THE SLICES OF $S^n \wedge H\underline{\mathbb{Z}}$ FOR CYCLIC p-GROUPS

CAROLYN YARNALL

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Abstract

The slice filtration is a filtration of equivariant spectra. While the tower is analogous to the Postnikov tower in the nonequivariant setting, complete slice towers are known for relatively few G-spectra. In this paper, we determine the slice tower for all G-spectra of the form $S^n \wedge H\underline{\mathbb{Z}}$ where $n \geqslant 0$ and G is a cyclic p-group for p an odd prime.

1. Introduction

The slice filtration is a filtration of equivariant spectra developed by Hill, Hopkins, and Ravenel in their solution to the Kervaire invariant one problem [**3**]. It is a generalization of Dugger's C_2-equivariant filtration [**1**] and is modeled on the motivic filtration of Voevodsky [**8**]. While it is an equivariant analogue of the Postnikov tower, there are some marked differences. These differences arise from the fact that we use representation spheres rather than ordinary spheres in the construction. If V is a G-representation, a representation sphere S^V is the one-point compactification of the associated representation space. The following definitions are given in [**2**] and are equivalent to the original definitions from [**3**].

Definition 1.1. For each integer n, let $\tau_{\geqslant n}$ denote the localizing subcategory of G-spectra of the form $G_+ \wedge_H S^{m\rho_H - \epsilon}$ where H ranges over all subgroups of G, ρ_H denotes the regular representation of H, $\epsilon = 0, 1$, and the dimension of the underlying sphere is $\geqslant n$.

What we mean by a *localizing subcategory* here is that it is the category of acyclics for a localization functor on G-spectra. More precisely we mean a full subcategory that is closed under weak equivalences, cofibers, extensions, retracts, wedges, and filtered colimits. The localizing subcategory $\tau_{\geqslant n}$ is the collection of slice cells used to construct the slice tower.

Definition 1.2. Let X be a G-spectrum. The *slice tower* of X is the tower consisting of the following data:

1. Stages $P^n(X)$ where $P^n(-)$ denotes the localization functor associated to $\tau_{\geqslant n}$.
2. Natural maps $P^n(X) \to P^{n-1}(X)$ with fiber denoted $P^n_n(X)$ and referred to as the *nth slice* of X.

From these definitions we see that the construction resembles that of the Postnikov

tower but differences arise due to the nature of spectra in $\tau_{\geqslant n}$ used. Most notably, these spectra are often less connected than ordinary spheres. For example, the C_2-representation sphere S^{ρ_2-1} is a slice-cell in $\tau_{\geqslant 0}$. However, the map $S^0 \hookrightarrow S^{\rho_2-1}$ is the inclusion of fixed points and thus the homotopy in dimension 0 is not trivial. In general, objects in $\tau_{\geqslant n}$ are often less than n connected and hence lower homotopy groups may be affected in forming each stage $P^n(X)$. Thus, unlike the fibers from the Postnikov tower, slices need not be Eilenberg–MacLane spectra.

As a result, the slice tower can be difficult to compute and thus far has been determined for relatively few genuine G-spectra. Hill, Hopkins, and Ravenel have determined the complete slice filtration for $MU_{(C_{2^k})}$ [3]. The slice tower for certain Eilenberg–MacLane spectra was given by Hill in [2] and for more general Eilenberg–MacLane spectra by Ullman in [7]. As the slice tower does not commute with all suspensions, knowing the towers for $H\underline{M}$ does not immediately give us the towers for $S^V \wedge H\underline{M}$ for representations V or even $S^n \wedge H\underline{\mathbb{Z}}$. On this front, Hill, Hopkins, and Ravenel have computed the slice tower for $S^V \wedge H\underline{\mathbb{Z}}$ where V is a multiple of a particular nontrivial subrepresentation of $\rho_{C_{p^k}}$ [5]. What we present here is a different flavor of the work in [5], namely, the positive integer-graded suspensions of $H\underline{\mathbb{Z}}$ for cyclic p-groups where p is an odd prime.

Main Theorem. *The slice tower for $S^n \wedge H\underline{\mathbb{Z}}$ for $G = C_{p^k}$ where p is an odd prime consists of the following data:*

- *The slice sections $P^m(S^n \wedge H\underline{\mathbb{Z}})$ are of the form $S^W \wedge H\underline{\mathbb{Z}}$ where W is a C_{p^k}-representation of dimension n and $n \leqslant m \leqslant (n-2)p^k - 1$.*
- *The nontrivial slices $P_m^m(S^n \wedge H\underline{\mathbb{Z}})$ are of the form $S^V \wedge H\underline{B}_{i,j}$ where $\underline{B}_{i,j}$ is a C_{p^k}-Mackey functor, V is a C_{p^k}-representation of dimension m, where $m \equiv -1 \pmod{p}$ and $n \leqslant m \leqslant (n-2)p^k - 1$. All other slices are trivial.*

The exact objects that appear in this theorem will be defined along the way to the final statement of the theorem in Section 4. In Section 2 we provide a definition of the Mackey functors $\underline{B}_{i,j}$ appearing in the slices of our tower. In Section 3 we present the collection of spectra of the form $S^V \wedge H\underline{B}_{i,j}$ from the statement of our Main Theorem and the fiber sequences that comprise our tower. After a more precise statement of the main theorem in Section 4, we provide a few examples. The final section consists of proofs of the essential theorems from Section 3.

Acknowledgments

2.   Background: Mackey functors

The definitions in this section are given by Hill, Hopkins, and Ravenel in [5]. We note that the Weyl action on these Mackey functors is trivial and thus we omit it from all definitions and discussions in this section.

First, we note that we will depict all Mackey functors using the diagrams of Lewis from [6]. Thus, if $\underline{M}$ is a C_p-Mackey functor, we will represent it as

$$\underline{M}(C_p/C_p)$$

$$res \left(\right) tr$$

$$\underline{M}(C_p/e).$$

For the constant C_{p^k}-Mackey functor $\underline{\mathbb{Z}}$, all restriction maps are the identity and transfer maps are multiplication by p. The restriction and transfer maps roles are reversed for its dual, $\underline{\mathbb{Z}}^*$. We can make strategic exchanges of the restriction and transfer maps to obtain similar Mackey functors.

Definition 2.1. Let $G = C_{p^k}$ and $0 \leqslant j < i \leqslant k$ be integers. For each pair, let $\underline{\mathbb{Z}}(i,j)$ denote the Mackey functor with constant value $\mathbb{Z}$ for which the restriction and transfer maps are

$$res^{C_{p^{m+1}}}_{C_{p^m}} = \begin{cases} 1 & m < j, \\ p & j \leqslant m < i, \\ 1 & i \leqslant m, \end{cases} \qquad tr^{C_{p^{m+1}}}_{C_{p^m}} = \begin{cases} p & m < j, \\ 1 & j \leqslant m < i, \\ p & i \leqslant m. \end{cases}$$

Notice that the restriction of $\underline{\mathbb{Z}}(i,j)$ to C_{p^j} is the constant Mackey functor $\underline{\mathbb{Z}}$. If we allow $i = j$ then $\underline{\mathbb{Z}} = \underline{\mathbb{Z}}(i,i)$. We also note that the dual to the constant Mackey functor $\underline{\mathbb{Z}}^*$ is equivalent to $\underline{\mathbb{Z}}(k,0)$.

*Example 2.2.* For $G = C_{p^2}$, we have the following $\underline{\mathbb{Z}}(i,j)$:
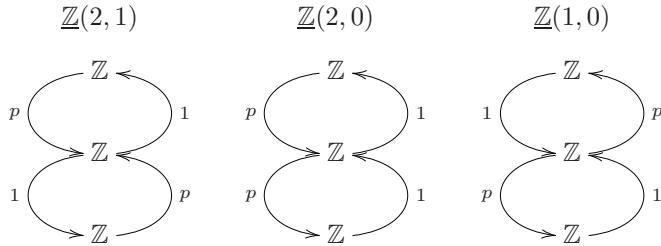
We can define maps from $\underline{\mathbb{Z}}(i,j)$ to $\underline{\mathbb{Z}}$ by choosing the image of the element 1 in $\underline{\mathbb{Z}}(i,j)(G/e)$. From such maps we obtain the following Mackey functors.

