)^*(E): \mathcal{Top}(X, B)/\simeq \longrightarrow {}_X\mathcal{Bun}_G/\simeq.$$

This map is injective, by Proposition 3.6, and surjective since its restriction to the subset \mathcal{Top}_*/\simeq is.

- [(2) \implies (1)]: It is an application of the Yoneda Lemma. Indeed, B represents the functor $X \mapsto {}_X\mathcal{Bun}_G/\simeq$ on the homotopy category of connected spaces that admit the structure of countable CW-complex.
- [(1) \implies (3)]: Let $f: B \rightarrow \tilde{B}G$ be a homotopy equivalence, and consider the pull-back $E := f^*\tilde{E}G$ of Milnor's model. It comes with the counit $\epsilon_f: E = f^*\tilde{E}G \rightarrow \tilde{E}G$, which is a morphism of G -bundles. There is a morphism of long exact sequences with identities on the component of the fibres and isomorphisms on the components of the base space. It follows that E is weakly equivalent to $\tilde{E}G$, which is contractible. \square

4. Classification of bundles with fixed base space

For a nice pointed space X , we now provide a classification of principal bundles over X , stated as Theorem 4.12. Milnor described in [5] a loop space model $\tilde{\Omega}$ that is in fact a group and plays a role dual to that of the classifying space. The loop space $\tilde{\Omega}X$ is the structure group of a universal bundle over X that generates all the others by *pushing forward* along some continuous group homomorphisms. Our contribution is a description of when two such continuous homomorphisms induce the same bundle. We also define a reasonable notion of $\tilde{\Omega}f$, for a pointed map f , so that $\tilde{\Omega}$ is pseudofunctorial, and the classification is natural with respect to X .

In Theorem 4.16 we put together the two sides of the coin to produce an adjunction in a homotopical context between the classifying space functor and Milnor's loop space. This adjunction is homotopically well-behaved, in the sense described by Farjoun and Hess in [2].

Although the structure of this section is at the beginning as similar as possible to the previous one, the logic of the arguments is quite different. Indeed, many of the proofs depend on results from the previous section. In particular, the following proposition allows some interaction between the context of the previous section and this section.

Proposition 4.1. *Given a morphism of topological groups $a: G \rightarrow H$, there is an equivalence of H -bundles over $\tilde{B}G$*

$$a_*\tilde{E}G \simeq (\tilde{B}a)^*\tilde{E}H.$$

Proof. The projection of $\tilde{E}G$ onto $\tilde{B}G$ and the map $\tilde{E}a$ induce a map $\phi: \tilde{E}G \rightarrow \tilde{B}a^*\tilde{E}H$, as indicated by the following diagram:

$$\begin{array}{ccccc}
 \tilde{E}G & & & & \\
 \downarrow \tilde{E}a & \searrow \phi & & \searrow \pi & \\
 & & \tilde{B}a^*\tilde{E}H & \xrightarrow{\pi^*} & \tilde{B}G \\
 & & \downarrow \epsilon_{\tilde{B}a} & \lrcorner & \downarrow \tilde{B}a \\
 & & \tilde{E}H & \longrightarrow & \tilde{B}H.
 \end{array}$$

By the commutativity of the upper triangle, it follows that ϕ respects the projection over $\tilde{B}G$. Also, one can check that ϕ induces a on each fibre, and it is therefore a -equivariant. As a consequence we obtain the desired equivalence. \square

Mimicking the approach we took in the dual case, we would like to describe the classification problem for principal bundles over a fixed base space X in terms of the following map:

$$(-)_*Q: \mathcal{G}p(\mathcal{T}op)(G, H) \longrightarrow {}_X\mathcal{B}un_G \twoheadrightarrow {}_X\mathcal{B}un_H/\simeq,$$

where Q is a G -bundle over X and $[a: G \rightarrow H] \mapsto a_*Q$. Unlike the dual case, this does not induce an interesting assignment on a quotient of $\mathcal{G}p(\mathcal{T}op)(G, H)$. We will see that the kernel is computable when $Q = \tilde{P}X$ and is given by the following equivalence relation.

Definition 4.2. Two continuous homomorphisms $a, b: G \rightarrow H$ are said to be **algebraically equivalent** if $\tilde{B}a$ is homotopic to $\tilde{B}b$, which we denote $a \equiv b$.

Algebraic equivalence of continuous group homomorphisms is easy to define, but hard to imagine. The feeling is that it should be somehow related to a notion of homotopy that respects the algebraic structure. A good candidate is the following.

