ON THE f-INVARIANT OF PRODUCTS

HANNO VON BODECKER

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Abstract

The f-invariant is a higher version of the e-invariant that takes values in the divided congruences between modular forms; in the situation of a cartesian product of two framed manifolds, the f-invariant can actually be computed from the e-invariants of the factors. The purpose of this note is to determine the f-invariant of all such products.

1.   Introduction and statement of the result

In order to gain a better understanding of the stable homotopy groups of the sphere, which, by the Pontrjagin–Thom construction, can be interpreted as the bordism groups of framed manifolds, it proves helpful to organize the information using suitable invariants. In his seminal work on the J-homomorphism, Adams considered the d-invariant, which is essentially the degree of a map, and introduced the notion of an e-invariant [**1**]. Using complex K-theory, he produced an invariant

$$e_{\mathbb{C}} \colon \pi_{2k+1}^{st} \to \mathbb{Q}/\mathbb{Z},$$

and determined its image; to be precise, $e_{\mathbb{C}}$ maps:

(i) π_{8k+1}^{st} onto integer multiples of $\frac{1}{2}$,

(ii) π_{8k+3}^{st} onto integer multiples of $\frac{B_{4k+2}}{4k+2}$,

(iii) π_{8k+5}^{st} onto integer multiples of 1,

(iv) π_{8k+7}^{st} onto integer multiples of $\frac{B_{4k+4}}{8k+8}$.

Moreover, he showed that the optimal values in (ii) and (iv) are attained on the generators of ImJ in the corresponding dimensions (although ImJ_{8k+3} actually has *twice* the order of the denominator of $B_{4k+2}/(4k+2)$, which can be seen by making use of KO-theory), and constructed an 8-periodic family μ_{8k+1} (in cokerJ_{8k+1} if $k > 1$) detected by (i). The subsequent work of Conner and Floyd using bordism theory allowed the $e_{\mathbb{C}}$-invariant to be computed via the Todd genus of a complex null-bordism of the framed manifold in question [**6**]; even later, the work of Atiyah, Patodi, and Singer on index theory on manifolds with boundary [**3**] led to an analytic formulation of the e-invariant [**4**], which was used by Deninger and Singhof to

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exhibit an infinite family of nilmanifolds representing (twice) the generator of $\mathrm{Im}J$ in dimensions $8k + 3$ $(8k + 7)$ [7].

Much of the recent progress on our knowledge of π_*^{st} can be accredited to the Adams–Novikov spectral sequence (ANSS),

$$E_2^{p,q} = \mathrm{Ext}_{MU_*MU}^{p,q}(MU_*, MU_*) \Rightarrow \pi_{q-p}^{st}$$

and the rich algebraic structure inherent in complex oriented cohomology theories (see, e.g., [12]). Within this framework, the complex e-invariant can be considered as taking values in the 1-line of the ANSS, i.e.,

$$e_{\mathbb{C}} \colon \pi_{2k+1}^{st} \to E_2^{1,2k+2} \subseteq \mathbb{Q}/\mathbb{Z}.$$

Working locally at a prime p and switching to BP, the 1-line is generated by the alpha elements $\alpha_{i/j} \in \mathrm{Ext}^{1,2(p-1)i}$ of order p^j; at odd primes, $j = 1 + \nu_p(i)$, and all these alphas are permanent and detect the generators of $\mathrm{Im}J_{(p)}$. At $p = 2$ however, the situation is a little bit more subtle: The elements α_{4k+3} are not permanent; furthermore $j = 2 + \nu_2(i)$ for $i = 2t > 2$, but for $t = 2k + 1$ it is actually $\alpha_{4k+2/2}$ (i.e., twice the generator of $\mathrm{Ext}^{1,8k+4}$ if $k > 0$) that is a permanent cycle represented by an element in $\mathrm{Im}J$ of order eight (cf. [12, Theorem 5.3.7]).

