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EQUIVARIANT Γ-SPACES

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(*communicated by J.P.C. Greenlees*)

Abstract

The aim of this note is to provide a comprehensive treatment of the homotopy theory of Γ-G-spaces for G a finite group. We introduce two level and stable model structures on Γ-G-spaces and exhibit Quillen adjunctions to G-symmetric spectra with respect to a flat level and a stable flat model structure, respectively. Then we give a proof that Γ-G-spaces model connective equivariant stable homotopy theory along the lines of the proof in the non-equivariant setting given by Bousfield and Friedlander. Furthermore, we study the smash product of Γ-G-spaces and show that the functor from Γ-G-spaces to G-symmetric spectra commutes with the derived smash product. Finally, we show that there is a good notion of geometric fixed points for Γ-G-spaces.

1. Introduction

In his seminal paper [**16**], Segal introduced Γ-spaces as a tool to produce infinite loop spaces. In fact, Segal showed that Γ-spaces model connective stable homotopy theory and later Bousfield and Friedlander proved this in the language of model categories [**2**].

For G a finite group, Segal developed the machinery of Γ-G-spaces in [**17**] and it is known that *very special* Γ-G-spaces give rise to equivariant infinite loop spaces (cf. [**17, 18, 19**]). Santhanam also proved that Γ-G-spaces with a suitable model structure are equivalent to equivariant E_∞-spaces (cf. [**18**]).

Using the results of Shimakawa [**19**], we give a proof along the lines of [**2**] that Γ-G-spaces model connective equivariant stable homotopy theory. Moreover, Γ-G-spaces possess a symmetric monoidal smash product as was shown by Lydakis [**7**], motivating the question if this equivalence can be realized by a Quillen functor to a symmetric monoidal category of G-spectra which commutes with the derived smash product. This turns out to be true, if one uses the flat model structure on G-symmetric spectra as constructed by Hausmann [**5**]. Even non-equivariantly, this might be of interest on its own right. In addition, we define a geometric fixed point functor for Γ-G-spaces which has all desirable properties.

The structure of the paper is as follows. Sections 2 and 3 contain a brief review of basic facts about G-equivariant homotopy theory and G-symmetric spectra. In particular, we will introduce the flat model structures. In Section 4, we briefly discuss basic

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definitions and constructions concerning Γ-G-spaces and introduce two level model structures. The projective model structure was employed by Santhanam in [**18**], too, but we also show how to generalize the *strict* model structure of [**2**] to the equivariant setting. In Section 5, we exhibit Quillen pairs between the level model structures on Γ-G-spaces and the flat level model structure on G-symmetric spectra. This requires a characterization of flat cofibrations of G-symmetric spectra which we carry out in the appendix. We also show that spectra obtained from Γ-G-spaces are equivariantly connective and, using the results of [**19**], we show that very special Γ-G-spaces give rise to $G\Omega$-symmetric spectra up to a level fibrant replacement. After these preparations, we show that the homotopy categories with respect to the level model structures of very special Γ-G-spaces and those connective spectra which are level equivalent to $G\Omega$-spectra are equivalent. In Section 6, we introduce stable equivalences of Γ-G-spaces and the stable model structures on Γ-G-spaces corresponding to the two level model structures. This leads to the equivalence of the homotopy categories of Γ-G-spaces and connective G-symmetric spectra with respect to the stable model structures. Section 7 contains a discussion of the smash product of Γ-G-spaces. Following the non-equivariant results from [**7**], we show that it is well-behaved with respect to the model structures and the functor from Γ-G-spaces to G-symmetric spetra commutes with the derived smash product. Finally, in Section 8, we define geometric fixed points for Γ-G-spaces with respect to a subgroup $H \leqslant G$. This is a lax symmetric monoidal functor which sends suspension spectra to suspension spectra and commutes with the derived smash product up to stable equivalence. We characterize stable equivalences of Γ-G-spaces as those maps which induce stable equivalences on all geometric fixed points.

For convenience, we work with simplicial sets in this paper. However, we want to point out that many results have a direct topological analogue.

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2. Recollections on equivariant homotopy theory

This section is intended to fix basic terminology about model categories and recollect facts about G-equivariant homotopy theory. If not stated otherwise, the source of material is [**10**, II.1, III.1] though we work with simplicial sets as opposed to topological spaces. Throughout this paper, the word "space" will mean simplicial set.

2.1. Model categorical notions

We will freely use the concepts of model category theory. A reference is [**3**] and we use the numbering therein when referring to the axioms **MC1** up to **MC5**. Recall that a model category is *proper* if weak equivalences are preserved under pullbacks along fibrations and under pushouts along cofibrations.

2.2. Model structures on G-spaces

The category of based spaces will be denoted by $\mathcal{S}_*$. It is closed symmetric monoidal under the smash product with unit S^0. If G is a finite group, we also have the category of based spaces with left G-action together with (not necessarily equivariant) maps $\mathcal{S}_{*G}$. It is enriched over $G\mathcal{S}_*$, the corresponding category enriched over $\mathcal{S}_*$ with the same objects but equivariant maps.

Several model structures on G-spaces will play a role in this paper. The following model structure on $G\mathcal{S}_*$ is the most important one. A morphism $f\colon X \to Y$ in $G\mathcal{S}_*$ is a *G-fibration* (resp. *G-equivalence*) if $f^H\colon X^H \to Y^H$ is a fibration (resp. weak equivalence) in $\mathcal{S}_*$ for all $H \leqslant G$. A map is a *G-cofibration* if it satisfies the left lifting property with respect to all acyclic G-fibrations. This is the case if and only if the map is levelwise injective. These notions of weak equivalences, fibrations and cofibrations make the category $G\mathcal{S}_*$ into a cofibrantly generated proper model category. In fact,

$$I = \{i\colon (G/H \times \partial\Delta^n)_+ \to (G/H \times \Delta^n)_+ |\ n \geqslant 0,\ H \leqslant G\}$$

and

$$J = \{j\colon (G/H \times \Delta^n)_+ \to (G/H \times \Delta^n \times \Delta^1)_+ |\ n \geqslant 0,\ H \leqslant G\}$$

are sets of generating cofibrations and acyclic cofibrations.

More generally, if $\mathcal{F}$ is a family of subgroups, by which we mean a collection of subgroups closed under conjugation and taking subgroups, there is a model structure relative to $\mathcal{F}$. A map $f\colon X \to Y$ in $G\mathcal{S}_*$ is an *$\mathcal{F}$-fibration* (resp. *$\mathcal{F}$-equivalence*) if $f^H\colon X^H \to Y^H$ is a fibration (resp. weak equivalence) in $G\mathcal{S}_*$ for all $H \in \mathcal{F}$. This yields a cofibrantly generated proper model structure (cf. [**10**, IV. Theorem 6.5]). As set of generating cofibrations (resp. acyclic cofibrations) we only take those maps in I (resp. J), where source and target have isotropy in $\mathcal{F}$. Note that if $\mathcal{F} = \mathcal{ALL}$, this reproduces the model structure introduced first and for arbitrary $\mathcal{F}$ the identity functor is a left Quillen functor from the $\mathcal{F}$-model structure to the $\mathcal{ALL}$-model structure, since every G-equivalence (resp. G-fibration) is an $\mathcal{F}$-equivalence (resp. $\mathcal{F}$-fibration). We want to point out that a map $A \to B$ is a cofibration in the $\mathcal{F}$-model structure if and only if it is a cofibration in the $\mathcal{ALL}$-model structure and all simplices not in the image have isotropy in $\mathcal{F}$.

If $\mathcal{F}$ is a family of subgroups of G, there is a *mixed* model structure on $G\mathcal{S}_*$ (cf. [**5**, Proposition 1.22]). The weak equivalences in this model structure are the $\mathcal{F}$-equivalences and the cofibrations are the G-cofibrations. The fibrations are defined by the appropriate lifting property and are called *mixed* G-fibrations.

Example 2.1. If the group in question is $G \times \Sigma_n$ for G an arbitrary finite group and Σ_n the symmetric group on n letters, there is a particularly important family denoted by $\mathcal{G}_n$. It consists of all subgroups $J \leqslant G \times \Sigma_n$ such that $J \cap \{1\} \times \Sigma_n = \{(1,1)\}$. Equivalently, those are the subgroups of the form $\{(h, \rho(h))|\ h \in H\}$ where H is a subgroup of G and $\rho\colon H \to \Sigma_n$ is a homomorphism.

2.3. Equivariant enrichments of model categories

We will now introduce the equivariant analogue of simplicial model categories. The difference is that in the equivariant setting there are usually function G-spaces, as opposed to simply function spaces. Let $\mathcal{C}_G$ be the category of G-objects in some category enriched over $\mathcal{S}_*$. Then $\mathcal{C}_G$ is enriched over $G\mathcal{S}_*$, where we equip the mapping space $\mathrm{Map}_{\mathcal{C}}(A,B)$ with the conjugation action. It follows formally that passing to G-fixed points on mapping spaces yields a category $G\mathcal{C}$ enriched over $\mathcal{S}_*$ with objects the G-objects and spaces of equivariant maps.

Assume now in addition that $G\mathcal{C}$ has a model structure and $\mathcal{C}_G$ is tensored and cotensored over $\mathcal{S}_{*G}$. The latter means that there are functors

$$\mathcal{S}_{*G} \times \mathcal{C}_G \longrightarrow \mathcal{C}_G,\ (X,C) \longmapsto X \otimes C,$$
$$(\mathcal{S}_{*G})^{\mathrm{op}} \times \mathcal{C}_G \longrightarrow \mathcal{C}_G,\ (X,D) \longmapsto \mathrm{map}_{\mathcal{C}}(X,D)$$

together with natural associativity isomorphisms and natural G-isomorphisms

$$\mathrm{Map}_{\mathcal{C}}(X\otimes C, D) \cong \mathrm{Map}_{\mathcal{S}_*}(X, \mathrm{Map}_{\mathcal{C}}(C,D)) \cong \mathrm{Map}_{\mathcal{C}}(C, \mathrm{map}_{\mathcal{C}}(X,D)).$$

Passing to G-fixed points shows that $G\mathcal{C}$ is automatically tensored and cotensored over $\mathcal{S}_*$, though in general not over $G\mathcal{S}_*$.

Given two maps $i\colon A \to X$ and $p\colon E \to B$ in $G\mathcal{C}$, there is a G-map

$$\mathrm{Map}_{\mathcal{C}}(i^*, p_*)\colon\ \mathrm{Map}_{\mathcal{C}}(X,E) \longrightarrow \mathrm{Map}_{\mathcal{C}}(A,E) \times_{\mathrm{Map}_{\mathcal{C}}(A,B)} \mathrm{Map}_{\mathcal{C}}(X,B)$$

and we have

Definition 2.2. In the situation above, $G\mathcal{C}$ is called *G-simplicial* if for all cofibrations i and all fibrations p the map $\mathrm{Map}_{\mathcal{C}}(i^*, p_*)$ is a G-fibration, which is in addition a G-equivalence if i or p is.

Example 2.3. The most elementary example is, of course, $G\mathcal{S}_*$ itself.

Lemma 2.4. *The following are equivalent:*

(a) $G\mathcal{C}$ is G-simplicial.

(b) For all cofibrations $f\colon A \to X$ in $G\mathcal{C}$ and all G-cofibrations $i : K \to L$ in $G\mathcal{S}_*$ the pushout product map

$$i \square f\colon K \otimes X \cup_{K\otimes A} L \otimes A \longrightarrow L \otimes X$$

is a cofibration, which is in addition acyclic if f or i is.

(c) For all fibrations $p\colon E \to B$ in $G\mathcal{C}$ and all G-cofibrations $i\colon K \to L$ in $G\mathcal{S}_*$ the map

$$\mathrm{map}_{\mathcal{C}}(i^*, p_*)\colon \mathrm{map}_{\mathcal{C}}(L,E) \longrightarrow \mathrm{map}_{\mathcal{C}}(K,E) \times_{\mathrm{map}_{\mathcal{C}}(K,B)} \mathrm{map}_{\mathcal{C}}(L,B)$$

is a fibration, which is in addition acyclic if i or p is.

Proof. See [**4**, Proposition 3.11, Proposition 3.13] for a proof in the non-equivariant case. The proof in the equivariant case is similar. □

We also have

Lemma 2.5. *Suppose $G\mathcal{C}$ is G-simplicial. A map $f\colon A \to B$ between cofibrant objects is a weak equivalence if and only if for all fibrant objects X the induced map*

$$\mathrm{Map}_{\mathcal{C}}(f^*, X)\colon\ \mathrm{Map}_{\mathcal{C}}(B, X) \longrightarrow \mathrm{Map}_{\mathcal{C}}(A, X)$$

is a G-equivalence.

Proof. See [**2**, Lemma 4.5] for instance. □

3. Recollections on G-symmetric spectra

From now on, G denotes a fixed finite group. In this section we give a brief account on the approach to G-symmetric spectra as developed in [**5**]. We refer to the same paper for a discussion of the relation to other models such as the equivariant symmetric spectra of [**9**].

3.1. G-symmetric spectra

If M is a finite set, we denote by S^M the M-fold smash product of the simplicial circle $S^1 = \Delta^1/\partial\Delta^1$. The set $\{1, \dots, n\}$ endowed with the trivial G-action will be denoted by $\mathbf{n}$.

Definition 3.1. A *symmetric spectrum* X consists of

(a) for all $n \geqslant 0$, a based Σ_n-space X_n and

(b) for all $n \geqslant 0$, a based structure map $\sigma_n\colon X_n \wedge S^1 \to X_{n+1}$.