Given a topological group H , the space H^I of paths in H is a topological group with respect to the pointwise multiplication.

Definition 4.3. Two continuous homomorphisms $a, b: G \rightarrow H$ are said to be **algebraically homotopic** if a is homotopic to b via an **algebraic homotopy**, i.e., a homotopy $F: G \rightarrow H^I$ that is also a homomorphism of topological groups.

Proposition 4.4. *If two continuous homomorphisms $a, b: G' \rightarrow G$ are algebraically homotopic, then they are algebraically equivalent.*

Proof. Consider the continuous map

$$\tilde{E}(G^I) \times I \longrightarrow \tilde{E}G$$

given by $(\sum_{i \in \mathbb{N}} t_i \cdot \alpha_i, t) \mapsto \sum_{i \in \mathbb{N}} t_i \cdot \alpha_i(t)$. This induces a map

$$\tilde{B}(G^I) \times I \longrightarrow \tilde{B}G,$$

whose adjoint is denoted

$$\rho_G: \tilde{B}(G^I) \longrightarrow (\tilde{B}G)^I.$$

Now, if $F: G' \rightarrow G^I$ is an algebraic homotopy between a and b , then

$$\rho_G \circ \tilde{B}F: \tilde{B}G' \xrightarrow{\tilde{B}F} \tilde{B}(G^I) \xrightarrow{\rho_G} (\tilde{B}G)^I$$

is a homotopy between $\tilde{B}a$ and $\tilde{B}b$. Therefore a and b are algebraically equivalent. \square

Remark 4.5 (Naturality of the classification of bundles with a fixed structure group). As a consequence of Proposition 4.1, the bijections described in Theorems 3.8 and 3.10 are **natural** in both the variables, with respect to (based) continuous maps up to homotopy and continuous homomorphisms up to algebraic equivalence. In fact, the following diagrams commute up to equivalence.

$$\begin{array}{ccc} X & \mathcal{T}op_{(*)}(X, \tilde{B}G) & \xrightarrow{(-)^* \tilde{E}G} X \mathcal{B}un_G \\ f \downarrow & \uparrow - \circ f & \uparrow f^*(-) \\ Y & \mathcal{T}op_{(*)}(Y, \tilde{B}G) & \xrightarrow{(-)^* \tilde{E}G} Y \mathcal{B}un_G \end{array}$$

$$\begin{array}{ccc} G & \mathcal{T}op_{(*)}(X, \tilde{B}G) & \xrightarrow{(-)^* \tilde{E}G} X \mathcal{B}un_G \\ a \downarrow & \tilde{B}a \circ - \downarrow & \downarrow a_*(-) \\ H & \mathcal{T}op_{(*)}(X, \tilde{B}H) & \xrightarrow{(-)^* \tilde{E}H} X \mathcal{B}un_H \end{array}$$

The first attempt to define a *universal bundle* for a base space K was for K a connected simplicial complex, pointed on one of its vertices. Milnor defines a group $\tilde{\Omega}K$, and a $\tilde{\Omega}K$ -bundle over K that are, respectively, a strictification of the loop space ΩK and the based path space PK . Indeed, using the cellular structure of K , Milnor selected a class of easy paths and loops, avoiding any issue related to the parametrization.

Let K be a connected simplicial complex, with a fixed vertex \bar{x} ,

- We denote by $L_n K$ the set of $n+1$ -uples (x_n, \dots, x_0) of points of K such that two consecutive elements lie in a common simplex of K , topologized as a subspace of K^{n+1} .
- We denote by $\tilde{L}K$ the set of **Milnor free paths**

$$\tilde{L}K := \left(\bigcup_{n \in \mathbb{N}} L_n K \right) / \sim,$$

where the relation is generated by the conditions

$$(x_n, \dots, x_i, \dots, x_0) \sim (x_n, \dots, \hat{x}_i, \dots, x_0),$$

when either $x_i = x_{i-1}$ or $x_{i-1} = x_{i+1}$.

- We denote by $\tilde{P}K$ the set of **based Milnor paths**, i.e., is the subset of $\tilde{L}K$, whose elements are of the form $[(x_n, x_{n-1}, \dots, x_1, \bar{x})]$.
- We denote by $\tilde{\Omega}K$ the set of **based Milnor loops**, i.e., the subset of $\tilde{L}K$, whose elements are of the form $[(\bar{x}, x_{n-1}, \dots, x_1, \bar{x})]$.