In order to detect second filtration phenomena, Laures introduced the f-invariant, which is a follow-up to the e-invariant and takes values in the divided congruences between modular forms [9, 10]. Let us briefly recall its definition: Considering the congruence subgroup $\Gamma = \Gamma_1(N)$, set $\mathbb{Z}^\Gamma = \mathbb{Z}[\zeta_N, 1/N]$ and denote by M_*^Γ the graded ring of modular forms w.r.t. Γ which expand integrally, i.e., which lie in $\mathbb{Z}^\Gamma[\![q]\!]$. The ring of *divided congruences* D^Γ consists of those rational combinations of modular forms which expand integrally; this ring can be filtered by setting

$$D_k^\Gamma = \left\{ f = \sum_{i=0}^k f_i \;\middle|\; f_i \in M_i^\Gamma \otimes \mathbb{Q}, \; f \in \mathbb{Z}^\Gamma[\![q]\!] \right\}.$$

Finally, introduce

$$\underline{D}_k^\Gamma = D_k^\Gamma + M_0^\Gamma \otimes \mathbb{Q} + M_k^\Gamma \otimes \mathbb{Q}.$$

Now, if Ell^Γ denotes the complex oriented elliptic cohomology theory associated to the universal curve over the ring of modular forms w.r.t. Γ, the composite

$$E_2^{2,2k+2}[MU] \to E_2^{2,2k+2}[Ell^\Gamma] \to \underline{D}_{k+1}^\Gamma \otimes \mathbb{Q}/\mathbb{Z}$$

is injective (away from primes dividing the level N) and can be combined with the composite $\pi_{2k}^{st} \to E_\infty^{2,2k+2}[MU] \to E_2^{2,2k+2}[MU]$ (for the second map, note that there are no differentials hitting the 2-line). Then, for a fixed level N (which we suppress from the notation), the f-invariant becomes a map

$$f \colon \pi_{2k}^{st} \to \underline{D}_{k+1}^\Gamma \otimes \mathbb{Q}/\mathbb{Z}.$$

In a more geometrical fashion, the f-invariant can be formulated as an elliptic genus of manifolds with corners of codimension two [10]; as shown in the author's thesis [13], this allows to use techniques from index theory to understand and, at least in some cases, to calculate the f-invariant analytically (see also [5] for a – slightly different – analytical approach to the f-invariant). In the situation of the product of two framed manifolds, the f-invariant can actually be computed from the

$e_{\mathbb{C}}$-invariants of the factors. The purpose of this note is to determine the f-invariant of all such products:

Theorem 1.1. *Let x_{4k-1} be a generator of $\mathrm{Im}J$ in dimension $4k-1$ and let μ_{8k+1} be a representative of the μ-family constructed by Adams. Then, for the level $N = 3$, we have:*

(i) $f(x_3^2) \equiv \frac{1}{2} \left(\frac{E_1^2-1}{12} \right)^2,$

(ii) $f(x_7^2) \equiv \frac{1}{2} \left(\frac{E_4-1}{240} \right)^2,$

(iii) $f(\mu_{8k+1}\mu_{8k'+1}) \equiv \frac{1}{2} \frac{E_1-1}{2},$

(iv) $f(\mu_{8k+1}x_{8k'-1}) \equiv \frac{1}{2} \frac{E_4-1}{240},$

(v) $f(\mu_{8k+1}x_{8k'+3}) \equiv 0.$

Furthermore, for any *level $N > 1$, we have:*

(vi) $f(x_{4k-1}x_{4k'-1}) \equiv 0,$

unless $k = k'$ equals one or two.

Some comments are in order before presenting the proof:

First of all, we have to admit that at *even* levels $N = 2l$, i.e., when two is inverted, our theorem does not provide anything new, since the products of alpha elements at odd primes are known to vanish [**11**] (although our result does not rely on this fact). At odd levels however, things become more interesting:

It might be somewhat amusing to find the two 8-periodic families (iii) and (iv), each (within its family) admitting the same representative of the f-invariant, but the experienced reader will surely recognize these as corresponding to $\mu_{8(k+k')+2}$ in $\mathrm{coker}J$ and $\eta x_{8(k+k')-1}$ in $\mathrm{Im}J$, respectively.

Furthermore, our result shows that beyond these two families, only the exceptional cases (i) and (ii) (corresponding to the Kervaire elements of product type) admit a non-trivial f-invariant.

Summarizing, and bearing in mind that we work on the level of manifolds, i.e., representatives of permanent cycles, this theorem may be thought of as an elliptic (and strengthened) analog of [**12**, Theorem 5.5.8].

