Homology, Homotopy and Applications, vol. 18(2), 2016, pp.267–281

PD_4-COMPLEXES: CONSTRUCTIONS, COBORDISMS AND SIGNATURES

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(*communicated by Maria Cristina Pedicchio*)

Abstract

The oriented topological cobordism group $\Omega_4(P)$ of an oriented PD_4-complex P is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$. The invariants of an element $\{f\colon X \to P\} \in \Omega_4(P)$ are the signature of X and the degree of f. We prove an analogous result for the Poincaré duality cobordism group $\Omega_4^{\mathrm{PD}}(P)$: If $\pi_1(P)$ does not contain nontrivial elements of order 2, then $\Omega_4^{\mathrm{PD}}(P)$ is isomorphic to $L^0(\Lambda) \oplus \mathbb{Z}$, where $L^0(\Lambda)$ is the Witt group of nondegenerated hermitian forms on finitely generated stably free Λ-modules. The component of an element $\{f\colon X \to P\} \in \Omega_4^{\mathrm{PD}}(P)$ in $L^0(\Lambda)$ is related to the symmetric signature of X. Then we construct explicitly PD_4-complexes, define the well-known map $L_4(\pi_1(P)) \to \Omega_4^{\mathrm{PD}}(P)$, and characterize the image of the map $\Omega_4^{\mathrm{PD}}(P) \to \Omega_4^N(P)$. The results are summarized in Theorems 1.1 and 1.2 stated in the introduction.

Prolog: background and motivation

We want to explain how the results fit in with the prior knowledge and the contributions in this area. The paper is occupied with 4-dimensional topology. An outstanding problem in 4-dimensional surgery theory is the following: *Given a degree-*1 *normal map $M \to P$, M a topological* 4*-manifold, can it be transformed by surgeries to a homotopy equivalence, provided an obstruction in the Wall group $L_4(\pi_1(P))$ vanishes*? The analogous problem is solved if the dimension of M is equal to or greater than 5. It is expressed in the exact surgery sequence

$$\mathcal{S}(P) \to [P, G/TOP] \to L_n(\pi_1(P)), \qquad (/)$$

where $\mathcal{S}(P)$ is the so-called structure set of P. It consists of equivalence classes of homotopy equivalences $N \to P$, where N is a topological n-manifold and $\dim M = n \geqslant 5$. Here P has to be a CW-complex satisfying n-Poincaré duality, called shortly a PD_n-*complex*. If there is at least one degree-1 normal map $M \to P$, then the homotopy classes $[P, G/TOP]$ classify all degree-1 normal maps up to normal cobordism. Ranicki

Received October 19, 2015, revised February 25, 2016; published on November 2, 2016.
2010 Mathematics Subject Classification: 57N65, 57R67, 57Q10.
Key words and phrases: Poincaré duality complex, signature, cobordism group, surgery sequence, Witt group, homotopy type, Whitehead quadratic group, spectral sequence, obstruction theory, homology with local coefficients, total surgery obstruction.
Article available at `http://dx.doi.org/10.4310/HHA.2016.v18.n2.a15`

[**16**] associates to a PD_n-complex P a total surgery obstruction $s(P)$ which belongs to an algebraically defined structure set $\mathcal{S}_n(P)$. A weaker problem as above is the following: *Given a* PD_4*-complex* P *with* $s(P) = 0$*, is* P *then cobordant to a topological* 4*-manifold*? We show that this is true and establish an exact sequence

$$(//) \qquad 0 \longrightarrow \Omega_4^{TOP}(P) \longrightarrow \Omega_4^{PD}(P) \xrightarrow{\ s\ } \mathcal{S}_4(P) \longrightarrow 0.$$

The decorations TOP resp. PD indicate topological resp. Poincaré duality cobordism groups. So, in particular, if $s(P) = 0$, then P is cobordant (in the PD sense) to a topological manifold. The sequence (/) can be extended on the right as $L_n(\pi_1(P)) \to \mathcal{S}_n(P)$ by the well-known Wall realization of surgery obstructions. However, this works only if $\dim M = n \geqslant 5$ for technical reasons and also because one needs $(n-1)$-manifolds with fundamental group isomorphic to $\pi_1(P)$. But $\pi_1(P)$ is not necessarily a 3-manifold fundamental group. We address this problem and define a map $L_4(\pi_1(P)) \to \Omega_4^{TOP}(P)$ by using a modified Wall realization. Moreover, with this map we establish the Levitt–Jones–Quinn exact sequence in the appropriate dimension. It also provides us with a tool to calculate $\Omega_4^{TOP}(P)$. For example, $\Omega_4^{TOP}(P) = \mathbb{Z}^{(p+3)/2}$ if $\pi_1(P) = \mathbb{Z}_p$, p an odd prime. First examples of such PD_4's were discovered by Wall [**19**]. Another result of the paper, which is substantial for calculations, is the isomorphism

$$\Omega_4^{TOP}(P) \cong L^0(\Lambda) \oplus \mathbb{Z}$$

provided $\pi_1(P)$ does not contain elements of order 2. Here $L^0(\Lambda)$ is the Witt group of non-degenerated hermitian forms on the group ring $\Lambda = \mathbb{Z}[\pi_1(P)]$. This generalizes the classical result $\Omega_4^{TOP}(P) \cong \mathbb{Z} \oplus \mathbb{Z}$, where an element $\{f\colon M \to P\}$ is determined by the pair (signature M, deg f).

1. Introduction

Let P^4 be an oriented closed Poincaré duality complex of dimension 4 in the sense of [**20**], shortly a PD_4-complex. The oriented topological cobordism group $\Omega_4(P)$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$. An element $\{f\colon M \to P\} \in \Omega_4(P)$ is determined by the signature of M and the degree of f. Our main result is an analogous characterization for the PD-cobordism group $\Omega_4^{\mathrm{PD}}(P)$.