Theorem 4.6 ([5, Sections 3,5]). *Let K be a connected simplicial complex with a fixed vertex. The space $\tilde{P}K$ is contractible and has the structure of a $\tilde{\Omega}K$ -bundle over X . Moreover, $\tilde{\Omega}K$ is a nice topological group. Furthermore, given a G -bundle P over K , there exists a morphism of bundles over K ,*

$$\phi: \tilde{P}K \longrightarrow P,$$

that is equivariant with respect to a homomorphism induced between the fibres

$$a: \tilde{\Omega}K \longrightarrow G.$$

As a consequence,

$$a_* \tilde{P}K \simeq P.$$

The constructions of \tilde{P} and $\tilde{\Omega}$ for simplicial complexes can be extended to connected countable CW-complexes, based at vertices, producing a bundle with the same properties. The idea is to first replace such a space by a homotopy equivalent simplicial complex.

Let X be a connected countable CW-complex, based at one of its vertices. As we see in [15, Theorem 1.3], there exists a locally finite simplicial complex K and a homotopy equivalence $s: X \rightarrow K$. Up to further subdivision of K , which does not change the topology because K is locally finite, we can assume that the image of the base point through s is a vertex of K . With a global definition of \tilde{P} and $\tilde{\Omega}$ on \mathcal{T}_* in mind, it is convenient to choose such a simplicial replacement for every space in \mathcal{T}_* .

Choice 4.7. Let $X \in \mathcal{T}_*$. We choose

- a countable CW-structure such that the base point is a vertex, and
- a homotopy equivalence $s_X: X \rightarrow K(X)$, where $K(X)$ is a locally finite simplicial complex, and the image of the base point through s_X is a vertex of $K(X)$. When X admits a simplicial structure such that the base point is a vertex we take s_X to be the identity.

By setting

$$\tilde{\Omega}X := \tilde{\Omega}(K(X)) \text{ and } \tilde{P}X := s_X^* \tilde{P}K(X),$$

we obtain the desired bundle.

Proposition 4.8 ([5, Corollary 3.7]). *Let X be a connected countable CW-complex, with a fixed vertex. The space $\tilde{P}X$ is contractible, and has the structure of a $\tilde{\Omega}X$ -bundle over X . Moreover, $\tilde{\Omega}X$ is a nice topological group. Furthermore, given a G -bundle P over X there exists a morphism of bundles over X ,*

$$\phi: \tilde{P}X \longrightarrow P,$$

that is equivariant with respect to a homomorphism induced between the fibres

$$a: \tilde{\Omega}X \longrightarrow G.$$

As a consequence,

$$a_* \tilde{P}X \simeq P.$$

The last part of the statement is dual to Proposition 3.7, and shows that $\tilde{P}X$ can be thought of as a **universal bundle** over X . It is important to note that this universal property is not valid (a priori) for other contractible bundles over X ; compare with Proposition 4.21.

Given a space $X \in \mathcal{T}_*$, since $\tilde{P}X$ is a $\tilde{\Omega}X$ -bundle over X , as a consequence of Theorem 3.7, there exists a morphism $\tilde{P}X \rightarrow \tilde{E}\tilde{\Omega}X$ of $\tilde{\Omega}X$ -bundles inducing a map on the orbit spaces $X \rightarrow B\tilde{\Omega}X$, whose pullback of $\tilde{E}\tilde{\Omega}X$ is equivalent to $\tilde{P}X$.

Choice 4.9 (Unit). Let $X \in \mathcal{T}_*$. We choose a morphism of $\tilde{\Omega}X$ -bundles

$$\tilde{P}X \longrightarrow \tilde{E}\tilde{\Omega}X,$$

and denote the induced weak equivalence

$$\eta_X: X \longrightarrow B\tilde{\Omega}X,$$

which induces an equivalence of $\tilde{\Omega}X$ -bundles over X

$$\eta_X^*(\tilde{E}\tilde{\Omega}X) \simeq \tilde{P}X.$$

The collection $\{\eta_X\}_{X \in \mathcal{T}_*}$ that we just chose will play the role of a **unit** of the hypothetical adjunction. Once proven that η is *homotopy universal*, the analogues of the other properties that characterize an adjunction will follow formally.