2. Proof of Theorem 1.1

The determination of the f-invariant of a product from the $e_{\mathbb{C}}$-invariants of its factors is made possible by the following result:

Lemma 2.1. [**13**] *Let Y_1, Y_2 be odd-dimensional framed manifolds, and let $m(Y_i)$ be* any *modular form of weight $(\dim Y_i + 1)/2$ w.r.t. the fixed congruence subgroup $\Gamma = \Gamma_1(N)$ such that $\bar{m}(Y_i) = m(Y_i) - e_{\mathbb{C}}(Y_i) \in \mathbb{Z}^{\Gamma}[\![q]\!]$. Then we have*

$$f(Y_1 \times Y_2) \equiv \bar{m}(Y_1)e_{\mathbb{C}}(Y_2) \equiv -\bar{m}(Y_2)e_{\mathbb{C}}(Y_1).$$

In particular, the f-invariant of a product is antisymmetric under exchange of the factors.

Besides the knowledge of the possible values of the $e_{\mathbb{C}}$-invariant, we need some elementary number theory:

Lemma 2.2. *For $n \geqslant 1$ and any positive integer d, we have:*

(i) $\left(d^{p^{n-1}(p-1)} - 1 \right) d^n \equiv 0 \mod p^n$,

(ii) $\left(d^{(2n'+1)2^n} - 1 \right) d^{n+2} \equiv 0 \mod 2^{n+2}$.

Proof. Part (i) is a simple consequence of the Euler–Fermat theorem (cf., e.g., [**2**]) which states that for $(a, m) = 1$, we have:

$$a^{\varphi(m)} \equiv 1 \mod m,$$

where $\varphi(m) = m \prod_{p|m} \left(1 - \frac{1}{p} \right)$ is the Euler totient.

The refinement (ii) stems from the fact that for odd $d^{2n'+1} = 2k + 1$:

$$(2k+1)^{2^n} = 1 + 2^n 2k + \frac{2^n (2^n - 1)}{2} 4k^2 + \sum_{i \geqslant 3} \binom{2^n}{i} (2k)^i$$

$$= 1 + 2^{n+1} \left(k + (2^n - 1) k^2 \right) + \sum_{i \geqslant 3} \binom{2^n}{i} (2k)^i$$

$$\equiv 1 \mod 2^{n+2},$$

since $\nu_2 (i!) < i$ and $\nu_2 ((2^n)! / (2^n - i)!) \geqslant n + 1$ for $i \geqslant 3$. $\qquad\square$

Concerning modular forms, recall that for even $k > 2$ the Eisenstein series

$$E_k = -\frac{2k}{B_k} G_k = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n$$

is a modular form of weight k w.r.t. the *full* modular group. Furthermore recall (cf., e.g., [**8**, Section 3.2]) that the ring of modular forms w.r.t. $\Gamma = \Gamma_1(3)$ is generated by

$$E_1 = 1 + 6 \sum_{n=1}^{\infty} \sum_{d|n} (\frac{d}{3}) q^n,$$

$$E_3 = 1 - 9 \sum_{n=1}^{\infty} \sum_{d|n} (\frac{d}{3}) d^2 q^n,$$

where $(\frac{\cdot}{\cdot})$ is the Legendre Symbol; in passing, we note that $(E_1^3 - E_3)/27 \in \mathbb{Z}[\![q]\!]$. These modular forms satisfy the following useful congruence:

Lemma 2.3.

$$\frac{1}{2} \left\{ \frac{E_1^2 - 1}{12} + (2k+1) \frac{E_3 - 1}{9} \right\} \in \mathbb{Z}[\![q]\!].$$

Proof. Since

$$E_1^2 = 1 + 12 \sum_{n=1}^{\infty} \sum_{3 \nmid d | n} d \, q^n,$$

we have to show that

$$\frac{1}{2} \left\{ \sum_{n=1}^{\infty} \sum_{3 \nmid d | n} d \, q^n - (2k+1) \sum_{n=1}^{\infty} \sum_{d|n} (\frac{d}{3}) d^2 q^n \right\} \in \mathbb{Z}[\![q]\!],$$

but obviously

$$\sum_{d|n}(\frac{d}{3})d^2 \equiv \sum_{3\nmid d|n} d^2 \equiv \sum_{3\nmid d|n} d \mod 2.$$

$\square$

Now let us prove the Theorem:

The exceptional cases (i) and (ii)

Looking at the list, we may assume that $e_{\mathbb{C}}(x_3)$ is given by $-\frac{1}{12}$; of course, this is in accordance with the well-known fact that x_3 may be represented by the sphere S^3 (with the framing induced by quaternion multiplication). So we apply Lemma 2.1, choose

$$-\bar{m}(x_3) = \frac{E_1^2 - 1}{12} = \sum_{n=1}^{\infty} \sum_{3\nmid d|n} d \; q^n \in \mathbb{Z}[\![q]\!],$$

and compute

$$\frac{1}{12}\frac{E_1^2 - 1}{12} \equiv -\frac{1}{2}\left(\frac{E_1^2 - 1}{12}\right)^2 \equiv \frac{1}{2}\left(\frac{E_1^2 - 1}{12}\right)^2 \mod \underline{\underline{D}}_4^{\Gamma};$$

this proves (i).