This is subject to the condition that for all n, $m \in \mathbb{N}$, the *iterated structure map*

$$\sigma_n^m\colon X_n \wedge S^{\mathbf{m}} \cong (X_n \wedge S^1) \wedge S^{\mathbf{m-1}} \longrightarrow X_{n+1} \wedge S^{\mathbf{m-1}} \longrightarrow \dots \longrightarrow X_{n+m}$$

is $\Sigma_n \times \Sigma_m$-equivariant. A morphism of symmetric spectra $f\colon X \to Y$ is a sequence of based Σ_n-maps $f_n\colon X_n \to Y_n$ such that, for all $n \in \mathbb{N}$, the square

$$\begin{tikzcd} X_n \wedge S^1 \arrow[r, "f_n \wedge S^1"] \arrow[d, "\sigma_n^X"'] & Y_n \wedge S^1 \arrow[d, "\sigma_n^Y"] \\ X_{n+1} \arrow[r, "f_{n+1}"'] & Y_{n+1} \end{tikzcd}$$

commutes. The category of symmetric spectra will be denoted by Sp^{Σ}.

Definition 3.2. A *G-symmetric spectrum* is a G-object in Sp^{Σ}. A morphism between G-symmetric spectra is a morphism of symmetric spectra commuting with the G-action. The category of G-symmetric spectra will be denoted by GSp^{Σ}.

Let M be a finite G-set of order m. We endow the space S^M with the G-action

$$g \cdot (\wedge_{l \in M} x_l) := \wedge_{l \in M} x_{g^{-1}l}.$$

The set of bijections $\mathrm{Bij}(m, M)$ carries a left G-action by postcomposition and a right Σ_m-action by precomposition. The value of a G-symmetric spectrum X at M

is defined to be the G-space

$$X(M) := X_m \wedge_{\Sigma_m} \mathrm{Bij}(m, M)^+,$$

where one identifies $x \wedge f\sigma$ with $\sigma_*(x) \wedge f$ whenever $\sigma \in \Sigma_m$ and G acts diagonally. The $G \times \Sigma_n$-space $X(\mathbf{n})$ is naturally isomorphic to X_n, by sending $[x \wedge \phi]$ to $\phi_*(x)$. In Appendix B below, it turns out to be more convenient to work with the values at $\mathbf{n}$, so we adopt this point of view from now on.

Given two finite G-sets M and N, there is a generalized structure map

$$\sigma_M^N \colon X(M) \wedge S^N \longrightarrow X(M \sqcup N).$$

Chosen any isomorphism $\psi \colon n \to N$, it is given by

$$([x \wedge f] \wedge s) \mapsto [\sigma_m^n(x \wedge \psi_*^{-1}(s)) \wedge (f \sqcup \psi)].$$

(The map σ_M^N does not depend on this choice.) The morphism spaces of symmetric spectra equipped with the conjugation action give rise to an enrichment of Sp_G^Σ over $G\mathcal{S}_*$ as described in Section 2.3 (cf. [**5**, Section 2.6]).

3.2. The flat model structures

We will now give a recollection on the flat model structures constructed in [**5**]. They are equivariant versions of the flat model structures of Shipley [**21**].

3.2.1. The flat level model structure

For the definition of the latching objects we refer the reader to Appendix B. Recall the definitions of the families $\mathcal{G}_n$ given in Example 2.1 and the various model structures on $G \times \Sigma_n$-spaces introduced in Section 2.2.

A morphism $f \colon X \to Y$ of G-symmetric spectra is a *G-level equivalence* (resp. *G-level fibration*) if, for all $n \in \mathbb{N}$, the $G \times \Sigma_n$-map $f(\mathbf{n}) \colon X(\mathbf{n}) \to Y(\mathbf{n})$ is a $\mathcal{G}_n$-equivalence (resp. mixed $G \times \Sigma_n$-fibration). The morphism f is a *G-flat cofibration* if, for all $n \in \mathbb{N}$, the pushout product map

$$\nu_n(f) \colon X(\mathbf{n}) \cup_{L_n(X)} L_n(Y) \longrightarrow Y(\mathbf{n})$$

is a $G \times \Sigma_n$-cofibration.

Proposition 3.3 (Flat level model structure)**.** *The classes of G-flat cofibrations, G-level fibrations and G-level equivalences define a proper G-simplicial model structure on the category of G-symmetric spectra.*

Proof. This is [**5**, Corollary 2.33, Proposition 2.38]. □

Remark 3.4. Cofibrant objects will be simply referred to as being *G-flat* in the following.

3.2.2. The stable flat model structure and π_*-isomorphisms

We will also need the *stable flat model structure* on G-symmetric spectra.

Definition 3.5. A G-symmetric spectrum X is *G-level fibrant* if, for all subgroups $H \leqslant G$ and all finite H-sets M, the space $X(M)^H$ is Kan.

Definition 3.6. A G-symmetric spectrum X is called a *$G\Omega$-spectrum* if it is G-level fibrant and, for all subgroups $H \leqslant G$ and all finite H-sets M and N, the adjoint of the generalized structure map

$$X(M) \longrightarrow \mathrm{Map}_{\mathcal{S}_*}(S^N, X(M \sqcup N))$$

is an H-equivalence.

Definition 3.7. A map $X \to Y$ of G-symmetric spectra is a *G-stable equivalence* if for all G-level fibrant $G\Omega$-spectra Z and for some G-flat replacement $X^c \to Y^c$ the induced map

$$[Y^c, Z]^G \longrightarrow [X^c, Z]^G$$

on based G-homotopy classes of G-maps is a bijection.

A *G-stable fibration* is a map that satisfies the right lifting property with respect to all G-flat cofibrations that are G-stable equivalences.

Theorem 3.8 (Stable flat model structure)**.** *The classes of G-flat cofibrations, G-stable equivalences and G-stable fibrations define a proper G-simplicial model category structure on the category of G-symmetric spectra. The fibrant objects are precisely the G-level fibrant $G\Omega$-spectra.*

Proof. This is [**5**, Theorem 4.10, Lemma 4.8]. □

An important fact is that π_*-isomorphisms as defined below are G-stable equivalences (cf. [**5**, Theorem 3.48]). To this end, let $\mathcal{U}$ be a complete G-set universe, that is a countably infinite G-set with the property that any finite G-set embeds infinitely often disjointly in it. We denote by $s(\mathcal{U})$ the set of all finite G-subsets of $\mathcal{U}$, partially ordered by inclusion. For all $n \geqslant 0$, a G-symmetric spectrum X gives rise to a functor

$$s(\mathcal{U}) \longrightarrow \mathrm{Sets}, \;\; M \mapsto [|S^{\mathbf{n}\sqcup M}|, |X(M)|]^G_*,$$

where $|-|$ denotes geometric realization. Here, an inclusion $M \subset N$ sends a map $f\colon |S^{\mathbf{n}\sqcup M}| \to |X(M)|$ to the composition

$$|S^{\mathbf{n}\sqcup N}| \cong |S^{\mathbf{n}\sqcup M}| \wedge |S^{N-M}| \longrightarrow |X(M)| \wedge |S^{N-M}| \longrightarrow |X(N)|,$$

where the last map is the geometric realization of the generalized structure map σ_M^{N-M}. For $n \geqslant 0$, we define

$$\pi_n^{G,\mathcal{U}}(X) := \operatorname*{colim}_{M \in s(\mathcal{U})} [|S^{\mathbf{n}\sqcup M}|, |X(M)|]^G_*.$$

In order to define the negative homotopy groups, for a G-symmetric spectrum X and a finite G-set M, we define the shift $sh^M X$ by

$$(sh^M X)(\mathbf{n}) := X(M \sqcup \mathbf{n}),$$

where the structure maps are induced by the structure maps of X (cf. [**5**, Definition 2.16]). Then, for $n < 0$, we set

$$\pi_n^{G,\mathcal{U}}(X) := \pi_0^{G,\mathcal{U}}(sh^{-\mathbf{n}} X).$$

Definition 3.9. A map $f\colon X \to Y$ of G-symmetric spectra is a *π_*-isomorphism* if for all subgroups $H \leqslant G$ and all $n \in \mathbb{Z}$ and some (hence any) complete G-set universe $\mathcal{U}$ the map

$$\pi_n^{H,\mathcal{U}} f\colon \pi_n^{H,\mathcal{U}}(X) \longrightarrow \pi_n^{H,\mathcal{U}}(Y)$$

is an isomorphism, where X and Y are considered as H-symmetric spectra via restriction and $\mathcal{U}$ is considered as a complete H-set universe via restriction.

Recall that the category of G-symmetric spectra is symmetric monoidal under the smash product with unit $\mathbb{S}$ [**5**, Section 2.6]. Then we have the following result.

Proposition 3.10. *Smashing with a G-flat G-symmetric spectrum preserves π_*-isomorphisms and G-stable equivalences.*

Proof. This is [**5**, Proposition 7.1]. □

4. Γ-G-spaces and two level model structures

In this section we introduce Γ-G-spaces and the projective and strict level model structures.

4.1. Generalities on Γ-G-spaces

Definition 4.1. We define Γ to be the category with objects the based finite sets $n^+ = \{1, \dots, n\} \sqcup \{+\}$ based at $\{+\}$ together with basepoint preserving maps.

Remark 4.2. The category Γ is the opposite of the category considered by Segal in [**16**].

Definition 4.3. A Γ-space is a functor $\Gamma \to \mathcal{S}_*$ such that $A(0^+) = *$. A map of Γ-spaces is a natural transformation. The category of Γ-spaces is denoted by $\Gamma(\mathcal{S}_*)$. A Γ-G-space is a G-object in the category of Γ-spaces. A map of Γ-G-spaces is a natural transformation of functors which commutes with the G-action. The category of Γ-G-spaces will be denoted by $G\Gamma(\mathcal{S}_*)$.

Given a Γ-G-space A, the value $A(X)$ at a based G-space X is defined by

$$A(X) = \bigsqcup_{n^+ \in \Gamma} \mathrm{Map}_{\mathcal{S}_*}(n^+, X) \times A(n^+) \;/ \sim,$$

where we divide out the equivalence relation generated by $(\phi^* f, a) \sim (f, A(\phi)(a))$ for $f\colon n^+ \to X$, $\phi\colon k^+ \to n^+$ and $a \in A(k^+)$. This space is based at the equivalence class of $(*, *)$.

Remark 4.4. It seems natural to define equivariant Γ-spaces as equivariant functors from the category Γ_G of finite based G-sets with based maps to the category $\mathcal{S}_{*G}$, where G acts on the morphisms of the categories by conjugation. But Shimakawa observed in [**20**, Theorem 1] that this gives a category equivalent to $G\Gamma(\mathcal{S}_*)$. The reason is precisely that the values of an equivariant functor $A\colon \Gamma_G \to \mathcal{S}_{*G}$ on a based finite G-set S^+ can be recovered by evaluating the underlying Γ-space on S^+.

If X is a based G-space and A is an object of $G\Gamma(\mathcal{S}_*)$, then, in level n^+, $X \wedge A$ and $\mathrm{map}_{\Gamma(\mathcal{S}_*)}(X, A)$ are given by $X \wedge A(n^+)$ with diagonal action and $\mathrm{Map}_{\mathcal{S}_*}(X, A(n^+))$ with conjugation action, respectively. The enrichment of $\Gamma(\mathcal{S}_*)_G$ in G-spaces is constructed as follows. Suppose A and B are Γ-G-spaces. $\mathrm{Map}_{\Gamma(\mathcal{S}_*)}(A, B)$ is the space with n-simplices consisting of the (not necessarily equivariant) natural transformations $(\Delta^n)^+ \wedge A \to B$ endowed with the conjugation action. Passing to fixed points yields a space $\mathrm{Map}_{\Gamma(\mathcal{S}_*)}(A, B)^G$ with 0-simplices $G\Gamma(\mathcal{S}_*)(A, B)$.

For a Γ-G-space A and an arbitrary based finite G-set S^+ we have a G-map

$$P_{S^+} \colon A(S^+) \longrightarrow A(1^+)^{S^+} = \mathrm{Map}_{\mathcal{S}_*}(S^+, A(1^+))$$

given by $(P_{S^+}(a))(s) = A(p_s)(a)$, where $p_s \colon S^+ \to 1^+$ maps s to 1 and everything else to the basepoint.

Definition 4.5. A Γ-G-space A is called *special* if P_{S^+} is a G-equivalence for all based finite G-sets S^+.

Let A be special and define the fold map $\nabla \colon 2^+ \to 1^+$ to be the map sending 1 and 2 to 1. The zigzag

$$A(1^+) \times A(1^+) \xleftarrow[\simeq_G]{p_1 \times p_2} A(2^+) \xrightarrow{\nabla} A(1^+)$$

induces the structure of a commutative monoid on the set $\pi_0^H(A(1^+))$ of path components of the H-fixed points of $A(1^+)$.

Definition 4.6. A special Γ-G-space A is called *very special* if, in addition, $A(1^+)$ is grouplike. That is, $\pi_0^H(A(1^+))$ with the composition law just defined is a group for all subgroups $H \leqslant G$.

4.2. Level model structures

Now we introduce two level model structures on the category of Γ-G-spaces.

4.2.1. The projective model structure

There is the projective model structure on $G\Gamma(\mathcal{S}_*)$, which was also employed in [**18**]. Recall the sets I and J defined in Section 2.2. A morphism of Γ-G-spaces $f \colon A \to B$ is called *level equivalence* (resp. *level fibration*) if $f(S^+)$ is a G-equivalence (resp. G-fibration) for all based finite G-sets S^+. A *projective cofibration* is a map which satisfies the left lifting property with respect to all level fibrations which are in addition level equivalences.