Definition 2.3. Let $G = C_{p^k}$ and $1 \leqslant j < i \leqslant k$ be integers such that $i + j \leqslant k$. Let $\underline{B}_{i,j}$ denote the quotient Mackey functor associated to the inclusion $\underline{\mathbb{Z}}(i+j,j) \to \underline{\mathbb{Z}}$. That is, the C_{p^k}-Mackey functor $\underline{B}_{i,j}$ is defined as follows:

$$\underline{B}_{i,j}(C_{p^k}/C_{p^m}) = \begin{cases} \mathbb{Z}/p^i & m \geqslant i + j, \\ \mathbb{Z}/p^{m-j} & j < m < i + j, \\ 0 & m \leqslant j, \end{cases}$$

where all restriction maps are the canonical quotients q and transfer maps are multiplication by p.

Example 2.4. For $G = C_{p^2}$, we have the following $\underline{B}_{i,j}$:
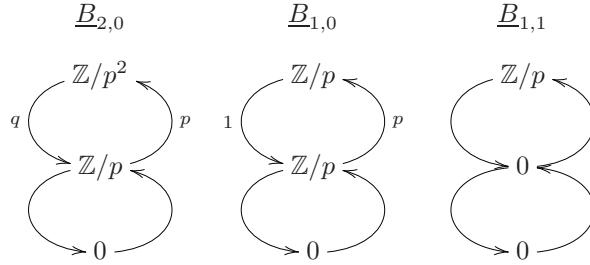
3.   Slice tower objects

In order to show that a tower is the slice tower for a G-spectrum X, by Proposition 4.45 in [**3**] it is sufficient to show that the limit of the tower is X, the colimit of the tower is contractible, and the mth fiber is an m-slice for all m. The towers that we present for $S^n \wedge H\underline{\mathbb{Z}}$ are finite so the interesting work amounts to determining the fiber sequences that make up the tower and showing that each of these fibers is m-slice.

3.1.   Slices

We will now define a collection of spectra that are m-slice for certain values of m. In Section 4 we will see that these are precisely the nontrivial slices in the tower for $S^n \wedge H\underline{\mathbb{Z}}$. For the remainder of this section, we assume $n \geqslant 3$.

Our spectra will all be of the form $S^V \wedge H\underline{M}$ where V is a sum of subrepresentations of the regular representation ρ_G. When $G = C_{p^k}$ we will often write ρ_G as ρ_{p^k}. This representation has the decomposition:

$$\rho_{p^k} = 1 + \bigoplus_{j=1}^{\frac{p^k-1}{2}} \lambda(j),$$

where $\lambda(j)$ is the composition of the inclusion of the p^kth roots of unity with a degree j map on S^1. All objects in our work are considered p-locally and thus $S^{\lambda(i)}$ and $S^{\lambda(j)}$ are equivalent whenever the p-adic orders of i and j agree. Thus, we need only keep track of $\lambda(p^j) = \lambda_j$. Working this way, we have an equivalent decomposition of ρ_{p^k}:

$$\rho_{p^k} = 1 + \frac{p-1}{2}\lambda_{k-1} + \frac{p(p-1)}{2}\lambda_{k-2} + \cdots + \frac{p^{k-1}(p-1)}{2}\lambda_0.$$

Let d_n denote the number of integers of the same parity as n that lie between $\frac{n}{p}$ and $n-2$, inclusive. Via a simple counting argument we obtain a useful way of writing d_n:

$$d_n = \frac{1}{2}\left(n - \left(\frac{n-n_0}{p}\right) - \delta\right),$$

where n_0 is the residue of n modulo p and

$$\delta = \begin{cases} 2 & \text{if } n_0 \text{ is even,} \\ 1 & \text{if } n_0 \text{ is odd,} \\ 0 & \text{if } n_0 = 0. \end{cases}$$

We label the integers between $\frac{n}{p}$ and $n-2$ that are of the same parity as n, $m_1 < m_2 < \cdots < m_{d_n}$ so $m_i = n - 2d_n + 2i - 2$.

Example 3.1. The following tables present a collection of d_n values for the given primes p and integers n.

$p = 3$			$p = 5$			$p = 7$	
n	d_n	$m_1, \ldots, m_{d_n}$	n	d_n	$m_1, \ldots, m_{d_n}$	n	d_n
3	1	1	3	1	1	3	1
4	1	2	4	1	2	4	1
5	1	3	5	2	1, 3	5	2
6	2	2, 4	6	2	2, 4	6	2
7	2	3, 5	7	2	3, 5	7	3
8	2	4, 6	8	3	2, 4, 6	8	3
9	3	3, 5, 7	9	3	3, 5, 7	9	3
10	3	4, 6, 8	10	4	2, 4, 6, 8	10	4

Definition 3.2. Let a, b be integers such that $1 \leqslant a \leqslant k$, $1 \leqslant b \leqslant d_n$. We use $V_{(a,b)}(n)$ to denote the G-representation:

$$V_{(a,b)}(n) = (n-2)\rho_G - 1 - \bigoplus_{i=1}^{\ell} \lambda(i),$$

where

$$\ell = \tfrac{1}{2}\big((n-2)p^k - m_b p^a\big).$$

We write $V_{(a,b)}$ whenever n is clear from the context.

It is immediate from the definition that the dimension of $V_{(a,b)}(n)$ is $m_b p^a - 1$. We now present a few examples of the representations $V_{(a,b)}(n)$ for given values of n and p^k.

Example 3.3. Directly from the definition, it is clear that when $a = k$ and $b = d_n$ we have $V_{(a,b)}(n) = (n-2)\rho - 1$. $V_{(a,b)}(n)$ for smaller values of a and b are obtained by stripping away various $\lambda(i)$. We present the tables in decreasing order of a and b to display this notion more clearly.

$$n = 7, \ p = 3$$

a	b	$m_b p^a - 1$	$V_{(a,b)}$ for $p^k = 3^1$	$V_{(a,b)}$ for $p^k = 3^2$
2	2	$5(3)^2 - 1$	–	$5\rho_9 - 1 = 4 + 5\lambda_1 + 15\lambda_0$
2	1	$3(3)^2 - 1$	–	$3\rho_9 - 1 = 2 + 3\lambda_1 + 9\lambda_0$
1	2	$5(3) - 1$	$5\rho_3 - 1 = 4 + 5\lambda_0$	$2 + \lambda_1 + 5\lambda_0$
1	1	$3(3) - 1$	$3\rho_3 - 1 = 2 + 3\lambda_0$	$\rho_9 - 1 = \lambda_1 + 3\lambda_0$

$$n = 8, \ p = 3$$

a	b	$m_b p^a - 1$	$V_{(a,b)}$ for $p^k = 3^1$	$V_{(a,b)}$ for $p^k = 3^2$
2	2	$6(3)^2 - 1$	–	$6\rho_9 - 1 = 5 + 6\lambda_1 + 18\lambda_0$
2	1	$4(3)^2 - 1$	–	$4\rho_9 - 1 = 3 + 4\lambda_1 + 12\lambda_0$
1	2	$6(3) - 1$	$6\rho_3 - 1 = 5 + 4\lambda_0$	$2\rho_9 - 1 = 1 + 2\lambda_1 + 6\lambda_0$
1	1	$4(3) - 1$	$4\rho_3 - 1 = 3 + 4\lambda_0$	$1 + \lambda_1 + 4\lambda_0$

$$n = 8, \ p = 5$$

a	b	$m_b p^a - 1$	$V_{(a,b)}$ for $p^k = 5^1$	$V_{(a,b)}$ for $p^k = 5^2$
2	3	$6(5)^2 - 1$	–	$6\rho_{25} - 1 = 5 + 12\lambda_1 + 60\lambda_0$
2	2	$4(5)^2 - 1$	–	$4\rho_{25} - 1 = 3 + 8\lambda_1 + 40\lambda_0$
2	1	$2(5)^2 - 1$	–	$2\rho_{25} - 1 = 1 + 4\lambda_1 + 20\lambda_0$
1	3	$6(5) - 1$	$6\rho_5 - 1 = 5 + 12\lambda_0$	$1 + 2\lambda_1 + 12\lambda_0$
1	2	$4(5) - 1$	$4\rho_5 - 1 = 3 + 8\lambda_0$	$1 + \lambda_1 + 8\lambda_0$
1	1	$2(5) - 1$	$2\rho_5 - 1 = 1 + 4\lambda_0$	$1 + 4\lambda_0$

We use the representations $V_{(a,b)}$ and the Mackey functors $\underline{B}_{i,j}$ from Definition 2.3 to form a collection of C_{p^k}-spectra that we later show are slices. In all that follows, we will let $\nu(b) = \min\{\nu_p(m_b), k - a\}$ where $\nu_p(m_b)$ is the p-adic order of m_b. We write ν when b is clear from context.

A proof of the following result may be found in Section 5.

