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ON THE COHOMOLOGY OF ORIENTED GRASSMANN MANIFOLDS

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Abstract

This paper presents a new approach to studying the kernel of the additive homomorphism from $H^q(G_{n,k})$ to $H^{q+1}(G_{n,k})$ given by the cup-product with the first Stiefel–Whitney class of the canonical k-plane bundle over the Grassmann manifold $G_{n,k}$ of all k-dimensional vector subspaces in Euclidean n-space. This method enables us to improve the understanding of the $\mathbb{Z}_2$-cohomology of the "oriented" Grassmann manifold $\widetilde{G}_{n,k}$ of oriented k-dimensional vector subspaces in Euclidean n-space. In particular, we derive new information on the characteristic rank of the canonical oriented k-plane bundle over $\widetilde{G}_{n,k}$ and the $\mathbb{Z}_2$-cup-length of $\widetilde{G}_{n,k}$. Our results on the cup-length for three infinite families of the manifolds $\widetilde{G}_{n,3}$ confirm the corresponding claims of Fukaya's conjecture from 2008.

Dedicated to Professor Ulrich Koschorke on the occasion of his 75-th birthday.

1. Introduction

The $\mathbb{Z}_2$-cohomology algebra of the "unoriented" Grassmann manifold $G_{n,k}$ ($k \leqslant n-k$) of k-dimensional vector subspaces in $\mathbb{R}^n$ has a simple description in terms of generators and relations [**3**]: we can write

$$H^*(G_{n,k}) = \mathbb{Z}_2[w_1, \dots, w_k]/I_{n,k}, \tag{1}$$

where $\dim(w_i) = i$ and the ideal $I_{n,k}$ is generated by the k homogeneous components of $(1 + w_1 + \cdots + w_k)^{-1}$ in dimensions $n-k+1, \dots, n$. If $\gamma_{n,k}$ (briefly γ) denotes the canonical k-plane bundle over $G_{n,k}$, then the indeterminate w_i is a representative of the ith Stiefel–Whitney class $w_i(\gamma)$ in the quotient algebra $H^*(G_{n,k})$. For $w_i(\gamma)$, we shall also use w_i as an abbreviation. Note that all cohomology in this paper will be taken with coefficients in $\mathbb{Z}_2$. Also note that $w_i = w_i(\gamma)$ should not be confused with the Stiefel–Whitney classes of the manifold, namely $w_i(G_{n,k}) = w_i(TG_{n,k})$, the Stiefel–Whitney classes of its tangent bundle.

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Besides $\gamma_{n,k}$ and its $(n-k)$-dimensional orthogonal complement $\gamma_{n,k}^{\perp}$ (briefly $\gamma^{\perp}$), over $G_{n,k}$ we have a nontrivial line bundle $\xi = \det(\gamma) = \det(\gamma^{\perp})$. Then the "oriented" Grassmann manifold $\widetilde{G}_{n,k}$ of *oriented* k-dimensional vector subspaces in $\mathbb{R}^n$ which, of course, is a double cover of $G_{n,k}$, can be interpreted as the 0-sphere bundle of ξ ([**9**, Corollary 12.3] or [**12**, Theorem 5.7.11]), to which one has an exact sequence of Gysin type,

$$\xrightarrow{\psi} H^{j-1}(G_{n,k}) \xrightarrow{w_1} H^j(G_{n,k}) \xrightarrow{p^*} H^j(\widetilde{G}_{n,k}) \xrightarrow{\psi} H^j(G_{n,k}) \xrightarrow{w_1} . \tag{2}$$

We write here and elsewhere $H^{j-1}(G_{n,k}) \xrightarrow{w_1} H^j(G_{n,k})$ for the homomorphism given by the cup-product with the Stiefel–Whitney class $w_1(\xi) = w_1 = w_1(\gamma^{\perp})$, and $p\colon \widetilde{G}_{n,k} \to G_{n,k}$ is the obvious covering projection. Note that $\widetilde{G}_{n,k}$ is always orientable as a manifold, whereas $G_{n,k}$ is an orientable manifold if and only if n is even.

It is known [**14**] that $\mathrm{Im}(p^*\colon H^*(G_{n,k}) \longrightarrow H^*(\widetilde{G}_{n,k}))$ is a self-annihilating subspace of $H^*(\widetilde{G}_{n,k})$ of half the dimension. Very little is known about the algebra $H^*(\widetilde{G}_{n,k})$, apart from the cases of $(n-1)$-dimensional spheres $\widetilde{G}_{n,1} \cong S^{n-1}$ and complex quadrics $\widetilde{G}_{n,2}$. This is due to the fact that it is difficult to obtain information on cohomology classes that generate

$$\mathrm{Im}(\psi) \cong H^*(\widetilde{G}_{n,k})/\mathrm{Ker}(\psi) = H^*(\widetilde{G}_{n,k})/\mathrm{Im}(p^*). \tag{3}$$

A reason for this is that, in general, it is hard to calculate explicitly in $H^*(G_{n,k})$ and determine the kernel of w_1; of course, by (2), the latter vector space is the same as $\mathrm{Im}(\psi)$.

Over $\widetilde{G}_{n,k}$ we have the canonical oriented k-plane bundle $\widetilde{\gamma}_{n,k}$ (briefly $\widetilde{\gamma}$), which is isomorphic to $p^*(\gamma)$. As a consequence, $p^*(w_i) = \widetilde{w}_i$ for all i, where $\widetilde{w}_i$ is an abbreviation, used throughout the paper, for the Stiefel–Whitney class $w_i(\widetilde{\gamma}_{n,k})$; note $\widetilde{w}_1 = 0$. The subspace $C(j;n,k) := \mathrm{Im}(p^*)$ of the $\mathbb{Z}_2$-vector space $H^j(\widetilde{G}_{n,k})$ is the *characteristic subspace* (all its elements can be expressed in the Stiefel–Whitney *characteristic* classes of $\widetilde{\gamma}_{n,k}$; that is why we call it by this name). If we denote $\dim(C(j;n,k))$ by $\chi_j(\widetilde{G}_{n,k})$ and $\dim(\mathrm{Im}(\psi))$ by $\alpha_j(\widetilde{G}_{n,k})$, then (see (3))

$$\chi_j(\widetilde{G}_{n,k}) + \alpha_j(\widetilde{G}_{n,k}) = b_j(\widetilde{G}_{n,k}),$$

the right-hand side being the jth $\mathbb{Z}_2$-Betti number of $\widetilde{G}_{n,k}$.

Recall [**10, 8**] that, for a real vector bundle α over a path-connected CW-complex X, its *characteristic rank*, $\mathrm{charrank}(\alpha)$, is defined to be the greatest integer q, $0 \leqslant q \leqslant \dim(X)$, such that every cohomology class in $H^j(X)$, $0 \leqslant j \leqslant q$, is a polynomial in the Stiefel–Whitney classes $w_i(\alpha) \in H^i(X)$. In particular (see [**7**]), if TM is the tangent bundle of a smooth closed connected manifold M, then $\mathrm{charrank}(TM)$ is the *characteristic rank of* M, denoted $\mathrm{charrank}(M)$.