We define ΓI and ΓJ to be the sets consisting of the maps $i \wedge \Gamma_G(S^+, -)$ and $j \wedge \Gamma_G(S^+, -)$, respectively, where S^+ runs over a set of representatives of isomorphism classes of based finite G-sets and $i \in I$ and $j \in J$, respectively.

The proof of the following result is standard.

Theorem 4.7 (Projective model structure)**.** *The classes of level fibrations, level equivalences and projective cofibrations define a cofibrantly generated proper G-simplicial model category structure on the category of Γ-G-spaces. The sets ΓI and ΓJ can be taken as sets of generating cofibrations and generating acyclic cofibrations, respectively.*

At last, we want to mention an important lemma about the coend $A(X)$ of $A \in G\Gamma(\mathcal{S}_*)$ and $X \in G\mathcal{S}_*$.

Lemma 4.8. *Suppose $f\colon A \to B$ in $G\Gamma(\mathcal{S}_*)$ is a level equivalence, then, for any based G-space X, $f(X)\colon A(X) \to B(X)$ is a G-equivalence.*

Proof. First of all, we observe that $A(X)$ is just the diagonal of the bisimplicial set $B_{n,m} = A(X_n)_m$. Since taking fixed points commutes with taking the diagonal, it suffices to show that $A(X_n)_*^H \to B(X_n)_*^H$ is an ordinary weak equivalence for all subgroups $H \leqslant G$ (cf. [**2**, Theorem B.2]). This holds true by assumption if all X_n are finite. Moreover, if $S^+ \to T^+$ is an injective map of based finite G-sets, then $A(S^+)^H \to A(T^+)^H$ is injective, too, hence is a cofibration upon geometric realization. The assertion now follows, because any based G-set is the filtered colimit of its based finite G-subsets and homotopy groups commute with filtered colimits along cofibrations. □

4.2.2. The strict model structure

We will now introduce a model structure which reduces to the model structure due to Bousfield and Friedlander (cf. [**2**]) if G is the trivial group. In order to introduce this model structure we need

Definition 4.9. The nth *skeleton* of a Γ-G-space A is the Γ-G-space given by

$$(\mathrm{sk}_n A)(m^+) := \operatorname*{colim}_{k^+ \to m^+,\ k \leqslant n} A(k^+)$$

in level m^+. Dually, the nth *coskeleton* of A is defined to be the Γ-G-space given by

$$(\mathrm{csk}_n A)(m^+) := \lim_{m^+ \to j^+,\ j \leqslant n} A(j^+).$$

Remark 4.10. More conceptual definitions of these two functors appear in Appendix A, where they occur naturally when maps between Γ-G-spaces are constructed inductively.

There are natural maps $(\mathrm{sk}_n A) \to A \to (\mathrm{csk}_n A)$. Hence, a map $f\colon X \to Y$ of Γ-G-spaces induces, for all $n \geqslant 0$, maps

$$i_n(f)\colon (\mathrm{sk}_{n-1} Y)(n^+) \cup_{(\mathrm{sk}_{n-1} X)(n^+)} X(n^+) \longrightarrow Y(n^+)$$

and

$$p_n(f)\colon X(n^+) \longrightarrow (\mathrm{csk}_{n-1} X)(n^+) \times_{(\mathrm{csk}_{n-1} Y)(n^+)} Y(n^+).$$

Then f is called a *strict cofibration* (resp. *strict fibration*) if, for all $n \geqslant 0$, the map $i_n(f)$ (resp. $p_n(f)$) is a $\mathcal{G}_n$-cofibration (resp. $\mathcal{G}_n$-fibration). The map f is called a *strict equivalence* if it is levelwise a $\mathcal{G}_n$-equivalence.

Remark 4.11. Note, for any map of Γ-G-spaces $f\colon A \to B$, being a strict equivalence amounts to saying that for any based finite H-set S^+, $f(S^+)$ is an H-equivalence. Yet, it turns out that this is a little bit redundant. Indeed, assume the seemingly weaker condition that $f(S^+)$ is a G-equivalence for all pointed G-sets S^+ and let T^+ be any based finite H-set. But T^+ is an H-retract of a G-set Q^+ and H-equivalences are closed under retracts, hence $f(T^+)$ is an H-equivalence. In particular, the strict equivalences coincide with the level equivalences defined in Section 4.2.1.

We have

Theorem 4.12 (Strict model structure)**.** *The classes of strict equivalences, strict fibrations and strict cofibrations define a proper G-simplicial model category structure on the category of Γ-G-spaces.*

Proof. Let $\mathcal{G}_n^1\mathcal{S}_*$ be the category of $G \times \Sigma_n$-spaces with the $\mathcal{G}_n$-model structure and let $\mathcal{G}_n^2\mathcal{S}_* := (G \times \Sigma_n)\mathcal{S}_*$ be the category of $G \times \Sigma_n$-spaces with the $\mathcal{ALL}$-model structure. Then the assumptions of Theorem A.1 are satisfied. The model structure is right proper by Lemma A.9 and the fact that $\mathcal{G}_n^1\mathcal{S}_*$ is right proper. It is left proper by Lemma A.6, the fact that fixed points respects pushouts along inclusions, and the fact that the usual model structure on $\mathcal{S}_*$ is left proper. □

4.2.3. Comparison of the projective and the strict model structure

We have the following basic result.

Proposition 4.13. *The identity functor induces a Quillen equivalence*

$$\mathrm{id} : G\Gamma(\mathcal{S}_*)^{projective} \rightleftarrows G\Gamma(\mathcal{S}_*)^{strict} : \mathrm{id}.$$

Proof. We have already seen that the weak equivalences coincide. Hence, we only have to check that the generating cofibrations are indeed strict cofibrations. In fact, if $i\colon A \to B$ is a G-cofibration of G-spaces and S^+ is any based finite G-set, then all simplices not in the image of $i_n(i \wedge \Gamma_G(S^+, -))$ have isotropy contained in $\mathcal{G}_n$, since the set $(\mathrm{sk}_{n-1}\,\Gamma_G(S^+, -))(n^+)$ consists precisely of the non-surjective maps. □

5. Unstable comparison of Γ-G-spaces and G-symmetric spectra

A Γ-G-space A gives rise to a G-symmetric spectrum $A(\mathbb{S})$. We show in Section 5.1 how this construction yields Quillen pairs between the strict and projective model structures on Γ-G-spaces and the flat level model structure on G-symmetric spectra. In Section 5.2, we show that the spectra obtained from Γ-G-spaces are connective and that very special Γ-G-spaces yield $G\Omega$-spectra upon a level fibrant replacement. Finally, in the last subsection we compare suitable subcategories of the homotopy categories of Γ-G-spaces and G-symmetric spectra with respect to the level model structures.

5.1. Quillen pairs between Γ-G-spaces and G-symmetric spectra

Let A be a Γ-G-space. The G-symmetric spectrum $A(\mathbb{S})$ is given by $A(S^{\mathbf{n}})$ in level n. Here G acts on A and Σ_n acts by permuting the sphere coordinates. The structure map $\sigma_n\colon A(S^{\mathbf{n}}) \wedge S^1 \to A(S^{\mathbf{n+1}})$ is defined by sending a class $[(v_1, \dots, v_n), a] \wedge w$ to the class $[(v_1 \wedge w, \dots, v_n \wedge w), a]$.

Conversely, given a G-symmetric spectrum, we may construct a Γ-G-space denoted $\Phi(\mathbb{S}, X)$ by setting $\Phi(\mathbb{S}, X)(n^+) := \mathrm{Map}_{Sp^\Sigma}(\mathbb{S}^{\times n}, X)$. With these definitions, we have

Proposition 5.1. *The functors*

$$(-)(\mathbb{S}) : G\Gamma(\mathcal{S}_*) \rightleftarrows GSp^\Sigma : \Phi(\mathbb{S}, -)$$

form a Quillen pair between the strict model structure on Γ-G-spaces and the flat level model structure on G-symmetric spectra.

Proof. It is well-known that this is an adjunction (cf. [**16**, Proposition 3.3], [**2**, Lemma 4.6]). We prove that $\Phi(\mathbb{S},-)$ sends G-level fibrations (resp. acyclic G-level fibrations) to strict fibrations (resp. acyclic strict fibrations). Note that, for any G-symmetric spectrum X, we have

$$(\mathrm{csk}_{n-1}\,\Phi(\mathbb{S},X))(n^+) = \lim_{n^+\to j^+,\ j\leqslant n-1} \mathrm{Map}_{Sp^\Sigma}(\mathbb{S}^{\times j},X) \cong \mathrm{Map}_{Sp^\Sigma}(\mathbb{S}^{\times n}_{\leqslant n-1},X),$$

where

$$(\mathbb{S}^{\times n}_{\leqslant n-1})_m = \{(x_1,\dots,x_n)\in (S^m)^{\times n} \mid x_i = x_j \text{ for some } i\neq j \text{ or } x_i = * \text{ for some } i\},$$

as a direct computation of colimits proves. So we have to show that the map

$$\mathrm{Map}_{Sp^\Sigma}(\mathbb{S}^{\times n},X) \longrightarrow \mathrm{Map}_{Sp^\Sigma}(\mathbb{S}^{\times n}_{\leqslant n-1},X)\times_{\mathrm{Map}_{Sp^\Sigma}(\mathbb{S}^{\times n}_{\leqslant n-1},Y)} \mathrm{Map}_{Sp^\Sigma}(\mathbb{S}^{\times n},Y)$$

induced by the inclusion $\mathbb{S}^{\times n}_{\leqslant n-1}\to \mathbb{S}^{\times n}$ and a G-level fibration (resp. acyclic G-level fibration) $f\colon X\to Y$ is a $\mathcal{G}_n$-fibration (resp. acyclic $\mathcal{G}_n$-fibration).

Equivalently, for all n and all based finite H-sets S^+ of order $n+1$, the map

$$\mathrm{Map}_{Sp^\Sigma}(\mathbb{S}^{\times S},X) \longrightarrow \mathrm{Map}_{Sp^\Sigma}(\mathbb{S}^{\times S}_{\leqslant n-1},X)\times_{\mathrm{Map}_{Sp^\Sigma}(\mathbb{S}^{\times S}_{\leqslant n-1},Y)} \mathrm{Map}_{Sp^\Sigma}(\mathbb{S}^{\times S},Y)$$

is an H-fibration. Since H-symmetric spectra are H-simplicial, it is, therefore, sufficient to show that $\mathbb{S}^{\times S}_{\leqslant n-1}\to\mathbb{S}^{\times S}$ is an H-flat cofibration of H-symmetric spectra. This is the content of Proposition B.1 in the appendix. □

Together with Proposition 4.13 this implies

Proposition 5.2. *The functors*

$$(-)(\mathbb{S}) : G\Gamma(\mathcal{S}_*) \rightleftarrows GSp^\Sigma : \Phi(\mathbb{S},-)$$

form a Quillen pair between the projective model structure on Γ-G*-spaces and the flat level model structure on* G*-symmetric spectra.*

5.2. Some properties of G-spectra of the form $A(\mathbb{S})$

5.2.1. Connectivity

We want to show that all the negative homotopy groups of a spectrum arising from a Γ-G-space vanish. This is accomplished by introducing a two sided bar construction. Recall the category Γ_G from Remark 4.4. By abuse of notation, we use the same symbol for a choice of small skeleton in the following.

Given a Γ-G-space A, we define a functor σA from Γ to G-spaces by setting

$$(\sigma A)(n^+) = B(n^+,\Gamma_G,A) = \mathrm{diag}(B_\bullet(n^+,\Gamma_G,A)),$$

where for a G-space X, $B_\bullet(X,\Gamma_G,A)$ denotes the simplicial G-space with k-simplices

$$B_k(X,\Gamma_G,A) = \textstyle\coprod_{s_0^+,\dots,s_k^+\in\Gamma_G} A(s_0^+)\times\Gamma_G(s_0^+,s_1^+)\times\cdots\times\Gamma_G(s_{k-1}^+,s_k^+)\times (X)^{s_k}.$$

There is a natural G-isomorphism $B(X,\Gamma_G,A)\to(\sigma A)(X)$ (cf. [**22**, Proof of Theorem 1.5]). Then $(\sigma A)/(\sigma A(0^+))$ is a Γ-G-space and the nth level of the G-symmetric spectrum $(\sigma A)(\mathbb{S})/(\sigma A(0^+))$ is G-isomorphic to $B(S^{\mathbf{n}},\Gamma_G,A)/B(*,\Gamma_G,A)$.

