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ON NONTRIVIALITY OF CERTAIN HOMOTOPY GROUPS OF SPHERES

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(*communicated by Graham Ellis*)

Abstract

We provide an alternative proof of Gray's result that, for an odd prime p, there is a non-trivial $\mathbb{Z}/p$-component in the homotopy group $\pi_{(2p-2)n+1}(S^3)$. As a corollary, it follows that, for $n \geqslant 2$, the homotopy groups $\pi_n(S^2)$ are non-zero.

1. Introduction

In [**5**], Curtis proved that $\pi_n(S^4) \neq 0$, for all $n \geqslant 4$. The main method from [**5**] of proving that a given element of the homotopy groups of spheres is non-zero is the analysis of Adams' d and e-invariants of the stabilization of either that element or its Hopf image. This method allowed Curtis to prove that (see [**5**])

$$\pi_n(S^2) \neq 0, \ n \not\equiv 1 \bmod 8.$$

The same results on non-vanishing terms of the homotopy groups of spheres were obtained with the help of the composition method by Mimura, Mori and Oda [**12**].

Using the methods of the stable homotopy theory, the analysis of the image of the J-homomorphism and K-theory, it was shown by Mahowald [**10, 11**] and Mori [**13**] that

$$\pi_n(S^5) \neq 0, \ n \geqslant 5.$$

On the other hand, since the fourth stable homotopy group of spheres is zero, one cannot get such a result for higher spheres, indeed $\pi_{n+4}(S^n) = 0, \ n \geqslant 6$. The only remaining case to consider when such phenomena can happen is the case of S^2 and S^3. The main result of this paper is the following:

Theorem 1.1. *For $n \geqslant 2$, the homotopy groups $\pi_n(S^2)$ are non-zero.*

Since $\pi_n(S^3) = \pi_n(S^2), \ n \geqslant 3$, the same result follows for the homotopy groups $\geqslant 3$ of the 3-sphere.

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In the proof of Theorem 1.1, we cover the gaps in dimensions $\equiv 1 \bmod 8$ by showing that, for any odd prime p and $n \geqslant 2$,

$$\mathbb{Z}/p \subseteq \pi_{(2p-2)n+1}(S^3).$$

In particular,

$$\mathbb{Z}/3 \subseteq \pi_{4n+1}(S^3), \quad \mathbb{Z}/15 \subseteq \pi_{8n+1}(S^3).$$

Let π_k^n denote the 2-component of $\pi_k(S^n)$. According to [**5**, Table on p. 543], the 2-component $\pi_k^4 \neq 0$ for $k > 4$. Mahowald [**11**, Theorem 1.6] and Mori [**13**, Corollary 5.12(iv)] also proved the stronger statement that $\pi_k^5 \neq 0$ for $k > 5$. For the 2-component π_k^3 of $\pi_*(S^3)$, Curtis proved that $\pi_n^3 \neq 0$, $n \not\equiv 1, 2 \bmod 8$. The non-triviality of these cases can be also read from the fact that the 2-local v_1-periodic homotopy group $v_1^{-1}\pi_n^3 \neq 0$ if and only if $n \not\equiv 1, 2 \bmod 8$ by [**6**, Theorem 4.2]. For the remaining cases of $\pi_n(S^3)$ with $n \equiv 1, 2 \bmod 8$, notice that the 2-components of $\pi_9(S^3)$ and $\pi_{10}(S^3)$ both vanish, and so it is necessary to fulfil odd primes for having the non-triviality. Indicated from [**15**, Figure 3.3.18], one could have the following conjecture:

Conjecture.[1] The 2-component of $\pi_n(S^3)$ is non-trivial for $n > 10$.