Now the greatest integer q such that

$$\alpha_0(\widetilde{G}_{n,k}) = \alpha_1(\widetilde{G}_{n,k}) = \cdots = \alpha_q(\widetilde{G}_{n,k}) = 0$$

is nothing but the characteristic rank of $\widetilde{\gamma}_{n,k}$, briefly $\mathrm{charrank}(\widetilde{\gamma}_{n,k})$.

Of course, we have

$$\alpha_{1+\mathrm{charrank}(\widetilde{\gamma}_{n,k})}(\widetilde{G}_{n,k}) \neq 0,$$

since $1 + \mathrm{charrank}(\widetilde{\gamma}_{n,k})$ is the least degree, in which an "anomalous" (not expressible exclusively in the Stiefel–Whitney classes of $\widetilde{\gamma}_{n,k}$) generator of $H^*(\widetilde{G}_{n,k})$ appears. Note that (see (2)) $\operatorname{charrank}(\widetilde{\gamma}_{n,k}) \geqslant j$ (for some j) if and only if $p^*\colon H^i(G_{n,k}) \longrightarrow H^i(\widetilde{G}_{n,k})$ is surjective or, equivalently, $w_1\colon H^i(G_{n,k}) \to H^{i+1}(G_{n,k})$ is injective, for all non-negative integers $i \leqslant j$.

When we know that some cohomology group of $\widetilde{G}_{n,k}$, in a degree not exceeding half of $\dim(\widetilde{G}_{n,k})$, does not vanish, we can use it to obtain an upper bound for $\operatorname{charrank}(\widetilde{\gamma}_{n,k})$. More precisely, due to the fact that the subspace $\operatorname{Im}(p^*) \subset H^*(\widetilde{G}_{n,k})$ is self-annihilating, one can easily adjust the proof of [**2**, Theorem 2.1] to verify that if $H^t(\widetilde{G}_{n,k}) \neq 0$ for some $t \leqslant \frac{1}{2}k(n-k)$, then we have

$$\operatorname{charrank}(\widetilde{\gamma}_{n,k}) \leqslant k(n-k) - t - 1.$$

Under certain conditions, the characteristic rank of a vector bundle over a smooth closed connected manifold M and the $\mathbb{Z}_2$-cup-length, denoted by $\operatorname{cup}(M)$, are nicely related, as shown in the following generalization of [**7**, Theorem 1.1] which, in particular, will be used (in Section 3) for deriving upper bounds or exact values for the cup-length of $\widetilde{G}_{n,k}$.

Theorem 1.1 (Naolekar and Thakur [**10**])**.** *Let M be a connected closed smooth d-dimensional manifold. Let α be a vector bundle over M satisfying the following: there exists j, $j \leqslant \operatorname{charrank}(\alpha)$, such that every monomial $w_{i_1}(\alpha)\cdots w_{i_s}(\alpha)$, $0 \leqslant i_t \leqslant j$, in dimension d vanishes. Then*

$$\operatorname{cup}(M) \leqslant 1 + \frac{d-j-1}{r_M},$$

where r_M is the smallest positive integer such that $\widetilde{H}^{r_M}(M) \neq 0$.

In addition, for any $j \leqslant \operatorname{charrank}(\widetilde{\gamma}_{n,k})$, one sees that both the w_1-homomorphisms in the Gysin sequence (2) are injective and the homomorphism $p^*\colon H^j(G_{n,k}) \longrightarrow H^j(\widetilde{G}_{n,k})$ is surjective, thus we have

$$H^j(\widetilde{G}_{n,k}) \cong H^j(G_{n,k})/\operatorname{Im}(w_1\colon H^{j-1}(G_{n,k}) \to H^j(G_{n,k})).$$

Of course, now $\dim(\operatorname{Im}(w_1\colon H^{j-1}(G_{n,k}) \to H^j(G_{n,k}))) = b_{j-1}(G_{n,k})$. Consequently, if $j \leqslant \operatorname{charrank}(\widetilde{\gamma}_{n,k})$, then we have for the Betti number $b_j(\widetilde{G}_{n,k})$ that

$$b_j(\widetilde{G}_{n,k}) = b_j(G_{n,k}) - b_{j-1}(G_{n,k}).$$

The difference of the $\mathbb{Z}_2$-Betti numbers of the Grassmann manifold $G_{n,k}$ on the right-hand side is readily calculable from the Poincaré polynomial, and is nothing but the number of linearly independent semi-invariants of degree k and weight j of a binary form of degree $n-k$, provided $j \leqslant \frac{k(n-k)}{2}$ (note that the latter number equals $\frac{1}{2}\dim(G_{n,k})$), by a theorem of Cayley and Sylvester (see [**11**, Satz 2.21]). This interesting interpretation of the Betti numbers $b_j(\widetilde{G}_{n,k})$ for

$$j \leqslant \min\{\operatorname{charrank}(\widetilde{\gamma}_{n,k}), \frac{k(n-k)}{2}\}$$

seems to have remained unnoticed thus far.

Theorem 2.1 in [**8**], on lower bounds or exact values for $\operatorname{charrank}(\widetilde{\gamma}_{n,k})$ ($3 \leqslant k \leqslant n-k$), gives information on the structure of the $\mathbb{Z}_2$-cohomology of the manifold $\widetilde{G}_{n,k}$.

In the present paper, we add further results. As compared to [**8**], we present a different approach to studying the kernel of w_1. Some of the new results on the characteristic rank presented here imply new exact values of the $\mathbb{Z}_2$-cup-length of $\widetilde{G}_{n,k}$. In particular, our results on the cup-length of three infinite families of the manifolds $\widetilde{G}_{n,3}$ in Theorem 3.6(2) confirm the corresponding claims of Fukaya's conjecture [**4**, p. 196].

2. An approach to studying the kernel of w_1

The aim of this section is to develop tools for studying the kernel of the homomorphism $w_1\colon H^j(G_{n,k}) \longrightarrow H^{j+1}(G_{n,k})$.

A key rôle will be played by the fact that, for the $\mathbb{Z}_2$-vector space $H^j(G_{n,k})$ ($k \leqslant n-k$), the set

$$\{w_1^{a_1}\cdots w_k^{a_k};\sum_{i=1}^k ia_i = j,\ \sum_{i=1}^k a_i \leqslant n-k\} \tag{4}$$

is an additive basis. This follows from [**9**, Corollary 6.7]; another proof can be found in [**5**]. We shall refer to the basis (4) as "*standard basis*" in this paper. We say that an element, $w_1^{a_1}\cdots w_k^{a_k} \in H^j(G_{n,k})$, of the standard basis is *regular* (with respect to the homomorphism $w_1\colon H^j(G_{n,k}) \longrightarrow H^{j+1}(G_{n,k})$) if its w_1-image is an element of the standard basis for $H^{j+1}(G_{n,k})$, that is, if $\sum_{i=1}^k a_i < n-k$. An element of the standard basis that is not regular is said to be *singular*.