Proposition 4.10 (Universality of the unit). *For every $X \in \mathcal{T}_*$, the unit η_X is **homotopy universal** among the continuous maps from X into a classifying space \tilde{B} . In other words, for any pointed continuous map $f: X \rightarrow \tilde{B}G$ there exists a unique (up to algebraic equivalence) homomorphism of topological groups $a: \tilde{\Omega}X \rightarrow G$ such that $Ba \circ \eta_X \simeq f$.*

$$\begin{array}{ccc} X & \xrightarrow{\eta_X} & B\tilde{\Omega}X & & \tilde{\Omega}X \\ & \searrow f & \downarrow Ba & & \downarrow a \\ & & BG & & G \end{array}$$

Proof. The pullback $f^*\tilde{E}G$ is a G -bundle over X . Thus, by Proposition 4.8, there exists $a: \tilde{\Omega}X \rightarrow G$ such that $a_*\tilde{P}X \simeq f^*\tilde{E}G$. As a consequence of Proposition 4.1 and Choice 4.9, we have the following equivalence of bundles:

$$(\tilde{B}a \circ \eta_X)^*\tilde{E}G \simeq \eta_X^*(\tilde{B}a)^*\tilde{E}G \simeq \eta_X^*(a_*(\tilde{E}\tilde{\Omega}X)) \simeq a_*(\eta_X^*(\tilde{E}\tilde{\Omega}X)) \simeq a_*\tilde{P}X \simeq f^*\tilde{E}G.$$

Thus, by Proposition 3.6, $\tilde{B}a \circ \eta_X \simeq f$.

For the uniqueness, suppose $a': \tilde{\Omega}X \rightarrow G$ also satisfies the condition. Then, if η^{-1} denotes a homotopy inverse for η_X , we have that

$$\tilde{B}a \simeq \tilde{B}a \circ \text{id}_{\tilde{\Omega}X} \simeq \tilde{B}a \circ \eta_X \circ \eta^{-1} \simeq f \circ \eta^{-1} \simeq \tilde{B}a' \circ \eta_X \circ \eta^{-1} \simeq \tilde{B}a' \circ \text{id}_{\tilde{\Omega}X} \simeq \tilde{B}a'.$$

So $a \equiv a'$. \square

Using the unit η_X we are finally able to describe when two group homomorphisms defined on $\tilde{\Omega}X$ induce the same bundle over X .

Proposition 4.11. *Let $X \in \mathcal{T}_*$. Two topological group morphisms $\tilde{\Omega}X \rightarrow G$ are algebraically equivalent if and only if their pushforwards of $\tilde{P}X$ are equivalent bundles of G -bundles over X , i.e.,*

$$a \equiv b: \tilde{\Omega}X \rightarrow G \iff a_*\tilde{P}X \simeq b_*\tilde{P}X.$$

Proof. As a consequence of Proposition 4.1 and Choice 4.9, we have the following equivalences of G -bundles over X ,

$$(\tilde{B}a \circ \eta_X)^*\tilde{E}G \simeq \eta_X^*(\tilde{B}a)^*\tilde{E}G \simeq \eta_X^*a_*\tilde{E}(\tilde{\Omega}X) \simeq a_*\eta_X^*\tilde{E}(\tilde{\Omega}X) \simeq a_*\tilde{P}X,$$

and, similarly,

$$(\tilde{B}b \circ \eta_X)^*\tilde{E}G \simeq b_*\tilde{P}X.$$

Thus, using Proposition 3.1,

$$\begin{aligned} a \equiv b &\iff Ba \simeq Bb \iff Ba \circ \eta_X \simeq Bb \circ \eta_X \iff \\ &\iff (Ba \circ \eta_X)^*\tilde{E}(\tilde{\Omega}X) \simeq (Bb \circ \eta_X)^*\tilde{E}(\tilde{\Omega}X) \iff a_*\tilde{P}X \simeq b_*\tilde{P}X. \quad \square \end{aligned}$$

Proposition 4.8 and Proposition 4.11 together imply the following.

Theorem 4.12 (Classification of bundles over X). *Let $X \in \mathcal{T}_*$.*

The map $(-)_\tilde{P}X$ induces a bijection*

$$(-)_*\tilde{P}X: \mathcal{G}p(\mathcal{T}op)(\tilde{\Omega}X, G)/\equiv \longrightarrow {}_X\mathcal{B}un_G/\simeq$$

between the equivalence classes of continuous homomorphisms $\tilde{\Omega}X \rightarrow G$ and the equivalence classes of G -bundles over X .

The next aim is to make $\tilde{\Omega}$ act on continuous maps, so that the classification described in Theorem 4.12 is *natural*. Unfortunately the nature of the values of $\tilde{\Omega}$ on objects involves arbitrary choices concerning the cell decomposition, and there is no canonical way to assign a homomorphism to a continuous function. Instead we require that $\tilde{\Omega}f$ be consistent with f in terms of the bundle they produce, imitating Proposition 4.1.