Theorem 3.4. *For all $1 \leqslant a \leqslant k$ and $1 \leqslant b \leqslant d_n$, the C_{p^k}-spectrum*

$$S^{V_{(a,b)}} \wedge H\underline{B}_{\nu+1,a-1}$$

is an $(m_b p^a - 1)$-slice.

We will see that the spectra in the theorem above make up all but one of the nontrivial slices in the tower for $S^n \wedge H\underline{\mathbb{Z}}$. The n-slice is of a slightly different form and moreover, its definition depends on whether n is a multiple of p.

Definition 3.5. If n not a multiple of p, let $W(n)$ be the C_{p^k}-representation

$$W(n) = V_{(1,1)} + 1 - \bigoplus_{i=1}^{\frac{1}{2}(m_1 p - n)} \lambda(i).$$

If n is a multiple of p, let $W'(n)$ be the C_{p^k}-representation

$$W'(n) = V_{(1,2)} + 1 - \bigoplus_{i=1}^{p-1} \lambda(i) - \lambda(1).$$

We write W and W' in place of $W(n)$ and $W'(n)$, respectively, when n is clear from the context. A straightforward computation gives that the dimensions of W and W' are both n.

A proof of the following result may be found in Section 5.

Theorem 3.6. $S^W \wedge H\mathbb{Z}$ and $S^{W'} \wedge H\mathbb{Z}$ are both n-slices.

3.2. Fiber sequences

We will use the slices given in Theorem 3.4 and the maps defined in this section to construct the slice tower of $S^n \wedge H\mathbb{Z}$.

Lemma 3.7. For $G = C_{p^k}$ and $i, j \in \mathbb{Z}$ such that $0 \leqslant j < i \leqslant k$ there are fiber sequences of the form

$$S^{\lambda_j} H\underline{B}_{i,j} \to S^{\lambda_{i+j}+1} \wedge H\mathbb{Z} \to S^{\lambda_j+1} \wedge H\mathbb{Z}.$$

Proof. Consider the fiber sequence

$$F \to H\mathbb{Z}(i+j, j) \to H\mathbb{Z},$$

where the map on the right is induced from the map from Definition 2.3. We then have the following long exact sequence

$$0 \to \underline{\pi_0(F)} \to \mathbb{Z}(i+j, j) \to \mathbb{Z} \to \underline{\pi_{-1}(F)} \to 0.$$

Thus by Definition 2.3 we obtain that $F = \Sigma^{-1} H\underline{B}_{i,j}$.

From [5, Theorem 3.1] we have that

$$S^{\lambda_{i+j} - \lambda_j} \wedge H\mathbb{Z} \simeq H\mathbb{Z}(i+j, j).$$

Implementing this equivalence and smashing with $S^{1+\lambda_j}$ yields the result. $\qquad \square$

In the following lemma and proof, we write $S(V, \underline{M})$ in place of $S^V \wedge H\underline{M}$.

Lemma 3.8. Let $1 \leqslant a \leqslant k$, $1 \leqslant b \leqslant d_n$, and $\widetilde{V}_{(a,b)} = V_{(a,b)} - U$ for any representation U that consists of a sum of representations of the form λ_r for any $r \leqslant a - 1$. Then the C_{p^k}-spectrum $S(V_{(a,b)}, \underline{B}_{\nu+1, a-1})$ is the fiber of any map of the form

$$S(\widetilde{V}_{(a,b)} + 1 + \lambda_{\nu+a}, \mathbb{Z}) \longrightarrow S(\widetilde{V}_{(a,b)} + 1 + \lambda_{a-1}, \mathbb{Z})$$

obtained from the fibration in Lemma 3.7 by setting $i = \nu + 1$, $j = a - 1$ and smashing with $S^{\widetilde{V}_{(a,b)}}$.

Proof. From Lemma 3.7 we have the following fiber sequence

$$S^{\lambda_j} H\underline{B}_{i,j} \to S^{\lambda_{i+j}+1} \wedge H\underline{\mathbb{Z}} \to S^{\lambda_j+1} \wedge H\underline{\mathbb{Z}},$$

for any $0 \leqslant j < i \leqslant k$. Since $\underline{B}_{i,j}$ is 0 on cells induced up from subgroups of C_{p^r} for all $r \leqslant j$, the inclusion of the fixed point sphere $S^0 \hookrightarrow S^{\lambda_r}$ induces an equivalence

$$H\underline{B}_{i,j} \simeq S^{\lambda_r} \wedge H\underline{B}_{i,j},$$

for all $r \leqslant j$. Thus we have the sequence

$$H\underline{B}_{i,j} \to S^{\lambda_{i+j}+1} \wedge H\underline{\mathbb{Z}} \to S^{\lambda_j+1} \wedge H\underline{\mathbb{Z}}.$$

Smashing with $S^{V_{(a,b)}}$ and setting $i = \nu + 1$ and $j = a - 1$ gives that $S(V_{(a,b)}, \underline{B}_{\nu+1,a-1})$ is the fiber of

$$S(V_{(a,b)} + 1 + \lambda_{\nu+a}, \mathbb{Z}) \to S(V_{(a,b)} + 1 + \lambda_{a-1}, \mathbb{Z}).$$

Lastly, since $H\underline{B}_{\nu+a,a-1}$ is equivalent to $S^{\lambda_r} \wedge H\underline{B}_{\nu+a,a-1}$ for all $r \leqslant a - 1$ shifting by representations U consisting solely of such λ_r will not change the fiber of the map and the result follows. □

4. Main result and discussion

We can now present the main result, namely, the entire slice tower for $S^n \wedge H\underline{\mathbb{Z}}$ and $G = C_{p^k}$ for odd primes p. Note that result below only holds for $n \geqslant 3$. The cases $n = 1, 2$ are addressed in Section 5.

Main Theorem. *The slice tower for $S^n \wedge H\underline{\mathbb{Z}}$ for $G = C_{p^k}$ for p an odd prime consists of the following data:*

- *The slice sections $P^m(S^n \wedge H\underline{\mathbb{Z}})$ are of the form $S^V \wedge H\underline{\mathbb{Z}}$ for $n \leqslant m \leqslant (n-2)p^k - 1$ and the C_{p^k}-representations V are determined by the maps from Theorem 3.8.*

- *If n is not divisible by p, the $(m_b p^a - 1)$-slices of $S^n \wedge H\underline{\mathbb{Z}}$ are the spectra $S^{V_{(a,b)}} \wedge H\underline{B}_{\nu+1,a-1}$ for $1 \leqslant a \leqslant k$ and $1 \leqslant b \leqslant d_n$. The n-slice is $S^W \wedge H\underline{\mathbb{Z}}$.*

- *If n is divisible by p, the $(m_b p^a - 1)$-slices of $S^n \wedge H\underline{\mathbb{Z}}$ are the spectra $S^{V_{(a,b)}} \wedge H\underline{B}_{\nu+1,a-1}$ for $1 \leqslant b \leqslant d_n$ when $2 \leqslant a \leqslant k$ and $2 \leqslant b \leqslant d$ when $a = 1$. The n-slice is $S^{W'} \wedge H\underline{\mathbb{Z}}$.*

All other sections and slices are trivial.

Proof. By Theorems 3.4 and 3.6 we know that the spectra listed in the statement above are slices. We also know that they are fibers of the maps from Lemma 3.8. We now show that the maps from Lemma 3.8 fit together sequentially so that the spectra $S^{V_{(a,b)}} \wedge H\underline{B}_{\nu+1,a-1}$ are the fibers in the appropriate order. In other words, we show that for $1 \leqslant a \leqslant k$ and $1 \leqslant b \leqslant d_n - 1$ there is a map from Lemma 3.8 with fiber $V_{(a,b+1)}$ that ends in a spectrum equivalent to the start of the map with fiber $V_{(a,b)}$. We will handle the case $b = d_n$ separately.

Recall that $V_{(a,b)} = (n-2)\rho_G - 1 - \bigoplus_{i=1}^{\ell} \lambda(i)$ where $\ell = \frac{1}{2}\big((n-2)p^k - m_b p^a\big)$. $V_{a,b+1}$ is defined similarly but we will write $\ell' = \frac{1}{2}\big((n-2)p^k - m_{b+1}p^a\big)$. Since

$m_{b+1} = m_b + 2$ a simple calculation gives that $\ell = \ell' + p^a$. Thus,

$$V_{(a,b+1)} - V_{(a,b)} = \bigoplus_{i=\ell'+1}^{\ell'+p^a} \lambda(i).$$

Since $p^a | \ell'$ we know that $\nu_p(i) < p^a$ for all $\ell' + 1 \leqslant j < \ell' + p^a$. Also, $\nu_p(\ell' + p^a) = \nu_p(\ell) = \nu(b) + a$. Thus

$$V_{(a,b+1)} - V_{(a,b)} = U + \lambda_{\nu+a},$$

where U consists solely of representations of the form λ_r for $r \leqslant a - 1$.