Of course,

$$\dim(\mathrm{Im}(w_1\colon H^j(G_{n,k}) \longrightarrow H^{j+1}(G_{n,k})))$$

is greater than or equal to the number of regular elements of the standard basis for $H^j(G_{n,k})$ and

$$\dim(\mathrm{Ker}(w_1\colon H^j(G_{n,k}) \longrightarrow H^{j+1}(G_{n,k}))) = \alpha_j(\widetilde{G}_{n,k})$$

does not exceed the number of singular elements of the standard basis for $H^j(G_{n,k})$. The latter inequality can be concretized. Indeed, let $p(\{1, 2, \ldots, k-1\}, x)$ denote the number of partitions of a non-negative integer x into parts, each taken from the set $\{1, 2, \ldots, k-1\}$; in particular, if $x \leqslant k-1$, then $p(\{1, 2, \ldots, k-1\}, x) = p(x)$ is the total number of partitions of x.

Proposition 2.1. *For the Grassmann manifold $G_{n,k}$ ($2 \leqslant k \leqslant n-k$), we have the following:*

(a) If $1 \leqslant x \leqslant n-k$, then all the elements in the standard basis for $H^{n-k-x}(G_{n,k})$ are regular, thus we have $\alpha_{n-k-x}(\widetilde{G}_{n,k}) = 0$.

(b) If $x \geqslant 0$, then precisely $p(\{1, 2, \ldots, k-1\}, x)$ elements of the standard basis for $H^{n-k+x}(G_{n,k})$ are singular; thus $\alpha_{n-k+x}(\widetilde{G}_{n,k}) \leqslant p(\{1, 2, \ldots, k-1\}, x)$.

Proof. Part (a). Let $w_1^{a_1}\cdots w_k^{a_k}$ be an element of the standard basis in $H^{n-k-x}(G_{n,k})$. We have

$$\sum_{i=1}^k a_i \leqslant \sum_{i=1}^k ia_i = n-k-x,$$

thus the equality $\sum_{i=1}^k a_i = n-k$ is impossible; in other words, each basis element is now regular.

Part (b). If $w_1^{a_1}\cdots w_k^{a_k}$ is a singular element of the standard basis in $H^{n-k+x}(G_{n,k})$, then clearly $a_1+a_2+\cdots+a_k=n-k$, whence

$$a_2+2a_3+\cdots+(k-1)a_k=x.$$

Since a_1 is uniquely determined by the equation $a_1=n-k-a_2-\cdots-a_k$, the number of singular elements of the standard basis is equal to $p(\{1,\,2,\,\dots,\,k-1\},x)$. □

Remark 2.2. Of course, the vanishing of $\alpha_{n-k-x}(\widetilde{G}_{n,k})$ for all x in Proposition 2.1(a) is equivalent to the known inequality [**8**, (2.5)] $\operatorname{charrank}(\widetilde{\gamma}_{n,k})\geqslant n-k-1$.

The next lemma is elementary and stated without proof.

Lemma 2.3. *Let $\vec{a}_1,\dots,\vec{a}_m,\vec{b}_1,\dots,\vec{b}_n$ be linearly independent vectors in a vector space V over a field K. If $\vec{c}_1,\dots,\vec{c}_s$ $(s\leqslant n)$ are linearly independent vectors in the linear span $[\vec{b}_1,\dots,\vec{b}_n]\subset V$, then also the vectors $\vec{a}_1,\dots,\vec{a}_m,\vec{c}_1,\dots,\vec{c}_s$ are linearly independent.*

In $\mathbb{Z}_2[w_1,\dots,w_k]$, let $\bar{w}_i(w_1,\dots,w_k)$ (briefly $\bar{w}_i$) denote the homogeneous component of $(1+w_1+\cdots+w_k)^{-1}=1+w_1+\cdots+w_k+(w_1+\cdots+w_k)^2+\dots$ in dimension i. Of course, in particular, $\bar{w}_i$ with $i=n-k+1,\,n-k+2,\,\dots,\,n$ are the generators of the ideal $I_{n,k}$; see (1). In addition, let $g_i(w_2,\dots,w_k)$ (briefly just g_i) denote the reduction of $\bar{w}_i(w_1,\dots,w_k)$ modulo w_1. We note that $\bar{w}_i$ is a representative of the Stiefel–Whitney class $w_i(\gamma^\perp)\in H^i(G_{n,k})$; we shall also use $\bar{w}_i$ as an abbreviation for $w_i(\gamma^\perp)$.

If $i=n-k+1,\,n-k+2,\,\dots,\,n$, then the polynomials $g_i(w_2,\dots,w_k)$ are representatives of some multiples (by an abuse of notation, also denoted by $g_i(w_2,\dots,w_k)$, briefly g_i) of the first Stiefel–Whitney class w_1 in the quotient algebra $H^*(G_{n,k})$. The singular elements of the standard basis in $H^{n-k+x}(G_{n,k})$ $(x\geqslant 0)$, when multiplied by w_1, do not produce elements of the standard basis in $H^{n-k+x+1}(G_{n,k})$. Combined with Lemma 2.3 (where we take those elements of the standard basis divisible by w_1 in the rôle of the vectors $\vec{a}_i$ and the others in the rôle of the vectors $\vec{b}_j$), this explains why the following proposition focuses on elements of the form $w_2^{b_2}\cdots w_k^{b_k}g_{n-k+1+i}\in H^{n-k+x+1}(G_{n,k})$ $(i=0,\,1,\dots,k-1)$.

Proposition 2.4. *For a non-negative integer x, we associate with $H^{n-k+x+1}(G_{n,k})$ $(2\leqslant k\leqslant n-k)$ the set*

$$N_x(G_{n,k}):=\bigcup_{i=0}^{k-1}\{w_2^{b_2}\cdots w_k^{b_k}g_{n-k+1+i};\,2b_2+3b_3+\cdots+kb_k=x-i\}.$$

(1) The cardinality of $N_x(G_{n,k})$ is equal to $p(\{1,\,2,\,\dots,\,k-1\},x)$, which is the same (by Proposition 2.1) as the number of singular elements in the standard basis for $H^{n-k+x}(G_{n,k})$.

(2) If $x\leqslant n-k-1$, then each element of $N_x(G_{n,k})$ consists exclusively of monomials $w_2^{c_2}\cdots w_k^{c_k}$ such that $c_2+\cdots+c_k\leqslant n-k$, thus of elements not divisible by w_1 and belonging to the standard basis of $H^{n-k+x+1}(G_{n,k})$.

(3) *If $x \leqslant n-k-1$ and there are t linearly independent (over $\mathbb{Z}_2$) elements in the set $N_x(G_{n,k})$, then*

$$\alpha_{n-k+x}(\widetilde{G}_{n,k}) \leqslant p(\{1, 2, \ldots, k-1\}, x) - t.$$

In particular, if $x \leqslant n-k-1$ and the set $N_x(G_{n,k})$ is linearly independent, then

$$w_1 \colon H^{n-k+x}(G_{n,k}) \longrightarrow H^{n-k+x+1}(G_{n,k})$$

is a monomorphism and $\alpha_{n-k+x}(\widetilde{G}_{n,k})$ vanishes.