Lemma 5.3. *The map*

$$B_k(X,\Gamma_G,A) \longrightarrow A(X),\ (a,f_0,\dots,f_{k-1},\phi) \mapsto (\phi_* f_{k-1*}\dots f_{0*})(a)$$

induces a natural level equivalence $(\sigma A)/(\sigma A(0^+)) \to A$ of Γ-G-spaces. In particular, it induces a G-level equivalence of G-symmetric spectra

$$(\sigma A)(\mathbb{S})/(\sigma A)(0^+) \longrightarrow A(\mathbb{S}).$$

Proof. As in [**7**, Proposition 5.19], one shows that the map induces, for any based finite G-set S^+, a G-equivalence $(\sigma A)(S^+) \to A(S^+)$. The first statement follows now, since $(\sigma A)(0^+) \to (\sigma A)(S^+)$ is a G-cofibration. The second part follows from Lemma 4.8. □

Lemma 5.4. *Let A be a Γ-G-space. If X is any pointed G-space such that, for all $K \leqslant L$, $\mathrm{conn}(X^K) \geqslant \mathrm{conn}(X^L) \geqslant 1$ then, for all $H \leqslant G$, we have*

$$\mathrm{conn}(B(X,\Gamma_G,A)^H/B(x_0,\Gamma_G,A)^H) \geqslant \mathrm{conn}(X^H).$$

Proof. We check the connectivity of

$$|B_\bullet(|X|,\Gamma_G,|A|)^H/B_\bullet(x_0,\Gamma_G,|A|)^H|.$$

This space is the geometric realization of the simplicial space with k-simplices

$$\bigvee_{s_0^+,\dots,s_k^+\in\Gamma_G} (|A(s_0^+)|^+)^H \wedge (\Gamma_G(s_0^+,s_1^+)^+)^H \wedge \dots \wedge (\Gamma_G(s_{k-1}^+,s_k^+)^+)^H \wedge (|X|^{s_k})^H.$$

Now, each wedge summand is at least as connected as $(|X|^{s_k})^H$. Writing $s_k^+ \cong_H H/L_1^+ \vee \dots \vee H/L_j^+$, we find $(X^{s_k})^H \cong X^{L_1} \times \dots \times X^{L_n}$, which is, by our assumptions, firstly, at least as connected as X^H and, secondly, simply connected. Hence, all simplicial levels are so. The space in question is a wedge of based G-CW-complexes of the form $|A(S^+)|^+ \wedge |X|^T$ and the degeneracy maps are just inclusions of certain wedge summands. One can now argue as in [**8**, Theorem 11.12]. □

Corollary 5.5. *The spectrum $A(\mathbb{S})$ is connective for all Γ-G-spaces A. That is for all $H \leqslant G$, for all complete G-set universes $\mathcal{U}$, and for all $n<0$, we have $\pi_n^{H,\mathcal{U}}(A(\mathbb{S}))=0$.*

Proof. Fix $n \geqslant 1$. The $(-n)$th homotopy group with respect to H is computed as a colimit over finite H-sets M of $[|S^M|,|A(S^{M\sqcup\mathbf{n}})|]_*^H$. So we may assume $|M^H| \geqslant 1$. For such M, we have, for all $K \leqslant H$,

$$\dim|S^M|^K = |M^K| \leqslant |M^K|+n-1 \leqslant \mathrm{conn}((A(S^{M\sqcup\mathbf{n}}))^K)$$

by Lemma 5.4 and hence $[|S^M|,|A(S^{M\sqcup\mathbf{n}})|]_*^H = 0$ by [**1**, Proposition 2.5]. □

5.2.2. Very special Γ-G-spaces and $G\Omega$-spectra

We briefly recall how one obtains $G\Omega$-spectra from very special Γ-G-spaces (cf. [**19**]). The next lemma and proposition are simplicial analogues of [**18**, Lemma 7.5, Theorem 7.6].

Lemma 5.6. *Suppose a Γ-G-space A is special. Then, for any based G-simplicial set X, the Γ-G-space $A(X)$, defined by $n^+ \mapsto A(n^+ \wedge X)$ is special, too. If A is very special, then $A(X)$ is very special, too.*

Proof. We first show that $A(X)$ is special again provided A is special. We need to show that the map $A(S^+ \wedge X) \to \mathrm{Map}_{\mathcal{S}_*}(S^+, A(X))$ is a G-equivalence for any based finite G-set S^+. This morphism is the diagonal of a morphism of bisimplicial G-sets, hence it suffices to check that, for all $n \geqslant 0$, $A(S^+ \wedge X_n) \to \mathrm{Map}_{\mathcal{S}_*}(S^+, A(X_n))$ is a G-equivalence. But both sides are G-equivalent to the weak product $\mathrm{Map}'_{\mathcal{S}_*}(S^+ \wedge X_n, A(1^+))$, since A is special.

Now, assume that A is very special. We have to show that $\pi_0(A(X))$ is a group. Equivalently, we have to show that the map $A(X \vee X) \to A(X) \times A(X)$ induced by retraction onto the first summand and the fold map is a G-equivalence. But again, it suffices to show that, for all $n \geqslant 0$, the map $A(X_n \vee X_n) \to A(X_n) \times A(X_n)$ is a G-equivalence. This is so because A is very special. □

Proposition 5.7. *Suppose A is a very special Γ-G-space. Then the G-symmetric spectrum $S(|A(\mathbb{S})|)$, where $|-|$ and $S(-)$ denote geometric realization and singular complex functor, respectively, is a $G\Omega$-spectrum.*

Proof. The proof is similar to the proof of [**18**, Theorem 7.6]. Fix a subgroup $H \leqslant G$.

Suppose T is any finite H-set. In view of Lemma 5.3 and Lemma 5.6, the cofiber sequence

$$S^T \wedge S^0 \longrightarrow S^T \wedge \Delta^1 \longrightarrow S^{T \sqcup \mathbf{1}}$$

induces an H-fibration sequence upon applying $A(-)$ by [**19**, Lemma 1.4. P2]. This implies that the adjoint of the geometric realization of the structure map $\sigma_T^{\mathbf{1}}$ is an H-equivalence. By [**19**, Theorem B], for any two H-sets M and N, the adjoint of the geometric realization of the structure map $\sigma_{M \sqcup \mathbf{1}}^N$ is an H-equivalence, too. It follows that in the diagram

$$\begin{tikzcd}
{|A(S^M)|} \arrow[r, "\simeq"] \arrow[d] & \mathrm{Map}_{\mathcal{T}_*}(S^1, |A(S^{M \sqcup \mathbf{1}})|) \arrow[d, "\simeq"] \\
\mathrm{Map}_{\mathcal{T}_*}(|S^N|, |A(S^{M \sqcup N})|) \arrow[r, "\simeq"] & \mathrm{Map}_{\mathcal{T}_*}(|S^{N \sqcup \mathbf{1}}|, |A(S^{M \sqcup N \sqcup \mathbf{1}})|)
\end{tikzcd}$$

the leftmost map is an H-equivalence, too. Finally, since $S(|A(\mathbb{S})|)$ is G-level fibrant, it is a $G\Omega$-spectrum. □

5.3. Comparison of very special Γ-G-spaces and connective $G\Omega$-spectra

The last thing we need before comparing suitable homotopy categories of very special Γ-G-spaces and $G\Omega$-spectra is a special instance of the Wirthmüller isomorphism.

Lemma 5.8. *For any finite G-set S the inclusion*

$$\bigvee_S \mathbb{S} \longrightarrow \prod_S \mathbb{S}$$

is a π_*-isomorphism, hence a G-stable equivalence, of G-symmetric spectra.

Proof. Both spectra are connective, so it suffices to show that all non-negative homotopy groups of the cofiber vanish. We choose a G-set M which contains every orbit type at least once and which admits an injective map $\iota\colon S \to M$. It suffices to show that $[|S^{k\cdot M+q}\wedge G/H_+|, |\prod_S S^{k\cdot M}/\vee_S S^{k\cdot M}|]_*^G = 0$ for all $H \leqslant G$, $q \in \mathbb{N}$ and $k \geqslant k_0$. This follows from [**1**, Prop. 2.5] if we can show that, for all L, eventually,

$$k|M^L| + q \leqslant \operatorname{conn}\left(\prod_S S^{k\cdot M}/\bigvee_S S^{k\cdot M}\right)^L. \tag{1}$$

Now, L-equivariantly we have a decomposition $S^+ \cong_L L/J_1^+ \vee \ldots \vee L/J_n^+$ and then

$$\left(\prod_S S^{k\cdot M}/\bigvee_S S^{k\cdot M}\right)^L \cong \prod_{i=1}^n S^{k|M^{J_i}|}/\bigvee_{i\colon J_i=L} S^{k|M^{J_i}|}.$$

There are two cases to distinguish. If $J_i = L$ for all i, the connectivity of this space is at least $2k|M^L| - 1$, so that (1) holds from some k_0 on. Otherwise, there is at least one i with $J_i \neq L$ and the connectivity is at least $k \cdot \min_{i\colon J_i \neq L}|M^{J_i}| - 1$. But $|M^{J_i}| > |M^L|$ for all i such that $J_i \neq L$, hence (1) holds for k large enough. □

We can now prove the equivariant analogue of [**2**, Theorem 5.1].

Theorem 5.9. *The derived adjoint functors*

$$\mathrm{Ho}(\mathrm{G}\Gamma(\mathcal{S}_*)^{\mathrm{strict}}) \rightleftarrows \mathrm{Ho}((\mathrm{GSp}^{\Sigma})^{\mathrm{flat\ level}})$$

restrict to mutually inverse equivalences of categories when restricted to the full subcategories given by very special Γ-*G-spaces and G-symmetric spectra which are G-level equivalent to connective G*Ω*-spectra, respectively.*

Proof. Given a very special Γ-G-space, we have seen that $A(\mathbb{S})$ is G-level equivalent to a $G\Omega$-spectrum. Suppose X is a G-symmetric spectrum X which is G-level equivalent to a connective $G\Omega$-spectrum. Then a fibrant replacement X_f in the flat level model structure is a connective $G\Omega$-spectrum, since it is G-level fibrant by [**5**, p. 11]. The Γ-G-space associated to X_f is special by Lemma 2.5, because for any finite G-set S the inclusion $\vee_S\mathbb{S} \to \prod_S \mathbb{S}$ is a G-stable equivalence by the Wirthmüller isomorphism and a G-flat cofibration between G-flat spectra (Proposition B.1, Proposition B.4). It is grouplike because $\pi_0^H(\Phi(\mathbb{S}, X)(1^+))$ is a group and the monoid structures on $\pi_0^H(\Phi(\mathbb{S}, X)(1^+)) \cong \pi_0^H(X_0)$ coincide. So the functors are well-defined.

Suppose A is very special and X is a G-level fibrant $G\Omega$-spectrum. If $A(\mathbb{S}) \to X$ is a G-level equivalence, then its adjoint $A \to \Phi(\mathbb{S}, X)$ is a level equivalence, since both are very special and

$$A(1^+) \simeq_G X_0 \cong \Phi(\mathbb{S}, X)(1^+).$$

Conversely, suppose $A \to \Phi(\mathbb{S}, X)$ is a level equivalence. Firstly, $A(\mathbb{S}) \to \Phi(\mathbb{S}, X)(\mathbb{S})$ is a G-level equivalence by Lemma 4.8. Secondly, the map $\Phi(\mathbb{S}, X)(\mathbb{S}) \to X$ is a π_*-isomorphism because $\Phi(\mathbb{S}, X)(1^+) \cong X_0$. And thirdly, a π_*-isomorphism of $G\Omega$-spectra is a G-level equivalence. A proof of this statement in the setting of G-orthogonal spectra can be found in [**10**, Section 9]. The arguments given there apply to our situation as well, because $G\Omega$-spectra are by assumption G-level fibrant. □

6. Stable comparison

We introduce stable model structures for the projective and the strict model structures. A map $f\colon A \to B$ of Γ-G-spaces is a *stable equivalence* if $f(\mathbb{S})$ is a π_*-isomorphism of G-symmetric spectra. A map between Γ-G-spaces is called a *stable fibration* if it satisfies the right lifting property with respect to all projective cofibrations which are stable equivalences. We define a map to be a *stable strict fibration* if it satisfies the right lifting property with respect to all strict cofibrations which are in addition stable equivalences.

Remark 6.1. It follows from [**5**, Proposition 2.47, Corollary 3.49], that f is a stable equivalence of Γ-G-spaces if and only if $f(\mathbb{S})$ is a G-stable equivalence of G-symmetric spectra.

Theorem 6.2 (Stable projective model structure)**.** *The classes of projective cofibrations, stable fibrations and stable equivalences define a left proper cofibrantly generated G-simplicial model category structure on the category of Γ-G-spaces. The stably fibrant objects are precisely the very special Γ-spaces X for which in addition $X(S^+)^H$ is Kan for all finite based G-sets S^+ and all subgroups $H \leqslant G$.*

Theorem 6.3 (Stable strict model structure)**.** *The classes of strict cofibrations, stable strict fibrations and stable equivalences define a left proper cofibrantly generated G-simplicial model category structure on the category of Γ-G-spaces. An object is stably strictly fibrant if and only if it is strictly fibrant and very special.*

Proofs of Theorems 6.2 and 6.3. The existence of the model structures, cofibrant generation, and the characterization of the stably fibrant objects follow in both cases along the lines of [**13**, Appendix A]. The model structures are left proper in view of the adjunction of Proposition 5.1 and left properness of the flat model structure on G-symmetric spectra. That the model structures are G-simplicial is a special case of the pushout product axiom which we prove in Proposition 7.7 below. □

Corollary 6.4. *The identity functor induces a Quillen equivalence between the stable projective and the stable strict model structures.*

Now we are in the position to prove the equivariant version of [**2**, Theorem 5.8].

Theorem 6.5. *The derived adjoint functors*

$$\mathrm{Ho}(G\Gamma(\mathcal{S}_*)^{\text{stable strict/stable projective}}) \rightleftarrows \mathrm{Ho}((G\mathrm{Sp}^{\Sigma})^{\text{flat stable}})$$

restrict to mutually inverse equivalences of categories when the right adjoint is restricted to the full subcategory given by connective G-symmetric spectra.

Proof. Consider a strictly cofibrant Γ-G-space A and a connective G-level fibrant $G\Omega$-spectrum X. Then $A \to \Phi(\mathbb{S}, X)$ is a stable equivalence if and only if $A(\mathbb{S}) \to X$ is a π_*-isomorphism (see the proof of Theorem 5.9). This implies that unit and counit of this adjunction are isomorphisms. □

For later usage we put the following on record.