*After writing this paper, the authors became aware of the result from the paper [**7**]. The gaps in dimensions* $\equiv 1 \bmod 8$ *have been covered by a result of Gray [**7**, Theorem 12(e)] although the result of Theorem 1.1 is not mentioned in [**7**]. We point out that the method of the present paper for proving Theorem 1.1 is different from that in [**7**].*

2. Lambda-algebra and Toda elements

Recall that, for any $k \geqslant 1$ and an odd prime p, the homotopy groups $\pi_{2(p-1)k+2}(S^3)$ contain non-trivial elements $\alpha_k(3)$ called the *Toda elements*. The elements $\alpha_k(3)$ have non-zero stable images in $\pi^S_{2(p-1)k-1}$. We will use the standard notation

$$\alpha_k(m) = \Sigma^{m-3}(\alpha_k(3)) \in \pi_{2(p-1)k+m-1}(S^m), \; m \geqslant 3.$$

There exists a p-local EHP sequence

$$J_{p-1}(S^4) \longrightarrow \Omega S^5 \xrightarrow{H_p} \Omega S^{4p+1},$$

where $J_{p-1}(S^4)$ is the $(2p-1)$-skeleton of ΩS^5, which implies the long exact sequence of homotopy groups [**17**, (2.11), p. 103]

$$\cdots \longrightarrow \pi_{n+1}(S^{4p+1}) \xrightarrow{P} \pi_{n-1}(J_{p-1}(S^4)) \xrightarrow{E} \pi_n(S^5) \xrightarrow{H_p} \pi_n(S^{4p+1}) \longrightarrow \cdots.$$

The following statement seems to be known. For example, there is a discussion of this result at the end of page 535 in [**2**]. However, we were not able to find an explicit reference to this statement and give here a proof.

[1] During the private circulation of this article, Doug Ravenel wrote a comment that it could be the case that all other 2-components of $\pi_*(S^3)$ are nontrivial except $\pi_9(S^3)$ and $\pi_{10}(S^3)$.

Proposition 2.1. *For $k \geqslant 2$, if the image of the map*

$$H_p\colon \pi_{2k(p-1)+4}(S^5) \to \pi_{2k(p-1)+4}(S^{4p+1})$$

contains the element $\alpha_{k-2}(4p+1)$, then $k \equiv 0 \bmod p$.

Let p be a fixed odd prime number. The mod-p lambda algebra ${}_{[p]}\Lambda = \Lambda$ is an $\mathbb{F}_p$-algebra generated by elements λ_i of degree $2(p-1)i-1$ for $i \geqslant 1$ and elements μ_j of degree $2(p-1)j$ for $j \geqslant 0$. We will use the following notations for $\mathtt{a}(k,j), \mathtt{b}(k,j) \in \mathbb{F}_p$

$$\mathtt{a}(k,j) = (-1)^{j+1}\binom{(p-1)(k-j)-1}{j},$$
$$\mathtt{b}(k,j) = (-1)^{j}\binom{(p-1)(k-j)}{j},$$

and for $N(k), N'(k) \in \mathbb{Z}$:

$$N(k) = \left\lfloor k - \frac{k+1}{p} \right\rfloor, \ N'(k) = \left\lfloor k - \frac{k}{p} \right\rfloor.$$

The ideal of relations in Λ is generated by the following relations:

$$\lambda_i\lambda_{pi+k} = \sum_{j=0}^{N(k)} \mathtt{a}(k,j)\lambda_{i+k-j}\lambda_{pi+j}, \ i \geqslant 1, k \geqslant 0,$$
$$\lambda_i\mu_{pi+k} = \sum_{j=0}^{N(k)} \mathtt{a}(k,j)\lambda_{i+k-j}\mu_{pi+j} + \sum_{j=0}^{N'(k)} \mathtt{b}(k,j)\mu_{i+k-j}\lambda_{pi+j}, \ i \geqslant 1, k \geqslant 0,$$
$$\mu_i\lambda_{pi+k+1} = \sum_{j=0}^{N(k)} \mathtt{a}(k,j)\mu_{i+k-j}\lambda_{pi+j+1}, \ i \geqslant 0, k \geqslant 0,$$
$$\mu_i\mu_{pi+k+1} = \sum_{j=0}^{N(k)} \mathtt{a}(k,j)\mu_{i+k-j}\mu_{pi+j+1}, \ i \geqslant 0, k \geqslant 0.$$