Let $f: X \rightarrow Y$ be a continuous map in \mathcal{T}_* . Then $f^*\tilde{P}Y$ is a $\tilde{\Omega}Y$ -bundle over X . As a consequence of Proposition 4.8, there exists a morphism $\tilde{P}X \rightarrow f^*\tilde{P}Y$ of bundles

over X inducing a homomorphism on the fibres $\tilde{\Omega}X \rightarrow \tilde{\Omega}Y$ whose pushforward of $\tilde{P}X$ is equivalent to $f^*\tilde{P}Y$.

Choice 4.13. Let $f: X \rightarrow Y$ be a continuous map in \mathcal{T}_* . We choose a morphism of bundles over X

$$\phi_f: \tilde{P}X \longrightarrow f^*\tilde{P}Y,$$

and denote the restriction to the fibre

$$\tilde{\Omega}f: \tilde{\Omega}X \longrightarrow \tilde{\Omega}Y,$$

which induces an equivalence

$$\phi_{f_*}: f^*\tilde{P}Y \simeq (\tilde{\Omega}f)_*(\tilde{P}X).$$

By Proposition 4.11, the choice of $\tilde{\Omega}f$ is determined up to algebraic equivalence.

Proposition 4.14 (Functoriality of the universal bundle over a fixed space). *The classifying group $\tilde{\Omega}$ gives a pseudofunctorial assignment*

$$\tilde{\Omega}: \mathcal{T}_* \longrightarrow \mathcal{G},$$

i.e., the following hold:

- Identity: $\tilde{\Omega}\text{id}_X \equiv \text{id}_{\tilde{\Omega}X}$, for any $X \in \mathcal{T}_*$;
- Composition: $\tilde{\Omega}(g \circ f) \equiv (\tilde{\Omega}g) \circ (\tilde{\Omega}f)$, for any $f: X \rightarrow Y$ and $g: Y \rightarrow Z$ in \mathcal{T}_* .

Proof. The assignment on objects is well defined, thanks to Proposition 4.8. As for the functorial properties, use Choice 4.13 to get the following equivalences of bundles:

$$\begin{aligned} (\tilde{\Omega}(g \circ f))_*\tilde{P}X &\simeq (g \circ f)^*\tilde{P}Z \simeq f^*g^*\tilde{P}Z \simeq f^*(\tilde{\Omega}g)_*\tilde{P}Y \simeq \\ &\simeq (\tilde{\Omega}g)_*f^*\tilde{P}Y \simeq (\tilde{\Omega}g)_*(\tilde{\Omega}f)_*\tilde{P}X = (\tilde{\Omega}g \circ \tilde{\Omega}f)_*\tilde{P}X. \end{aligned}$$

Thus, by Proposition 4.11, $\tilde{\Omega}(g \circ f) \simeq \tilde{\Omega}g \circ \tilde{\Omega}f$. The argument for the unit is analogous. \square

Remark 4.15 (Naturality of the classification of bundles over a fixed base space). The bijection described in Theorem 4.12 is **natural** in both the variables, with respect to pointed continuous maps and continuous homomorphisms. Indeed, the following diagrams commute up to equivalence.

$$\begin{array}{ccc} X & \mathcal{G}p(\mathcal{T}op)(\tilde{\Omega}X, G) & \xrightarrow{(-)_*\tilde{P}X} \mathcal{X}\mathcal{B}un_G \\ f \downarrow & \uparrow -\circ\tilde{\Omega}f & \uparrow f^*(-) \\ Y & \mathcal{G}p(\mathcal{T}op)(\tilde{\Omega}Y, G) & \xrightarrow{(-)_*\tilde{P}Y} \mathcal{Y}\mathcal{B}un_G \\ \\ G & \mathcal{G}p(\mathcal{T}op)(\tilde{\Omega}X, G) & \xrightarrow{(-)_*\tilde{P}X} \mathcal{X}\mathcal{B}un_G \\ a \downarrow & \downarrow a\circ- & \downarrow a_*(-) \\ H & \mathcal{G}p(\mathcal{T}op)(\tilde{\Omega}X, H) & \xrightarrow{(-)_*\tilde{P}X} \mathcal{X}\mathcal{B}un_H \end{array}$$