Theorem 1.1. *Let* P *be an oriented* PD_4*-complex. Then*

(1) *There is a morphism* $\Omega_4^{\mathrm{PD}}(P) \to L^0(\Lambda) \oplus \mathbb{Z}$*;*

(2) *If* $\pi_1(P)$ *does not contain elements* $g \neq 1$ *of order* 2*, then the morphism in* (1) *is an isomorphism.*

Here $\Lambda = \mathbb{Z}[\pi_1(P)]$ *and* $L^0(\Lambda)$ *is the Witt group of non-degenerated hermitian forms on finitely generated stably free* Λ*-modules.*

The analogous Witt group $L^0_{pr}(\Lambda)$ based on finitely generated projective Λ-modules is the zero-th symmetric L-group [**17**]. Under composition with the periodicity map $L^0_{pr}(\Lambda) \to L^4_{pr}(\Lambda)$, which in general is not an isomorphism, the element

$$\{f\colon X \to P\} \in \Omega_4^{\mathrm{PD}}(P)$$

corresponds to the pair

$$(\sigma^*(X) - \sigma^*(P),\, \deg f) \in L^4_{pr}(\Lambda) \oplus \mathbb{Z},$$

where $\sigma^*(X)$ and $\sigma^*(P)$ denote the symmetric signatures of X and P, respectively. The proof of Theorem 1.1 is given in Section 2. Section 3 is concerned with the Poincaré surgery sequence of Levitt, Jones and Quinn:

$$L_4(\pi_1(P)) \to \Omega_4^{\mathrm{PD}}(P) \to \Omega_4^N(P),$$

where $\Omega_4^N(P)$ is the cobordism group of normal spaces (see [**11, 12, 15**]). We study the map $\Omega_4^{\mathrm{PD}}(P) \to \Omega_4^N(P)$ in Section 3 (II). In part (III) of Section 3 we construct PD_4-complexes X_G and degree 1 maps $\Phi_G\colon X_G \to X$ for any non-singular quadratic form (G, λ, μ), where G is stably Λ-free. The triple (G, λ, μ) represents an element $\{G, \lambda, \mu\} \in L_4(\pi_1(X))$. The construction of X_G uses a modified Wall realization construction on a certain 3-manifold. The usual construction as described in [**18**, p. 203], is not applicable since $\pi_1(P)$ is not necessarily the fundamental group of a 3-manifold. The results of Section 3 are summarized in the following

Theorem 1.2. *Let P be an oriented* PD_4*-complex. Then*

(1) There is an action

$$L_4(\pi_1(P)) \times \Omega_4^{\mathrm{PD}}(P) \to \Omega_4^{\mathrm{PD}}(P)$$

given by

$$(\{G, \lambda, \mu\}, \{f\colon X \to P\}) \to \{f \circ \Phi_G\colon X_G \to P\}.$$

(2) If $L_4(\pi_1(P)) \to \Omega_4^{\mathrm{PD}}(P)$ is defined by

$$\{G, \lambda, \mu\} \to \{\Phi_G\colon P_G \to P\} - \{\mathrm{Id}\colon P \to P\},$$

then the sequence

$$L_4(\pi_1(P)) \to \Omega_4^{\mathrm{PD}}(P) \to \Omega_4^N(P)$$

is exact.

(3) The image of $\{f\colon X \to P\} \in \Omega_4^{\mathrm{PD}}(P)$ in $\Omega_4^N(P)$ is essentially the obstruction $\omega_X \in H^3(X, \pi_2(G/TOP))$ to a TOP-reduction of the Spivak normal fibration ν_X. "Essentially" means that besides ω_X there is the degree of f and the ordinary signature of X mod 8 *appearing in the image of the element $\{f\colon X \to P\}$ in $\Omega_4^N(P)$.*

We recall some properties and constructions. Basic in all proofs are the following two facts proved in [**19**].

Let X be a PD_4-complex. Then

(I) X is homotopy equivalent to $K \cup_\varphi D^4$, where K is a 3-complex and $\varphi\colon \mathbb{S}^3 \to K$ is the attaching map of the (unique) 4-cell (called *disc property*);

(II) X is homotopy equivalent to $Z \cup_{\partial H} H$, where Z is dominated by a 2-complex and $H = \#_1^r \mathbb{S}^1 \times D^3$, hence $\partial H = \#_1^r \mathbb{S}^1 \times \mathbb{S}^2$. Moreover, the homomorphism $\pi_1(\partial H) \to \pi_1(Z) \cong \pi_1(X)$ is surjective (manifold structure of the 1-skeleton). There is also a relative version of it (see [**20**, remark after 2.8]).

If $\{f\colon X \to P\} \in \Omega_4^{\mathrm{PD}}(P)$ is given, we can write it as a sum

$$\{f'\colon X' \to P\} - (d-1)\{Id\colon P \to P\},$$

where $\deg f = d$ and $\deg f' = 1$ (obvious but this is described at the end of Section 2). Elements $\{f\colon X \to P\} \in \Omega_4^{\mathrm{PD}}(P)$ with $\deg f = 1$ are important for the following reasons. First, $f_*\colon \pi_1(X) \to \pi_1(P)$ is surjective [**1**]. Then one uses the manifold structure of the 2-skeleton of X to obtain by surgeries a cobordant 2-connected map $g\colon Y \to P$. Hence, given $\{f\colon X \to P\} \in \Omega_4^{\mathrm{PD}}(P)$ with $\deg f = 1$, we can assume that f is 2-connected. Then $H_2(X,\Lambda) \to H_2(P,\Lambda)$ is surjective with kernel a stably Λ-free module $K_2(f,\Lambda)$. Moreover, the intersection form $\lambda_X\colon H_2(X,\Lambda) \times H_2(X,\Lambda) \to \Lambda$ is non-degenerated when restricted to $K_2(f,\Lambda)$, denoted by λ_f [**20**]. We obtain the class $\{K_2(f,\Lambda),\lambda_f\} \in L^0(\Lambda)$. Then the map $\Omega_4^{\mathrm{PD}}(P) \to L^0(\Lambda) \oplus \mathbb{Z}$ is defined by sending $\{f\colon X \to P\}$ to $(\{K_2(f',\Lambda),\lambda_{f'}\},\deg f)$ using the above decomposition. The disc property is used to form connected sums as $P\#(\mathbb{S}^2 \times \mathbb{S}^2)$ or, for example, $X\#P$.

We recall now the semi-simplicially defined spectra $\underline{\Omega}^{\mathrm{PD}}$ and $\underline{\Omega}^{N}$ (also denoted MSG) of adic PD-complexes and adic normal complexes, respectively (see [**18**], for details). Transversality of normal complexes implies $\Omega_*^N(Y) = H_*(Y, MSG)$ for any complex Y. Moreover, MSG is a product of Eilenberg–Mc Lane spectra. In the PD-case, the assembly map $H_*(Y,\underline{\Omega}^{\mathrm{PD}}) \to \Omega_*^{\mathrm{PD}}(Y)$ is not an isomorphism. This map fits into a long exact sequence

$$\cdots \longrightarrow \mathcal{S}_{n+1}(Y) \longrightarrow H_n(Y,\underline{\Omega}^{\mathrm{PD}}) \longrightarrow \Omega_n^{\mathrm{PD}}(Y) \xrightarrow{\ s\ } \mathcal{S}_n(Y) \longrightarrow \cdots,$$

where $\mathcal{S}_*(Y)$ are structure sets (see [**18**]) and $s\left(\{f\colon X \to Y\}\right) = f_*(s(X))$ with $s(X) \in \mathcal{S}_n(X)$ the total surgery obstruction of the PD_n-complex X.