Similarly, x_7 may be represented by the sphere S^7 (see, e.g., [6]), resulting in $e_{\mathbb{C}}(x_7) = \frac{1}{240}$, so we choose

$$\bar{m}(x_7) = \frac{E_4 - 1}{240} = \sum_{n=1}^{\infty} \sum_{d|n} d^3 q^n \in \mathbb{Z}[\![q]\!],$$

and 'complete the square', hence establishing (ii).

Remark 2.4. Of course, (i) and (ii) correspond to the Kervaire elements of product type, i.e., $\alpha_{2/2}^2 = \beta_{2/2}$ and $\alpha_{4/4}^2 = \beta_{4/4}$ at the prime $p = 2$. In order to see this on the level of f-invariants, we may compare (i) and (ii) to the results of [8], i.e.,

$$f\left(\beta_{2/2}\right) \equiv \frac{1}{2}\left(\frac{E_1^2 - 1}{4}\right)^2 \in \underline{\underline{D}}_4^{\Gamma} \otimes \mathbb{Q}/\mathbb{Z},$$

$$f\left(\beta_{4/4}\right) \equiv \frac{1}{2}\left(\frac{E_1^2 - 1}{4}\right)^4 + \frac{1}{2}\left(\frac{E_1^2 - 1}{4}\right)^3 \in \underline{\underline{D}}_8^{\Gamma} \otimes \mathbb{Q}/\mathbb{Z}.$$

Concerning (ii), the fourth Eisenstein series is a modular form of weight four, hence it follows that $E_4 = 9E_1^4 - 8E_1 E_3$ by comparing the first two terms of the q-expansions.

Consequently, a short computation modulo $\underline{\underline{D}}_8^\Gamma$ reveals:

$$\frac{1}{2}\left(\frac{E_4-1}{16}\right)^2 = \frac{1}{2}\left(\frac{1}{2}\frac{E_1^4-1}{8}+\frac{1}{2}\left(E_1^4-E_1E_3\right)\right)^2$$

$$= \frac{1}{2}\left(\frac{1}{4}\left(\frac{E_1^4-1}{8}\right)^2+\frac{1}{2}\frac{E_1^4-1}{8}\left(E_1^4-E_1E_3\right)+\frac{1}{4}\left(E_1^4-E_1E_3\right)^2\right)$$

$$\equiv \frac{1}{32}E_1E_3+\frac{1}{2}\left(\frac{E_1^2-1}{4}\right)^4+\frac{1}{2}\left(\frac{E_1^2-1}{4}\right)^3+\frac{1}{8}\left(\frac{E_1^2-1}{4}\right)^2$$

$$\equiv \frac{1}{16}\frac{E_1^2-1}{4}+\frac{1}{2}\left(\frac{E_1^2-1}{4}\right)^4+\frac{1}{2}\left(\frac{E_1^2-1}{4}\right)^3+\frac{1}{16}\frac{E_1^4-1}{8}-\frac{1}{16}\frac{E_1^2-1}{4}$$

$$\equiv \frac{1}{2}\left(\frac{E_1^2-1}{4}\right)^4+\frac{1}{2}\left(\frac{E_1^2-1}{4}\right)^3,$$

since

$$-\frac{1}{32}E_1E_3 \equiv \frac{1}{2}\frac{E_4-1}{16}E_1E_3 \equiv \frac{1}{2}\frac{E_4-1}{16}E_3$$

$$\equiv \frac{E_4-1}{16}\frac{1}{2}\left(E_3-1\right) \equiv \frac{1}{2}\frac{E_1^2-1}{4}\frac{E_4-1}{16}$$

$$\equiv \frac{1}{2}\frac{E_6-1}{8}\frac{E_1^2-1}{4} \equiv -\frac{1}{16}\frac{E_1^2-1}{4}.$$

Remark 2.5. From the point of view of arithmetics, the product-type Kervaire elements $(x_3^2=\nu^2$ and $x_7^2=\sigma^2)$ are lucky exceptions: They happen to occur in the dimensions just low enough to be left unscathed by the congruence (ii) of Lemma 2.2.