Theorem 4.16 (Looping–Delooping adjunction). *For every $X \in \mathcal{T}_*$ and $G \in \mathcal{G}$ there is a bijection*

$$\mathcal{G}(\tilde{\Omega}X, G)/\equiv \longrightarrow \mathcal{T}_*(X, \tilde{B}G)/\simeq,$$

given by $a \mapsto \tilde{B}a \circ \eta_X$. It is natural as explained in Remarks 4.5 and 4.15. In this sense Milnor’s loop space pseudofunctor $\tilde{\Omega}$ and the classifying space functor \tilde{B} form a **homotopy adjunction**

$$\tilde{\Omega}: \mathcal{T}_* \begin{array}{c} \xrightarrow{\quad} \\ \xleftarrow{\quad} \end{array} \mathcal{G} : \tilde{B}.$$

Proof. Let $a, b: \tilde{\Omega}X \rightarrow G$ be continuous homomorphisms. Since η_X is a homotopy equivalence, we have the following equivalences

$$a \equiv b \iff Ba \simeq Bb \iff Ba \circ \eta_X \simeq Bb \circ \eta_X.$$

This shows that the assignment is well defined and injective. The surjectivity is a consequence of Proposition 4.10. \square

Proposition 4.17 (Naturality of the unit). *The unit is a **homotopy natural transformation***

$$\eta: \text{id}_{\mathcal{T}_*} \longrightarrow \tilde{B}\tilde{\Omega},$$

i.e., for any $f: X \rightarrow Y$ in \mathcal{T}_* , the following square commutes up to homotopy.

$$\begin{array}{ccc} X & \xrightarrow{\eta_X} & \tilde{B}\tilde{\Omega}X \\ f \downarrow & & \downarrow \tilde{B}\tilde{\Omega}f \\ Y & \xrightarrow{\eta_Y} & \tilde{B}\tilde{\Omega}Y \end{array}$$

Proof. Using Proposition 4.1, Choice 4.9 and Choice 4.13, we have the following equivalences of bundles:

$$\begin{aligned} (\tilde{B}\tilde{\Omega}f \circ \eta_X)^* \tilde{E}(\tilde{\Omega}Y) &\simeq \eta_X^* (\tilde{B}\tilde{\Omega}f)^* \tilde{E}(\tilde{\Omega}Y) \simeq \eta_X^* (\tilde{\Omega}f)_* \tilde{E}(\tilde{\Omega}X) \simeq \\ &\simeq (\tilde{\Omega}f)_* \eta_X^* \tilde{E}(\tilde{\Omega}X) \simeq (\tilde{\Omega}f)_* \tilde{P}X \simeq f^* \tilde{P}Y \simeq f^* \eta_Y^* \tilde{E}(\tilde{\Omega}Y) \simeq (\eta_Y \circ f)^* \tilde{E}(\tilde{\Omega}Y). \end{aligned}$$

Thus, by Proposition 3.6, $\tilde{B}\tilde{\Omega}f \circ \eta_X \simeq \eta_Y \circ f$. \square

Let $G \in \mathcal{G}$. Then $\tilde{E}G$ is a G -bundle over $\tilde{B}G$. As a consequence of Proposition 4.8, there exists a morphism $\tilde{P}\tilde{B}G \rightarrow \tilde{E}G$ of bundles over $\tilde{B}G$ inducing a homomorphism on the fibre $\tilde{\Omega}\tilde{B}G \rightarrow G$, whose pushforward of $\tilde{P}\tilde{B}G$ is equivalent to $\tilde{E}G$.

Choice 4.18 (Counit). Let $G \in \mathcal{G}$. We choose a morphism of bundles over $\tilde{B}G$

$$\tilde{P}\tilde{B}G \longrightarrow \tilde{E}G,$$

and denote the restriction to the fibre

$$\epsilon_G: \tilde{\Omega}\tilde{B}G \longrightarrow G,$$

which induces an equivalence

$$\epsilon_{G*} \tilde{P}(\tilde{B}G) \simeq \tilde{E}G.$$

The collection $\{\epsilon_G\}_{G \in \mathcal{G}}$ that we just chose is the **counit** of the homotopy adjunction.