Lemma 6.6. *Fix a complete G-set universe $\mathcal{U}$. If $i\colon A \to B$ is a map of Γ-G-spaces which is levelwise injective, then, for all subgroups $H \leqslant G$, there is a long exact sequence*

$$\cdots \longrightarrow \pi_n^{H,\mathcal{U}}(A(\mathbb{S})) \longrightarrow \pi_n^{H,\mathcal{U}}(B(\mathbb{S})) \longrightarrow \pi_n^{H,\mathcal{U}}(B/A(\mathbb{S})) \to \cdots \to \pi_0^{H,\mathcal{U}}(B/A(\mathbb{S})) \longrightarrow 0.$$

Proof. Indeed, being a colimit, taking the cone of a map commutes with the left adjoint $(-)(\mathbb{S})$. Moreover, the map $C(i) \to B/A$ is a level equivalence of Γ-G-spaces and so the map $C(i)(\mathbb{S}) \to B/A(\mathbb{S})$ is a G-level equivalence (cf. Lemma 4.8), in particular it is a π_*-isomorphism. The result thus follows from the usual long exact sequence of homotopy groups (cf. [**5**, Proposition 3.8]) and the connectivity of the spectra obtained from Γ-G-spaces. □

7. The smash product of Γ-G-spaces

7.1. The definition of the smash product

In [**7**], Lydakis defined a smash product for Γ-spaces. To begin with, we choose a smash product functor $\wedge\colon \Gamma \times \Gamma \to \Gamma$ (for example by identifying the usual smash product $m^+ \wedge n^+$ with $(mn)^+$ via $(i,j) \mapsto (i-1)n + j$). Given two Γ-spaces F and F' the nth level of the smash product $F \wedge F'$ is given by

$$(F \wedge F')(n^+) = \operatorname{colim}_{k^+ \wedge l^+ \to n^+} F(k^+) \wedge F(l^+).$$

$F \wedge F'$ is characterized by the property that maps of Γ-spaces $F \wedge F' \to T$ correspond bijectively to maps of $\Gamma \times \Gamma$-spaces $F(-) \wedge F'(-) \to T(- \wedge -)$. Elements in the smash product are represented by triples $[f, x \wedge y]$, where $f\colon k^+ \wedge l^+ \to n^+$ is a morphism in Γ and $x \wedge y \in F(k^+) \wedge F(l^+)$. We define the internal mapping object to be the Γ-space $Hom(F, F')(m^+) := Map_{\Gamma(\mathcal{S}_*)}(F, F'(m^+ \wedge -))$. Recall from [**7**, Theorem 2.18] that with these definitions, the category of Γ-spaces is closed symmetric monoidal with unit $\Gamma(1^+, -)$.

Consequently, the category of Γ-G-spaces is a closed symmetric monoidal category with unit $\Gamma(1^+, -)$ by defining the smash product of two Γ-G-spaces to be the smash product of the underlying Γ-spaces endowed with the diagonal G-action and equipping the internal mapping object with the conjugation action.

7.2. Smash product and cofibrations

We study the pushout product of two strict cofibrations (resp. projective cofibrations).

Lemma 7.1. *If F and F' are strictly cofibrant (resp. projectively cofibrant), then so is $F \wedge F'$.*

Proof. A Γ-G-space is strictly cofibrant if and only if its underlying Γ-space is strictly cofibrant in the sense of Bousfield and Friedlander [**2**]. So in the case of strict cofibrations, this follows from the non-equivariant case [**7**, Lemma 4.5].

In the case of projective cofibrations, this follows from the fact that $Hom(F, -)$ preserves level fibrations which are level equivalences if F is projectively cofibrant because the projective model structure is G-simplicial. □

Proposition 7.2. *If $F \to F'$, $\tilde{F} \to \tilde{F}'$ are two strict cofibrations (resp. projective cofibrations) of Γ-G-spaces, then the pushout product map*

$$F \wedge \tilde{F}' \cup_{F \wedge \tilde{F}} F' \wedge \tilde{F} \longrightarrow F' \wedge \tilde{F}'$$

is a strict cofibration (resp. projective cofibration).

Proof. The pushout product map is injective by [**7**, Proposition 4.4] and has cofiber isomorphic to $F'/F \wedge \tilde{F}'/\tilde{F}$ which is strictly cofibrant (resp. projectively cofibrant) by the previous lemma. This implies the claim, since a map is a strict cofibration (resp. projective cofibration) if and only if it is injective and its cofiber is strictly cofibrant (resp. projectively cofibrant). For strict cofibrations this statement follows directly from the definition. For projective cofibrations one can use an equivariant version of [**13**, Lemma A3]. In the equivariant situation, the *free* Γ-G-spaces are those of the form $\bigvee_i G^+ \wedge_{H_i} \Gamma_{H_i}(S_i^+, -)$ as defined below. □

7.3. Smash product and level equivalences

We show that smashing with a strictly cofibrant Γ-G-space preserves level equivalences.

Proposition 7.3. *For any Γ-G-space F and any positive integer m, there is a pushout square of Γ-G-spaces*

$$\begin{array}{ccc} \partial(\Gamma(m^+, -) \wedge_{\Sigma_m} F(m^+)) & \longrightarrow & \Gamma(m^+, -) \wedge_{\Sigma_m} F(m^+) \\ \downarrow & & \downarrow \\ (\mathrm{sk}_{m-1} F) & \longrightarrow & (\mathrm{sk}_m F), \end{array}$$

where $\partial(\Gamma(m^+, -) \wedge_{\Sigma_m} F(m^+))$ is defined as the pushout

$$\begin{array}{ccc} (\mathrm{sk}_{m-1}\Gamma(m^+, -)) \wedge_{\Sigma_m} (\mathrm{sk}_{m-1} F)(m^+) & \longrightarrow & (\mathrm{sk}_{m-1}\Gamma(m^+, -)) \wedge_{\Sigma_m} F(m^+) \\ \downarrow & & \downarrow \\ \Gamma(m^+, -) \wedge_{\Sigma_m} (\mathrm{sk}_{m-1} F)(m^+) & \longrightarrow & \partial(\Gamma(m^+, -) \wedge_{\Sigma_m} F(m^+)). \end{array}$$

Proof. This follow from the nonequivariant case [**7**, Theorem 3.10], because the forgetful functor from G-spaces to spaces detects pushouts. □

Next we prove an equivariant analogue of [**7**, Proposition 3.11]. Given a subgroup $H \leqslant G$ and a based finite H-set S^+ we will encounter Γ-G-spaces of the form $G^+ \wedge_H \Gamma_H(S^+, -)$. Here in the quotient we identify $gh \wedge \phi$ and $g \wedge \phi(h^{-1})$. The group G acts on the left smash factor.

Proposition 7.4. *For any strictly cofibrant Γ-G-set F and any $n \geqslant 0$ there exists a* *pushout diagram*

$$\begin{tikzcd} \bigvee_{i\in I} G^+ \wedge_{H_i} (\mathrm{sk}_{n-1}\,\Gamma_{H_i}(S_i^+,-)) \arrow[r] \arrow[d] & \bigvee_{i\in I} G^+ \wedge_{H_i} \Gamma_{H_i}(S_i^+,-) \arrow[d] \\ (\mathrm{sk}_{n-1} F) \arrow[r] & (\mathrm{sk}_n F), \end{tikzcd}$$

where I is a set and for $i \in I$, $H_i \leqslant G$ is a subgroup and S_i^+ is based finite H_i-set.

Proof. There are elements $s_i \in F(n^+)$ and subgroups $\Gamma_i = \{(h, \rho_i(h)) |\ H_i \leqslant G,$ $\rho_i \colon H_i \to \Sigma_n$ homomorphism$\}$, such that

$$(F/(\mathrm{sk}_{n-1} F))(n^+) \cong \bigvee_{i\in I} (G \times \Sigma_n/\Gamma_i) \cdot s_i.$$

For each i, we have the following isomorphism of Γ-G-spaces

$$\Gamma(n^+,-) \wedge_{\Sigma_n} (G \times \Sigma_n/\Gamma_i)^+ \longrightarrow G^+ \wedge_{H_i} \Gamma_{H_i}(S_i^+,-),\ [f \wedge (g,\sigma)\Gamma_i] \mapsto [g \wedge (f\sigma)],$$

where on the right S_i^+ denotes the set $\{1,\dots,n\}$ equipped with the H_i-action coming from ρ_i. So the s_i give rise to a map

$$\bigvee_{i\in I} G^+ \wedge_{H_i} \Gamma_{H_i}(S_i^+,-) \longrightarrow \Gamma(n^+,-) \wedge_{\Sigma_n} F(n^+).$$

The image of this map intersects $\partial(\Gamma(n^+,-) \wedge_{\Sigma_n} F(n^+))$ precisely in $\bigvee_{i\in I} G^+ \wedge_{H_i}$ $(\mathrm{sk}_{n-1}\,\Gamma_{H_i}(S_i^+,-))$ and any element can be lifted either to $\partial(\Gamma(n^+,-) \wedge_{\Sigma_n} F(n^+))$ or to $\bigvee_{i\in I} G^+ \wedge_{H_i} \Gamma_{H_i}(S_i^+,-)$. It follows that we have a pushout square

$$\begin{tikzcd} \bigvee_{i\in I} G^+ \wedge_{H_i} (\mathrm{sk}_{n-1}\,\Gamma_{H_i}(S_i^+,-)) \arrow[r] \arrow[d] & \bigvee_{i\in I} G^+ \wedge_{H_i} \Gamma_{H_i}(S_i^+,-) \arrow[d] \\ \partial(\Gamma(n^+,-) \wedge_{\Sigma_n} F(n^+)) \arrow[r] & (\Gamma(n^+,-) \wedge_{\Sigma_n} F(n^+)). \end{tikzcd}$$

Together with Proposition 7.3, this proves the result. □

Proposition 7.5. *Smashing with a strictly cofibrant Γ-G-space preserves level equivalences.*

Proof. Consider a strictly cofibrant Γ-G-space F and a level equivalence $f \colon A \to B$. The map $F \wedge A \to F \wedge B$ is the diagonal of the map of bisimplicial Γ-G-sets $(F_n \wedge A)_m \to (F_n \wedge B)_m$. Here the subscript denotes the simplicial degree. So we may assume that F is a strictly cofibrant Γ-G-set. Suppose for a moment that, for any $H \leqslant G$ and any based H-set S^+, smashing with $G^+ \wedge_H \Gamma_H(S^+,-)$ preserves level equivalences. In view of Proposition 7.4 it follows inductively that $(\mathrm{sk}_n F) \wedge f$ is a level equivalence for all $n \geqslant 0$. Then $F \wedge f$ is a level equivalence, because homotopy groups commute with filtered colimits along G-cofibrations and the maps $(\mathrm{sk}_n F) \wedge X \to (\mathrm{sk}_{n+1} F) \wedge X$ are strict cofibrations by Proposition 7.2.

We now prove that smashing with $G^+ \wedge_H \Gamma_H(S^+,-)$ preserves level equivalences. Let X be an arbitrary Γ-G-space and let T^+ be any based finite G-set. Then we have an isomorphism (natural in T^+)

$$(G^+ \wedge_H \Gamma_G(S^+, -) \wedge X)(T^+) \longrightarrow G^+ \wedge_H X(\Gamma_H(S^+, T^+))$$

given by mapping a tuple $[f, [\utilde{g \wedge \phi}] \wedge x]$ consisting of $\utilde{f\colon k^+} \wedge l^+ \to T^+$, $\phi\colon S^+ \to k^+$ and $x \in X(l^+)$ to $[g \wedge X(f \circ \widetilde{(\phi \wedge l^+)})(x)]$, where $f \circ \widetilde{(\phi \wedge l^+)} : l^+ \to \Gamma_H(S^+, T^+)$ is the adjoint of the composition $S^+ \wedge l^+ \to k^+ \wedge l^+ \to T^+$ indicated. G acts from the left by

$$g' \cdot [g \wedge x] = [g'g \wedge X(\Gamma_H(S^+, g'))(x)],$$

where $\Gamma_H(S^+, g')$ denotes postcomposition of maps with the action of G on T^+.

Now let $K \leqslant G$ be a subgroup. We choose a set of representatives $\{g_i\}$ of $(G/H)^K = \{gH\colon K^g \leqslant H\}$. Then

$$(G^+ \wedge_H X(\Gamma_H(S^+, T^+)))^K = \bigvee_i X(\Gamma_H(S^+, T^+))^{K^{g_i}}.$$

This implies the claim. □

7.4. The functor $(-)(\mathbb{S})$ and smash products

The main result of this section is Theorem 7.6 below, which states that $(-)(\mathbb{S})$ takes smash products to smash products up to π_*-isomorphism at least when one of the factors is strictly cofibrant.

The functor $(-)(\mathbb{S})\colon G\Gamma(\mathcal{S}_*) \to GSp^\Sigma$ is lax symmetric monoidal. Indeed, given two Γ-G-spaces F and F', the natural maps

$$F(n^+) \wedge F'(m^+) \longrightarrow (F \wedge F')(n^+ \wedge m^+)$$

induce a map

$$F(X) \wedge F'(Y) \longrightarrow (F \wedge F')(X \wedge Y)$$

natural in based G-spaces X and Y. This in turn induces a bimorphism of spectra (cf. [**14**, I.3]) from the pair $(F(\mathbb{S}), F'(\mathbb{S}))$ to $(F \wedge F')(\mathbb{S})$ which gives rise to the natural transformation

$$a_{F,F'}\colon F(\mathbb{S}) \wedge F'(\mathbb{S}) \longrightarrow (F \wedge F')(\mathbb{S}).$$

Moreover, sending $x \in S^M$ to $[x, \mathrm{id}_{1^+}]$ induces an isomorphism $\lambda\colon \mathbb{S} \to \Gamma(1^+, -)(\mathbb{S})$. Now several coherence diagrams have to be checked, which we skip (cf. [**11**, Proposition 3.3 and p. 442]).

