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KHOVANOV HOMOTOPY TYPES AND THE DOLD–THOM FUNCTOR

BRENT EVERITT, ROBERT LIPSHITZ, SUCHARIT SARKAR AND PAUL TURNER

(*communicated by Ralph Cohen*)

Abstract

We show that the spectrum constructed by Everitt and Turner as a possible Khovanov homotopy type is a product of Eilenberg–MacLane spaces and is thus determined by Khovanov homology. By using the Dold–Thom functor it can therefore be obtained from the Khovanov homotopy type constructed by Lipshitz and Sarkar.

A *Khovanov homotopy type* is a way of associating a (stable) space to each link L so that the classical invariants of the space yield the Khovanov homology of L. There are two recent constructions of Khovanov homotopy types, using different techniques and giving different results [**ET14, LS14a**]. In [**ET14**] homotopy limits were employed to build an Ω-spectrum $\mathbf{X}_\bullet L = \{X_k(L)\}$ with the following properties:

1. The homotopy type is a link invariant, and
2. the homotopy groups are Khovanov homology:

$$\pi_i(\mathbf{X}_\bullet(L)) = Kh^{-i}(L).$$

The goal of this note is to prove the following result:

Main Theorem. *Each of the spaces $X_k(L)$ is homotopy equivalent to a product of Eilenberg–MacLane spaces.*

In [**LS14a**] the programme of Cohen, Jones and Segal [**CJS95**] was generalized to produce a suspension spectrum $\mathcal{X}_{Kh}(L)$ with the following properties:

1. The homotopy type is a link invariant, and
2. the reduced cohomology is Khovanov homology:

$$\widetilde{H}^i(\mathcal{X}_{Kh}(L)) = Kh^i(L).$$

As a corollary we obtain that $\mathbf{X}_\bullet(L)$ is homotopy equivalent to the infinite symmetric product of $\mathcal{X}_{Kh}(L)$.

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To prove Theorem 1 we use an explicit model, due to McCord [**McC69**], of the Eilenberg–MacLane spaces. Given a monoid G and a based topological space X, let $B(G,X)$ denote the set of maps $u\colon X\to G$ such that $u(x)=0$ for all but finitely many $x\in X$. Then $B(G,X)$ is a monoid, and if G is a group (the case of interest) then $B(G,X)$ is a group. Moreover, when G is an abelian topological group the set $B(G,X)$ can be topologized in a natural way so that the group operation is continuous. This construction has nice functoriality: letting $\mathsf{Ab}, \mathsf{Top}_*$ and AbTop denote, respectively, the categories of abelian groups, based topological spaces and topological abelian groups, one has the following result [**McC69**, Proposition 6.7]:

Proposition 1. *McCord's construction is a bifunctor*

$$B(-,-)\colon \mathsf{Ab}\times\mathsf{Top}_*\to\mathsf{AbTop}.$$

Furthermore, as a special case of [**McC69**, Theorem 11.4], for an abelian group G the space $B(G,S^n)$ is the Eilenberg–MacLane space $K(G,n)$. Thus we may take as *the* Eilenberg–MacLane space functor:

$$B(-,S^n)\colon \mathsf{Ab}\to\mathsf{AbTop}.$$

Conversely, the following is [**Hat02**, Corollary 4K.7, p. 483] (apparently originally due to Moore; cf. [**McC69**, p. 295]):

Proposition 2. *A path-connected, commutative topological monoid is a product of Eilenberg–MacLane spaces.*

The spaces $X_k(L)$ are built as homotopy limits of diagrams of spaces. Recall that given a small category C and a (covariant) functor $D\colon\mathsf{C}\to\mathsf{Top}_*$ (a diagram), that $\operatorname{holim}_{\mathsf{C}} D$ is constructed as follows (see, e.g., [**BK72**, Section 11.5]). Consider the product

$$\prod_{\sigma\in N(\mathsf{C})}\operatorname{Hom}(\Delta^n,D(c_n))=\prod_{n\geqslant 0}\prod_{c_0\to\cdots\to c_n}\operatorname{Hom}(\Delta^n,D(c_n)),\tag{1}$$