Proof. Part (1). The set $\{w_2^{b_2} \cdots w_k^{b_k} g_{n-k+1+i}; 2b_2 + 3b_3 + \cdots + kb_k = x - i\}$ consists of $p(\{2, \ldots, k\}, x-i)$ elements. Thus the cardinality of $N_x(G_{n,k})$ is equal to

$$\sum_{i=0}^{k-1} p(\{2, \ldots, k\}, x-i). \tag{5}$$

If S is a set of positive integers and $p(S, j)$ is the number of partitions of j whose parts are from S, then (see [**1**, Theorem 1.1] if needed) we have, for $|q| < 1$,

$$\sum_{i \geqslant 0} p(S, i) q^i = \prod_{i \in S} (1 - q^i)^{-1}.$$

Since

$$(1 + q + q^2 + \cdots + q^{k-1}) \prod_{i=2}^{k} (1 - q^i)^{-1} = \prod_{i=1}^{k-1} (1 - q^i)^{-1},$$

the cardinality of $N_x(G_{n,k})$ (the sum in (5)) is $p(\{1, 2, \ldots, k-1\}, x)$.

Part (2). Since each element of $N_x(G_{n,k})$ is some $w_2^{b_2} \cdots w_k^{b_k} g_{n-k+1+i}$ such that $2b_2 + 3b_3 + \cdots + kb_k = x - i$, it consists of monomials of the form

$$w_2^{b_2} \cdots w_k^{b_k} w_2^{c_2} \cdots w_k^{c_k} = w_2^{b_2+c_2} \cdots w_k^{b_k+c_k},$$

where $\sum_{i=2}^{k} i(b_i + c_i) = n - k + x + 1$. Thus, if $x \leqslant n-k-1$, then

$$\sum_{i=2}^{k} (b_i + c_i) \leqslant \frac{n-k+x+1}{2} \leqslant n - k.$$

Part (3). This is obviously implied by Lemma 2.3 and the first two parts of this proposition. □

3. Results on the characteristic rank and cup-length

In this section, the tools developed in Section 2 yield new bounds or exact results on the characteristic rank of $\widetilde{\gamma}_{n,k}$ (for odd n, also on the characteristic rank of $\widetilde{G}_{n,k}$). These lead to obtaining infinitely many new exact values of the cup-length of $\widetilde{G}_{n,3}$, regarded as likely in Fukaya's conjecture [**4**, p. 196].

Theorem 3.1. *For the oriented Grassmann manifold $\widetilde{G}_{n,k}$ $(4 \leqslant 2k \leqslant n)$, with the unique integer t such that $2^{t-1} < n \leqslant 2^t$, we have the following:*

(1) *If n is odd, then* $\operatorname{charrank}(\widetilde{\gamma}_{n,2}) \geqslant n-2$*, and if n is even, then* $\operatorname{charrank}(\widetilde{\gamma}_{n,2}) \geqslant n-3$.

(2) *If $s=1$ or $s=2$, r is a non-negative integer, and $2^{t-1}+\lfloor\frac{s-1}{2}\rfloor < n < 2^t-s-2$, then* $\operatorname{charrank}(\widetilde{\gamma}_{n+r,3+r}) \geqslant n+s-2$*. If $3 \leqslant s \leqslant 6$ and $2^{t-1}+\lfloor\frac{s-1}{2}\rfloor < n < 2^t-s-2$, then* $\operatorname{charrank}(\widetilde{\gamma}_{n,3}) \geqslant n+s-2$.

In addition, if n is odd, then the replacement of the canonical bundle $\widetilde{\gamma}_{n,k}$ by the corresponding manifold $\widetilde{G}_{n,k}$ gives the corresponding result on $\operatorname{charrank}(\widetilde{G}_{n,k})$.

Proof. It is known ([**8**, Theorem 2.1] and the final part of its proof) that if n is odd, then $\operatorname{charrank}(\widetilde{\gamma}_{n,k}) = \operatorname{charrank}(\widetilde{G}_{n,k})$. Thus it suffices to prove Parts (1) and (2).

Part (1). By Remark 2.2, $\operatorname{charrank}(\widetilde{\gamma}_{n,2}) \geqslant n-3$ for all n. If n is odd, then $N_0(G_{n,2})$ (see Proposition 2.4) only contains g_{n-1}. From $(1+w_2)^{-1} = 1+w_2+w_2^2+w_2^3+\cdots$, one sees that $g_{n-1} \neq 0$. Thus, by Proposition 2.4(3), $\alpha_{n-2}(\widetilde{G}_{n,2})$ vanishes and we have

$$\operatorname{charrank}(\widetilde{\gamma}_{n,2}) \geqslant n-2.$$

Part (2) We shall repeatedly use the fact that, by [**8**, Lemma 2.3(i)],

$$g_i(w_2,w_3) = 0 \text{ if and only if } i = 2^j-3 \text{ for some } j \geqslant 2. \tag{6}$$

The following lemma (when combined with (6)) will also be useful; cf. each of the four tables that occur in the proof of Theorem 3.1.

Lemma 3.2. *For $G_{n,3}$, let g_i denote the same polynomial in $\mathbb{Z}_2[w_2,w_3]$ as in the rest of this paper.*

(a) *If $m \neq 4$ is such that $g_{m-1} \neq 0$ and $g_m \neq 0$, then $w_2^2 g_{m-1} + w_3 g_m \neq 0$.*

(b) *If $m \neq 9$ is such that $g_{m-2} \neq 0$ and $g_{m+1} \neq 0$, then $w_2^3 g_{m-2} + w_3 g_{m+1} \neq 0$.*

Proof of the lemma. We know [**6**], for all $j \geqslant 1$, that

$$g_j = \sum_{\frac{j}{3} \leqslant i \leqslant \frac{j}{2}} \binom{i}{3i-j} w_2^{3i-j} w_3^{j-2i}. \tag{7}$$

Part (a). Of course, a necessary condition for $w_2^2 g_{m-1} + w_3 g_m = 0$ is that

$$w_2^2 \mid g_m \text{ and } w_3 \mid g_{m-1}. \tag{8}$$

Writing $m = 6a+b$ $(0 \leqslant b \leqslant 5)$, from (7), one either directly sees that $w_2^2 g_{m-1} + w_3 g_m \neq 0$, or that the divisibility condition (8) is not satisfied.

Indeed, if $b=0$, then g_m, and if $b=1$, then g_{m-1} is equal to $w_3^{2a}+\cdots+w_2^{3a}$, thus the condition (8) fails for $b=0,1$. Similarly, if $b=2$, then g_m, and if $b=3$, then g_{m-1} is equal to $w_2 w_3^{2a}+\cdots+w_2^{3a+1}$, thus the condition (8) fails for $b=2,3$.

If $b=4$, then we see that the condition (8) is fulfilled. But one calculates that $w_2^2 g_{m-1} + w_3 g_m$ is equal to

$$a w_2^2 w_3^{2a+1} + \cdots + \alpha w_2^{3a-4} w_3^5 + \beta w_2^{3a-1} w_3^3 + a w_2^{3a+2} w_3,$$

where the third last and second last coefficients are abbreviated,

$$\alpha = \binom{3a-1}{5} + \binom{3a}{4}, \qquad \beta = \binom{3a}{3} + \binom{3a+1}{2}.$$

Thus, of course, $w_2^2 g_{m-1} + w_3 g_m \neq 0$ if a is odd. For even a, one readily verifies that if $a = 8l, 8l+4$, then $\alpha = 1$, and if $a = 8l+2, 8l+6$, then $\beta = 1$, thus again $w_2^2 g_{m-1} + w_3 g_m \neq 0$.

Finally, if $b = 5$, then $g_{m-1} = \cdots + w_2^{3a+2}$, thus $w_3 \nmid g_{m-1}$.