Theorem 7.6. *The map $a_{X,Y}$ is a π_*-isomorphism, in particular a G-stable equivalence, if X or Y is strictly cofibrant.*

Proof. In view of Propositions 3.10 and 7.5 and Lemma 4.8 we may assume that X and Y are projectively cofibrant. If we fix Y, then the class of Γ-G-spaces X for which the assembly map is a π_*-isomorphism is closed under pushouts along generating projective cofibrations, filtered colimits along projective cofibrations and retracts. This reduces to consider $X = \Gamma_G(S_1^+, -)$ for some based finite G-set S_1^+ and applying the same reasoning again reduces to $Y = \Gamma_G(S_2^+, -)$ for some based finite G-set S_2^+. In this case we have to show that

$$\mathbb{S}^{\times S_1} \wedge \mathbb{S}^{\times S_2} \longrightarrow \mathbb{S}^{\times(S_1 \times S_2)}$$

induced by the bimorphism

$$(S^{\mathbf{n}})^{\times S_1} \wedge (S^{\mathbf{m}})^{\times S_2} \longrightarrow (S^{\mathbf{n}\sqcup\mathbf{m}})^{\times (S_1 \times S_2)}, ((x_i),(y_j)) \mapsto (x_i \wedge y_j)$$

is a π_*-isomorphism. Precomposition with the π_*-isomorphism (Proposition 3.10, Proposition B.4 and Lemma 5.8) $\mathbb{S}^{\vee S_1} \wedge \mathbb{S}^{\vee S_2} \to \mathbb{S}^{\times S_1} \wedge \mathbb{S}^{\vee S_2} \to \mathbb{S}^{\times S_1} \wedge \mathbb{S}^{\times S_2}$ is a π_*-isomorphism by Lemma 5.8. Hence the map is a π_*-isomorphism as well. □

Proposition 7.7.

(a) Smashing with a strictly cofibrant Γ-G-space preserves stable equivalences.

(b) (Pushout product axiom) If $F \to F'$, $\tilde{F} \to \tilde{F}'$ are two strict cofibrations (resp. projective cofibrations) of Γ-G-spaces, then the pushout product map

$$F \wedge \tilde{F}' \cup_{F \wedge \tilde{F}} F' \wedge \tilde{F} \longrightarrow F' \wedge \tilde{F}'$$

is a strict cofibration (resp. projective cofibration). If in addition one of the former maps is a stable equivalence, then so is the pushout product.

(c) (Monoid Axiom) Let I denote the smallest class of maps of Γ-G-spaces which contains the maps of the form $A \wedge Z \to B \wedge Z$, where $A \to B$ is a stable equivalence and a projective cofibration (resp. strict cofibration) and which is closed under cobase change and transfinite composition. Then every map in I is a stable equivalence.

Proof. The first part follows from Theorem 7.6 and Proposition 3.10. The second part follows from Proposition 7.2, Lemma 6.6 and the first part. It remains to prove the third part. This is in analogy with [**13**, Lemma 1.7]. □

Remark 7.8. Define a (commutative) Γ-G-ring to be a (commutative) monoid in the symmetric monoidal category $G\Gamma(\mathcal{S}_*)$. A left R-module is a Γ-G-space M together with a map $R \wedge M \to M$ satisfying associativity and unit conditions. Defining weak equivalences (resp. fibrations) to be stable equivalences (resp. stable fibrations or stable strict fibrations) and cofibrations by the adequate lifting property, it follows essentially from the previous proposition (cf. [**13**, Theorem 2.2]) that, for any Γ-G-ring R, the category of left R-modules becomes a cofibrantly generated closed G-simplicial model category.

Suppose k is a commutative Γ-G-ring. The category of left k-modules is a symmetric monoidal category with respect to the smash product $A \wedge_k B$ which is the coequalizer of the two actions $A \wedge k \wedge B \rightrightarrows A \wedge B$ given by multiplication.

A k-algebra is then a monoid in k-modules and the category of k-algebras is a closed G-simplicial model category when defining a map to be a weak equivalence (resp. fibration) if the underlying map of k-modules has this property (cf. [**13**, Theorem 2.5]).

8. Geometric fixed points of Γ-G-spaces

In this section we construct a geometric fixed points functor

$$\Phi^G \colon G\Gamma(\mathcal{S}_*) \longrightarrow \Gamma(\mathcal{S}_*).$$

Given a Γ-G-space A, $\Phi^G A$ is defined to be the Γ-space given by $(\Phi^G A)(k^+) = A((k^+)^{\wedge G})^G$. This is in fact a lax symmetric monoidal functor. The transformation $(\Phi^G X) \wedge (\Phi^G Y) \to \Phi^G(X \wedge Y)$ is induced by the map

$$X((k^+)^{\wedge G})^G \wedge Y((l^+)^{\wedge G})^G \longrightarrow (X \wedge Y)((kl^+)^{\wedge G})^G,\ (x \wedge y) \mapsto [\mathrm{id}, x \wedge y]$$

and the map $\Gamma(1^+, -) \to \Phi^G\Gamma(1^+, -)$ is defined to be the isomorphism

$$\Gamma(1^+, k^+) \cong \Gamma(1^+, ((k^+)^{\wedge G})^G) \cong \Gamma(1^+, (k^+)^{\wedge G})^G.$$

The functor $\Phi^G(-)$ enjoys several good properties, which we collect in the next two propositions.

Proposition 8.1. *A map $f\colon A \to B$ of Γ-G-spaces is a stable equivalence if and only if, for all $H \leqslant G$, the map $\Phi^H(f)\colon \Phi^H A \to \Phi^H B$ is a stable equivalence.*

Proof. Given a Γ-G-space the G-symmetric spectrum (of spaces) $|A(\mathbb{S})|$ is the underlying G-symmetric spectrum of a G-orthogonal spectrum $|A|(\mathbb{S})$ (by abuse of notation, we denote the topological sphere spectrum by $\mathbb{S}$, too). Moreover, $|(\Phi^H A)(\mathbb{S})|$ is naturally isomorphic to the geometric fixed point spectrum $\Phi^H(|A|(\mathbb{S}))$ of the G-orthogonal spectrum $|A|(\mathbb{S})$ [**15**]. This follows from the fact that $|(S^{\mathbf{n}})^{\wedge H}|$ is isomorphic to the one point compactification $S^{n\rho_H}$ of n copies of the regular representation ρ_H of H. Now, a morphism $f\colon A \to B$ of Γ-G-spaces is a G-stable equivalence if and only if $A(\mathbb{S}) \to B(\mathbb{S})$ is a π_*-isomorphism of G-symmetric spectra by definition. This is the case if and only if $|A|(\mathbb{S}) \to |B|(\mathbb{S})$ is a π_*-isomorphism of G-orthogonal spectra [**5**, Remark 3.5]. Equivalently, $\Phi^H(|A|(\mathbb{S})) \to \Phi^H(|B|(\mathbb{S}))$ is a π_*-isomorphism of orthogonal spectra for all subgroups $H \leqslant G$ [**15**, Theorem 7.12]. And this is the case if and only if $\Phi^H A \to \Phi^H B$ is a stable equivalence of Γ-spaces for all $H \leqslant G$. □

Proposition 8.2.

(a) *For any based finite G-set S^+, the map*

$$S^+ \wedge \Gamma(1^+, -) \longrightarrow \Gamma_G(S^+, -),\ s \wedge \phi \mapsto (\phi \circ p_s)$$

induces a stable equivalence $(S^+)^G \wedge \Gamma(1^+, -) \simeq \Phi^G(\Gamma_G(S^+, -))$.

(b) *$(\Phi^G A)_c \wedge (\Phi^G B) \to \Phi^G(A \wedge B)$ is a stable equivalence whenever A or B is strictly cofibrant. Here, X_c denotes a cofibrant replacement in the stable strict model structure.*

Proof. Part (a) follows from the Wirthmüller isomorphism Lemma 5.8 and the previous proposition. This implies that (b) holds for $A = \Gamma_G(S_1^+, -)$ and $B = \Gamma_G(S_2^+, -)$. If we fix this B, then the class of Γ-G-space for which (b) holds is closed under pushouts along generating projective cofibrations ($\Phi^G(-)$ takes pushouts along cofibrations to pushouts), filtered colimits along projective cofibrations (since $\Phi^G(-)$ commutes with such colimits) and retracts. Thus A may be an arbitrary projectively cofibrant Γ-G-space and the same argument shows that B can be an arbitrary projectively cofibrant Γ-G-space. This finishes the proof in view of Proposition 7.5. □

Appendix A. The strict model structure for Γ-G-spaces

The aim of this section is to prove Theorem A.1 below. We start by observing that we have the following adjunction for a based right Σ_n- and left Σ_l-space A:

$$A \wedge_{\Sigma_n} - : (G \times \Sigma_n)\mathcal{S}_* \rightleftarrows (G \times \Sigma_l)\mathcal{S}_* : \mathrm{Map}_{\mathcal{S}_*}(A, -)^{\Sigma_l}.$$

Here, $G\times\Sigma_n$ acts on $\mathrm{Map}_{\mathcal{S}_*}(A,-)^{\Sigma_l}$ by $((\sigma,g)\cdot f)(a):=gf(a\sigma)$. For the pointed sets n^+ and m^+, $\mathrm{Inj}_*(n^+,m^+)$ (resp. $\mathrm{Surj}_*(n^+,m^+)$) denotes the set of based injective (resp. surjective) maps $n^+\to m^+$ endowed with the trivial G-action. We make the following assumptions.

Assumptions. (a) There are structures of model categories on $G\times\Sigma_n$-spaces denoted by $\mathcal{G}^1_n\mathcal{S}_*$ and $\mathcal{G}^2_n\mathcal{S}_*$, respectively, such that the first one is $G\times\Sigma_n$-simplicial.

(b) The class of $\mathcal{G}^2_n$-equivalences is included in the class of $\mathcal{G}^1_n$-equivalences for all $n\geqslant 0$.

(c) The adjoint pairs

$$\mathrm{Inj}_*(l^+,n^+)^+\wedge_{\Sigma_l}-:\mathcal{G}^1_l\mathcal{S}_*\rightleftarrows\mathcal{G}^2_n\mathcal{S}_*:\mathrm{Map}_{\mathcal{S}_*}(\mathrm{Inj}_*(l^+,n^+)^+,-)^{\Sigma_n},$$
$$\mathrm{Surj}_*(n^+,l^+)^+\wedge_{\Sigma_n}-:\mathcal{G}^1_n\mathcal{S}_*\rightleftarrows\mathcal{G}^1_l\mathcal{S}_*:\mathrm{Map}_{\mathcal{S}_*}(\mathrm{Surj}_*(n^+,l^+)^+,-)^{\Sigma_l}$$

are Quillen adjunctions.

Let $\Gamma_{\leqslant n}$ denote the full subcategory of Γ with objects the sets l^+, $l\leqslant n$. As in [**2**], the truncation functor $T_n\colon G\Gamma(\mathcal{S}_*)\to\Gamma_{\leqslant n}(G\mathcal{S}_*)$ has both a left and a right adjoint denoted by sk_n and csk_n, respectively. By abuse of notation, we will usually write sk_n (resp. csk_n) for the composition $\mathrm{sk}_n\circ T_n$ (resp. $\mathrm{csk}_n\circ T_n$), too.

Consider a map $f\colon X\to Y$ between Γ-G-spaces. Then f is a *strict cofibration* if, for all $n\geqslant 0$, the map

$$i_n(f)\colon(\mathrm{sk}_{n-1}Y)(n^+)\cup_{(\mathrm{sk}_{n-1}X)(n^+)}X(n^+)\longrightarrow Y(n^+)$$

is a $\mathcal{G}^1_n$-cofibration. Dually, f is a *strict fibration* if, for all $n\geqslant 0$, the map

$$p_n(f)\colon X(n^+)\longrightarrow(\mathrm{csk}_{n-1}X)(n^+)\times_{(\mathrm{csk}_{n-1}Y)(n^+)}Y(n^+)$$

is a $\mathcal{G}^1_n$-fibration. Finally, f is a *strict weak equivalence* if it is levelwise a $\mathcal{G}^1_n$-equivalence.

We prove

Theorem A.1. *Under these assumptions, the strict notions of weak equivalences, fibrations and cofibrations make the category $G\Gamma(\mathcal{S}_*)$ into a G-simplicial model category.*

Example A.2.
- Suppose G is the trivial group. We may take $\mathcal{G}^1_n\mathcal{S}_*$ to be the model structure on Σ_n-spaces where weak equivalences and fibrations are defined by the forgetful functor to spaces and $\mathcal{G}^2_n$ to be the usual model structure on Σ_n-spaces where weak equivalences and fibrations are detected on all fixed points. This recovers the model structure by Bousfield and Friedlander (cf. [**2**]).
- More generally, taking $\mathcal{G}^1_n\mathcal{S}_*$ (resp. $\mathcal{G}^2_n\mathcal{S}_*$) to be the model structure with respect to the family of subgroups of $G\times\Sigma_n$ that intersect $\{1\}\times\Sigma_n$ trivially (resp. the family of all subgroups of $G\times\Sigma_n$) yields the model structure applied throughout this paper.

Before proving the theorem, we need a few preparations.