For other relevant papers on the homotopy classification of PD_4-complexes and the s-cobordism classification of 4-manifolds through the group of homotopy self-equivalences see [**6, 7, 9, 10**].

2. Proof of Theorem 1.1

We fix a PD_4-complex P. Let $L^0(\Lambda)$ be the Witt group of $\Lambda = \mathbb{Z}[\pi_1(P)]$ based on finitely generated stably free Λ-modules. There is an obvious map $L^0(\Lambda) \to L^0_{pr}(\Lambda)$ into the Witt group based on finitely generated projective Λ-modules (see [**17**, I, Chp. 9], for intermediate L-groups). Recall that two non-singular forms $\varphi\colon M \times M \to \Lambda$ and $\varphi'\colon M' \times M' \to \Lambda$ define the same element in $L^0(\Lambda)$ if there are metabolic forms

$$(N,\Theta) = \left(R^* \oplus R, \begin{pmatrix} 0 & 1 \\ 1 & \theta \end{pmatrix}\right), \qquad (N',\Theta') = \left(R'^* \oplus R', \begin{pmatrix} 0 & 1 \\ 1 & \theta' \end{pmatrix}\right),$$

such that

$$(M,\varphi) \oplus (N,\Theta) \cong (M',\varphi') \oplus (N',\Theta').$$

Notation. If there is no confusion, in the above matrices, $1 = \mathbf{I}_m$ the $m \times m$ identity matrix, $0 = \mathbf{0}_m$ the $m \times m$ zero matrix, and θ is an $m \times m$ matrix with entries in Λ (here $m = \operatorname{rank} R$).

The element $\{M,\varphi\} \in L^0(\Lambda)$ is trivial if (M,φ) admits a Lagrangian $H \subset M$, that is, $\varphi(H \times H) = 0$ and the morphism $\bar{\varphi}\colon M/H \to \mathrm{Hom}_\Lambda(H,\Lambda) = H^*$, induced from the adjoint of φ, is an isomorphism. To prove Theorem 1 we have to show that if $f\colon X \to P$ and $g\colon Y \to P$ are cobordant 2-connected degree 1 maps, then

$$\{K_2(f,\Lambda),\lambda_f\} = \{K_2(g,\Lambda),\lambda_g\} \in L^0(\Lambda).$$

Let $F\colon (W,\partial W) \to (P\times I, \partial(P\times I))$ be a PD-cobordism which can be assumed to be 2-connected. Observe that F is of degree 1. One can write down the following diagram:

$$\begin{array}{ccccccccccc}
0 \to & K_3(W) & \to & K_3(W,\partial W) & \to & K_2(\partial W) & \to & K_2(W) & \to & K_2(W,\partial W) & \to 0\\
 & \downarrow\cong & & \downarrow\cong & & \downarrow\cong & & \downarrow\cong & & \downarrow\cong & \\
0 \to & K^2(W,\partial W) & \to & K^2(W) & \to & K^2(\partial W) & \to & K^3(W,\partial W) & \to & K^3(W) & \to 0\\
 & \downarrow\cong & & \downarrow\cong & & \downarrow\cong & & \downarrow\cong & & \downarrow\cong & \\
0 \to & (K_2(W,\partial W))^* & \to & (K_2(W))^* & \to & (K_2(\partial W))^* & \to & (K_3(W,\partial W))^* & \to & (K_3(W))^* & \to 0
\end{array}$$

where $K_*(\ )$ and $K^*(\ )$ denote various kernels and cokernels of F_* and F^* (see [**20**]). All Λ-modules in the diagram turn out to be stably Λ-free and

$$H = \operatorname{Im}(K_3(W,\partial W) \to K_2(\partial W)) \subset K_2(\partial W) = K_2(f,\Lambda) \oplus K_2(g,\Lambda)$$

is a Lagrangian, that is,

$$\{(K_2(f,\Lambda),\lambda_f) \oplus (-K_2(g,\Lambda),\lambda_g)\} = 0 \in L^0(\Lambda).$$

Therefore, the correspondence $\{f\colon X\to P\} \to \{K_2(f',\Lambda),\lambda_{f'}\}$ is well-defined. We shall now prove surjectivity. Let $\{G,\lambda\} \in L^0(\Lambda)$ be given. We assume first that $G \cong \oplus_1^r\Lambda$, that is, G is Λ-free. Our hypothesis implies that the Tate cohomology group

$$\hat{H}^1(\mathbb{Z}_2, G, -1) = \{a \in G\colon a = \bar{a}\}/\{b+\bar{b}\colon b\in G\} \cong \hat{H}^1(\mathbb{Z}_2, \oplus_1^r\mathbb{Z}, -1) \cong \oplus_1^r\mathbb{Z}_2$$

(see [**17**, I, p. 168]). Let $e_1, \ldots, e_r \in G$ be a base, and set $\lambda_{ij} = \lambda(e_i,e_j)$, where $\lambda_{ii} = \bar{\lambda}_{ii}$. Therefore, it can be written as $\lambda_{ii} = b_i + \bar{b}_i + \delta_i$, with $\delta_i \in \mathbb{Z}_2 = \{0,1\}$. We consider $K' = K \vee \{\vee_1^r \mathbb{S}^2\}$, where $P = K \cup_\varphi D^4$ (disc property of P). Let

$$i_j\colon \mathbb{S}^2 \to K \vee \{\vee_1^r\mathbb{S}^2\}$$

be the jth inclusion, and consider

$$\sum_{1\leqslant i<j\leqslant r} [\lambda_{ij}\, i_i, i_j] + \sum_{j=1}^{r} ([b_j\, i_j, i_j] + \delta_j\, \eta_j) \in \Gamma(\pi_2(K')) \subset \pi_3(K').$$

Here $\Gamma(\pi_2)$ is the Whitehead quadratic group of $\pi_2(P)$, $[\,,\,]$ denotes the Whitehead bracket, and $\eta_j\colon \mathbb{S}^3 \to \mathbb{S}^2 \xrightarrow{i_j} K'$ is the Hopf map composed with i_j. If $\varphi_0\colon \mathbb{S}^3 \to K'$ represents this element, then $X = K' \cup_{\varphi+\varphi_0} D^4$ is a PD_4-complex. There is a 2-connected map $f\colon X\to P$ of degree 1 such that $(K_2(f,\Lambda),\lambda_f) \cong (G,\lambda)$ (see [**9**]). In general, G is stably Λ-free. We can stabilize it with

$$\left(\Lambda^s \oplus \Lambda^s, \begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix}\right),$$

for s sufficiently large. Then we have

$$\left\{G\oplus\Lambda^s\oplus\Lambda^s, \lambda \oplus \begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix}\right\} = \{G,\lambda\} \in L^0(\Lambda).$$