Part (b). We proceed similarly as in the proof of Part (a). A necessary condition for $w_2^3 g_{m-2} + w_3 g_{m+1} = 0$ is that

$$w_2^3 \mid g_{m+1} \text{ and } w_3 \mid g_{m-2}. \tag{9}$$

Writing $m = 6a + b$ $(0 \leqslant b \leqslant 5)$, one either directly sees, from (7), that $w_2^3 g_{m-2} + w_3 g_{m+1} \neq 0$, or that the divisibility condition (9) is not satisfied. Indeed, if $b = 0$, then $g_{m-2} = g_{6a-2} = g_{6(a-1)+4} = w_2^{3a-1} + \cdots$, and if $b = 4$, then $g_{m-2} = g_{6a+2} = \cdots + w_2^{3a+1}$, thus $w_3 \nmid g_{m-2}$; the condition (9) fails for $b = 0$, 4. If $b = 1$, then $g_{m+1} = g_{6a+2} = w_2 w_3^{2a} + \cdots$, if $b = 2$, then $g_{m+1} = g_{6a+3} = w_3^{2a+1} + \cdots$, and if $b = 5$, then $g_{m+1} = g_{6(a+1)} = w_3^{2a+2} + \cdots$, thus $w_2^3 \nmid g_{m+1}$; the condition (9) fails for $b = 1$, 2, 5. Finally, let us suppose that $b = 3$. Then $g_{m+1} = g_{6a+4} = (a+1) w_2^2 w_3^{2a} + \cdots + w_2^{3a+2}$, thus $w_2^3 \nmid g_{m+1}$ if a is even. It remains to see what happens for odd a. If $a = 8l+1$ $(l \geqslant 1)$, then one calculates that $w_2^3 g_{m-2} + w_3 g_{m+1} = w_2^3 g_{6a+1} + w_3 g_{6a+4} = \cdots + w_2^{3a-10} w_3^9 + \cdots \neq 0$, if $a = 8l+3$, then one calculates that $w_2^3 g_{m-2} + w_3 g_{m+1} = \cdots + w_2^{3a-1} w_3^3 + \cdots \neq 0$, if $a = 8l+5$, then we have $w_2^3 g_{m-2} + w_3 g_{m+1} = \cdots + w_2^{3a-7} w_3^7 + \cdots \neq 0$ and, finally, if $a = 8l+7$, then we have $w_2^3 g_{m-2} + w_3 g_{m+1} = \cdots + w_2^{3a-1} w_3^3 + \cdots \neq 0$. This proves the lemma.

Now we are ready to verify the claims of Theorem 3.1(2).

Case $s = 1$. We have $2^{t-1} < n < 2^t - 3$ and assumptions of the theorem imply that $n \geqslant 9$. By [**8**, Theorem 2.1], we know that $\operatorname{charrank}(\widetilde{\gamma}_{n+r,3+r}) \geqslant n-2$. One readily calculates (see Proposition 2.4) that

$$N_2(G_{n,3}) = \{w_2 g_{n-2}(w_2, w_3), g_n(w_2, w_3)\}.$$

Since (see (6)) $w_2 g_{n-2} \neq 0$, $g_n \neq 0$ and, since $w_2 g_{n-2} + g_n = w_3 g_{n-3} \neq 0$, the set $N_2(G_{n,3})$ is linearly independent. At the same time,

$$N_2(G_{n+r,3+r}) = \{w_2 g_{n-2}(w_2, w_3, \ldots, w_{3+r}), g_n(w_2, w_3, \ldots, w_{3+r})\}.$$

By iterating the obvious "inclusion" $G_{n,k} \to G_{n+1,k+1}$ $(D \mapsto D \oplus \mathbb{R})$, we obtain an inclusion

$$j \colon G_{n,k} \to G_{n+r,k+r} \tag{10}$$

such that, for the pullbacks, we have $j^*(\gamma) \cong \gamma \oplus r\varepsilon$ (here $r\varepsilon$ is the trivial r-plane bundle) and $j^*(\gamma^\perp) \cong \gamma^\perp$. Of course, for the induced cohomology homomorphism, we have that $j^*(w_i) = w_i$ (with the right-hand side zero when $k = 3$ and $i \geqslant 4$) and $j^*(\bar{w}_i) = \bar{w}_i$. Thus, since the set $N_2(G_{n,3}) = j^*(N_2(G_{n+r,3+r}))$ is linearly independent, $N_2(G_{n+r,3+r})$ has this property as well. Proposition 2.4(3) implies that $\alpha_{n-1}(\widetilde{G}_{n+r,3+r}) = 0$ and $\operatorname{charrank}(\widetilde{\gamma}_{n+r,3+r}) \geqslant n-1$.

Case $s = 2$. Now $2^{t-1} < n < 2^t - 4$ and assumptions of the theorem imply that $n \geqslant 9$. By the result for $s = 1$, we know that $\operatorname{charrank}(\widetilde{\gamma}_{n+r,3+r}) \geqslant n-1$. Since $w_3 g_{n-2} \neq 0$, $w_2 g_{n-1} \neq 0$, and $w_3 g_{n-2} + w_2 g_{n-1} = g_{n+1} \neq 0$, the set

$$N_3(G_{n,3}) = \{w_3 g_{n-2}, w_2 g_{n-1}\}$$

is linearly independent. Similarly to the case of $s = 1$, one sees for $r > 0$ that the set $N_3(G_{n+r,3+r}) = \{w_3 g_{n-2}, w_2 g_{n-1}, g_{n+1}\}$ is independent. Thus Proposition 2.4(3) implies that we have $\alpha_n(\widetilde{G}_{n+r,3+r}) = 0$ and $\mathrm{charrank}(\widetilde{\gamma}_{n+r,3+r}) \geqslant n$.

Case $s = 3$. We have $2^{t-1} + 1 < n < 2^t - 5$; assumptions of the theorem imply that $n \geqslant 10$. By the result for $s = 2$, we know that $\mathrm{charrank}(\widetilde{\gamma}_{n,3}) \geqslant n$. One verifies that $N_4(G_{n,3})$ consists precisely of the obviously nonvanishing elements $w_2^2 g_{n-2}$, $w_3 g_{n-1}$, and $w_2 g_n$; they are linearly independent, as the following table shows.

h_1	h_2	h_3	$h_1 w_2^2 g_{n-2} + h_2 w_3 g_{n-1} + h_3 w_2 g_n$
0	1	1	$g_{n+2} \neq 0$
1	0	1	$w_2 w_3 g_{n-3} \neq 0$
1	1	0	$w_2^2 g_{n-2} + w_3 g_{n-1} \neq 0$, Lemma 3.2(a)
1	1	1	$w_3^2 g_{n-4} \neq 0$

By Proposition 2.4(3), now $\alpha_{n+1}(\widetilde{G}_{n,3}) = 0$ and $\mathrm{charrank}(\widetilde{\gamma}_{n,3}) \geqslant n + 1$.