The 8-periodic families (iii) and (iv)

According to [**1**], we have $e_{\mathbb{C}}(\mu_{8k+1})=1/2$, and this is the only possible non-trivial value of $e_{\mathbb{C}}$ in dimension $8k+1$. Thus, we judiciously choose

$$f(\mu_{8k+1}\mu_{8k'+1}) \equiv \frac{1}{2}\frac{E_1E_4^k-1}{2}\in \underline{\underline{D}}_{4(k+k')+2}^\Gamma \otimes \mathbb{Q}/\mathbb{Z},$$

but

$$\tfrac{1}{2}\tfrac{E_1E_4^k-1}{2}-\tfrac{1}{2}\tfrac{E_1-1}{2}=\tfrac{E_4^k-1}{4}E_1\in\mathbb{Z}[\![q]\!]$$

yields (iii). For (iv), we may assume that $e_{\mathbb{C}}(x_{8k'-1})\equiv B_{4k'}/8k' \mod \mathbb{Z}$, hence

$$f(\mu_{8k+1}x_{8k'-1}) \equiv \tfrac{1}{2}\tfrac{B_{4k'}}{8k'}\left(E_{4k'}-1\right)\equiv\tfrac{1}{2}\sum\sigma_{4k'-1}(n)q^n\in\underline{\underline{D}}_{4(k+k')+1}^\Gamma\otimes\mathbb{Q}/\mathbb{Z},$$

which is congruent to the desired result by virtue of Lemma 2.2.

Remark 2.6. In order to express (iv) in terms of E_1 and E_3, it is useful to work out

the following relation modulo $\underline{\underline{D}}^{\Gamma}_{4k+1}$:

$$\frac{1}{2}\frac{E_4-1}{16} = \frac{1}{2}\left[\left(\frac{E_1^2-1}{4}\right)^2 + \frac{1}{2}\frac{E_1^2-1}{4} + \frac{1}{2}\left(E_1^4 - E_1 E_3\right)\right]$$

$$= \frac{1}{2}\left(\frac{E_1^2-1}{4}\right)^2 + \frac{1}{2}\left[\frac{1}{2}\frac{E_1^2-1}{4} - \frac{1}{2}\left(E_3-1\right)\right] - \frac{1}{2}\frac{E_1-1}{2}E_3 + \frac{E_1^4-1}{4}$$

$$\equiv \frac{1}{2}\left(\frac{E_1^2-1}{4}\right)^2 + \frac{1}{2}\left[\frac{1}{2}\frac{E_1^2-1}{4} - \frac{1}{2}\left(E_3-1\right)\right] + \frac{1}{2}\frac{E_1-1}{2}\frac{E_1^2-1}{4};$$

furthermore, we observe that for $k \geqslant 2$ the first summand can be dropped due to
$\frac{1}{2}\left(\frac{E_1^2-1}{4}\right)^2 \equiv \frac{1}{2}(E_3-1)^2 \equiv \frac{1}{2}E_3^2 \equiv \frac{1}{2}E_1^{4k-5}E_3^2$.

Remark 2.7. The elements occurring in part (iii) and (iv) of the theorem also allow detection via the invariants $d_{\mathbb{R}}$ and $e_{\mathbb{R}}$, respectively, cf. [**1**]. While for low values of k, k', the non-triviality of f in these cases is easily checked by hand, the author does not know how to establish non-triviality for *all* values of k, k' in a direct manner.

The generic situation

For (v), we may assume that $e_{\mathbb{C}}(x_{8k+3})$ is represented by $B_{4k+2}/(4k+2)$, so, for $k > 0$, we have

$$f(x_{8k+3}\mu_{8k'+1}) \equiv \frac{1}{2}\frac{B_{4k+2}(1 - E_{4k+2})}{4k+2} = \sum \sigma_{4k+1}(n)q^n,$$

whereas for $k = 0$ we have

$$f(x_3\mu_{8k'+1}) \equiv \frac{1}{2}\frac{E_1^2-1}{12} \equiv \frac{1}{2}\left\{\frac{E_1^2-1}{12} + E_4^{k'}E_3 - 1\right\}$$

$$\equiv \frac{1}{2}\left\{\frac{E_1^2-1}{12} + E_3 - 1\right\} \quad \mod \underline{\underline{D}}^{\Gamma}_{4k'+3},$$

which expands integrally by Lemma 2.3.