Proposition 4.19 (Universality of the counit). *For every $G \in \mathcal{G}$, the counit ϵ_G is **homotopy universal** among the homomorphisms of topological groups from Milnor’s loop space $\tilde{\Omega}$ to G . In other words, for any continuous homomorphism $a: \tilde{\Omega}X \rightarrow G$ there exists a unique (up to homotopy) pointed continuous map $f: X \rightarrow \tilde{B}G$ such that $\epsilon_G \circ \tilde{\Omega}f \equiv a$.*

$$\begin{array}{ccc} \tilde{B}G & & \tilde{\Omega}\tilde{B}G \xrightarrow{\epsilon_G} G \\ \uparrow f & & \uparrow \tilde{\Omega}f \nearrow a \\ X & & \tilde{\Omega}X \end{array}$$

Proof. The pushforward $a_*\tilde{P}X$ is a G -bundle over X . Thus, by Theorem 3.7, there exists $f: X \rightarrow \tilde{B}G$ such that $f^*\tilde{E}G \simeq a_*\tilde{P}X$. Using Choice 4.13 and Choice 4.18, we have the following equivalences of bundles:

$$(\epsilon_G \circ \tilde{\Omega}f)_*\tilde{P}X \simeq \epsilon_{G*}\tilde{\Omega}f_*\tilde{P}X \simeq \epsilon_{G*}f^*\tilde{P}(\tilde{B}G) \simeq f^*\epsilon_{G*}\tilde{P}(\tilde{B}G) \simeq f^*\tilde{E}G \simeq a_*\tilde{P}X.$$

Thus, by Proposition 4.11, $\epsilon_G \circ \tilde{\Omega}f \simeq a$.

For the uniqueness, we first remark that $\tilde{B}\epsilon_G \circ \eta_{\tilde{B}G} \simeq \text{id}_{\tilde{B}G}$. Indeed,

$$\begin{aligned} (\tilde{B}\epsilon_G \circ \eta_{\tilde{B}G})^*\tilde{E}G &\simeq (\eta_{\tilde{B}G})^*(\tilde{B}\epsilon_G)^*\tilde{E}G \simeq (\eta_{\tilde{B}G})^*(\epsilon_G)_*\tilde{E}(\tilde{\Omega}\tilde{B}G) \simeq \\ &\simeq (\epsilon_G)_*(\eta_{\tilde{B}G})^*\tilde{E}(\tilde{\Omega}\tilde{B}G) \simeq (\epsilon_G)_*\tilde{P}(\tilde{B}G) \simeq \tilde{E}G \simeq \text{id}_{\tilde{B}G}^*\tilde{E}G. \end{aligned}$$

Using this identity and the naturality of the unit, we see that f is determined by a as follows:

$$f = \text{id}_{\tilde{B}G} \circ f \simeq \tilde{B}\epsilon_G \circ \eta_{\tilde{B}G} \circ f \simeq \tilde{B}\epsilon_G \circ \tilde{B}\tilde{\Omega}f \circ \eta_X = \tilde{B}(\epsilon_G \circ \tilde{\Omega}f) \circ \eta_X \simeq Ba \circ \eta_X,$$

and therefore f is unique up to homotopy. \square

Proposition 4.20 (Naturality of the counit). *The counit is a **homotopy natural transformation***

$$\epsilon: \tilde{\Omega}\tilde{B} \longrightarrow \text{id}_{\mathcal{G}},$$

i.e., for any $a: G \rightarrow H$ in \mathcal{G} , the following square commutes up to algebraic equivalence.

$$\begin{array}{ccc} \tilde{\Omega}\tilde{B}G & \xrightarrow{\epsilon_G} & G \\ \tilde{\Omega}\tilde{B}a \downarrow & & \downarrow a \\ \tilde{\Omega}\tilde{B}H & \xrightarrow{\epsilon_H} & H \end{array}$$

Proof. Using Choice 4.13, Choice 4.18, and Proposition 4.1, we have the following equivalences of bundles:

$$\begin{aligned} (\epsilon_H \circ \tilde{\Omega}Ba)_*\tilde{P}(\tilde{B}G) &\simeq \epsilon_{H*}(\tilde{\Omega}Ba)_*\tilde{P}(\tilde{B}G) \simeq \epsilon_{H*}(Ba)_*\tilde{P}(\tilde{B}H) \simeq \\ &\simeq (\tilde{B}a)^*\epsilon_{H*}\tilde{P}(\tilde{B}H) \simeq (\tilde{B}a)^*\tilde{E}H \simeq a_*\tilde{E}G \simeq a_*\epsilon_{G*}\tilde{P}(\tilde{B}G) \simeq (a \circ \epsilon_G)_*\tilde{P}(\tilde{B}G). \end{aligned}$$

Thus, by Proposition 4.11, $\epsilon_H \circ \tilde{\Omega}Ba \equiv a \circ \epsilon_G$. \square

Thanks to Proposition 3.11(2), every contractible G -bundle gives a classification of G -bundles. Moreover, the specific model that Milnor suggests for the classifying space is not special; equivalent results hold for any model of the classifying space.