where $N(\mathsf{C})$ is the nerve of C consisting of all sequences of composable morphisms $\sigma=(c_0\xrightarrow{\alpha_1}c_1\xrightarrow{\alpha_2}\cdots\xrightarrow{\alpha_n}c_n)$ and Hom denotes the space of continuous maps from the standard n-simplex. The homotopy limit $\operatorname{holim}_{\mathsf{C}} D$ is the subspace of this product consisting of those tuples $(f_\sigma)_{\sigma\in N(\mathsf{C})}$ such that the following diagrams commute:

$$\begin{array}{ccc}\Delta^{n-1}&\xrightarrow{f_{d_i\sigma}}&D(c_n)\\ {\scriptstyle d^i}\downarrow&&\downarrow{\scriptstyle \mathrm{Id}}\\ \Delta^n&\xrightarrow{f_\sigma}&D(c_n)\end{array}\qquad\begin{array}{ccc}\Delta^{n-1}&\xrightarrow{f_{d_n\sigma}}&D(c_{n-1})\\ {\scriptstyle d^n}\downarrow&&\downarrow{\scriptstyle D(\alpha_n)}\\ \Delta^n&\xrightarrow{f_\sigma}&D(c_n)\end{array}\qquad\begin{array}{ccc}\Delta^{n+1}&\xrightarrow{f_{s_i\sigma}}&D(c_n)\\ {\scriptstyle s^i}\downarrow&&\downarrow{\scriptstyle \mathrm{Id}}\\ \Delta^n&\xrightarrow{f_\sigma}&D(c_n),\end{array}\tag{2}$$

with $0\leqslant i<n$ on the left and $0\leqslant i\leqslant n$ on the right. Here the d^i and s^i are coface and codegeneracy maps and d_i and s_i are the face and degeneracy maps of the nerve

given by

$$d_i\sigma = (c_0 \xrightarrow{\alpha_1} \cdots c_{i-1} \xrightarrow{\alpha_{i+1}\alpha_i} c_{i+1} \cdots \xrightarrow{\alpha_n} c_n)$$

and

$$s_i\sigma = (c_0 \xrightarrow{\alpha_1} \cdots c_i \xrightarrow{\mathrm{Id}} c_i \cdots \xrightarrow{\alpha_n} c_n),$$

with d_0, d_n similarly.

The following is well-known, but for completeness we give its (short) proof.

Proposition 3. *Let $D\colon \mathsf{C} \to \mathsf{Top}_*$ be a diagram of topological abelian groups and continuous group homomorphisms. Then the homotopy limit of D is a topological abelian group.*

Proof. Pointwise addition makes the set $\mathrm{Hom}(\Delta^n, D(c_n))$ into an abelian group, and the product in formula (1) is the product (topological abelian) group. It remains to see that the diagrams (2) describe a subgroup of this product. Suppose that tuples (f_σ) and (g_σ) make these diagrams commute. Then the first and last diagrams automatically commute for the pointwise sum $(f_\sigma + g_\sigma)$. The middle diagram for the pointwise sum becomes,

$$\begin{array}{ccccccc}
\Delta^{n-1} & \longrightarrow & \Delta^{n-1}\times\Delta^{n-1} & \xrightarrow{f_{d_n\sigma}\times g_{d_n\sigma}} & D(c_{n-1})\times D(c_{n-1}) & \xrightarrow{+} & D(c_{n-1}) \\
\big\downarrow{\scriptstyle d^n} & & \big\downarrow{\scriptstyle d^n\times d^n} & & \big\downarrow{\scriptstyle D(\alpha_n)\times D(\alpha_n)} & & \big\downarrow{\scriptstyle D(\alpha_n)} \\
\Delta^n & \longrightarrow & \Delta^n\times\Delta^n & \xrightarrow{f_\sigma\times g_\sigma} & D(c_n)\times D(c_n) & \xrightarrow{+} & D(c_n)
\end{array}$$

for which the first square obviously commutes, the second commutes since f and g are in the prescribed subspace and the third commutes from the fact that $D(\alpha_n)$ is a group homomorphism. The inverse operation is similarly seen to be closed, hence the subspace defined above is a subgroup. □