The differential $\partial\colon \Lambda \to \Lambda$ is given by

$$\partial\lambda_k = \sum_{j=1}^{N(k)} \mathtt{a}(k,j)\lambda_{k-j}\lambda_j,$$
$$\partial\mu_k = \sum_{j=0}^{N(k)} \mathtt{a}(k,j)\lambda_{k-j}\mu_j + \sum_{j=1}^{N'(k)} \mathtt{b}(k,j)\mu_{k-j}\lambda_j.$$

Further by ν_i we denote an element of $\{\lambda_i, \mu_i\}$. A monomial $\nu_{i_1}\cdots\nu_{i_l}$ is said to be *admissible* if $i_{k+1} \leqslant pi_k - 1$ whenever $\nu_{i_k} = \lambda_{i_k}$ and if $i_{k+1} \leqslant pi_k$ whenever $\nu_{i_k} = \mu_{i_k}$. The set of admissible monomials is a basis of Λ. The *unstable lambda algebra* $\Lambda(n)$ is a dg-subalgebra of Λ generated by admissible elements $\nu_{i_1}\cdots\nu_{i_l}$ such that $i_1 \leqslant n$. We denote by $\Lambda(n)_m$ the subspace generated by monomials of degree m in $\Lambda(n)$ and by $\Lambda(n)_{m,l}$ the vector space generated by monomials of length l in $\Lambda(n)_m$. Then

$$\Lambda(n) = \bigoplus_{m,l} \Lambda(n)_{m,l}, \qquad \Lambda(n)_m = \bigoplus_{l} \Lambda(n)_{m,l}.$$

Consider the left ideal $\Lambda\lambda = \sum_i \Lambda\lambda_i$ of Λ. The set of all admissible monomials $\nu_{i_1} \cdots \nu_{i_l}$ such that $\nu_{i_l} = \lambda_{i_l}$ forms a basis of $\Lambda\lambda$. Further we put

$$\Lambda\lambda(n) = \Lambda\lambda \cap \Lambda(n).$$

There exists a spectral sequence which converges to the p-primary components of the homotopy groups of spheres, whose E^1-page is the lambda-algebra and d^1-differential is the differential in the lambda algebra:

$$E^1(n) = \Lambda\lambda(n) \Rightarrow {}_{(p)}\pi_*(S^{2n+1}).$$

This is an integral version of the well-known lower central series spectral sequence of six authors [**3**]. This spectral sequence was considered in details in the thesis of Leibowitz [**9**].

In the language of the lambda-algebra, the elements α_k can be presented as (see, for example, [**16**, 2.9]) $\mu_1^{k-1}\lambda_1$. The map $H_p\colon \Omega S^5 \to \Omega S^{4p+1}$ induces a map h_p on the level of E^1-terms of the spectral sequence (see [**16**, p. 23], and also [**19, 8**]) with the short exact sequence

$$0 \to \Lambda(1) \oplus \lambda_2\Lambda(5) \longrightarrow \Lambda(2) \xrightarrow{h_p} \Lambda(2p) \to 0,$$
$$h_p(\mu_1\alpha) = h_p(\lambda_1\alpha) = h_p(\lambda_2\alpha) = 0,$$

for any α and

$$h_p(\mu_2\alpha) = \alpha \in \Lambda(4p+1).$$

Lemma 2.2. *The linear map*

$$d_1 : \mathsf{span}(\mu_1^k\lambda_2, \{\mu_1^{k-i}\mu_2\mu_1^{i-1}\lambda_1\}_{i=1}^k) \longrightarrow \mathsf{span}(\{\mu_1^{k-i}\lambda_1\mu_1^i\lambda_1\}_{i=0}^k)$$

is an isomorphism if and only if $k+2 \not\equiv 0 \pmod p$.