Now we show that a map from Lemma 3.8 with fiber $V_{(a+1,1)}$ connects exactly to a map with fiber $V_{(a,d_n)}$. We write ℓ_{a,d_n} and $\ell_{a+1,1}$ in place of ℓ from Definition 3.2 for clarity. More precisely,

$$\ell_{a,d_n} = \frac{1}{2}\big((n-2)p^k - (n-2)p^a\big)$$

and

$$\ell_{a+1,1} = \frac{1}{2}\big((n-2)p^k - (n-2d_n)p^{a+1}\big).$$

Thus,

$$\ell_{a,d_n} - \ell_{a+1,1} = \frac{(n-2d_n)p^{a+1}}{2} - \frac{(n-2)p^a}{2} = \frac{p^a}{2}\big((n-2d_n)p - (n-2)\big)$$

$$= \frac{p^a}{2}\big((n - (n - \frac{n-n_0}{p} - \delta))p - n + 2\big) = \frac{p^a}{2}(\delta p - n_0 + 2).$$

We write $L = \ell_{a,d} - \ell_{a+1,1}$. Since $\delta p - n_0 + 2$ is always less than $2p$, we have that $L < p^{a+1}$. By Definition 3.2 we have

$$V_{a+1,1} - V_{a,d_n} = \bigoplus_{i=\ell_{a+1,1}+1}^{\ell_{a+1,1}+L} \lambda(i).$$

Since $p^{a+1} | \ell_{a+1,1}$ and $L < p^{a+1}$ as shown above, we know that $\nu_p(i) < p^{a+1}$ for all $\ell_{a+1,1} + 1 \leqslant i < \ell_{a+1,1} + L$. Also, $\nu_p(\ell_{a+1,1} + L) = \nu_p(\ell_{a,d_n}) = \nu(d_n) + a$. Thus

$$V_{a+1,1} - V_{a,d_n} = U + \lambda_{\nu(d_n)+a},$$

where U consists solely of representations of the form λ_r where $r \leqslant a$.

Thus, the maps from Lemma 3.8 form a tower with fibers that are the slices listed in the statement of the theorem. Moreover, the slices are in the appropriate order by inspection. All that is left to show is that the tower in fact begins at the bottom with the n-slice and ends at the top with $S^n \wedge H\mathbb{Z}$.

When n is not divisible by p, it is clear from the definition of $S^W \wedge H\mathbb{Z}$ and the fact that $H\underline{B}_{i,j}$ is equivalent to $S^{\lambda_r} \wedge H\underline{B}_{i,j}$ whenever $r \leqslant j$ that we have a fiber sequence:

$$S^{V_{(1,1)}} \wedge H\underline{B}_{\nu(1)+1,0} \longrightarrow S^{\widetilde{V}_{(1,1)}-1+\lambda_{\nu(1)+a}} \wedge H\mathbb{Z}$$

$$\downarrow$$

$$S^W \wedge H\mathbb{Z},$$

where $\widetilde{V}_{(1,1)} = V_{(1,1)} - \bigoplus_{i=1}^{\frac{1}{2}(m_1 p - n) - 1} \lambda(i)$. Thus our $(m_1 p^1 - 1)$-slice fits in as the next nontrivial slice after the n-slice. Note that $\dim(\widetilde{V}_{(1,1)} - 1 + \lambda_{\nu(1)+a}) = n$. After this stage, we already know each of our maps fit together. Additionally, each map evenly exchanges λ_i representations, so we have $S^V \wedge H\underline{\mathbb{Z}}$ at each stage with $\dim(V) = n$. Thus, at the top of the tower we have

$$S^{(n-2)\rho_G - 1} \wedge H\underline{B}_{1,k-1} \longrightarrow S^n \wedge H\underline{\mathbb{Z}}$$
$$\downarrow$$
$$S^{n-2+\lambda_{k-1}} \wedge H\underline{\mathbb{Z}},$$

since $V_{(k,d_n)} = m_{d_n} \rho_G - 1$ and we must have that the representation in the $(m_{d_n} p^k - 1)$-stage contains n copies of the trivial representation and be of dimension n.

When n is divisible by p we have an identical argument but begin at the bottom of the tower with

$$S^{V_{(1,2)}} \wedge H\underline{B}_{\nu(2)+1,0} \longrightarrow S^{\widetilde{V}_{(1,2)} - 1 + \lambda_{\nu(2)+a}} \wedge H\underline{\mathbb{Z}}$$
$$\downarrow$$
$$S^{W'} \wedge H\underline{\mathbb{Z}},$$

where $\widetilde{V}_{(1,2)} = V_{(1,2)} - \bigoplus_{i=1}^{\frac{1}{2}(m_1 p - n) - 1} \lambda(i)$. $\square$

Example 4.1. When $G = C_p$ and n is not divisible by p we have the following tower for $S^n \wedge H\underline{\mathbb{Z}}$:

$$S^{n-3} \wedge H\underline{B}_{1,0} \longrightarrow S^n \wedge H\underline{\mathbb{Z}}$$
$$\downarrow$$
$$S^{n-5} \wedge H\underline{B}_{1,0} \longrightarrow S^{(n-2)+\lambda_0} \wedge H\underline{\mathbb{Z}}$$
$$\downarrow$$
$$S^{n-7} \wedge H\underline{B}_{1,0} \longrightarrow S^{(n-4)+2\lambda_0} \wedge H\underline{\mathbb{Z}}$$
$$\vdots$$
$$S^{\lceil \frac{n}{p} \rceil + \epsilon - 1} \wedge H\underline{B}_{1,0} \longrightarrow S^{(n-2d_n+2)+(d_n-1)\lambda_0} \wedge H\underline{\mathbb{Z}}$$
$$\downarrow$$
$$S^{(n-2d_n)+d_n\lambda_0} \wedge H\underline{\mathbb{Z}},$$

where $\lceil \frac{n}{p} \rceil + \epsilon$ has the same parity as n.

Example 4.2. When $G = C_9$ the slice tower for $S^7 \wedge H\underline{\mathbb{Z}}$ is below. We indicate the slice dimension at the side.

$$
\begin{array}{cl}
(5(3)^2 - 1) & S^{5\rho-1} \wedge H\underline{B}_{1,1} \longrightarrow S^7 \wedge H\underline{\mathbb{Z}} \\[2.5em]
(3(3)^2 - 1) & S^{3\rho-1} \wedge H\underline{B}_{1,1} \longrightarrow S^{5+\lambda_1} \wedge H\underline{\mathbb{Z}} \\[2.5em]
(5(3) - 1) & S^{2+\lambda_1} \wedge H\underline{B}_{1,0} \longrightarrow S^{3+2\lambda_1} \wedge H\underline{\mathbb{Z}} \\[2.5em]
(3(3) - 1) & S^{\rho-1} \wedge H\underline{B}_{2,0} \longrightarrow S^{3+\lambda_1+\lambda_0} \wedge H\underline{\mathbb{Z}} \\[2.5em]
& S^{1+\lambda_1+2\lambda_0} \wedge H\underline{\mathbb{Z}}
\end{array}
$$

Example 4.3. When $G = C_9$ the slice tower for $S^{16} \wedge H\underline{\mathbb{Z}}$ is below. We break the tower in half for ease of arrangement.

$$
\begin{array}{ll}
S^{14\rho-1} \wedge H\underline{B}_{1,1} \longrightarrow S^{16} \wedge H\underline{\mathbb{Z}} & \qquad S^{5+4\lambda_1} \wedge H\underline{B}_{1,0} \longrightarrow S^{6+5\lambda_1} \wedge H\underline{\mathbb{Z}} \\[2em]
S^{12\rho-1} \wedge H\underline{B}_{1,1} \longrightarrow S^{14+\lambda_1} \wedge H\underline{\mathbb{Z}} & \qquad S^{4\rho-1} \wedge H\underline{B}_{2,0} \longrightarrow S^{6+4\lambda_1+\lambda_0} \wedge H\underline{\mathbb{Z}} \\[2em]
S^{10\rho-1} \wedge H\underline{B}_{1,1} \longrightarrow S^{12+2\lambda_1} \wedge H\underline{\mathbb{Z}} & \qquad S^{3+3\lambda_1} \wedge H\underline{B}_{1,0} \longrightarrow S^{4+4\lambda_1+2\lambda_0} \wedge H\underline{\mathbb{Z}} \\[2em]
S^{8\rho-1} \wedge H\underline{B}_{1,1} \longrightarrow S^{10+3\lambda_1} \wedge H\underline{\mathbb{Z}} & \qquad S^{3+2\lambda_1} \wedge H\underline{B}_{1,0} \longrightarrow S^{4+3\lambda_1+3\lambda_0} \wedge H\underline{\mathbb{Z}} \\[2em]
S^{6\rho-1} \wedge H\underline{B}_{1,1} \longrightarrow S^{8+4\lambda_1} \wedge H\underline{\mathbb{Z}} & \qquad S^{2\rho-1} \wedge H\underline{B}_{2,0} \longrightarrow S^{4+2\lambda_1+4\lambda_0} \wedge H\underline{\mathbb{Z}} \\[2em]
& \qquad S^{2+2\lambda_1+5\lambda_0} \wedge H\underline{\mathbb{Z}}
\end{array}
$$