We are now ready for the main theorem:

Theorem 1. *Each of the spaces $X_k(L)$ is homotopy equivalent to a product of Eilenberg–MacLane spaces.*

Proof. Let L be an oriented link diagram with c negative crossings. The space $X_k(L)$ is constructed as follows. Let I denote the category with objects $\{0, 1\}$ and a single morphism from 0 to 1, and I^n the product of I with itself n times. Let $\overline{0}$ be the initial object in I^n, and let $\mathbf{P}$ be the result of adjoining one more object to I^n and a single morphism from the new object to every object except $\overline{0}$.

In [**ET14**] it is shown that there is a functor $F\colon \mathbf{P} \to \mathsf{Ab}$ such that the ith derived functor of the inverse limit, $\varprojlim_{\mathbf{P}}{}^i F$, is isomorphic to the ith unreduced Khovanov homology of L. The space $X_k(L)$ is constructed by composing this functor with the Eilenberg–MacLane space functor $K(-, k+c)$ and taking the homotopy limit of the resulting diagram of spaces.

We may now use the explicit model for Eilenberg–MacLane spaces given by McCord. By applying Proposition 1 we define a diagram $D\colon \mathbf{P} \to \mathsf{AbTop}$ as the composition $B(-, S^{k+c}) \circ F : \mathbf{P} \to \mathsf{Ab} \to \mathsf{AbTop}$. By the homotopy invariance property of the homotopy limit construction we have

$$X_k(L) \simeq \operatorname{holim}_{\mathbf{P}} D.$$

By Proposition 3, the homotopy limit on the right is itself a topological abelian group, and hence, by Proposition 2, a product of Eilenberg–MacLane spaces. ☐

Corollary 1. *The homotopy type of* $\mathbf{X}_\bullet(L)$ *is determined by* $Kh(L)$.

The spectrum $\mathcal{X}_{Kh}(L) = \{\mathcal{X}_{Kh}^{(k)}(L)\}$ constructed in [**LS14a**] has the additional property that the cellular cochain complex of the space $\mathcal{X}_{Kh}^{(k)}(L)$ is isomorphic to the Khovanov complex of L (up to shift). It follows from the description of the Khovanov homology of the mirror image (see [**Kho00**]) that

$$\widetilde{H}_i(\mathcal{X}_{Kh}(L)) = Kh^{-i}(-L)$$

where $-L$ denotes the mirror of L. The infinite symmetric product $\operatorname{Sym}^\infty \mathcal{X}_{Kh}^{(k)}(L)$ is seen from the Dold–Thom theorem to be

$$\operatorname{Sym}^\infty \mathcal{X}_{Kh}^{(k)}(L) = \prod_n K(\widetilde{H}_n(\mathcal{X}_{Kh}^{(k)}(L)), n)$$

from which we have the following:

Corollary 2. *For large enough* k*, the space* $X_k(-L)$ *is homotopy equivalent to the infinite symmetric product* $\operatorname{Sym}^\infty \mathcal{X}_{Kh}^{(k)}(L)$.

We end by noting that the analogue of Theorem 1 for the spectra $\mathcal{X}_{Kh}(L)$ is not true. For all alternating knots $\mathcal{X}_{Kh}(L)$ is a wedge of Moore spaces [**LS14a**], however there are examples of non-alternating knots for which $\mathcal{X}_{Kh}(L)$ is not a wedge of Moore spaces (see [**LS14b**]).

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Brent Everitt `brent.everitt@york.ac.uk`

Department of Mathematics, University of York, York, YO10 5DD, UK

Robert Lipshitz `lipshitz@uoregon.edu`

Department of Mathematics, University of Oregon, Eugene, OR 97403, USA

Sucharit Sarkar `sucharit@math.ucla.edu`

Department of Mathematics, University of California, Los Angeles, CA 90095 USA

Paul Turner `Paul.Turner@unige.ch`

Section de mathématiques, Université de Genève, 2-4 rue du Lièvre, CH-1211, Geneva, Switzerland