Proposition A.3. *Suppose B, $X\in\Gamma_{\leqslant n}(G\mathcal{S}_*)$ and $u_{n-1}\colon T_{n-1}B\to T_{n-1}X$ is a map in $\Gamma_{\leqslant n-1}(G\mathcal{S}_*)$. A map $u^n\colon B(n^+)\to X(n^+)$ in $G\mathcal{S}_*$ determines a prolongation of*

u_{n-1} to $u\colon B \to X$ in $\Gamma_{\leqslant n}(G\mathcal{S}_*)$ if and only if u^n is $G \times \Sigma_n$-equivariant and fills in the following commutative diagram in $G \times \Sigma_n \mathcal{S}_*$:

$$\begin{tikzcd}
(\mathrm{sk}_{n-1} B)(n^+) \arrow[r] \arrow[d] & B(n^+) \arrow[r] \arrow[d] & (\mathrm{csk}_{n-1} B)(n^+) \arrow[d] \\
(\mathrm{sk}_{n-1} X)(n^+) \arrow[r] & X(n^+) \arrow[r] & (\mathrm{csk}_{n-1} X)(n^+).
\end{tikzcd}$$

Proof. See Proposition 3.4 from [**2**]. □

Proposition A.4. *Consider a diagram*

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & X \arrow[d] \\
B \arrow[r] & Y
\end{tikzcd} \tag{2}$$

in $\Gamma_{\leqslant n}(G\mathcal{S}_*)$ and a map $T_{n-1}B \to T_{n-1}X$ which makes the diagram

$$\begin{tikzcd}
T_{n-1}A \arrow[r] \arrow[d] & T_{n-1}X \arrow[d] \\
T_{n-1}B \arrow[r] \arrow[ur] & T_{n-1}Y
\end{tikzcd}$$

commute. Then, the diagram (2) *has a lift $B \to X$ if there is a lift in the diagram of $G \times \Sigma_n$-spaces*

$$\begin{tikzcd}
(\mathrm{sk}_{n-1} B)(n^+) \cup_{(\mathrm{sk}_{n-1} A)(n^+)} A(n^+) \arrow[r] \arrow[d] & X(n^+) \arrow[d] \\
B(n^+) \arrow[r] & (\mathrm{csk}_{n-1} X)(n^+) \times_{(\mathrm{csk}_{n-1} Y)(n^+)} Y(n^+).
\end{tikzcd}$$

Proof. This is a direct consequence of the preceding proposition. □

Proposition A.5. *For any Γ-G-space X and any positive integers m, $n \geqslant 0$, there is a pushout square of $G \times \Sigma_n$-spaces*

$$\begin{tikzcd}
\mathrm{Inj}_*(l^+, n^+)^+ \wedge_{\Sigma_l} (\mathrm{sk}_{l-1} X)(l^+) \arrow[r] \arrow[d] & (\mathrm{sk}_{l-1} X)(n^+) \arrow[d] \\
\mathrm{Inj}_*(l^+, n^+)^+ \wedge_{\Sigma_l} X(l^+) \arrow[r] & (\mathrm{sk}_l X)(n^+).
\end{tikzcd} \tag{3}$$

Here the top and bottom horizontal maps are given by pushing forward along an element of $\mathrm{Inj}_*(l^+, n^+)$, where one uses the canonical isomorphism $(\mathrm{sk}_l X)(l^+) \to X(l^+)$ for the lower one, and the left and right vertical maps are induced by the canonical maps $(\mathrm{sk}_{l-1} X)(l^+) \to X(l^+)$ and $(\mathrm{sk}_{l-1} X)(n^+) \to (\mathrm{sk}_l X)(n^+)$.

Proof. The diagram is a commutative diagram of $G \times \Sigma_n$-spaces and its underlying diagram of spaces is isomorphic to

$$\begin{tikzcd}
\binom{l}{n}^+ \wedge (\mathrm{sk}_{l-1} X)(l^+) \arrow[r] \arrow[d] & (\mathrm{sk}_{l-1} X)(n^+) \arrow[d] \\
\binom{l}{n}^+ \wedge X(l^+) \arrow[r] & (\mathrm{sk}_l X)(n^+),
\end{tikzcd}$$

where $\binom{l}{n}$ denotes the set of order-preserving injections of the set $\{1, \ldots, l\}$ into the set $\{1, \ldots, n\}$ both endowed with the natural ordering. Lydakis shows (cf. [**7**, Proposition 3.8]) that for any Γ-space X and any positive integers m, $n \geqslant 0$ this is a pushout diagram, hence (3) is a pushout diagram in $G \times \Sigma_n$-spaces. □

Lemma A.6. *[2, Lemma 3.7] Let n be a non-negative integer and fix $N \leqslant n$. Consider a map $f\colon A \to B$ in $G\Gamma(\mathcal{S}_*)$. If the maps $i_m(f)$ are $\mathcal{G}^1_m$-cofibrations (resp. acyclic $\mathcal{G}^1_m$-cofibrations) for all $m \leqslant N$, then the maps $(\mathrm{sk}_l A)(n^+) \to (\mathrm{sk}_l B)(n^+)$ are $\mathcal{G}^2_n$-cofibrations (resp. acyclic $\mathcal{G}^2_n$-cofibrations) for all $l \leqslant N$.*

Proof. The case $l = 0$ is trivial. Assume inductively that the assertion holds true for all $l - 1 \leqslant N - 1$. By the first part of assumption (c), $\mathrm{Inj}_*(l^+, n^+)^+ \wedge_{\Sigma_l} i_l(f)$ is a $\mathcal{G}^2_n$-cofibration (resp. acyclic $\mathcal{G}^2_n$-cofibration). In view of Proposition A.5, the inductive step can now be finished by applying Reedy's patching lemma (cf. [**2**, 3.8]) to the diagram

$$\begin{tikzcd}
\mathrm{Inj}_*(l^+, n^+)^+ \wedge_{\Sigma_l} A(l^+) \arrow[d] & \mathrm{Inj}_*(l^+, n^+)^+ \wedge_{\Sigma_l} (\mathrm{sk}_{l-1} A)(l^+) \arrow[l] \arrow[r] \arrow[d] & (\mathrm{sk}_{l-1} A)(n^+) \arrow[d] \\
\mathrm{Inj}_*(l^+, n^+)^+ \wedge_{\Sigma_l} B(l^+) & \mathrm{Inj}_*(l^+, n^+)^+ \wedge_{\Sigma_l} (\mathrm{sk}_{l-1} B)(l^+) \arrow[l] \arrow[r] & (\mathrm{sk}_{l-1} B)(n^+).
\end{tikzcd}$$

□

Lemma A.7. *If $f\colon A \to B$ is an acyclic strict cofibration, then the maps*

$$i_n(f)\colon A(n^+) \cup_{\mathrm{sk}_{n-1} A(n^+)} (\mathrm{sk}_{n-1} B)(n^+) \longrightarrow B(n^+)$$

are in fact acyclic $\mathcal{G}^1_n$-cofibrations.

Proof. The case $n = 0$ is trivial. Assume inductively that $i_m(f)$ is an acyclic $\mathcal{G}^1_m$-cofibration for all $m \leqslant n - 1$. We show that $i_n(f)$ is a $\mathcal{G}^1_n$-equivalence. To this end it suffices to show that $(\mathrm{sk}_{n-1} A)(n^+) \to (\mathrm{sk}_{n-1} B)(n^+)$ is an acyclic $\mathcal{G}^2_n$-cofibration, because this implies that $A(n^+) \to A(n^+) \cup_{\mathrm{sk}_{n-1} A(n^+)} (\mathrm{sk}_{n-1} B)(n^+)$ is an acyclic $\mathcal{G}^2_n$-cofibration and, since $\mathcal{G}^2_n$-equivalences are in particular $\mathcal{G}^1_n$-equivalences by assumption (b), the assertion follows then from two out of three for weak equivalences. But the map in question is an acyclic $\mathcal{G}^2_n$-cofibration by Lemma A.6 applied to the case $N = n - 1$. □

There are dual results for fibrations.

Proposition A.8. *For any Γ-G-space X and any positive integers m, $n \geqslant 0$, there is a pullback square of $G \times \Sigma_n$-spaces*

$$\begin{tikzcd}
(\mathrm{csk}_l X)(n^+) \arrow[r] \arrow[d] & \mathrm{Map}_{\mathcal{S}_*}(\mathrm{Surj}_*(n^+, l^+)^+, X(l^+))^{\Sigma_l} \arrow[d] \\
(\mathrm{csk}_{l-1} X)(n^+) \arrow[r] & \mathrm{Map}_{\mathcal{S}_*}(\mathrm{Surj}_*(n^+, l^+)^+, (\mathrm{csk}_{l-1} X)(l^+))^{\Sigma_l}.
\end{tikzcd} \tag{4}$$

Here the top and bottom horizontal maps are given by pushing forward along an element of $\mathrm{Surj}_*(l^+, n^+)$*, where one uses the canonical identification* $(\mathrm{csk}_l X)(l^+) \to X(l^+)$ *for the top map, and the left and right vertical maps are induced by the canonical maps* $X(l^+) \to (\mathrm{csk}_{l-1} X)(l^+)$ *and* $(\mathrm{csk}_l X)(n^+) \to (\mathrm{csk}_{l-1} X)(n^+)$*, respectively.*

Proof. The proof is analogous to the proof of [**7**, Proposition 3.8]. □

Lemma A.9. *[2, Lemma 3.7] Consider a map of* Γ*-*G*-spaces* $f\colon A \to B$ *such that the maps* $p_m(f)$ *are* $\mathcal{G}^1_m$*-fibrations (resp. acyclic* $\mathcal{G}^1_m$*-fibrations) for all* $m \leqslant N$*, where* $N \leqslant n$ *is fixed. Then the maps* $(\mathrm{csk}_l A)(n^+) \to (\mathrm{csk}_l B)(n^+)$ *are* $\mathcal{G}^1_n$*-fibrations (resp. acyclic* $\mathcal{G}^1_n$*-fibrations) for all* $l \leqslant N$*.*

Proof. The case $l = 0$ is trivial. Assume inductively that the assertion holds true for $l - 1 \leqslant N - 1$. It suffices to know that $\mathrm{Map}_{\mathcal{S}_*}(\mathrm{Surj}_*(n^+, l^+)^+, p_l)^{\Sigma_l}$ is a $\mathcal{G}^1_n$-fibration (resp. acyclic $\mathcal{G}^1_n$-fibration) by Reedy's patching lemma. This follows from the second part of assumption (c). □

Lemma A.10. *If* $f\colon A \to B$ *is an acyclic strict fibration, then the maps*

$$p_n(f)\colon A(n^+) \longrightarrow (\mathrm{csk}_{n-1} A)(n^+) \times_{(\mathrm{csk}_{n-1} B)(n^+)} B(n^+)$$

are in fact acyclic $\mathcal{G}^1_n$*-fibrations.*

Proof. The case $n = 0$ is trivial. Assume inductively that $p_m(f)$ are acyclic $\mathcal{G}^1_m$-fibrations for $m \leqslant n-1$. We show that $p_n(f)$ is an acyclic $\mathcal{G}^1_n$-fibration. By the previous lemma in the case $N = n - 1$, we have that $(\mathrm{csk}_{n-1} A)(n^+) \to (\mathrm{csk}_{n-1} B)(n^+)$ is an acyclic $\mathcal{G}^1_n$-fibration. Hence $(\mathrm{csk}_{n-1} A)(n^+) \times_{(\mathrm{csk}_{n-1} B)(n^+)} B(n^+) \to B(n^+)$ is an acyclic $\mathcal{G}^1_n$-fibration as well. □

Proof of Theorem A.1. **MC 1**, **MC 2** and **MC 3** are clear. **MC 4** follows immediately from Lemma A.7, Lemma A.10 and Proposition A.4. So we only have to show **MC 5**, the existence of factorizations. Given a map $f\colon A \to B$ in $G\Gamma(\mathcal{S}_*)$, assume inductively that it has already been factored up to level $n-1$ as an acyclic strict cofibration followed by a strict fibration (resp. strict cofibration followed by an acyclic strict fibration) $T_{n-1}A \to C_{\leqslant n-1} \to T_{n-1}B$. Then, as in [**2**], we obtain a diagram

$$\begin{tikzcd}
(\mathrm{sk}_{n-1} A)(n^+) \arrow[r] \arrow[d] & A(n^+) \arrow[r] \arrow[d] & (\mathrm{csk}_{n-1} A)(n^+) \arrow[d] \\
(\mathrm{sk}_{n-1} C_{\leqslant n-1})(n^+) \arrow[r] \arrow[d] & K \arrow[r] \arrow[d] & (\mathrm{csk}_{n-1} C_{\leqslant n-1})(n^+) \arrow[d] \\
(\mathrm{sk}_{n-1} B)(n^+) \arrow[r] & B(n^+) \arrow[r] & (\mathrm{csk}_{n-1} B)(n^+),
\end{tikzcd} \tag{5}$$

where K comes from a factorization

$$(\mathrm{sk}_{n-1} C_{\leqslant n-1})(n^+) \cup_{(\mathrm{sk}_{n-1} A)(n^+)} A(n^+) \longrightarrow K \longrightarrow (\mathrm{csk}_{n-1} C_{\leqslant n-1})(n^+) \times_{(\mathrm{csk}_{n-1} B)(n^+)} B(n^+)$$

of the canonical map into an acyclic cofibration followed by a fibration (resp. cofibration followed by an acyclic fibration) in $\mathcal{G}^1_n\mathcal{S}_*$. The $G \times \Sigma_n$-space K gives rise to an object $C_{\leqslant n} \in \Gamma_{\leqslant n}(G\mathcal{S}_*)$ with $C_{\leqslant n}(k^+) = C_{\leqslant n-1}(k^+)$ for all $k \leqslant n-1$ and

$C_{\leqslant n}(n^+) = K$, such that the canonical factorization

$$(\mathrm{sk}_{n-1} C_{\leqslant n})(n^+) \longrightarrow K \longrightarrow (\mathrm{csk}_{n-1} C_{\leqslant n})(n^+)$$

equals the factorization in (5).

