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# AN ALTERNATE APPROACH TO THE LIE BRACKET ON HOCHSCHILD COHOMOLOGY

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(*communicated by Claude Cibils*)

*Abstract*

We define Gerstenhaber's graded Lie bracket directly on complexes other than the bar complex, under some conditions, resulting in a practical technique for explicit computations. The Koszul complex of a Koszul algebra in particular satisfies our conditions. As examples we recover the Schouten–Nijenhuis bracket for a polynomial ring and the Gerstenhaber bracket for a group algebra of a cyclic group of prime order.

## 1. Introduction

Hochschild cohomology incorporates useful information about an algebra, the study of which was begun by Hochschild [13] and Gerstenhaber [8]. In low degrees one finds the center of the algebra, derivations, infinitesimal deformations, and obstructions in algebraic deformation theory. Vanishing in high degrees is equivalent to smoothness in commutative settings [1, 2]. Noncommutative algebras can behave quite differently [3], yet analogous notions have also been explored in noncommutative settings [14]. Hochschild cohomology is used in support variety theory, a tool for studying representations of some types of finite dimensional algebras [21].

In spite of its many uses, some of the structure of Hochschild cohomology remains elusive. It is a Gerstenhaber algebra, that is, it has both a cup product and a graded Lie bracket, and the bracket induces graded derivations with respect to the product. Products are defined in any number of equivalent ways: as Yoneda composition of $n$-extensions of bimodules, as composition of maps in arbitrary projective bimodule resolutions, or as application of a diagonal map to the tensor product over the algebra of two copies of an arbitrary bimodule resolution. This freedom of choice makes the product quite tractable for many algebras. Brackets have not been so amenable to study on resolutions other than the bar resolution where they were historically defined, and thus they are more difficult to compute and to use. Typically one computes cohomology with a resolution other than the bar resolution, and then translates the bracket from the bar resolution using explicit comparison maps. These maps are nearly always very cumbersome, and beg for a better approach.

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The question about realizing the Gerstenhaber bracket on other resolutions was raised by Gerstenhaber and Schack [**9**]. An elegant such realization was given by Schwede [**20**], based on Retakh's description of categories of extensions [**18**]. Hermann [**12**] generalized Schwede's construction of brackets as loops in an extension category to other suitable exact monoidal categories. Yet it seems difficult to translate these beautiful constructions into practical techniques for explicit computations of brackets as may be required, for example, to answer some questions in algebraic deformation theory. Our paper takes a different route to computational techniques.

We begin with the observation that there is more than one way to define the graded Lie bracket on the bar resolution $B$ of an algebra: We show in Section 2 that a particular class of chain maps, of graded degree 1, from the tensor product of two copies of $B$ to $B$, gives rise to many brackets at the chain level. These all induce the Gerstenhaber bracket on cohomology. We mimic this construction in Section 3 for other resolutions satisfying some hypotheses. We define brackets and prove that these brackets also induce Gerstenhaber brackets on cohomology. One useful condition in particular is when the resolution embeds into the bar resolution in such a way that the diagonal maps commute with the embedding, and the strongest results follow from this condition (Subsection 3.2). Koszul resolutions of Koszul algebras in particular satisfy this hypothesis.

We illustrate our techniques by recovering the Schouten–Nijenhuis bracket on polynomial rings in Section 4. We also give some results under weaker conditions (Subsection 3.4) that still may be useful but have the disadvantage of requiring a more detailed comparison with the bar resolution. In Section 5 we show that these techniques may be used to recover Gerstenhaber brackets for a group algebra of a cyclic group of prime order $p$ over a field of characteristic $p$. (Expressions for such brackets were originally given in the work of Sanchez-Flores [**19**].) These two well known classes of examples, in Sections 4 and 5, serve merely to illustrate our techniques here. A new class of examples is given in [**10**]: Brackets are computed there for the quantum complete intersections $\Lambda_q := k\langle x, y\rangle/(x^2, y^2, xy + qyx)$ for various (nonzero) values of a parameter $q$ in a field $k$. The algebra structure of Hochschild cohomology of $\Lambda_q$ had been computed by Buchweitz, Green, Madsen, and Solberg [**3**]. Grimley, Nguyen, and the second author [**10**] used the techniques of the current paper to compute Gerstenhaber brackets directly on the Koszul resolution of $\Lambda_q$. They did not need to know explicit formulas for chain maps between the bar and Koszul resolutions, as these were not used; it suffices to know existence of such maps satisfying some conditions. Also in [**10**] is a general result about the Gerstenhaber algebra structure of the Hochschild cohomology of a twisted tensor product of algebras. Its proof uses techniques from the current paper, showing that these techniques can be useful as well for algebras that are not Koszul.