Case $s = 4$. Now $2^{t-1} + 1 < n < 2^t - 6$ and, by assumptions of the theorem, we have $n \geqslant 18$. By the result for $s = 3$, we know that $\mathrm{charrank}(\widetilde{\gamma}_{n,3}) \geqslant n + 1$. We see that $N_5(G_{n,3})$ consists precisely of the obviously nonvanishing elements $w_2 w_3 g_{n-2}$, $w_2^2 g_{n-1}$, and $w_3 g_n$; they are linearly independent, as the following table shows.

h_1	h_2	h_3	$h_1 w_2 w_3 g_{n-2} + h_2 w_2^2 g_{n-1} + h_3 w_3 g_n$
0	1	1	$w_2^2 g_{n-1} + w_3 g_n \neq 0$, Lemma 3.2(a)
1	0	1	$w_3^2 g_{n-3} \neq 0$
1	1	0	$w_2 g_{n+1} \neq 0$
1	1	1	$g_{n+3} \neq 0$

So we have proved that $\alpha_{n+2}(\widetilde{G}_{n,3}) = 0$, and Proposition 2.4(3) implies that now $\mathrm{charrank}(\widetilde{\gamma}_{n,3}) \geqslant n + 2$.

Case $s = 5$. We have $2^{t-1} + 2 < n < 2^t - 7$ and assumptions of the theorem imply that $n \geqslant 19$. By the result for $s = 4$, we know that $\mathrm{charrank}(\widetilde{\gamma}_{n,3}) \geqslant n + 2$. One calculates that $N_6(G_{n,3})$ consists precisely of the obviously nonvanishing elements $w_2^3 g_{n-2}$, $w_3^2 g_{n-2}$, $w_2 w_3 g_{n-1}$, and $w_2^2 g_n$. The following table shows that they are linearly independent.

h_1	h_2	h_3	h_4	$h_1 w_2^3 g_{n-2} + h_2 w_3^2 g_{n-2} + h_3 w_2 w_3 g_{n-1} + h_4 w_2^2 g_n$
0	0	1	1	$w_2 g_{n+2} \neq 0$
0	1	0	1	$g_{n+4} \neq 0$
1	0	0	1	$w_2^2 w_3 g_{n-3} \neq 0$
0	1	1	0	$w_3 g_{n+1} \neq 0$
1	0	1	0	$w_2(w_2^2 g_{n-2} + w_3 g_{n-1}) \neq 0$, Lemma 3.2(a)
1	1	0	0	$(w_2^3 + w_3^2) g_{n-2} \neq 0$
0	1	1	1	$w_2^2 g_n + w_3 g_{n+1} \neq 0$, Lemma 3.2(a)
1	0	1	1	$w_2 w_3^2 g_{n-4} \neq 0$
1	1	0	1	$w_3(w_2^2 g_{n-3} + w_3 g_{n-2}) \neq 0$, Lemma 3.2(a)
1	1	1	0	$w_2^3 g_{n-2} + w_3 g_{n+1} \neq 0$, Lemma 3.2(b)
1	1	1	1	$w_3^3 g_{n-5} \neq 0$

So we have proved that now $\alpha_{n+3}(\widetilde{G}_{n,3}) = 0$. By Proposition 2.4(3), we have

$$\operatorname{charrank}(\widetilde{\gamma}_{n,3}) \geqslant n+3.$$

Case $s = 6$. We have $2^{t-1}+2 < n < 2^t - 8$ and assumptions of the theorem imply that $n \geqslant 19$. By the result for $s = 5$, we know that $\operatorname{charrank}(\widetilde{\gamma}_{n,3}) \geqslant n+3$. One calculates that $N_7(G_{n,3})$ consists precisely of the obviously nonvanishing elements $w_2^2 w_3 g_{n-2}$, $w_2^3 g_{n-1}$, $w_3^2 g_{n-1}$, and $w_2 w_3 g_n$. The following table shows that they are linearly independent.

h_1	h_2	h_3	h_4	$h_1 w_2^2 w_3 g_{n-2} + h_2 w_2^3 g_{n-1} + h_3 w_3^2 g_{n-1} + h_4 w_2 w_3 g_n$
0	0	1	1	$w_3 g_{n+2} \neq 0$
0	1	0	1	$w_2(w_2^2 g_{n-1} + w_3 g_n) \neq 0$, Lemma 3.2(a)
1	0	0	1	$w_2 w_3^2 g_{n-3} \neq 0$
0	1	1	0	$(w_2^3 + w_3^2) g_{n-1} \neq 0$
1	0	1	0	$w_3(w_2^2 g_{n-2} + w_3 g_{n-1}) \neq 0$, Lemma 3.2(a)
1	1	0	0	$w_2^2 g_{n+1} \neq 0$
0	1	1	1	$w_2^3 g_{n-1} + w_3 g_{n+2} \neq 0$, Lemma 3.2(b)
1	0	1	1	$w_3^3 g_{n-4} \neq 0$
1	1	0	1	$w_2 g_{n+3} \neq 0$
1	1	1	0	$w_2^2 g_{n+1} + w_3^2 g_{n-1} = g_{n+5} \neq 0$
1	1	1	1	$w_2^2 g_{n+1} + w_3 g_{n+2} \neq 0$, Lemma 3.2(a)

So we have proved that now $\alpha_{n+4}(\widetilde{G}_{n,3}) = 0$. By Proposition 2.4(3), we have

$$\operatorname{charrank}(\widetilde{\gamma}_{n,3}) \geqslant n+4.$$

The proof of Theorem 3.1 is finished. □

Conjecture 3.3. *We conjecture that Theorem 3.1(2) holds true for all* $s \geqslant 1$ *such that* $2^{t-1} + \lfloor \frac{s-1}{2} \rfloor < n < 2^t - s - 2$, *and not just when* $s \leqslant 6$.

In the proof of Theorem 3.1(2) for $s = 1$ or $s = 2$, we have extended a specific lower bound for the characteristic rank of $\widetilde{G}_{n,3}$ to a lower bound for the characteristic rank of $\widetilde{G}_{n+r,3+r}$ ($r \geqslant 0$). The following theorem brings an additional piece of information on the homomorphism w_1 and offers further possibilities for extensions of results on the characteristic rank of the vector bundle $\widetilde{\gamma}_{n,k}$.

Proposition 3.4. *For the Grassmann manifold* $G_{n,k}$ ($1 \leqslant k \leqslant n-k$) *and any positive integer* l *not exceeding* $n-1$, *we have the following:*

(1) If the homomorphism $w_1\colon H^l(G_{n,k}) \longrightarrow H^{l+1}(G_{n,k})$ *is injective, then also* $w_1\colon H^l(G_{n+1,k+1}) \longrightarrow H^{l+1}(G_{n+1,k+1})$ *is injective.*

(2) An obvious consequence of (1) *is that if* $\operatorname{charrank}(\widetilde{\gamma}_{n,k}) \geqslant l$ *then, for any nonnegative integer* r, *we have* $\operatorname{charrank}(\widetilde{\gamma}_{n+r,k+r}) \geqslant l$.

Proof. Let $j\colon G_{n,k}\to G_{n+1,k+1}$ denote the inclusion described in (10). The diagram

$$\begin{array}{ccc} H^l(G_{n+1,k+1}) & \xrightarrow{w_1} & H^{l+1}(G_{n+1,k+1}) \\ \downarrow j^* & & \downarrow j^* \\ H^l(G_{n,k}) & \xrightarrow{w_1} & H^{l+1}(G_{n,k}) \end{array}$$

obviously commutes. Let us suppose that the lower homomorphism w_1 is injective; we should prove that the upper homomorphism w_1 is injective as well.