In any case, this produces a factorization $A \to C \to B$ as a strict cofibration followed by a strict fibration. Assume that K was always obtained by a factorization as an acyclic cofibration followed by a fibration. We show that $A \to C$ is acyclic. But this follows from Lemma A.6 for $l = n$. In the other case, K was always obtained by a factorization as cofibration followed by an acyclic fibration. Then $C \to B$ is acyclic by Lemma A.9 in the case $l = n$.

Finally, this model structure is G-simplicial since, for all $n \geqslant 0$, $i_n(j \square f)$ is isomorphic to $j \square i_n(f)$ for any strict cofibration f and any G-cofibration j of G-spaces. $\square$

Appendix B. A characterization of flat cofibrations

The aim of this section is to prove

Proposition B.1. *For any G-flat G-symmetric spectrum X and any finite G-set S of cardinality n, the spectrum $X^{\times S}$ is G-flat and the inclusion*

$$X^{\times S}_{\leqslant n-1} \longrightarrow X^{\times S}$$

is a G-flat cofibration of G-symmetric spectra. Here $X^{\times S}_{\leqslant n-1}$ is the subspectrum which is levelwise given by those tuples in the product such that either two entries coincide or one of them equals the basepoint. In particular, the G-symmetric spectrum $\mathbb{S}^{\times S}$ is G-flat and the inclusion

$$\mathbb{S}^{\times S}_{\leqslant n-1} \longrightarrow \mathbb{S}^{\times S}$$

is a G-flat cofibration of G-symmetric spectra.

A $G \times \Sigma_n$-map is a $G \times \Sigma_n$-cofibration if and only if its underlying map is a cofibration. Therefore, a map of G-symmetric spectra is a G-flat cofibration if and only if its underlying morphism of symmetric spectra is a flat cofibration and hence it suffices to prove the above proposition for G the trivial group.

B.1. Latching objects of symmetric spectra

Let $\mathbf{k}$ denote the set $\{1, \ldots, k\}$. Those are the objects of the category $\mathcal{I}$, where morphisms are injective maps of sets. The category $\mathcal{I}$ has a symmetric monoidal structure $\sqcup$ given by concatenation $\mathbf{m} \sqcup \mathbf{n} = \mathbf{m} + \mathbf{n}$ with unit the empty set $\mathbf{0}$. Let $(\sqcup \downarrow \mathbf{n})$ be the category with objects consisting of tuples $(\mathbf{k}, \mathbf{k}', \alpha\colon \mathbf{k} \sqcup \mathbf{k}' \to \mathbf{n})$ with α injective. A morphism $(\mathbf{k}, \mathbf{k}', \alpha\colon \mathbf{k} \sqcup \mathbf{k}' \to \mathbf{n}) \to (\mathbf{l}, \mathbf{l}', \beta\colon \mathbf{l} \sqcup \mathbf{l}' \to \mathbf{n})$ is a tuple of morphisms $(\gamma\colon \mathbf{k} \to \mathbf{l}, \gamma'\colon \mathbf{k}' \to \mathbf{l}')$ in $\mathcal{I}$ such that $\beta \circ (\gamma \sqcup \gamma') = \alpha$.

Given two symmetric spectra E and F, their smash product is given in level n by

$$\mathrm{colim}_{\alpha\colon \mathbf{k} \sqcup \mathbf{k}' \to \mathbf{n}}\, E(\mathbf{k}) \wedge F(\mathbf{k}') \wedge S^{\mathbf{n}-\alpha},$$

where the colimit is taken over the category $(\sqcup \downarrow \mathbf{n})$ and we use α as a shorthand for the image of α. A map (γ, γ') in this category induces the map

$$E(\mathbf{k}) \wedge F(\mathbf{k}') \wedge S^{\mathbf{n}-\alpha} \cong E(\mathbf{k}) \wedge S^{\mathbf{l}-\gamma} \wedge F(\mathbf{k}') \wedge S^{\mathbf{l}'-\gamma'} \wedge S^{\mathbf{n}-\beta} \to E(\mathbf{l}) \wedge F(\mathbf{l}') \wedge S^{\mathbf{n}-\beta},$$

where one uses γ and γ' to identify $\mathbf{n}-\alpha$ with $(\mathbf{n}-\beta)\sqcup(\mathbf{l}-\gamma)\sqcup(\mathbf{l}'-\gamma')$ in the first isomorphism and the second map uses the isomorphisms $E(\mathbf{k})\cong E(\gamma)$, $F(\mathbf{k}')\cong F(\gamma')$ given by γ and γ' and the generalized structure maps.

Define $\overline{\mathbb{S}}$ to be the truncated sphere spectrum, i.e., $\overline{\mathbb{S}}_0 = *$ and $\overline{\mathbb{S}}_n = S^n$ if $n \geqslant 1$. The structure maps are the evident maps. The nth latching object of a symmetric spectrum X is now defined to be the nth level of the smash product of X with $\overline{\mathbb{S}}$,

$$L_n(X) = (X \wedge \overline{\mathbb{S}})_n.$$

More generally for a morphism $f\colon X \to Y$ of symmetric spectra we set

$$L_n(f) = X(\mathbf{n}) \cup_{L_n(X)} L_n(Y).$$

The generalized structure maps induce $\nu_n(X)\colon L_n(X) \to X(\mathbf{n})$ and $\nu_n(f)\colon L_n(f) \to Y(\mathbf{n})$, which are the maps that appear in the definition of the (G-)flat model structure.

For our purpose, it is convenient to use a slightly different model for the latching morphisms. To this end, we define $\mathcal{P}(\mathbf{n})$ to be the poset of subsets of $\mathbf{n}$. Given a symmetric spectrum X and a morphism $f\colon X \to Y$ we get two functors $\mathbf{L}_n(X)$ and $\mathbf{L}_n(f)$ from $\mathcal{P}(\mathbf{n})$ to $\mathcal{S}_*$. On objects, these are given by $U \mapsto X(U)\wedge S^{\mathbf{n}-U}$ and $U \mapsto X(\mathbf{n}) \cup_{X(U)\wedge S^{\mathbf{n}-U}} Y(U)\wedge S^{\mathbf{n}-U}$, respectively. For an inclusion $\iota\colon U \subset V$ we let $\mathbf{L}_n(X)(\iota)$ be the composite

$$X(V)\wedge S^{\mathbf{n}-V} \cong X(V)\wedge S^{U-V}\wedge S^{\mathbf{n}-U} \longrightarrow X(U)\wedge S^{\mathbf{n}-U},$$

where the second map is given by the generalized structure map σ_V^{U-V} smashed with the identity on $S^{\mathbf{n}-U}$ and similarly for $\mathbf{L}_n(f)$. The generalized structure maps induce $\tilde{\nu}_n(X)\colon \operatorname{colim}_{U\subsetneq\mathbf{n}} \mathbf{L}_n(X) \to X(\mathbf{n})$ and $\tilde{\nu}_n(f)\colon \operatorname{colim}_{U\subsetneq\mathbf{n}} \mathbf{L}_n(f) \to Y(\mathbf{n})$ and we have

Lemma B.2. *The spaces $L_n(X)$ and $\operatorname{colim}_{U\subsetneq\mathbf{n}} \mathbf{L}_n(X)$ are naturally isomorphic as Σ_n-spaces over $X(\mathbf{n})$. Similarly, the spaces $L_n(f)$ and $\operatorname{colim}_{U\subsetneq\mathbf{n}} \mathbf{L}_n(f)$ are naturally isomorphic as Σ_n-spaces over $Y(\mathbf{n})$.*

Proof. Indeed, we define a Σ_n-map

$$L_n(X) \to \operatorname*{colim}_{U\subsetneq\mathbf{n}} \mathbf{L}_n(X)$$

by mapping the pair $(\alpha\colon \mathbf{k}\sqcup\mathbf{k}' \to \mathbf{n},\, x\wedge y\wedge z \in X(\mathbf{k})\wedge S^{\mathbf{k}'}\wedge S^{\mathbf{n}-\alpha})$ to $(\alpha(\mathbf{k}), [x,\alpha|_{\mathbf{k}}]\wedge((\alpha|_{\mathbf{k}'})_*(y)\wedge z)) \in X(\alpha(\mathbf{k}))\wedge S^{\mathbf{n}-\alpha|_{\mathbf{k}}}$. By abuse of notation, we secretly identified $X(\mathbf{k})$ with X_k via the isomorphism $[x\wedge f]\mapsto f_*(x)$. The inverse is then given by $(U,[x,\alpha]\wedge y)\mapsto(\alpha\colon \mathbf{k}\to U\subset\mathbf{n}, x\wedge y)$. The second part follows since colimits commute with each other. $\square$

B.2. A characterization of flat cofibrations

In order to give a characterization of flat cofibrations, we need the following lemma.

Lemma B.3. *Given a functor $C\colon \mathcal{P}(\mathbf{n}) \to \mathcal{S}_*$, the induced map $\operatorname{colim}_{V\subsetneq U} C(V) \to C(U)$ is a cofibration for all $U \subset \mathbf{n}$ if and only if*

(a) for all inclusions $V \subset U \subset \mathbf{n}$, the map $C(V)\to C(U)$ is a cofibration and

(b) for all $U, V \subset \mathbf{n}$, the intersection of the images of $C(U)$ and $C(V)$ in $C(U\cup V)$ equals the image of $C(U\cap V)$.

Proof. This appears in the proof of [**12**, Proposition 3.11]. $\square$

We can now prove

Proposition B.4. *A map $f\colon X \to Y$ of symmetric spectra is a flat cofibration if and only if*

(a) for all k, $l \geqslant 0$ the map $X(\mathbf{k} \sqcup \mathbf{l}) \cup_{X(\mathbf{k}) \wedge S^{\mathbf{l}}} Y(\mathbf{k}) \wedge S^{\mathbf{l}} \to Y(\mathbf{k} \sqcup \mathbf{l})$ is a cofibration and

(b) for all integers k, l, $m \geqslant 0$, we have that

$$\begin{tikzcd}
X(\mathbf{k} \sqcup \mathbf{l} \sqcup \mathbf{m}) \cup_{X(\mathbf{l}) \wedge S^{\mathbf{k} \sqcup \mathbf{m}}} Y(\mathbf{l}) \wedge S^{\mathbf{k} \sqcup \mathbf{m}} \arrow[r] \arrow[d] & X(\mathbf{k} \sqcup \mathbf{l} \sqcup \mathbf{m}) \cup_{X(\mathbf{l} \sqcup \mathbf{m}) \wedge S^{\mathbf{k}}} Y(\mathbf{l} \sqcup \mathbf{m}) \wedge S^{\mathbf{k}} \arrow[d] \\
X(\mathbf{k} \sqcup \mathbf{l} \sqcup \mathbf{m}) \cup_{X(\mathbf{k} \sqcup \mathbf{l}) \wedge S^{\mathbf{m}}} Y(\mathbf{k} \sqcup \mathbf{l}) \wedge S^{\mathbf{m}} \arrow[r] & Y(\mathbf{k} \sqcup \mathbf{l} \sqcup \mathbf{m})
\end{tikzcd}$$

is a pullback.

In particular, if Y is a flat symmetric spectrum and $X \subset Y$ is a subspectrum, then the inclusion $X \to Y$ is a flat cofibration if and only if for all k, $l \geqslant 0$ the intersection of the images of $X(\mathbf{k} \sqcup \mathbf{l})$ and $Y(\mathbf{k}) \wedge S^{\mathbf{l}}$ in $Y(\mathbf{k} \sqcup \mathbf{l})$ equals the image of $X(\mathbf{k}) \wedge S^{\mathbf{l}}$.

Proof. In view of Lemma B.2, a map of symmetric spectra $f\colon X \to Y$ is flat if and only if for all $n \geqslant 0$ and all subsets $U \subset \mathbf{n}$ the maps $\operatorname{colim}_{V \subsetneq U} X(U) \cup_{X(V) \wedge S^{U-V}} Y(V) \wedge S^{U-V} \to Y(U)$ are cofibrations. By Lemma B.3, this is equivalent to conditions (a) and (b). □

We can now give a proof of the result we are after.

Proof of Proposition B.1. Suppose X is a flat symmetric spectrum. We prove first that $X^{\times N}$ is flat, provided that X is flat. Condition (a) in Proposition B.4 requires the map $X(\mathbf{n})^{\times N} \wedge S^{\mathbf{k}} \to X(\mathbf{n} \sqcup \mathbf{k})^{\times N}$ to be a cofibration. But this map factors as the composition of two cofibrations

$$X(\mathbf{n})^{\times N} \wedge S^{\mathbf{k}} \to (X(\mathbf{n}) \wedge S^{\mathbf{k}})^{\times N} \to X(\mathbf{n} \sqcup \mathbf{k})^{\times N}.$$

Condition (b) requires

$$\begin{tikzcd}
X(\mathbf{l})^{\times N} \wedge S^{\mathbf{k} \sqcup \mathbf{m}} \arrow[r] \arrow[d] & X(\mathbf{l} \sqcup \mathbf{m})^{\times N} \wedge S^{\mathbf{k}} \arrow[d] \\
X(\mathbf{k} \sqcup \mathbf{l})^{\times N} \wedge S^{\mathbf{m}} \arrow[r] & X(\mathbf{k} \sqcup \mathbf{l} \sqcup \mathbf{m})^{\times N}
\end{tikzcd}$$

to be a pullback, which is readily checked. It follows now from the second part of Proposition B.4 that $X^{\times N}_{\leqslant N-1} \to X^{\times N}$ is a flat cofibration. □

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