By above we obtain a 2-connected degree 1 map $g\colon Y \to P$ with

$$(K_2(g,\Lambda),\lambda_g) \cong \left(G \oplus \Lambda^s \oplus \Lambda^s, \lambda \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right).$$

Along the generators $\{e_i, f_j\}$ of $\Lambda^s \oplus \Lambda^s \subset K_2(g,\Lambda) \subset \pi_2(Y)$, we attach 3-cells obtaining $X = Y \cup \{\cup_{i,j} D^3 \cup D^3\}$. Now X is a PD_4-complex, and there is a 2-connected degree 1 map $q\colon Y \to X$ with

$$(K_2(q,\Lambda),\lambda_q) \cong \left(\Lambda^s \oplus \Lambda^s, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right).$$

Moreover, $g\colon Y \to P$ factorizes over a 2-connected degree 1 map $f\colon X \to P$, that is, $f \circ q \sim g$ (see [**8**]). Hence $\{f\colon X \to P\} \in \Omega_4^{\mathrm{PD}}(P)$ maps to $\{G,\lambda\} \oplus 1 \in L^0(\Lambda) \oplus \mathbb{Z}$. Forming connected sums $X \# \#_1^d(\pm P) \to P$, we obtain all elements of $L^0(\Lambda) \oplus \mathbb{Z}$.

Proof of injectivity. Let $\{f\colon X \to P\} \in \Omega_4^{\mathrm{PD}}(P)$ be given such that it maps to the pair $(0,1) \in L^0(\Lambda) \oplus \mathbb{Z}$.

Claim. We obtain $\{f\colon X \to P\} = \{Id\colon P \to P\} \in \Omega_4^{\mathrm{PD}}(P)$. Injectivity will follow from this claim. After taking connected sums with $\mathbb{S}^2 \times \mathbb{S}^2$, we can assume that $K_2(f,\Lambda)$ is Λ-free. Now, since $\{K_2(f,\Lambda),\lambda_f\} = 0 \in L^0(\Lambda)$, we obtain isomorphisms

$$(K_2(f,\Lambda),\lambda_f) \oplus \left(\Lambda^k \oplus \Lambda^k, \begin{pmatrix} 0 & 1 \\ 1 & \theta \end{pmatrix}\right) \cong \left(\Lambda^m \oplus \Lambda^m, \begin{pmatrix} 0 & 1 \\ 1 & \omega \end{pmatrix}\right).$$

First step. By our hypothesis we have $\theta_{ii} = b_i + \bar{b}_i + \delta_i$ as above, so we can construct a PD_4-complex Y from $K \vee \{\vee_1^k(\mathbb{S}^2 \vee \mathbb{S}^2)\}$ with attaching map $\varphi + \varphi_0$, where φ_0 is determined by $\begin{pmatrix} 0 & 1 \\ 1 & \theta \end{pmatrix}$, and a 2-connected degree 1 map $g\colon Y \to P$ with

$$(K_2(g,\Lambda),\lambda_g) \cong (K_2(f,\Lambda),\lambda_f) \oplus \left(\Lambda^k \oplus \Lambda^k, \begin{pmatrix} 0 & 1 \\ 1 & \theta \end{pmatrix}\right) \cong \left(\Lambda^m \oplus \Lambda^m, \begin{pmatrix} 0 & 1 \\ 1 & \omega \end{pmatrix}\right).$$

Hence Y can also be obtained from $K \vee \{\vee_1^m(\mathbb{S}^2 \vee \mathbb{S}^2)\}$ by attaching a 4-cell with $\varphi + \varphi_1$, where φ_1 is determined by $\begin{pmatrix} 0 & 1 \\ 1 & \omega \end{pmatrix}$. Here it follows that $\omega_{ii} = c_i + \bar{c}_i + \delta_i$, for $i = 1, \dots, m$, as a consequence of our hypothesis. Let $\{e_1, \dots, e_m, f_1, \dots, f_m\} \subset K_2(Y,g)$ be the base according to

$$\left(\Lambda^m \oplus \Lambda^m, \begin{pmatrix} 0 & 1 \\ 1 & \omega \end{pmatrix}\right),$$

that is, $\lambda_g(e_i, f_j) = \delta_{ij}$, $\lambda_g(f_r, f_s) = \omega_{rs}$, and λ_g is zero otherwise. If we perform the transformation of the base $\{e_1, \dots, e_m, f_1, \dots, f_m\}$ given by $e_i \to e_i$, for $i = 1, \dots, m$, and $f_r \to f'_r = f_r - \sum_{s<r} \omega_{rs}\, e_s - c_r\, e_r$, then the intersection form with respect to $\{e_1, \dots, e_m, f'_1, \dots, f'_m\}$ is defined by the $(2m) \times (2m)$ matrix $\begin{pmatrix} \mathbf{0}_m & \mathbf{I}_m \\ \mathbf{I}_m & \mathbf{\Delta}_m \end{pmatrix}$, where $\mathbf{0}_m$ is the $m \times m$ null matrix, $\mathbf{I}_m$ is the $m \times m$ identity matrix, and $\mathbf{\Delta}_m = \mathrm{diag}(\delta_1, \dots, \delta_m)$. Since this Λ-intersection form is extended from the corresponding $\mathbb{Z}$-intersection form, Y is homotopy equivalent to a connected sum (see [**8**]). More

precisely, since

$$\begin{pmatrix} \mathbf{0}_m & \mathbf{I}_m \\ \mathbf{I}_m & \mathbf{\Delta}_m \end{pmatrix} \cong \begin{pmatrix} 0 & 1 \\ 1 & \delta_1 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} 0 & 1 \\ 1 & \delta_m \end{pmatrix}$$

we get $P\#(\#_{i=1}^m Q_i)$, where $Q_i = \mathbb{S}^2 \times \mathbb{S}^2$ if $\delta_i = 0$, and $Q_i = \mathbb{C}P^2\#(-\mathbb{C}P^2)$ if $\delta_i = 1$. Note that $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \cong \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Now $\#_{i=1}^m Q_i$ bounds obviously a 5-manifold V. Connected sum with $P \times I$ along $P \times 1$ gives a PD-cobordism W between P and $P\#(\#_{i=1}^m Q_i)$. But the last one is homotopy equivalent to Y over P. So we have proved that $\{g\colon Y \to P\} = \{Id\colon P \to P\} \in \Omega_4^{\mathrm{PD}}(P)$.