Proof. Using the definition of $d_1 =: d$ we get

$$d(\lambda_1) = 0, \quad d(\mu_1) = -\lambda_1\mu_0, \quad d(\lambda_2) = -2\lambda_1^2, \quad d(\mu_2) = -\lambda_2\mu_0 - 2\lambda_1\mu_1 + \mu_1\lambda_1.$$

Using the relations $\mu_0\mu_1 = 0 = \mu_0\lambda_1$ and $\mu_0\lambda_2 = -\mu_1\lambda_1$, $\mu_0\mu_2 = -\mu_1\mu_1$, it is easy to compute that

$$\begin{aligned}
d(\mu_2\lambda_1) &= -2\lambda_1\mu_1\lambda_1 + \mu_1\lambda_1^2,\\
d(\mu_1\lambda_2) &= \lambda_1\mu_1\lambda_1 - 2\mu_1\lambda_1^2,\\
d(\mu_1\mu_2\lambda_1) &= \lambda_1\mu_1^2\lambda_1 - 2\mu_1\lambda_1\mu_1\lambda_1 + \mu_1^2\lambda_1^2.
\end{aligned}$$

Moreover, we obtain $d(\mu_1)\mu_1 = 0$ and $d(\mu_1)\lambda_1 = 0$. It follows that

$$\begin{aligned}
d(\mu_1^k\lambda_2) &= \mu_1^{k-1}d(\mu_1\lambda_2) = \mu_1^{k-1}\lambda_1\mu_1\lambda_1 - 2\mu_1^k\lambda_1^2,\\
d(\mu_1^{k-1}\mu_2\lambda_1) &= \mu_1^{k-2}d(\mu_1\mu_2\lambda_1) = \mu_1^{k-2}\lambda_1\mu_1^2\lambda_1 - 2\mu_1^{k-1}\lambda_1\mu_1\lambda_1 + \mu_1^k\lambda_1^2,\\
d(\mu_1^{k-i-1}\mu_2\mu_1^i\lambda_1) &= \mu_1^{k-i-2}d(\mu_1)\mu_2\mu_1^i\lambda_1 + \mu_1^{k-i-1}d(\mu_2)\mu_1^i\lambda_1\\
&= \mu_1^{k-i-2}\lambda_1\mu_1^{i+2}\lambda_1 - 2\mu_1^{k-i-1}\lambda_1\mu_1^{i+1}\lambda_1 + \mu_1^{k-i}\lambda_1\mu_1^i\lambda_1,
\end{aligned}$$

for $1 \leqslant i \leqslant k-2$ and

$$d(\mu_2\mu_1^{k-1}\lambda_1) = d(\mu_2)\mu_1^{k-1}\lambda_1 = -2\lambda_1\mu_1^k\lambda_1 + \mu_1\lambda_1\mu_1^{k-1}\lambda_1.$$

If we denote $v_i := \mu_1^{k-i}\lambda_1\mu_1^i\lambda_1$ for $0 \leqslant i \leqslant k$, and $u_i = \mu_1^{k-i}\mu_2\mu_1^{i-1}\lambda_1$ for $1 \leqslant i \leqslant k$ and $u_0 = \mu_1^k\lambda_2$, then

$$d(u_0) = v_1 - 2v_0, \quad d(u_i) = v_{i+1} - 2v_i + v_{i-1}, \quad d(u_k) = -2v_k + v_{k-1}.$$

The matrix corresponding to this linear map is the following matrix:

$$\begin{pmatrix} -2 & 1 & 0 & 0 & \cdots & 0 \\ 1 & -2 & 1 & 0 & \cdots & 0 \\ 0 & 1 & -2 & 1 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 1 & -2 \end{pmatrix}.$$

It is easy to check by induction that its determinant is equal to $(-1)^{k+1}(k+2)$. It follows that $d\colon \mathsf{span}(u_0,\ldots,u_k) \to \mathsf{span}(v_0,\ldots,v_k)$ is an isomorphism if and only if $k+2 \not\equiv 0 \pmod p$. □

Now we are ready to prove Proposition 2.1.