In the dual picture, having a contractible bundle over a space does not guarantee (a priori) any universal property among bundles over the same space.

Proposition 4.21 (Uniqueness of the classifying group). *Let $X \in \mathcal{T}_*$.*

For $\Omega \in \mathcal{G}$ the following are equivalent:

- (1) Equivalence: Ω is **equivalent** to $\tilde{\Omega}X$, i.e., there exist continuous homomorphisms

$$\Omega \begin{array}{c} \xrightarrow{b} \\ \xleftarrow{a} \end{array} \tilde{\Omega}X$$

such that $a \circ b \equiv \text{id}_\Omega$ and $b \circ a \equiv \text{id}_{\tilde{\Omega}X}$;

- (2) Universal property: there exists a natural bijection

$$\mathcal{G}(\Omega, G)/\equiv \longrightarrow {}_X\mathcal{Bun}_G/\simeq$$

between the equivalence classes of continuous homomorphisms $\Omega \rightarrow G$ and the equivalence classes of G -bundles over X , for every G .

Moreover, (1) and (2) imply the following equivalent conditions.

- (3) Contractibility: there exists an Ω -bundle P over X that is contractible;
 (1') Milnor's equivalence: there exists a nice group $A \in \mathcal{G}$ and continuous homomorphisms that are homotopy equivalences

$$\Omega \xleftarrow{\simeq} A \xrightarrow{\simeq} \tilde{\Omega}X.$$

Proof. • [(1) \iff (2)]: It is an application of the Yoneda Lemma. Indeed, $\tilde{\Omega}X$ represents the functor $G \mapsto {}_X\mathcal{Bun}_G/\simeq$ on the category \mathcal{G} up to equivalence.

- [(1') \iff (3)]: See [6, Theorem 5.2.(4)].
- [(1) \implies (1')]: It is enough to take

$$\Omega \xleftarrow[\simeq]{a} \tilde{\Omega}X \xlongequal{\quad} \tilde{\Omega}X. \quad \square$$

References

- [1] A. Dold. Partitions of unity in the theory of fibrations. *Ann. of Math. (2)*, 78:223–255, 1963.
- [2] E.D. Farjoun and K. Hess. Normal and conormal maps in homotopy theory. *Homology Homotopy Appl.*, 14(1):79–112, 2012.
- [3] R. Lashof. Classification of fibre bundles by the loop space of the base. *Ann. of Math. (2)*, 64:436–446, 1956.
- [4] J. Lurie. *Higher Topos Theory*, volume 170 of *Ann. Math. Stud.* Princeton University Press, Princeton, NJ, 2009.
- [5] J.W. Milnor. Construction of universal bundles. I. *Ann. of Math. (2)*, 63:272–284, 1956.

- [6] J.W. Milnor. Construction of universal bundles. II. *Ann. of Math. (2)*, 63:430–436, 1956.
- [7] J.W. Milnor and J.D. Stasheff. *Characteristic Classes*, volume 76 of *Ann. Math. Stud.* Princeton University Press, Princeton, NJ; University of Tokyo Press, Tokyo, 1974.
- [8] S. Mitchell. Notes on principal bundles and classifying spaces, available at <http://www.math.washington.edu/~mitchell/Atopc/prin.pdf>
- [9] R.A. Piccinini. *Lectures on Homotopy Theory*, volume 171 of *North-Holland Math. Stud.* North-Holland Publishing Co., Amsterdam, 1992.
- [10] D.G. Quillen. *Homotopical Algebra*, volume 43 of *Lecture Notes in Math.* Springer-Verlag, Berlin, New York, 1967.
- [11] M. Rovelli. A characteristic long exact sequence for principal bundles, [arXiv:1610.08893](https://arxiv.org/abs/1610.08893)
- [12] E.H. Spanier. *Algebraic Topology*. Springer-Verlag, New York, Berlin, 1981. Corrected reprint.
- [13] J. Stasheff. A classification theorem for fibre spaces. *Topology*, 2:239–246, 1963.
- [14] N. Steenrod. *The Topology of Fibre Bundles*. Princeton Landmarks in Math. Princeton University Press, Princeton, NJ, 1999. Reprint of the 1957 edition, Princeton Paperbacks.
- [15] J.H.C. Whitehead. Combinatorial homotopy. I. *Bull. Amer. Math. Soc.*, 55:213–245, 1949.

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