Second step. It remains to prove that

$$\{g\colon Y \to P\} = \{f\colon X \to P\} \in \Omega_4^{\mathrm{PD}}(P).$$

We proceed as in the first step with the form

$$\left(\Lambda^k \oplus \Lambda^k, \begin{pmatrix} 0 & 1 \\ 1 & \theta \end{pmatrix}\right)$$

and X instead of P. One obtains Y homotopy equivalent to $X\#(\#_{i=1}^k Q_i)$, where Q_i is as above according to $\delta_i \in \{0, 1\}$. This proves the above claim. To prove injectivity of $\Omega_4^{\mathrm{PD}}(P) \to L^0(\Lambda) \oplus \mathbb{Z}$, which sends $\{f\colon X \to P\}$ to $(\{K_2(f', \Lambda), \lambda_{f'}\}, \deg f)$, we recall that

$$f'\colon X\#(\#_1^{|d-1|} P) \to X \vee (\vee_1^{|d-1|} P) \to \vee_1^d P \to P$$

is cobordant to $\{f\colon X \to P\} - (d-1)\{Id\colon P \to P\}$. If $d - 1 < 0$, it means that $X\#(\#_1^{|d-1|} (-P))$. If $\{f\colon X \to P\}$ maps to $(0, 0) \in L^0(\Lambda) \oplus \mathbb{Z}$, then we get $f'\colon X' = X\#P \to P$, that is,

$$\{f'\colon X' \to P\} = \{f\colon X \to P\} + \{Id\colon P \to P\}.$$

From the above claim we have $\{f'\colon X' \to P\} = \{Id\colon P \to P\}$, hence $\{f\colon X \to P\} = 0 \in \Omega_4^{\mathrm{PD}}(P)$. This proves Theorem 1.1 (2). □

3. The geometric Poincaré surgery sequence of Levitt, Jones and Quinn revisited

(I) The map $\Omega_4^{\mathrm{PD}}(P) \to \Omega_4^N(P)$

We fix again an oriented PD$_4$-complex P with $\pi_1 = \pi_1(P)$. For $n \geqslant 4$, there is an exact sequence

$$(3.1) \qquad \cdots \to \Omega_{n+1}^N(P) \to L_n(\pi_1) \to \Omega_n^{\mathrm{PD}}(P) \to \Omega_n^N(P) \to L_{n-1}(\pi_1) \to \cdots$$

(see [**11, 12, 15**]). Here $\Omega_n^N(P) = H_n(P, MSG)$ is the cobordism group of normal spaces X and reference maps $f\colon X \to P$. We recall that X comes with an oriented $(k-1)$-spherical fibration ξ over X, and a map

$$\alpha\colon \mathbb{S}^{n+k} \to T(\xi),$$

where $T(\xi)$ is the Thom space of ξ. One obtains a "fundamental class" $[X] = \mathcal{U}_\xi \cap \alpha_*[\mathbb{S}^{n+k}] \in H_n(X, \mathbb{Z})$, where $\mathcal{U}_\xi \in H^k(T(\xi), \mathbb{Z})$ is the Thom class. Of course, the map

$\cap[X]\colon H^p(X,\Lambda)\to H_{n-p}(X,\Lambda)$ is not required to be an isomorphism. If it is, then X is a PD_4-complex and $\xi\cong\nu_X$ is the Spivak fibration over X. The sequence (3.1) is derived from the sequence

$$\cdots\to\Omega^N_{n+1}(P)\to\Omega^{\mathrm{NPD}}_{n+1}(P)\to\Omega^{\mathrm{PD}}_n(P)\to\Omega^N_n(P)\to\Omega^{\mathrm{NPD}}_n(P)\to\cdots.$$

Namely, it is proved that for $n\geqslant 3$, $\Omega^{\mathrm{NPD}}_n(P)\cong L_n(\pi_1)$ (see references above). A proof in the critical dimension $n=3$ resp. 4 can also be found in [**5**] by using Poincaré surgery and stabilization as in [**3**]. In the above, P can be replaced by a finite complex K but our interest concerns the sequence

$$\cdots\to\Omega^N_5(P)\to L_4(\pi_1)\to\Omega^{\mathrm{PD}}_4(P)\to\Omega^N_4(P)\to L_3(\pi_1)\to\cdots.$$

Since the differentials in the Atiyah–Hirzebruch spectral sequence for the groups $H_*(P,MSG)$ are trivial, we get

$$\begin{aligned}\Omega^N_4(P)&=H_4(P,MSG)\\&=H_0(P,\pi_4(MSG))\oplus H_1(P,\pi_3(MSG))\oplus H_4(P,\pi_0(MSG))\\&=\mathbb{Z}_8\oplus H_1(P,\mathbb{Z}_2)\oplus\mathbb{Z}.\end{aligned}$$

For calculations of $\pi_q(MSG)$ we refer to [**14, 12**] in case $q\leqslant 3$. For $q=4$, it is shown in [**2**]. We determine next the image of an element

$$\{f\colon X\to P\}\in\Omega^{\mathrm{PD}}_4(P),$$

in $\mathbb{Z}_8\oplus H_1(P,\mathbb{Z}_2)\oplus\mathbb{Z}$. By naturality, it is the image of $\{Id\colon X\to X\}\in\Omega^{\mathrm{PD}}_4(X)$ in $\mathbb{Z}_8\oplus H_1(X,\mathbb{Z}_2)\oplus\mathbb{Z}$ under the induced map f_* of f.

Lemma 3.1. *The $\mathbb{Z}_8$ component of $\{Id\colon X\to X\}$ in $\Omega^N_4(X)$ is the signature of X* mod 8.

Proof. This follows from naturality, that is, from the diagram:

$$\begin{array}{ccccc}
L_4(\pi_1) & \longrightarrow & \Omega^{\mathrm{PD}}_4(X) & \longrightarrow & \Omega^N_4(P)\\
\downarrow & & \downarrow & & \downarrow\\
L_4(1) & \longrightarrow & \Omega^{\mathrm{PD}}_4(\{*\}) & \longrightarrow & \Omega^N_4(\{*\})=\pi_4(MSG)\\
\| & & \| & & \|\\
\mathbb{Z} & \xrightarrow{\times 8} & \mathbb{Z} & \longrightarrow & \mathbb{Z}_8.
\end{array}$$

Here $\Omega^{\mathrm{PD}}_4(\{*\})\cong\Omega_4(\{*\})\cong\mathbb{Z}$ is given by the signature (see [**2**]). □

To determine the second component, we note

$$H_1(X,\pi_3(MSG))\cong H^3(X,\pi_3(MSG)).$$

Since $\pi_q(MSO)=0$ for $q=1,2,3$, one obtains as in [**14**]

$$\begin{aligned}\pi_3(MSG)&\cong\pi_3(MSG,MSO)\cong H_3(MSG,MSO)\\&\cong H_3(BSG,BSO)\cong\pi_3(BSG,BSO)\\&\cong\pi_2(G/O)\cong\pi_2(G/TOP).\end{aligned}$$

The third isomorphism is the Thom isomorphism. The others are the Hurewicz isomorphisms. Note that $\pi_1(MSG)=\{1\}$.