In comparing the towers from Examples 4.2 and 4.3 we see a similarity in pattern. The bottom half of the towers consist of alternating slices formed from $\underline{B}_{1,0}$ and $\underline{B}_{2,0}$. Both towers begin with $\underline{B}_{1,0}$ and end with $\underline{B}_{2,0}$. The reason for this is that 16 and 7 differ by the order of our group $|C_9|$. In the tower for $S^{16} \wedge H\underline{\mathbb{Z}}$ the spectra in between form a 3-adic pattern of $\underline{B}_{2,0}$ and $\underline{B}_{1,0}$. If we computed the tower for $S^{25} \wedge H\underline{\mathbb{Z}}$ we would see the same start and end and a longer alternating pattern of

one $\underline{B}_{2,0}$ followed by two $\underline{B}_{1,0}$. This pattern, owing to our result relying on the p-adic orders of the integers m_b, occurs more generally.

5. Verification of slices

In order to prove Theorems 3.4 and 3.6 we require a result about homology.

5.1. Preliminaries

We will need to use precise formulations of restriction and induction of Mackey functors to perform homology computations. The following are given in [5] and are equivalent to Webb's definitions in [9]. In each definition below, G is an abelian group with subgroup H.

Definition 5.1. If $\underline{M}$ is any G-Mackey functor then the restriction of $\underline{M}$ to H, denoted $\downarrow_H^G \underline{M}$, is given by

$$\downarrow_H^G \underline{M}(H/K) = \underline{M}(G/K),$$

for any subgroup K of H.

Definition 5.2. If $\underline{M}$ is an H-Mackey functor the induction of $\underline{M}$ up to G, denoted $\uparrow_H^G \underline{M}$, is given by

$$\uparrow_H^G \underline{M}(G/K) = \underline{M}(i_H^* G/K),$$

for any subgroup K of G where i_H^* is the forgetful functor from G-sets to H-sets.

While the definition of the restricted Mackey functor $\downarrow_H^G \underline{M}$ is obvious for any Mackey functor or group, it is useful to unpack the definition for $\uparrow_H^G \underline{M}$ further. In our work, we only consider G to be a cyclic p-group and thus the following equivalent way of writing $\uparrow_H^G \underline{M}$ will be useful. It follows immediately from the definition of Mackey functors and properties of G-sets.

Proposition 5.3. If G is a cyclic p-group with subgroups H and K and $\underline{M}$ is a G-Mackey functor then

$$\uparrow_H^G \underline{M}(G/K) = \begin{cases} \mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} \underline{M}(H/K) & K \subseteq H, \\ (\mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} \underline{M}(H/H))^K & H \subseteq K. \end{cases}$$

This tells us that for $G = C_{p^k}$ the Mackey functor $\uparrow_H^G \underline{M}$ is the fixed point Mackey functor for the G/H-module $\mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} \underline{M}(G/H)$ on subgroups containing H and is a direct sum of $|G/H|$ copies of the restriction to H on subgroups contained in H.

In performing equivariant computations, we make use of the composition of the restriction and induction operations on Mackey functors.

Proposition 5.4. Let G be a cyclic p-group and H and K be subgroups of G. If $\underline{M}$ is a G-Mackey functor then the composition of restriction followed by induction is given by

$$\uparrow_H^G \downarrow_H^G \underline{M}(G/K) = \begin{cases} \mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} \underline{M}(G/K) & K \subseteq H, \\ (\mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} \underline{M}(G/H))^K & H \subseteq K. \end{cases}$$

We will use these observations to describe the homology of certain representation spheres taken with coefficients in $\underline{B}_{i,j}$. First, we note that if

$$V = m_k + m_{k-1}\lambda_{k-1} + \cdots m_1\lambda_1 + m_0\lambda_0$$

a cellular decomposition of S^V arises as a union of cells of the following form in appropriate dimensions.

Proposition 5.5. *There is a cellular decomposition*

$$S^{\lambda_j} = (C_{p^k}/C_{p^j})_+ \wedge e^1 \underset{1-\gamma}{\cup} (C_{p^k}/C_{p^j})_+ \wedge e^2,$$

where γ is a generator of C_{p^k}.

For details concerning this cellular decomposition, the reader is encouraged to see [**4**, Section 6] or [**5**, Proposition 1.2].

Lemma 5.6. *Let $1 \leqslant j < i \leqslant k$ where $i + j \leqslant k$. If W is a representation of $G = C_{p^k}$ and $C_{p^{i+j}} \subseteq H$ then the restriction map*

$$\underline{H}_{-\epsilon}(S^{-W}; \underline{B}_{i,j})(G/G) \to \underline{H}_{-\epsilon}(S^{-W}; \underline{B}_{i,j})(G/H)$$

is an injection for $\epsilon = 0, 1$.

Proof. If W has at least 2 trivial summands, then the homology in dimensions 0 and -1 is trivial so the result holds. If W contains exactly 1 trivial representation then the homology in dimension 0 is trivial. Furthermore,

$$H_{-1}(S^{-W}; \underline{B}_{i,j}) = H_0(S^{-W+1}; \underline{B}_{i,j}),$$

so it will suffice to consider the case when W has no trivial summands.

Let $G(W)$ denote the maximal proper subgroup of G such that the restriction of W to $G(W)$ has a trivial summand. Then the cellular decomposition of S^{-W} is of the form

$$S^0 \cup (G/G(W))_+ \wedge e^{-1} \underset{1-\gamma}{\cup} (G/G(W))_+ \wedge e^{-2} \cup \cdots,$$

where γ is a generator of G. The associated chain complex is of the form

$$\underline{B}_{i,j} \longrightarrow \uparrow^G_{G(W)}\downarrow^G_{G(W)}\underline{B}_{i,j} \xrightarrow{1-\gamma} \uparrow^G_{G(W)}\downarrow^G_{G(W)}\underline{B}_{i,j} \longrightarrow \cdots,$$

where the first map is the restriction on subgroups containing $G(W)$ and is a composition of restriction maps with the diagonal map on subgroups contained in $G(W)$. From Proposition 5.4 it follows that our complex on G-fixed points is

$$\underline{B}_{i,j}(G/G) \longrightarrow \underline{B}_{i,j}(G/G(W)) \longrightarrow \underline{B}_{i,j}(G/G(W)) .$$

If $C_{p^{i+j}} \subseteq G(W)$, by Definition 2.3 this sequence is exact so the statement is trivially true.

If $G(W)$ does not contain $C_{p^{i+j}}$, then $G(W) \subset H$ since $C_{p^{i+j}} \subseteq H$. Then by Proposition 5.4 the restriction map on homology in dimension 0 is of the form

$$\ker(\underline{B}_{i,j}(G/G) \to \underline{B}_{i,j}(G/G(W)))$$

$$\downarrow$$

$$\ker(\underline{B}_{i,j}(G/H) \to (\mathbb{Z}[G] \otimes_{\mathbb{Z}[G(W)]} \underline{B}_{i,j}(G/G(W)))^H).$$

Since $C_{p^{i+j}} \subseteq H$, we know that we have an isomorphism given by the composition of restriction maps

$$\underline{B}_{i,j}(G/G) \to \underline{B}_{i,j}(G/H).$$

Therefore, the restriction map on $H_0(S^{-W}; \underline{B}_{i,j})$ is always an injection.

Additionally, since $G(W) \subset H$ we know that

$$\underline{\ker(1-\gamma)}(G/G) \cong \underline{\ker(1-\gamma)}(G/H) = \underline{B}_{i,j}(G/G(W)),$$

and since $\underline{B}_{i,j}(G/G)$ is also isomorphic to $\underline{B}_{i,j}(G/H)$, we have the desired injection in dimension -1 as well.   □

5.2. Slice results

To assist in proving Theorems 3.4 and 3.6 we establish a more computationally accessible way to classify n-slices.