Proof of Proposition 2.1. Indeed, if the Toda element $\alpha_{k-2} = \mu_1^{k-3}\lambda_1$ lies in the H_p image, then there must be some term on E^2-page like $C\mu_2\mu_1^{k-3}\lambda_1 + \sum\cdots$, $C \not\equiv 0 \bmod p$, which maps onto α_{k-2} by h_p. However, by Lemma 2.2, this is possible only in the case $k \equiv 0 \bmod p$, in all other cases the corresponding E^2-term of the spectral sequence for S^5 is zero. □

3. Proof of Theorem 1.1

For the proof of Theorem 1.1, we will use the following classical results.

(1) [**1**] or [**17**, (4.3), p. 112]. The element $\alpha_k \in \pi^S_{2(p-1)k-1}$ is not divisible by p for $k \not\equiv 0 \bmod p$.

(2) Let $p > 2$. By the classical work of Cohen, Moore and Neisendorfer [**4**], there exists a map $\pi\colon \Omega^2 S^{2n+1} \to S^{2n-1}$ such that the composite

$$\Omega^2 S^{2n+1} \xrightarrow{\pi} S^{2n-1} \xrightarrow{\Sigma^2} \Omega^2 S^{2n+1}$$

is homotopic to the p-th power map $p\colon \Omega^2 S^{2n+1} \to \Omega^2 S^{2n+1}$, where the case $p = 3$ is given in [**14**, Theorem 4.1]. Following the notation in [**4**], let $D(n)$ be the homotopy fibre of $\pi\colon \Omega^2 S^{2n+1} \to S^{2n-1}$. According to [**4**, Section 6], $D(p) \simeq \Omega^2 S^3\langle 3\rangle$ and so there is a fibre sequence

$$\Omega S^{2p-1} \xrightarrow{\tau} \Omega^2 S^3\langle 3\rangle \xrightarrow{\theta} \Omega^2 S^{2p+1} \xrightarrow{\pi} S^{2p-1}$$

that implies a long exact sequence

$$\cdots \longrightarrow \pi_{n+1}(S^{2p+1}) \xrightarrow{\pi_*} \pi_{n-1}(S^{2p-1}) \xrightarrow{\tau_*} \pi_n(S^3) \xrightarrow{\theta_*} \pi_n(S^{2p+1}) \longrightarrow \cdots,$$

with the property that, for every i, the composition

$$\pi_{i+2}(S^{2p+1}) \xrightarrow{\pi_*} \pi_i(S^{2p-1}) \xrightarrow{\Sigma^2} \pi_{i+2}(S^{2p+1})$$

is the multiplication by p.

(3) [**17**, (2.12), p. 104]. For $m \geqslant 2$, denote by Q_2^{2m-1}, the homotopy fibre of the double suspension map $S^{2m-1} \to \Omega^2 S^{2m+1}$. We will use the notation from [**17**]. The natural map $Q_2^{2m-1} \to S^{2m-1}$ induces the map on homotopy groups p_*. There is a natural map

$$I\colon \pi_i(Q_2^{2m-1}) \to \pi_{i+3}(S^{2mp+1}),$$

such that the composition

$$\pi_{i+3}(S^{2m+1}) \to \pi_i(Q_2^{2m-1}) \xrightarrow{I} \pi_{i+3}(S^{2mp+1})$$

is the Hopf map H_p.

For a given $k \not\equiv 1 \bmod p$, consider the element

$$\alpha_{k-1} \in \pi_{2(p-1)(k-1)+2p-2}(S^{2p-1}) = \pi_{2(p-1)k}(S^{2p-1}).$$

Suppose that $\alpha_{k-1}(2p-1) \in im\{\pi_*\colon \pi_{2(p-1)k+2}(S^{2p+1}) \to \pi_{2(p-1)k}(S^{2p-1})\}$. Then the element $\Sigma^2\alpha_{k-1}(2p-1) = \alpha_{k-1}(2p+1)$ is p-divisible by (2), hence its stable image is p-divisible. But this is not possible by (1). We conclude that

$$\tau_*(\alpha_{k-1}(2p-1)) \neq 0$$

by the long exact sequence in (2), and so

$$(A) \qquad\qquad \mathbb{Z}/p \subseteq \pi_{2(p-1)k+1}(S^3),\ \ k \not\equiv 1 \bmod p.$$