Lemma 3.2. *Under the above identification*

$$H_1(X, \pi_3(MSG)) \cong H^3(X, \pi_2(G/TOP)),$$

the component of $\{Id\colon X \to X\}$ *is the obstruction to lifting the Spivak fibration* ν_X *to* $BTOP$.

Proof. This can be seen from the diagram

$$\begin{array}{ccc} \Omega_4^{\mathrm{PD}}(X) & \longrightarrow & \Omega_4^N(X) \\ {\scriptstyle s}\big\downarrow & & \big\downarrow \\ \mathcal{S}_4(X) & \longrightarrow & H_3(X, \mathbb{L}) \end{array}$$

(see [**16**]), where $\mathbb{L}$ is the connected $\mathbb{L}$-spectrum and s is the total surgery obstruction. Hence we have

$$H_3(X, \mathbb{L}) \cong H_1(X, \pi_2(G/TOP)) \cong H_1(X, \pi_3(MSG)).$$

The image of $s(X)$ in $H_3(X, \mathbb{L})$ is the obstruction to lift ν_X to the map $\widetilde{\nu}_X\colon X \to BTOP$. □

The component of $\{Id\colon X \to X\} \in \Omega_4^{\mathrm{PD}}(X)$ in $H_4(X, \pi_0(MSG)) = H_4(X, \mathbb{Z})$ is the fundamental class. If we denote by

$$\omega_X \in H_1(X, \pi_3(MSG)) \cong H^3(X, \pi_2(G/TOP))$$

the obstruction of Lemma 3.2, then we have the following result:

Corollary 3.3. *The morphism*

$$\Omega_4^{\mathrm{PD}}(P) \to \Omega_4^N(P) = \mathbb{Z}_8 \oplus H_1(P, \pi_3(MSG)) \oplus H_4(P, \mathbb{Z})$$

sends $\{f\colon X \to P\}$ *to* $(\operatorname{sign} X \pmod 8), f_*(\omega_X), f_*[X])$.

Remark 3.4. Because $\pi_3(G/TOP) = 0$, ω_X is the unique obstruction for a TOP-reduction of ν_X.

(II) The map $L_4(\pi_1) \to \Omega_4^{\mathrm{PD}}(P)$

In higher dimensions this map is defined by the Wall realization construction on an $(n-1)$-manifold M^{n-1} (see [**16**]). This cannot be assumed for $n = 4$ because π_1 is not supposed to be a 3-manifold fundamental group. We use instead the manifold structure of the 1-skeleton of P to realize elements of $L_4(\pi_1)$. Let $P = Z \cup H$, where $H = \#_1^r (\mathbb{S}^1 \times D^3)$, and Z is a complex dominated by a 2-complex. Moreover, $\pi_1(\partial H) \to \pi_1(Z)$ is surjective. Let be given an element of $L_4(\pi_1)$ represented by (G, λ, μ). As in [**20**, pp. 53–54], one can realize (G, λ, μ) on the part $\partial H \times 0 \subset Z \cup (\partial H \times I)$. More precisely, we may add a small collar to Z, say $Z \cup (\partial H \times [0, \epsilon])$. Using the collar we write $\partial Z = \partial H$. Then we realize (G, λ, μ) in the part $\partial H \times [\epsilon, 1]$. This is possible because loops in Z can be lifted to ∂H. A very good description of Wall's realization construction is given in [**13**, pp. 141–142]. It applies to our situation. One gets framed embeddings $t_j\colon \mathbb{S}^1 \times D^2 \to \partial H \times \{1\}$ on which one attaches

2-handles h_j^2, for $j = 1, \ldots, \ell$. Here ℓ is the rank of G which we can assume to be Λ-free. We write

$$Z_G = Z \cup (\partial H \times I) \cup (\cup_1^\ell h_j^2),$$

hence

$$\partial Z_G = \partial H \times \{1\} \backslash \left(\cup_{j=1}^\ell t_j(\mathbb{S}^1 \times D^2)\right) \cup \left(\cup_{j=1}^\ell (D^2 \times \mathbb{S}^1)\right)$$

is obtained from $\partial Z = \partial H$ by surgeries on the link $\cup_{j=1}^\ell t_j(\mathbb{S}^1 \times 0) \subset \partial H$. Contrary to the higher dimensional case, we have $\pi_1(\partial Z_G) \neq \pi_1(\partial Z)$; in particular, ∂Z_G is not homotopy equivalent to ∂Z. However, let

$$\phi\colon (Z_G, \partial Z_G) \to (Z \cup (\partial H \times I), \partial H \times 1)$$

be the map coming from the above construction. Then we prove

Lemma 3.5. *The map ϕ induces an isomorphism*

$$H_*(\partial Z_G, \Lambda) \xrightarrow[\cong]{} H_*(\partial Z, \Lambda).$$

Note that $\Lambda = \mathbb{Z}[\pi_1(P)]$ is a module over $\mathbb{Z}[\pi_1(\partial Z_G)]$, and over $\mathbb{Z}[\pi_1(\partial Z)]$, induced via $\pi_1(\partial Z_G) \to \pi_1(Z) = \pi_1(P)$, and via $\pi_1(\partial Z) \to \pi_1(Z) = \pi_1(P)$.