Turning to (vi) and excluding the cases $k = k' = 1$ and $k = k' = 2$, we assume that $k \leqslant k'$; thus

$$f(x_{4k-1}x_{4k'-1}) \equiv \left(\epsilon(k')\sum \sigma_{2k'-1}(n)q^n\right) \cdot \left(\epsilon(k)\frac{B_{2k}}{4k}\right),$$

where $\epsilon(k) = 1$ for k even and two otherwise.

The theorem of von Staudt–Clausen [**2**] allows the computation of the denominator of the Bernoulli numbers. More precisely, let j_{2k} denote the denominator of $B_{2k}/2k$; if $(p-1)p^{n-1}|2k$, then $p^n|j_{2k}$, and this result is sharp. But by Lemma 2.2 (i) we have:

$$p^{-n}\sum \sigma_{2k'-1+2k}(r)q^r \equiv p^{-n}\sum \sigma_{2k'-1}(r)q^r \text{ if } (p-1)p^{n-1}|2k,$$

and the LHS is $p^{-n}G_{2k+2k'}$ plus a constant, hence vanishes mod $\underline{\underline{D}}_{2(k+k')}$.

Similarly, if $k = 2l = (2n'+1)2^{m+1}$, $2^{m+4}|2j_{4l}$ (and $2j_{4l}$ is precisely the order of

ImJ in dimension $8l-1$), and part (ii) of Lemma 2.2 yields:

$$\frac{1}{2^{4+m}} \left(\sum \sigma_{2k'-1+4l}(r)q^r - \sum \sigma_{2k'-1}(r)q^r \right) \equiv 0 \mod \mathbb{Z}[\![q]\!].$$

This completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

References

[1]   J.F. Adams, *On the groups $J(X)$. IV*, Topology **5** (1966), 21–71.

[2]   T.M. Apostol, *Introduction to Analytic Number Theory*, Undergrad. Texts Math., Springer-Verlag, New York, 1976.

[3]   M.F. Atiyah, V.K. Patodi, and I.M. Singer, *Spectral asymmetry and Riemannian geometry. I*, Math. Proc. Cambridge Philos. Soc. **77** (1975), 43–69.

[4]   M.F. Atiyah, V.K. Patodi, and I.M. Singer, *Spectral asymmetry and Riemannian geometry. II*, Math. Proc. Cambridge Philos. Soc. **78** (1975), no. 3, 405–432.

[5]   U. Bunke and N. Naumann, *The f-invariant and index theory*, Manuscripta Math. 132 (2010) 365–397.

[6]   P.E. Conner and E.E. Floyd, *The Relation of Cobordism to K-Theories*, Lect. Notes Math., No. 28, Springer-Verlag, Berlin, 1966.

[7]   Ch. Deninger and W. Singhof, *The e-invariant and the spectrum of the Laplacian for compact nilmanifolds covered by Heisenberg groups*, Invent. Math. **78** (1984), no. 1, 101–112.

[8]   J. Hornbostel and N. Naumann, *Beta-elements and divided congruences*, Amer. J. Math. **129** (2007), no. 5, 1377–1402.

[9]   G. Laures, *The topological q-expansion principle*, Topology **38** (1999), no. 2, 387–425.

[10] G. Laures, *On cobordism of manifolds with corners*, Trans. Amer. Math. Soc. **352** (2000), no. 12, 5667–5688 (electronic).

[11] H.R. Miller, D.C. Ravenel, and W.S. Wilson, *Periodic phenomena in the Adams-Novikov spectral sequence*, Ann. of Math. (2) **106** (1977), no. 3, 469–516.

[12] D.C. Ravenel, *Complex Cobordism and Stable Homotopy Groups of Spheres. 2nd ed.*, AMS Chelsea Publishing, Providence, RI, 2004.

[13] H. von Bodecker, *On the geometry of the f-invariant*, Ph.D. thesis, Ruhr-Universität Bochum, 2008.

Hanno von Bodecker   `hvb@math.uni-bielefeld.de`

Fakultät für Mathematik, Universität Bielefeld, Postfach 100131, D-33501 Bielefeld, Germany