Now we recall the following statement of Toda ([**17**, Theorem 5.2(ii)], case $m = 1$). For $k \geqslant 2$, there exist an element

$$\gamma' \in \pi_{2p+2k(p-1)-1}(Q_2^3) = \pi_{2(p-1)(k+1)+1}(Q_2^3),$$

such that

$$I(\gamma') = \alpha_{k-1}(4p+1) \in \pi_{4p+2(k-1)(p-1)}(S^{4p+1}).$$

Here $I\colon \pi_{2p+2k(p-1)-1}(Q_2^3) \to \pi_{2p+2k(p-1)+2}(S^{4p+1})$. Suppose that $p_*(\gamma') = 0$, then

$$\gamma' \in im\{H^{(2)}\colon \pi_{2(p-1)(k+1)+4}(S^5) \to \pi_{2(p-1)(k+1)+1}(Q_2^3)\}.$$

In this case, we get

$$\alpha_{k-1}(4p+1) \in im\{H_p\colon \pi_{2(k+1)(p-1)+4}(S^5) \to \pi_{2(k+1)(p-1)+4}(S^{4p+1})\}.$$

This is possible only for $k+1 \equiv 0 \bmod p$ by Proposition 2.1. For $k+1 \not\equiv 0 \bmod p$, we get $0 \neq p_*(\gamma') \in \pi_{2(p-1)(k+1)+1}(S^3)$. Therefore,

$$(B) \qquad\qquad \mathbb{Z}/p \subseteq \pi_{2(p-1)k+1}(S^3),\ \ k \not\equiv 0 \bmod p.$$

The statements (A) and (B) together give the needed statement:

$$\mathbb{Z}/p \subseteq \pi_{2(p-1)k+1}(S^3),\ \ k \geqslant 1.$$

Theorem 1.1 now follows, since all dimensions $\equiv 1 \bmod 8$ are covered, moreover there is a $\mathbb{Z}/15$-summand in homotopy groups $\pi_{8l+1}(S^2),\ l \geqslant 2$. $\square$

As a final remark we observe that homotopy groups of S^2 in certain dimension $\equiv 1 \bmod 8$ can be covered in another way. For that, we recall the results from [**17, 13**].

(4) [**18**, Lemma 15.3(i)]. Let $y \in \pi_i(S^{2p-1})$ be an element of order p. There exists an element $a \in \pi_{i+2}(S^3)$, such that

$$H_p(a) = x\Sigma^2 y \in \pi_{i+2}(S^{2p+1}),$$

for some $x \not\equiv 0 \bmod p$.

(5) For $f \geqslant 0$, there is a family of elements $\alpha_i^{(f)} \in \pi_{2i(p-1)p^f+2f+2}(S^{2f+3})$ of order p^f, which have non-zero stable image in $\pi^S_{2i(p-1)p^f-1}$. The e-invariants of these elements are the following: $e_C(\alpha_i^{(f)}) = -p^{-f-1}$.

(6) [**13**, Lemma 4.1]. Let $f, g \geqslant 0,\ i, j \geqslant 1$ and

$$\alpha\colon S^{2n+2i(p-1)p^f-1} \to S^{2n},$$
$$\beta\colon S^{2n+2i(p-1)p^f+2j(p-1)p^g-2} \to S^{2n+2i(p-1)p^f-1}.$$

Assume that $e_C(\alpha)e_C(\beta) = p^{-u}$ and

$$\nu_p(j) + g + 1 < u \leqslant \nu_p(i) + f + 1 + i(p-1)p^f,$$
$$u + \nu_p(ip^f + jp^g) - \nu_p(i) - f - i(p-1)p^f \leqslant n < u + \nu_p(ip^f + jp^g) - \nu_p(j) - g,$$

then $\alpha \circ \beta$ non-zero.