Definition 5.7. Let X be a G-spectrum. If $X \in \tau_{\geqslant n}$ we write $X \geqslant n$ and if the localization map $X \to P^n(X)$ is an equivalence we write $X \leqslant n$.

In order to show that a G spectrum X is an n-slice, we must show that $X \leqslant n$ and $X \geqslant n$. We will use the following result to assist in showing $X \geqslant n$. It is equivalent to [2, Corollary 3.9] where we let $Y = H\underline{M}$.

Proposition 5.8. Let W be a representation of G and $\underline{M}$ be a G-Mackey functor such that $S^W \wedge H\underline{M} \geqslant dim(W)$. If V is a subrepresentation of W such that

1. $V^G = W^G$ and

2. for all proper subgroups $H \subset G$, the restriction $i_H^*(S^V \wedge H\underline{M}) \geqslant dim(V)$,

then $S^V \wedge H\underline{M} \geqslant dim(V)$.

We now describe a straightforward way to show that a certain collection of G-spectra are slices.

Theorem 5.9. Let $G = C_{p^k}$. Let V be a G-representation and $\underline{M}$ be a G-Mackey functor. In order to show that the G-spectrum $S^V \wedge H\underline{M}$ is a $dim(V)$-slice we must show that

$$V \subset m\rho_G - \epsilon \qquad and \qquad V^G = (m\rho_G - \epsilon)^G,$$

for some $m > 0$ and

$$[S^{-\epsilon}, S^{V-m\rho_G} \wedge H\underline{M}] = 0,$$

for all $mp^k - \epsilon > \dim(V)$.

Proof. We induct on the subgroups of C_{p^k}. Our underlying spectrum is $S^{\dim(V)} \wedge H\underline{M}$ which is $\geqslant \dim(V)$ by definition. Since there are no nontrivial maps from spectra of the form $C_{p^k_+} \wedge S^m$ to our spectrum whenever $m > \dim(V)$ we know it is $\leqslant \dim(V)$ as well.

By our inductive hypothesis and Theorem 5.8 with $W = m\rho_G - \epsilon$ and our inductive hypothesis, we need only show that $V \subset m\rho_G - \epsilon$ and $V^G = (m\rho_G - \epsilon)^G$ in order to show that $S^V \wedge H\underline{M} \geqslant \dim(V)$.

To show $S^V \wedge H\underline{M} \leqslant \dim(V)$, we must show that

$$[G/H_+ \wedge S^{t\rho_H - \epsilon}, S^V \wedge H\underline{M}] = 0,$$

for all $H \subseteq G$ and $t|H| - \epsilon > \dim(V)$. By our inductive hypothesis and Spanier–Whitehead duality, it will be sufficient to show that

$$[S^{-\epsilon}, S^{V - t\rho_G} \wedge H\underline{M}] = 0,$$

for $tp^k - \epsilon > \dim(V)$. $\qquad\square$

As a first example using the above methodology, we compute the slice tower for $S^n \wedge H\mathbb{Z}$ for small n. We know by [**2**, Corollary 2.16] that $H\mathbb{Z}$ is itself a 0-slice and thus the slice tower is immediately determined to be $H\mathbb{Z} \to H\mathbb{Z} \to 0$. We show below that this is also the case for $S^1 \wedge H\mathbb{Z}$ and $S^2 \wedge H\mathbb{Z}$.

Proposition 5.10. $S^n \wedge H\mathbb{Z}$ *is* n-*slice for* $n = 1, 2$ *when* $G = C_{p^k}$.

Proof. It is clear that $S^n \wedge H\mathbb{Z} \geqslant n$ for all n. By Theorem 5.9, it will suffice to show that

$$[S^{-\epsilon}, S^{n - t\rho_G} \wedge H\mathbb{Z}] = 0,$$

when $n = 1, 2$ and $t > \frac{n + \epsilon}{p^k}$. The C_{p^k}-fixed point part of the chain complex associated to $S^{n - t\rho_G}$ ending in dimension $n - t$ is as follows:

$$\mathbb{Z} \xrightarrow{\ 1\ } \mathbb{Z} \xrightarrow{\ 0\ } \mathbb{Z} \xrightarrow{\ p\ } \mathbb{Z} \xrightarrow{\ 0\ } \cdots.$$

Since $t > \frac{n + \epsilon}{p} > 0$ and t is an integer, $t \geqslant 1$. Thus, when $n = 1$, $n - t \leqslant 0$ and when $n = 2$, $n - t \leqslant 1$. In either case, we can see that the homology in dimensions 0 and -1 are trivial so $S^n \wedge H\mathbb{Z} \leqslant n$ for $n = 1, 2$ on C_{p^k}. $\qquad\square$

We are now in a position to show that the spectra we used in the construction of our slice tower are slices.

Theorem 3.4. *For all* $1 \leqslant a \leqslant k$ *and* $1 \leqslant b \leqslant d_n$, *the* C_{p^k}-*spectrum*

$$S^{V_{(a,b)}} \wedge H\underline{B}_{\nu+1, a-1}$$

is an $(m_b p^a - 1)$-*slice.*

Proof. First, note that if $a = k$, then $V_{(k,b)} = (n - 2 - 2d_n + 2b)\rho_G - 1 = m_b\rho_G - 1$ so we only need to consider $1 \leqslant a \leqslant k - 1$. Recall that

$$V_{(a,b)} = (n-2)\rho_G - 1 - \bigoplus_{i=1}^{\ell} \lambda(i),$$

for $\ell = \frac{1}{2}\big((n-2)p^k - m_b p^a\big)$. Let L be the residue of ℓ modulo p^k. So we may write $\ell = \ell_k p^k + L$. If $L = 0$, then $V_{(a,b)} = (n - 2 - 2\ell_k)\rho_G - 1$ so we are done. Otherwise,

$0 < L < p^k$ and so

$$V_{(a,b)} = (n - 2 - 2\ell_k)\rho_G - 1 - \bigoplus_{i=1}^{L}\lambda(i) \subset (n - 2 - 2\ell_k)\rho_G - 1.$$

Furthermore, $\lambda(i)$ is not trivial for $0 \leqslant i \leqslant L$ and $L < p^k$ so

$$(V_{(a,b)})^G = ((n - 2 - 2\ell_k)\rho_G - 1)^G.$$

We now show the second part of Theorem 5.9. That is, we show

$$[S^{-\epsilon}, S^{V_{(a,b)} - t\rho_G}H\underline{B}_{\nu+1,a-1}] = 0,$$

for $tp^k - \epsilon > \dim(V_{(a,b)})$. Again, $\ell = \ell_k p^k + L$ so we can rewrite $V_{(a,b)} - t\rho_G$ as

$$V_{(a,b)} - t\rho_G = (n - 2 - 2\ell_k - t)\rho_G - 1 - \bigoplus_{i=1}^{L}\lambda(i),$$

and the condition $tp^k - \epsilon > \dim(V_{(a,b)})$ is equivalent to

$$tp^k > (n - 2 - 2\ell_k)p^k - 1 - 2L + \epsilon.$$

We will now show that $V_{(a,b)} - t\rho_G$ has only non-positive cells for such t. We break the proof into two cases depending on the size of L.

Case 1: $L \leqslant \frac{p^k - 1}{2}$. In this case, $tp^k - \epsilon > \dim(V_{(a,b)})$ is equivalent to $tp^k \geqslant (n - 2 - 2\ell_k)p^k - p^k + \epsilon = (n - 3 - 2\ell_k)p^k + \epsilon$ and hence $t \geqslant n - 2 - 2\ell_k$. Thus, $V_{(a,b)} - t\rho_G$ is a negative representation.

Case 2: $L > \frac{p^k - 1}{2}$. Since we also have that $L < p^k$, $tp^k - \epsilon > \dim(V_{(a,b)})$ is equivalent to $tp^k > (n - 2 - 2\ell_k)p^k - 2p^k - 1 + \epsilon = (n - 4 - 2\ell_k)p^k - 1 + \epsilon$ and hence $t \geqslant n - 3 - 2\ell_k$. Now, we may rewrite $V_{(a,b)} - t\rho_G$ as

$$V_{(a,b)} - t\rho_G = (n - 3 - 2\ell_k - t)\rho_G + \rho_G - 1 - \bigoplus_{i=1}^{L}\lambda(i)$$

$$= (n - 3 - 2\ell_k - t)\rho_G - \bigoplus_{i=\frac{p^k+1}{2}}^{L}\lambda(i),$$

and so again, we have that $V_{(a,b)} - t\rho_G$ contains only non-positive cells.