The standard basis vectors (see (4)) in $H^l(G_{n+1,k+1})$ are

$$w_1^{a_1}(\gamma_{n+1,k+1})\cdots w_k^{a_k}(\gamma_{n+1,k+1})w_{k+1}^{a_{k+1}}(\gamma_{n+1,k+1}), \tag{11}$$

such that $a_1+2a_2+\cdots+ka_k+(k+1)a_{k+1}=l$ and $a_1+a_2+\cdots+a_k+a_{k+1}\leqslant n-k$. The images, under $w_1\colon H^l(G_{n+1,k+1})\to H^{l+1}(G_{n+1,k+1})$, of those vectors (11) having $a_{k+1}=0$ are linearly independent. Indeed, these images are

$$w_1^{1+a_1}(\gamma_{n+1,k+1})\cdots w_k^{a_k}(\gamma_{n+1,k+1});$$

the vectors $j^*(w_1^{1+a_1}(\gamma_{n+1,k+1})\cdots w_k^{a_k}(\gamma_{n+1,k+1}))=w_1^{1+a_1}(\gamma_{n,k})\cdots w_k^{a_k}(\gamma_{n,k})$ (being images of the standard basis vectors $w_1^{a_1}(\gamma_{n,k})\cdots w_k^{a_k}(\gamma_{n,k})\in H^l(G_{n,k})$ under the *injective* linear map $w_1\colon H^l(G_{n,k})\to H^l(G_{n,k})$) are linearly independent. Thus also $w_1^{1+a_1}(\gamma_{n+1,k+1})\cdots w_k^{a_k}(\gamma_{n+1,k+1})$ are linearly independent. In addition, the images under $w_1\colon H^l(G_{n+1,k+1})\to H^{l+1}(G_{n+1,k+1})$ of those vectors (11) having $a_{k+1}\geqslant 1$ are also linearly independent, because all the standard basis vectors (11) having $a_{k+1}\geqslant 1$ are regular. Indeed, we have for any of these standard basis vectors in $H^l(G_{n+1,k+1})$ that

$$(a_1+a_2+\cdots+a_k+a_{k+1})+ka_{k+1}\leqslant a_1+2a_2+\cdots+ka_k+(k+1)a_{k+1}=l,$$

thus

$$a_1+a_2+\cdots+a_k+a_{k+1}\leqslant l-ka_{k+1}\leqslant l-k\leqslant n-k-1.$$

Finally, the w_1-images of all the standard basis vectors (11) of $H^l(G_{n+1,k+1})$ are linearly independent. Indeed, let us suppose that a linear combination of all these images vanishes, that is,

$$\sum\alpha_{(a_1,\ldots,a_k,0)}w_1^{1+a_1}w_2^{a_2}\cdots w_k^{a_k}+\sum_{a_{k+1}\geqslant 1}\alpha_{(a_1,\ldots,a_k,a_{k+1})}w_1^{1+a_1}w_2^{a_2}\cdots w_k^{a_k}w_{k+1}^{a_{k+1}}=0. \tag{12}$$

When mapped by $j^*\colon H^{l+1}(G_{n+1,k+1})\to H^{l+1}(G_{n,k})$, this gives that

$$\sum\alpha_{(a_1,\ldots,a_k,0)}w_1^{1+a_1}(\gamma_{n,k})w_2^{a_2}(\gamma_{n,k})\cdots w_k^{a_k}(\gamma_{n,k})=0,$$

implying that all the coefficients $\alpha_{(a_1,\ldots,a_k,0)}$ vanish, since $w_1^{1+a_1}(\gamma_{n,k})\cdots w_k^{a_k}(\gamma_{n,k})$ are linearly independent. So the left-hand side of (12) is reduced to a linear combination of vectors already known to be linearly independent, we have

$$\sum_{a_{k+1}\geqslant 1}\alpha_{(a_1,\ldots,a_k,a_{k+1})}w_1^{1+a_1}(\gamma_{n+1,k+1})w_2^{a_2}(\gamma_{n+1,k+1})\cdots w_{k+1}^{a_{k+1}}(\gamma_{n+1,k+1})=0,$$

thus also all the coefficients $\alpha_{(a_1,\ldots,a_k,a_{k+1})}$ $(a_{k+1}\geqslant 1)$ must vanish. This finishes the proof of Proposition 3.4. □

Remark 3.5. The assumption $l \leqslant n-1$ in Proposition 3.4(1) is the best possible, in the sense that the claim is false, in general, for $l = n$. Indeed, $w_1 \colon H^7(G_{7,2}) \to H^8(G_{7,2})$ is readily seen to be a monomorphism (apply Proposition 2.4(3); the set $N_2(G_{7,2}) = \{w_2 g_6\}$ is linearly independent), but the homomorphism $w_1 \colon H^7(G_{8,3}) \to H^8(G_{8,3})$ is not injective (by a calculation in the cohomology algebra $H^*(G_{8,3})$, (1), or consulting Stong's result on the height of w_1 in [**13**], one sees that the kernel of this homomorphism contains $w_1^7 \neq 0$).

Theorem 3.1 enables us to derive, among others, new exact results on the characteristic rank and $\mathbb{Z}_2$-cup-length of three infinite families of the manifolds $\widetilde{G}_{n,3}$.

Theorem 3.6. *For the oriented Grassmann manifolds $\widetilde{G}_{n,k}$ $(4 \leqslant 2k \leqslant n)$ we have the following:*

(1) If n is odd, then

$$\operatorname{charrank}(\widetilde{\gamma}_{n,2}) = n-2, \ \operatorname{cup}(\widetilde{G}_{n,2}) = \frac{n-1}{2},$$

and if n is even, then

$$\operatorname{charrank}(\widetilde{\gamma}_{n,2}) = n-3, \ \operatorname{cup}(\widetilde{G}_{n,2}) = \frac{n}{2}.$$

(2) If $q \geqslant 4$, then

$$\operatorname{charrank}(\widetilde{\gamma}_{2^{q-1}+1,3}) = 2^{q-1}+1, \ \operatorname{cup}(\widetilde{G}_{2^{q-1}+1,3}) = 2^{q-1}-3,$$
$$\operatorname{charrank}(\widetilde{\gamma}_{10,3}) = 11, \ \operatorname{cup}(\widetilde{G}_{10,3}) = 5,$$

and, if $q \geqslant 5$, then

$$\operatorname{charrank}(\widetilde{\gamma}_{2^{q-1}+2,3}) = 2^{q-1}+4, \ \operatorname{cup}(\widetilde{G}_{2^{q-1}+2,3}) = 2^{q-1}-3,$$
$$\operatorname{charrank}(\widetilde{\gamma}_{2^{q-1}+3,3}) = 2^{q-1}+7, \ \operatorname{cup}(\widetilde{G}_{2^{q-1}+3,3}) = 2^{q-1}-3.$$

Remark 3.7. The results on the cup-length in Theorem 3.6(2) confirm the corresponding claims of Fukaya's conjecture [**4**, p. 196]; another claim contained in this conjecture was proved in [**8**].