Proof of Lemma 3.5. Note first that $\phi_*\colon H_k(Z_G, \Lambda) \to H_k(Z, \Lambda)$ is an isomorphism for $k \neq 2$. By construction, Z_G is a PD_4-complex with boundary ∂Z_G and the map $\phi\colon (Z_G, \partial Z_G) \to (Z, \partial Z)$ is of degree 1. Hence by Poincaré duality we have

$$\begin{array}{ccc} H_3(Z_G, \partial Z_G, \Lambda) & \xrightarrow{\phi_*} & H_3(Z, \partial Z, \Lambda) \\ \cong \uparrow & & \uparrow \cong \\ H^1(Z_G, \Lambda) & \xleftarrow[\phi^*]{} & H^1(Z, \Lambda) \end{array}$$

so $H_3(Z_G, \partial Z_G, \Lambda) \to H_3(Z, \partial Z, \Lambda)$ is an isomorphism. Note that the Λ-modules

$$G = K_2(Z_G, \Lambda) = \mathrm{Ker}\,(H_2(Z_G, \Lambda) \to H_2(Z, \Lambda))$$

and

$$\mathrm{Ker}\,(H_2(Z_G, \partial Z_G, \Lambda) \to H_2(Z, \partial Z, \Lambda)) = K_2(Z_G, \partial Z_G, \Lambda)$$

can be identified with

$$\mathrm{Hom}_\Lambda(G, \Lambda) \cong K^2(Z_G, \Lambda) = \mathrm{Coker}\left(H^2(Z, \Lambda) \to H^2(Z_G, \Lambda)\right),$$

via duality

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^2(Z, \Lambda) & \xrightarrow{\phi^*} & H^2(Z_G, \Lambda) & \longrightarrow & K^2(Z_G, \Lambda) & \longrightarrow & 0 \\ & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \\ 0 & \longleftarrow & H_2(Z, \partial Z, \Lambda) & \xleftarrow[\phi_*]{} & H_2(Z_G, \partial Z_G, \Lambda) & \longleftarrow & K_2(Z_G, \partial Z_G, \Lambda) & \longleftarrow & 0. \end{array}$$

Consider now the diagram of exact sequences of rows and vertical lines

$$\begin{array}{ccccccccc}
 & & 0 & \longrightarrow & H_3(Z_G,\Lambda) & \xrightarrow[\cong]{} & H_3(Z,\Lambda) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & \longrightarrow & H_3(Z_G,\partial Z_G,\Lambda) & \xrightarrow[\cong]{} & H_3(Z,\partial Z,\Lambda) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & K_2(\partial Z_G,\Lambda) & \longrightarrow & H_2(\partial Z_G,\Lambda) & \longrightarrow & H_2(\partial Z,\Lambda) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & G & \longrightarrow & H_2(Z_G,\Lambda) & \longrightarrow & H_2(Z,\Lambda) & \longrightarrow & 0 \\
 & & \hat{\lambda}\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathrm{Hom}_\Lambda(G,\Lambda) & \longrightarrow & H_2(Z_G,\partial Z_G,\Lambda) & \longrightarrow & H_2(Z,\partial Z,\Lambda) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K_1(\partial Z_G,\Lambda) & \longrightarrow & H_1(\partial Z_G,\Lambda) & \longrightarrow & H_1(\partial Z,\Lambda) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & \longrightarrow & H_1(Z_G,\Lambda) & \xrightarrow[\cong]{} & H_1(Z,\Lambda) & \longrightarrow & 0.
\end{array}$$

Note that $\hat{\lambda}$ (i.e., the adjoint of $\lambda\colon G\times G\to\Lambda$) is an isomorphism. Hence $H_q(\partial Z_G,\Lambda)\to H_q(\partial Z,\Lambda)$ is an isomorphism for $q=1,2$. For $q=3$ it follows from

$$\begin{array}{ccc}
H_4(Z_G,\partial Z_G,\Lambda) & \xrightarrow[\cong]{} & H_4(Z,\partial Z,\Lambda) \\
\downarrow & & \downarrow \\
H_3(\partial Z_G,\Lambda) & \longrightarrow & H_3(\partial Z,\Lambda) \\
\downarrow & & \downarrow \\
H_3(Z_G,\Lambda) & \xrightarrow[\cong]{} & H_3(Z,\Lambda).
\end{array}$$

This completes the proof of the lemma. □

We form now the complex

$$Z'_G = Z_G \cup (\partial Z_G\times[0,1]) \cup_\varphi (\partial H\times[1,2]),$$

where $\varphi\colon \partial Z_G\to\partial Z=\partial H$ is the restriction of ϕ, that is, we identify the pair $(z,1)\in \partial Z_G\times\{1\}$ with $(\varphi(z),1)\in\partial H\times\{1\}$. Obviously, $Z_G\subset Z'_G$ is a deformation retract. Lemma 3.5 implies that

$$H_*(Z_G,\partial Z_G,\Lambda) \xrightarrow[\cong]{} H_*(Z'_G,\partial H,\Lambda).$$

By the Wall realization construction, the maps $\phi\colon (Z_G,\partial Z_G)\to(Z,\partial Z)$ and

$\varphi\colon \partial Z_G \to \partial Z$ are of degree 1, so we get

$$H_4(Z_G, \partial Z_G, \mathbb{Z}) \xrightarrow[\cong]{} H_4(Z'_G, \partial H, \mathbb{Z}).$$

The fundamental class of Z_G determines a (fundamental) class of Z'_G inducing the commutative diagram

$$\begin{array}{ccc} H^k(Z'_G, \Lambda) & \xrightarrow[\cong]{} & H^k(Z_G, \Lambda) \\ \downarrow & & \downarrow \cong \\ H_{4-k}(Z'_G, \partial Z'_G, \Lambda) & \xleftarrow[\cong]{} & H_{4-k}(Z_G, \partial Z_G, \Lambda) \end{array}$$

that is, Z'_G is a PD_4-complex too, with boundary $\partial Z'_G = \partial H$.

Corollary 3.6. *The space $P_G = Z'_G \cup_{\partial H} H$ is a PD_4-complex. The map ϕ extends to a degree 1 map $\phi_G\colon P_G \to Z \cup H = P$.*

Supplement. The class $\{\phi_G\colon P_G \to P\} \in \Omega_4^{\mathrm{PD}}(P)$ depends only on the class $\{G, \lambda, \mu\} \in L_4(\pi_1)$. For this, one notes that equivalent quadratic forms are distinguished by stabilization with hyperbolic forms

$$\left(\Lambda^r \oplus (\Lambda^r)^*, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right),$$

which are realized by connected sums with $\#_1^r(\mathbb{S}^2 \times \mathbb{S}^2)$.