Now, since $V_{(a,b)} - t\rho_G$ contains only non-positive cells, by Lemma 5.6 we have that the restriction

$$H_{-\epsilon}(S^{V_{(a,b)} - t\rho_G}; \underline{B}_{\nu+1,a-1})(G/G) \rightarrow H_{-\epsilon}(S^{V_{(a,b)} - t\rho_G}; \underline{B}_{\nu+1,a-1})(G/C_{p^{\nu+a}})$$

is an injection. So if we can show that the homology on $G/C_{p^{\nu+a}}$ is trivial, we will have the result.

It is a simple exercise to show that for $H = C_{p^{\nu+a}}$

$$i_H^* V_{(a,b)} = \frac{m_b}{p^\nu}\rho_H - 1.$$

Thus, restricting $V_{(a,b)} - t\rho_G$ to $H = C_{p^{\nu+a}}$ yields $(\frac{m_b}{p^\nu} - t[G : H])\rho_H - 1$. Now recall that we are only considering $1 \leqslant a \leqslant k - 1$ as $a = k$ yields an appropriate slice in a

straightforward manner. Thus for $tp^k > \dim(V_{(a,b)}) + \epsilon$ we have

$$H_{-\epsilon}(S^{V_{(a,b)}-t\rho_G}; \underline{B}_{\nu+1,a-1})(G/C_{p^{\nu+a}}) = 0. \qquad \square$$

We will use the following in showing that $S^W \wedge H\underline{\mathbb{Z}}$ and $S^{W'} \wedge H\underline{\mathbb{Z}}$ are slices.

Lemma 5.11. *Let $G = C_{p^k}$. Then*

$$W(n) + \rho_G = W(n + p^k),$$

when n is not a multiple of p and

$$W'(n) + \rho_G = W'(n + p^k),$$

when n is a multiple of p.

Proof. We first show that

$$V_{(a,b)}(n) + \rho_G = V_{(a,b)}(n + p^{k-a+1}).$$

From Definition 3.2 we have that the left hand side is equivalent to

$$V_{(a,b)}(n) + \rho_G = (n-1)\rho_G - 1 - \bigoplus_{i=1}^{\ell} \lambda(i),$$

where $\ell = \frac{1}{2}\big((n-2)p^k - m_b p^a\big)$. Similarly, the right hand side is

$$V_{(a,b)}(n + p^{k-a+1}) = (n + p^{k-a+1} - 2)\rho_G - 1 - \bigoplus_{i=1}^{\ell'} \lambda(i)$$

$$= (n-1)\rho_G - 1 + (p^{k-a+1} - 1)\rho_G - \bigoplus_{i=1}^{\ell'} \lambda(i),$$

where $\ell' = \frac{1}{2}\big((n + p^{k-a+1} - 2)p^k - m_b' p^a\big)$ and m_b' is the number m_b associated to $n + p^{k-a+1}$ rather than n.

Since $2\rho_G = \bigoplus_{i=1}^{p^k} \lambda(i)$ we see that to show the left and right hand sides are equal, we must show that $\ell' = \frac{1}{2}(p^{k-a+1} - 1)p^k + \ell$. A simple computation gives that $m_b' = m_b + p^{k-a}$ and thus

$$\ell' = \frac{1}{2}\big((n + p^{k-a+1} - 2)p^k - (m_b + p^{k-a})p^a\big)$$

$$= \frac{1}{2}\big((n-2)p^k - m_b p^a + (p^{k-a+1})p^k - p^k\big)$$

$$= \ell + \frac{1}{2}(p^{k-a+1} - 1)p^k.$$

Now we can prove the lemma:

$$W(n + p^k) = V_{(1,1)}(n + p^k) + 1 - \bigoplus_{i=1}^{\frac{1}{2}(m_1' p - (n + p^k))} \lambda(i),$$

where m_1' is the number m_1 associated to $n + p^k$. Since $m_1' = m_1 + p^{k-1}$ then $m_1' p - (n + p^k) = m_1 p - n$. Since $V_{(a,b)}(n) + \rho_G = V_{(a,b)}(n + p^{k-a+1})$, the proof for $W(n)$ is complete. $W'(n) + \rho_G = W'(n + p^k)$ is an immediate consequence. $\qquad \square$

Theorem 3.6. $S^W \wedge H\mathbb{Z}$ and $S^{W'} \wedge H\mathbb{Z}$ are both n-slices.

Proof. We first show the statement is true for $S^W \wedge H\mathbb{Z}$. So for now we assume n is not a multiple of p. By Lemma 5.11 we may additionally assume that $2 < n < p^k$. We have

$$W = (n-2)\rho_G - \bigoplus_{i=1}^{\ell} \lambda(i) - \bigoplus_{i=1}^{\frac{1}{2}(m_1 p - n)} \lambda(i),$$

where ℓ can be written as $\ell_k p^k + L$ for L the residue of ℓ modulo p^k. Thus, W may also be written as

$$W = (n - 2 - 2\ell_k)\rho_G - \bigoplus_{i=1}^{L} \lambda(i) - \bigoplus_{i=1}^{\frac{1}{2}(m_1 p - n)} \lambda(i).$$

Since $L < p^k$ and $\frac{1}{2}(m_1 p - n) < p$ then using $\overline{\rho_G}$ to denote the reduced regular representation of G we have

$$\bigoplus_{i=1}^{L} \lambda(i) \subset 2\overline{\rho_G} \quad \text{and} \quad \bigoplus_{i=1}^{\frac{1}{2}(m_1 p - n)} \lambda(i) \subset \overline{\rho_G}.$$

So $W \subset (n - 2 - 2\ell_k)\rho_G$ and $W^G = ((n - 2 - 2l_k)\rho_G)^G$.

Now we must show

$$[S^{-\epsilon}, S^{W - t\rho_G} \wedge H\mathbb{Z}] = 0,$$

when $tp^k - \epsilon > n$. Since $n < p^k$ then when $\epsilon = 0$ we are considering $t \geqslant 1$ and when $\epsilon = 1$, $t \geqslant 2$. We have that

$$W - t\rho_G = (n - 2 - t)\rho_G - \bigoplus_{i=1}^{\ell} \lambda(i) - \bigoplus_{i=1}^{\frac{1}{2}(m_1 p - n)} \lambda(i),$$

where $\ell = \frac{1}{2}((n-2)p^k - m_1 p)$. We break our argument into two cases: n is even and n is odd.

Case 1: If n is even, we can write

$$\ell = \tfrac{1}{2}(n-4)p^k + p^k - \tfrac{1}{2}m_1 p = \tfrac{1}{2}(n-4)p^k + L.$$

Recall that we use m_1 to denote the integer after $\frac{n}{p}$ that is of the same parity as n. So, since n is not a multiple of p, $\frac{n}{p} < m_1 < \frac{n}{p} + 2$. Furthermore, since $n \leqslant p^k - 1$, then $m_1 \leqslant \frac{p^k - 1}{p} + 2$. Simplifying gives us that $m_1 \leqslant p^{k-1} + 1$. Then we see that we can bound $L = p^k - \frac{1}{2}m_1 p$ as follows

$$\frac{p^k - p}{2} < L < p^k - \frac{n}{2}.$$

In particular, $L > 0$ and so we may rewrite W as

$$W = (n - 2 - (n - 4))\rho_G - \bigoplus_{i=1}^{L} \lambda(i) - \bigoplus_{i=1}^{\frac{1}{2}(m_1 p - n)} \lambda(i)$$

$$= 2\rho_G - \left(\bigoplus_{i=1}^{L} \lambda(i) + \bigoplus_{i=1}^{\frac{1}{2}(m_1 p - n)} \lambda(i) \right).$$

By the bounds given for m_1, we know that $\frac{1}{2}(m_1 p - n) < p$. Also, since $S^{\lambda(i)}$ and $S^{\lambda(j)}$ are equivalent whenever i and j have the same p-adic valuation, for our purposes we can rewrite

$$\bigoplus_{i=1}^{\frac{1}{2}(m_1 p - n)} \lambda(i) = \bigoplus_{i=p^k - \frac{1}{2}(m_1 p - n)}^{p^k - 1} \lambda(i).$$

This and the fact that $L = p^k - \frac{1}{2} m_1 p$ give us that

$$\bigoplus_{i=1}^{L} \lambda(i) + \bigoplus_{i=1}^{\frac{1}{2}(m_1 p - n)} \lambda(i) \subset 2\overline{\rho_G}.$$