Proof. Part (1). Let us first suppose that n is odd. It is clear (for instance, from (1)) that $w_2^{\frac{n-3}{2}} \in H^{n-3}(G_{n,2})$ is not a multiple of w_1, thus we have $\widetilde{w}_2^{\frac{n-3}{2}} \neq 0$ and $\operatorname{cup}(\widetilde{G}_{n,2}) \geqslant \frac{n-1}{2}$. We know, from Theorem 3.1, that $\operatorname{charrank}(\widetilde{\gamma}_{n,2}) \geqslant n-2$. Thus Theorem 1.1 implies that $\operatorname{cup}(\widetilde{G}_{n,2}) \leqslant \frac{n-1}{2}$, and we see that $\operatorname{cup}(\widetilde{G}_{n,2}) = \frac{n-1}{2}$, as claimed. At the same time, this shows that $\operatorname{charrank}(\widetilde{\gamma}_{n,2}) \leqslant n-2$ ($\operatorname{charrank}(\widetilde{\gamma}_{n,2}) \geqslant n-1$ would imply a false inequality, $\operatorname{cup}(\widetilde{G}_{n,2}) \leqslant \frac{n-2}{2}$), and so $\operatorname{charrank}(\widetilde{\gamma}_{n,2}) = n-2$. [To see that $\operatorname{charrank}(\widetilde{\gamma}_{n,2}) \leqslant n-2$, it also suffices to compare the Betti numbers $b_{n-1}(G_{n,2}) = \frac{n-1}{2}$ and $b_n(G_{n,2}) = \frac{n-3}{2}$, readily calculated from the Poincaré polynomial.]

Now let us suppose that n is even. First, we note that $\widetilde{G}_{4,2} \cong S^2 \times S^2$; clearly $\chi_2(\widetilde{G}_{4,2}) = 1 = \alpha_2(\widetilde{G}_{4,2})$, $\operatorname{charrank}(\widetilde{\gamma}_{4,2}) = 1$, and $\operatorname{cup}(\widetilde{G}_{4,2}) = 2$, as claimed. So we may suppose that $n \geqslant 6$. Then $w_2^{\frac{n-2}{2}} \in H^{n-2}(G_{n,2})$ cannot be a multiple of w_1, thus we have $\widetilde{w}_2^{\frac{n-2}{2}} \neq 0$ and $\mathrm{cup}(\widetilde{G}_{n,2}) \geqslant \frac{n}{2}$. We know, from Theorem 3.1, that

$$\mathrm{charrank}(\widetilde{\gamma}_{n,2}) \geqslant n-3;$$

Theorem 1.1 gives $\mathrm{cup}(\widetilde{G}_{n,2}) = \frac{n}{2}$. We know, from Theorem 3.1, that

$$\mathrm{charrank}(\widetilde{\gamma}_{n,2}) \geqslant n-3.$$

Admitting that $\mathrm{charrank}(\widetilde{\gamma}_{n,2}) \geqslant n-2$ implies a false inequality, $\mathrm{cup}(\widetilde{G}_{n,2}) \leqslant \frac{n-1}{2}$. [An alternative: since $b_{n-2}(G_{n,2}) = \frac{n}{2}$ and $b_{n-1}(G_{n,2}) = \frac{n-3}{2}$, the homomorphism $w_1 \colon H^{n-2}(G_{n,2}) \longrightarrow H^{n-1}(G_{n,2})$ is not injective, and we conclude that $\mathrm{charrank}(\widetilde{\gamma}_{n,2}) \leqslant n-3$.] Thus $\mathrm{charrank}(\widetilde{\gamma}_{n,2}) = n-3$, as claimed.

Part (2). We first note that, for any non-negative integer x, one has an obvious "inclusion" $\tilde{j} \colon \widetilde{G}_{2^{q-1},3} \to \widetilde{G}_{2^{q-1}+x,3}$, such that $\tilde{j}^*(\widetilde{\gamma}_{2^{q-1}+x,3}) \cong \widetilde{\gamma}_{2^{q-1},3}$. Thus, in cohomology, $\tilde{j}^*(w_2^{2^{q-1}-4}(\widetilde{\gamma}_{2^{q-1}+x,3})) = w_2^{2^{q-1}-4}(\widetilde{\gamma}_{2^{q-1},3})$. It was proved in [**7**, p. 77] that the latter cohomology class does not vanish. As a consequence, we have that

$$\mathrm{cup}(\widetilde{G}_{2^{q-1}+x,3}) \geqslant 2^{q-1}-3. \tag{13}$$

For $\widetilde{G}_{2^{q-1}+1,3}$ ($q \geqslant 4$), Theorem 3.1(2) with $s=2$ implies that

$$\mathrm{charrank}(\widetilde{\gamma}_{2^{q-1}+1,3}) \geqslant 2^{q-1}+1.$$

Then, from Theorem 1.1, we obtain that $\mathrm{cup}(\widetilde{G}_{2^{q-1}+1,3}) \leqslant 2^{q-1}-3$, thus we have (see (13)) $\mathrm{cup}(\widetilde{G}_{2^{q-1}+1,3}) = 2^{q-1}-3$ and $\mathrm{charrank}(\widetilde{\gamma}_{2^{q-1}+1,3}) = 2^{q-1}+1$.

For $\widetilde{G}_{2^{q-1}+2,3}$ with $q=4$, that is, for $\widetilde{G}_{10,3}$, Theorem 3.1(2) with $s=3$ applies and gives that $\mathrm{charrank}(\widetilde{\gamma}_{10,3}) = 11$. Thus from Theorem 1.1, we obtain that $\mathrm{cup}(\widetilde{G}_{10,3}) \leqslant 5$ which, when combined with (13), implies that $\mathrm{cup}(\widetilde{G}_{10,3}) = 5$ and $\mathrm{charrank}(\widetilde{\gamma}_{10,3}) = 11$. Let us continue with $\widetilde{G}_{2^{q-1}+2,3}$, $q \geqslant 5$. Then Theorem 3.1(2) with $s=4$ implies that $\mathrm{charrank}(\widetilde{\gamma}_{2^{q-1}+2,3}) \geqslant 2^{q-1}+4$. From Theorem 1.1, we see that $\mathrm{cup}(\widetilde{G}_{2^{q-1}+2,3}) \leqslant 2^{q-1}-3$; this, jointly with (13), yields

$$\mathrm{cup}(\widetilde{G}_{2^{q-1}+2,3}) = 2^{q-1}-3 \text{ and } \mathrm{charrank}(\widetilde{\gamma}_{2^{q-1}+2,3}) = 2^{q-1}+4,$$

as claimed.

For $\widetilde{G}_{2^{q-1}+3,3}$ with $q \geqslant 5$, we apply Theorem 3.1(2) with $s=6$ and see that $\mathrm{charrank}(\widetilde{\gamma}_{2^{q-1}+3,3}) \geqslant 2^{q-1}+7$. Theorem 1.1 implies that $\mathrm{cup}(\widetilde{G}_{2^{q-1}+3,3}) \leqslant 2^{q-1}-3$ which, when combined with (13), shows that

$$\mathrm{cup}(\widetilde{G}_{2^{q-1}+3,3}) = 2^{q-1}-3 \text{ and } \mathrm{charrank}(\widetilde{\gamma}_{2^{q-1}+3,3}) = 2^{q-1}+7,$$

indeed. The proof of Theorem 3.6 is finished. □

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