Now we will show that, for any $k \geqslant 1$, there is a non-zero p-torsion element in $\pi_{2(p-1)(p^pk+1)+1}(S^3)$. For that, consider the case $g = 0, f = p-2, i = p^2k-1, j = p^{p-2}$. By (6), we see that, $\alpha_i^{(p-2)} \circ \alpha_j^{(0)}$ is a non-zero element in the homotopy group $\pi_{2(p-1)p^pk+2p-1}(S^{2p+1})$ which is equal to the image of the double suspension of an element of order p from $\pi_{2(p-1)p^pk+2p-3}(S^{2p-1})$. Hence, by (4), there is an element in $\pi_{2(p-1)p^pk+2p-1}(S^3) = \pi_{2(p-1)(p^pk+1)+1}(S^3)$ whose H_p-image gives a non-zero multiple of this element.

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References

[**1**] J.F. Adams, On the groups $J(X)$. IV. *Topology* **5** (1966), 21–71.

[**2**] M. Bendersky, Unstable towers in the odd primary groups of spheres, *Trans. Amer. Math. Soc.* **287** (1985), 529–542.

[**3**] A.K. Bousfield, E.B. Curtis, D.M. Kan, D.G. Quillen, D.L. Rector, J.W. Schlesinger, The mod-p lower central series and the Adams spectral sequence, *Topology* **5** (l966), 331–342.

[**4**] F.R. Cohen, J.C. Moore, J.A. Neisendorfer, The double suspension and exponents of the homotopy groups of spheres, *Ann. of Math.* **110** (1979), no. **3**, 549–565.

[**5**] E. Curtis, Some nonzero homotopy groups of spheres, *Bull. Amer. Math. Soc. (N.S.)* **75** (1969), 541–544.

[6] D.M. Davis, Computing v_1-periodic homotopy groups of spheres and some compact Lie groups, *Handbook of Algebraic Topology*, 993–1048, North-Holland, Amsterdam, 1995.

[7] B. Gray, Unstable families related to the image of J, *Math. Proc. Cambridge Philos. Soc.* **96** (1984), no. **1**, 95–113.

[8] J. Harper, H. Miller, On the double suspension homomorphism at odd primes, *Trans. Amer. Math. Soc.* **273** (1982), 319–331.

[9] D. Leibowitz, The E^1 term of the lower central series spectral sequence for the homotopy of spaces, PhD thesis, Brandeis University (1972).

[10] M. Mahowald, Description homotopy of the elements in the image of the J-homomorphism, *Manifolds Tokyo*, 255–264, Univ. of Tokyo Press, 1975.

[11] M. Mahowald, The image of J in the EHP sequence. *Ann. of Math. (2)* **116** (1982), no. **1**, 65–112.

[12] M. Mimura, M. Mori, N. Oda, On the homotopy groups of spheres, *Proc. Japan Acad.* **50** (1974), 277–280.

[13] M. Mori, Applications of secondary e-invariants to unstable homotopy theory groups of spheres, *Mem. Fac. Sci., Kyushu Univ. Ser. A* **29** (1975), 59–87.

[14] J.A. Neisendorfer, 3-primary exponents, *Math. Proc. Cambridge Philos. Soc.* **90** (1981), 63–83.

[15] D.G. Ravenel, *Complex Cobordism and Stable Homotopy Groups of Spheres*, 2nd edn., AMS Chelsea Publishing, American Mathematical Society, 2003.

[16] M.C. Tangora, Computing the Homology of the Lambda Algebra, *Mem. Amer. Math. Soc.* **337** (1985), v+163 pp.

[17] H. Toda, On iterated suspensions I, *J. Math. Kyoto Univ.* **5** (1965), 87–142.

[18] H. Toda, On iterated suspensions III, *J. Math. Kyoto Univ.* **8** (1968), 101–130.

[19] R. Thompson, The ν_1-periodic homotopy groups of an unstable sphere at odd primes, *Trans. Amer. Math. Soc.* **319** (1990), 535–559.

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