Furthermore, since $L > \frac{p^k - p}{2}$ we know that

$$\frac{p^k - 1}{2} - L < \frac{p - 1}{2} < p.$$

Thus, since $\overline{\rho_G} = \bigoplus_{i=1}^{\frac{p^k - 1}{2}} \lambda(i)$ and

$$L + \frac{1}{2}(m_1 p - n) = p^k - \frac{n}{2} \geqslant \frac{p^k + 1}{2}$$

we have that

$$\overline{\rho_G} \subset \bigoplus_{i=1}^{L} \lambda(i) + \bigoplus_{i=1}^{\frac{1}{2}(m_1 p - n)} \lambda(i). \tag{1}$$

Now we know we may write

$$W - t\rho_G = (2 - t)\rho_G - \left(\bigoplus_{i=1}^{L} \lambda(i) + \bigoplus_{i=1}^{\frac{1}{2}(m_1 p - n)} \lambda(i) \right),$$

and we are considering $tp^k > n + \epsilon$. Since $n < p^k$, we are really just considering $t \geqslant 1$. If $t > 1$ then clearly $W - t\rho_G$ contains only negative cells. If $t = 1$, $W - t\rho_G$ also contains only negative cells by (1). In either case, considering the chain complex associated to $S^{W - t\rho_G}$ gives the result:

$$0 \xrightarrow{} \mathbb{Z} \xrightarrow{1} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{p} \cdots.$$

$$2 - t \qquad 1 - t \qquad -t$$

So by Theorem 5.9 $S^W \wedge H\underline{\mathbb{Z}}$ is an n-slice when n is even.

Case 2: If n is odd we write $\ell = \frac{1}{2}(n-3)p^k - \frac{1}{2}(p^k - m_1 p)$. Since $n < p^k$, the largest odd m_1 is the next odd integer after $\frac{p^k - 2}{p}$. So $m_1 \leqslant p^{k-1}$ and thus, $m_1 p \leqslant p^k$. Since we also have $m_1 p > 0$, this gives us that $0 \leqslant \frac{1}{2}(p^k - m_1 p) < \frac{p^k}{2}$. So we may rewrite

$$W = (n - 2 - (n-3))\rho_G - \bigoplus_{i=1}^{L} \lambda(i) - \bigoplus_{i=1}^{\frac{1}{2}(m_1 p - n)} \lambda(i)$$

$$= \rho_G - \Big(\bigoplus_{i=1}^{L} \lambda(i) + \bigoplus_{i=1}^{\frac{1}{2}(m_1 p - n)} \Big),$$

where $L = \frac{1}{2}(p^k - m_1 p)$. Since $m_1 < \frac{n}{p} + 2$ and $n \geqslant 3$, we have that

$$L < \frac{p^k - (n + 2p)}{2} \leqslant \frac{p^k - 2p - 3}{2}.$$

Then since $\frac{1}{2}(m_1 p - n) < p$ we have that

$$\bigoplus_{i=1}^{L} \lambda(i) + \bigoplus_{i=1}^{\frac{1}{2}(m_1 p - n)} \lambda(i) \subset \overline{\rho_G}.$$

So $W \subset t\rho_G$ whenever $t \geqslant 1$ completing the proof for $S^W \wedge H\mathbb{Z}$.

We now show that $S^{W'} \wedge H\mathbb{Z}$ is an n-slice and assume n is a multiple of p. By Lemma 5.11 we again need only consider the case when $2 < n \leqslant p^k$.

Recall that

$$W'(n) = V_{(1,2)} + 1 - \bigoplus_{i=1}^{p-1} \lambda(i) - \lambda(1),$$

or equivalently

$$W'(n) = (n-2)\rho_G - \bigoplus_{i=1}^{\ell} \lambda(i) - \bigoplus_{i=1}^{p-1} \lambda(i) - \lambda(1),$$

where now $\ell = \frac{1}{2}\big((n-2)p^k - m_2 p\big)$. Since n is divisible by p, we can write

$$\ell = \frac{1}{2}(n-2)(p^k - p) + \big(\frac{np - n}{2} - 2\big)p = \frac{1}{2}(n-2)p^k - \frac{n}{2} - p.$$

We will again consider the cases where n is even and n is odd separately.

Case 1: If n is even we can write $\ell = \frac{1}{2}(n-4)p^k + p^k - \frac{n+2p}{2}$. This way we have

$$W' = 2\rho_G - \bigoplus_{i=1}^{L} \lambda(i) - \bigoplus_{i=1}^{p-1} \lambda(i) - \lambda(1),$$

where $L = p^k - \frac{n+2p}{2}$. Since n is even and a multiple of p, then $2 < n \leqslant p^k - p$ and we can see that we have the following bound on L:

$$\frac{p^k - p}{2} \leqslant L < p^k - (p+1).$$

By the upper bound, we see that

$$\bigoplus_{i=1}^{L} \lambda(i) + \bigoplus_{i=1}^{p-1} \lambda(i) + \lambda(1) \subset 2\rho_G - 2$$

so $W' \subset 2\rho_G$ and $W'^G = (2\rho_G)^G$.

By the lower bound on L, we see that

$$\overline{\rho_G} \subset \bigoplus_{i=1}^{L} \lambda(i) + \bigoplus_{i=1}^{p-1} \lambda(i).$$

Thus for all $t > \frac{n+\epsilon}{p^k}$, the C_{p^k}-fixed point part of the chain complex associated to $W' - t\rho_G$ is of the form

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\;1\;} \mathbb{Z} \xrightarrow{\;0\;} \mathbb{Z} \xrightarrow{\;p\;} \cdots.$$

$$2 - t \qquad\quad 1 - t \qquad\quad -t$$

Since for $n < p^k$ we are looking at $t \geqslant 1$, it is easy to see from the complex that

$$[S^{-\epsilon}, S^{W'-t\rho_G} \wedge H\underline{\mathbb{Z}}] = 0.$$

Case 2: When n is odd, we have one extreme case: $n = p^k$. So we will deal with this case separately.

Case 2a: $n < p^k$. In this case, we will write $l = \frac{1}{2}(n-3)p^k + \frac{p^k-n-2p}{2}$ so we can see that

$$W' = \rho_G - \bigoplus_{i=1}^{L} \lambda(i) - \bigoplus_{i=1}^{p-1} \lambda(i) - \lambda(1),$$

where $L = \frac{p^k-n-2p}{2}$. Since we have assumed that $n < p^k$, n is odd, and n is divisible by p, we know that $p \leqslant n \leqslant p^k - 2p$. So we get the following bound on L:

$$0 \leqslant L \leqslant \frac{p^k - 3p}{2}.$$

By the upper bound we have that $L + (p-1) + 1 \leqslant \frac{p^k-p}{2}$ so

$$\bigoplus_{i=1}^{L} \lambda(i) + \bigoplus_{i=1}^{p-1} \lambda(i) + \lambda(1) \subset \overline{\rho_G}.$$

Thus, $W' \subset \rho_G$ and $W'^G = (\rho_G)^G$. Also then $W' \subset t\rho_G$ for $t \geqslant 1$ and so

$$[S^{-\epsilon}, S^{W'-t\rho_G} \wedge H\underline{\mathbb{Z}}] = 0.$$

Thus by Theorem 5.9 this case is complete.

Case 2b: When $n = p^k$ we have

$$V_{(1,2)} + 1 - \bigoplus_{i=1}^{p-1} \lambda(i) - \lambda(1) = (n-2)\rho_G - \bigoplus_{i=1}^{\ell} \lambda(i) - \bigoplus_{i=1}^{p-1} \lambda(i) - \lambda(1)$$

$$= (p^k - 2)\rho_G - \bigoplus_{i=1}^{\ell+p-1} \lambda(i) - \lambda(1),$$

since by Definition 3.2

$$\ell = (p^k - 2)\frac{p^k - p}{2} + \big(\frac{p^k - p^{k-1}}{2} - 2\big)p = p^k\big(\frac{p^k - 3}{2}\big) - p.$$

Thus,

$$W'(n) = (p^k - 2 - (p^k - 3))\rho_G + 2 - \lambda(1) = \rho_G + 2 - \lambda(1).$$

Now we also know by [2] that $H\underline{\mathbb{Z}}^*$ is a 0-slice and so $S^{\rho_G} \wedge H\underline{\mathbb{Z}}^*$ is a p^k-slice. Furthermore, $H\underline{\mathbb{Z}}^* = S^{2-\lambda} \wedge H\underline{\mathbb{Z}}$ and so we see that $S^{W'} \wedge H\underline{\mathbb{Z}}$ is also an n-slice in this case. $\square$

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Carolyn Yarnall   carolyn.yarnall@uky.edu

Department of Mathematics, University of Kentucky, Lexington, KY 40506, USA