(III) Exactness

The map $L_4(\pi_1(P)) \to \Omega_4^{\mathrm{PD}}(P)$ is defined by sending the element

$$\{G, \lambda, \mu\} \in L_4(\pi_1(P))$$

to $\{\phi_G\colon P_G \to P\} - \{Id\colon P \to P\}$. Then the sequence

$$L_4(\pi_1(P)) \to \Omega_4^{\mathrm{PD}}(P) \to \mathbb{Z}_8 \oplus H_1(P, \pi_3(MSG)) \oplus \mathbb{Z}$$

is exact. The composition is the zero-map. In fact, the first component is the difference of the signatures of P_G and P, hence the signature of the pair $(G \otimes_\Lambda \mathbb{Z}, \lambda \otimes_\Lambda 1)$, which is divisible by 8. The third component is the difference of the degrees of ϕ_G and Id, hence it is zero. The second component vanishes since $(\phi_G)_*(\omega_{P_G}) = \omega_P \in H_1(P, \pi_3(MSG))$. This follows because $\deg \phi_G = 1$ and the obstructions are natural with respect to maps, and from the isomorphism

$$(\phi_G)_*\colon H_1(P_G, \pi_3(MSG)) \to H_1(P, \pi_3(MSG)).$$

Conversely, suppose that $\{f\colon X \to P\} \in \Omega_4^{\mathrm{PD}}(P)$ maps to

$$(0,0,0) \in \mathbb{Z}_8 \oplus H_1(P, \pi_3(MSG)) \oplus \mathbb{Z}.$$

The Spivak fibration of X has a BTOP–reduction, hence there is a degree 1 normal map $(g, c)\colon M^4 \to X$, where M is an oriented TOP 4-manifold. The image of the surgery obstruction $\sigma(g, c) \in L_4(\pi_1(P))$ under the map $\partial\colon L_4(\pi_1(P)) \to \mathcal{S}_4(P)$ (where $\mathcal{S}_4(P)$ is the structure set) is $-f_* s(X)$, that is, the total surgery obstruction $s(X) \in \mathcal{S}_4(X)$ is mapped into $\mathcal{S}_4(P)$ (see [**18**, Corollary 17.7]). If $-\sigma(g, c)$ is represented by (G, λ, μ), we may construct $\{\phi_G\colon P_G \to P\} \in \Omega_4^{\mathrm{PD}}(P)$.

Lemma 3.7. *With the above notation, we have*

$$\{f\colon X\to P\} = \{\phi_G\colon P_G\to P\} - \{Id\colon P\to P\}\in \Omega_4^{\mathrm{PD}}(P)$$

Proof. Let $\underline{\Omega}^{\mathrm{PD}}$ be the semisimplicial spectrum of adic PD-complexes (see [**18**]). From the Atiyah–Hirzebruch spectral sequence one gets

$$H_4(P,\underline{\Omega}^{\mathrm{PD}}) = H_0(P,\Omega_4^{\mathrm{PD}}(*))\oplus H_4(P,\Omega_0^{\mathrm{PD}}(*)) = \Omega_4^{\mathrm{PD}}(*)\oplus H_4(P,\mathbb{Z}) = \mathbb{Z}\oplus\mathbb{Z}.$$

The sequence

$$H_4(P,\underline{\Omega}^{\mathrm{PD}}) = \mathbb{Z}\oplus\mathbb{Z} \longrightarrow \Omega_4^{\mathrm{PD}}(P) \xrightarrow{\ s\ } \mathcal{S}_4(P)$$

is exact, where the map $\mathbb{Z}\oplus\mathbb{Z}\to\Omega_4^{\mathrm{PD}}(P)$ is (split) injective, and s can be proved to be surjective. The first component of $\mathbb{Z}\oplus\mathbb{Z}$ is the signature and the second one is the degree. The claim of Lemma 3.7 can be deduced from the following diagram

$$\begin{array}{ccccc}
 & & H_4(P,\underline{\Omega}^{\mathrm{PD}}) & & \\
 & & \| & & \\
L_4(1) & \xrightarrow{\times 8} & \Omega_4^{\mathrm{PD}}(*)\oplus H_4(P,\mathbb{Z}) & & \\
\downarrow & & \downarrow & & \\
L_4(\pi_1(P)) & \longrightarrow & \Omega_4^{\mathrm{PD}}(P) & \longrightarrow & \mathbb{Z}_8\oplus H_1(P,\pi_3(MSG))\oplus H_4(P,\mathbb{Z}). \\
\partial\downarrow & & \downarrow s & & \\
\mathcal{S}_4(P) & = & \mathcal{S}_4(P) & &
\end{array}$$

Now $\Omega_4^{\mathrm{PD}}(*)=\mathbb{Z}$ maps onto $\mathbb{Z}_8$ under the composition, and $H_4(P,\mathbb{Z})$ maps onto $H_4(P,\mathbb{Z})$. The map $\times 8$ is the multiplication by 8 into $\Omega_4^{\mathrm{PD}}(*)$ and it is zero into $H_4(P,\mathbb{Z})$. Moreover, the element

$$\{\phi_G\colon P_G\to P\} - \{Id\colon P\to P\} - \{f\colon X\to P\}\in\Omega_4^{\mathrm{PD}}(P)$$

maps to zero under s using $\partial\sigma(g,c) = -f_*s(X)$. □

Remark to Theorem 1.2(1). The action

$$L_4(\pi_1(P))\times\Omega_4^{\mathrm{PD}}(P)\to\Omega_4^{\mathrm{PD}}(P)$$

is analogous to the well-known action

$$L_n(\pi_1(P))\times\mathcal{S}_n(P)\to\mathcal{S}_n(P),$$

where $\mathcal{S}_n(P)$ is the n-dimensional topological structure set of P when $n>4$. To the pair $(\{G,\lambda,\mu\},\{f\colon X\to P\})\in L_4(\pi_1(P))\times\Omega_4^{\mathrm{PD}}(P)$ is associated the element $\{f\circ\phi_G\colon X_G\to P\}\in\Omega_4^{\mathrm{PD}}(P)$.

(IV) An example

First, let π be a finitely presented group realized as fundamental group of a 2-complex C. The boundary of a regular neighbourhood of $C\subset\mathbb{R}^5$ is a 4-manifold P with $\pi_1(P)=\pi$.

Consider now $\pi = \mathbb{Z}_p$, where p is an odd prime. Since

$$\Omega_5^N(P) = H_1(P, \pi_4(MSG)) \oplus H_2(P, \pi_3(MSG))$$

is finite and $L_4(\pi) = \mathbb{Z} \oplus \mathbb{Z}^{(p-1)/2}$, the map $L_4(\pi) \to \Omega_4^{\mathrm{PD}}(P)$ is injective. Moreover, the image of the map

$$\Omega_4^{\mathrm{PD}}(P) \to \Omega_4^N(P) = \mathbb{Z}_8 \oplus H_2(P, \mathbb{Z}_2)$$

is $\mathbb{Z}_8$. Hence, any PD_4-complex X with $\pi_1(X) = \pi$ is up to cobordism of type P_G for some $\{G, \lambda, \mu\} \in L_4(\pi)$.

Acknowledgments

Work performed under the auspices of the scientific group G.N.S.A.G.A. of the C.N.R (National Research Council) of Italy and partially supported by the MIUR (Ministero per la Ricerca Scientifica e Tecnologica) of Italy within the project *Strutture Geometriche, Combinatoria e loro Applicazioni.*

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