## 2. Alternate brackets on the Hochschild complex

Let $k$ be a field of arbitrary characteristic and let $A$ be a $k$-algebra. Let us recall the definitions of the bar resolution $B$ of $A$ and the Hochschild cochain complex. We write $\otimes$ to mean $\otimes_k$.

Let $TA = T(A)$ denote the graded tensor coalgebra, that is, $TA = \oplus_{r\geqslant 0}(TA)_r$

where $(TA)_r = A^{\otimes r}$ and the coproduct $\Delta\colon TA \to TA \otimes TA$ is the $k$-linear map defined by

$$\Delta(a_1 \otimes \cdots \otimes a_r) = \sum_{i=0}^{r} (a_1 \otimes \cdots \otimes a_i) \otimes (a_{i+1} \otimes \cdots \otimes a_r),$$

for $a_1, \ldots, a_r \in A$. As a graded $A$-bimodule, we have $B = A \otimes TA \otimes A$, with $B_r = A^{\otimes(r+2)}$ for each $r \geqslant 0$. We may use the notation $a \otimes x \otimes a'$ to denote monomials in $B$, where $a, a' \in A$, and $x \in TA$. The differential on $B$ is

$$a_0 \otimes \ldots \otimes a_{r+1} \mapsto \sum_{0 \leqslant i \leqslant r} (-1)^i a_0 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_{r+1}. \tag{2.1}$$

Note that the comultiplication on $TA$ induces a quasi-isomorphism

$$\begin{gathered} \Delta\colon B \to B \otimes_A B \\ a_0 \otimes \ldots \otimes a_{r+1} \mapsto \sum_i (a_0 \otimes \ldots \otimes a_i \otimes 1) \otimes (1 \otimes a_{i+1} \otimes \ldots \otimes a_{r+1}). \end{gathered} \tag{2.2}$$

The map $\Delta$ is coassociative by construction, that is, $(\Delta \otimes id)\Delta = (id \otimes \Delta)\Delta$ as chain maps from $B$ to $B \otimes_A B \otimes_A B$. On monomials, we can write this map symbolically as

$$a \otimes x \otimes a' \mapsto \sum (a \otimes x_1 \otimes 1) \otimes (1 \otimes x_2 \otimes a'),$$

where the sum runs over all possible ways to factor the monomial $x$. Let

$$C(A) := \mathrm{Hom}_{A^e}(B, A),$$

where $A^e = A \otimes A^{op}$. The cup product may be defined at the cochain level via the diagonal map $\Delta$: If $f \in \mathrm{Hom}_{A^e}(B_r, A)$, $g \in \mathrm{Hom}_{A^e}(B_s, A)$, then

$$(f \smile g)(a_0 \otimes \ldots \otimes a_{r+s+1}) = f(a_0 \otimes \ldots \otimes a_r \otimes 1) g(1 \otimes a_{r+1} \otimes \ldots \otimes a_{r+s+1}).$$

We use the notation $|f| = r$ for the degree of $f$ in this case. The Gerstenhaber bracket is defined as follows, where we employ for the moment the canonical identifications $\mathrm{Hom}_{A^e}(B_n, A) = \mathrm{Hom}_k(A^{\otimes n}, A)$ via freeness of $B_n$.

**Definition 2.1** (Standard Gerstenhaber Bracket [8])**.** Let $\circ$ denote the operation $C(A) \otimes C(A) \to C(A)$ given on homogeneous elements $f$ and $g$ by

$$\begin{aligned} &f \circ g(a_1 \otimes \ldots \otimes a_n) \\ &= \sum_{j=1}^{|f|} (-1)^{(|g|-1)(j-1)} f(a_1 \otimes \ldots \otimes a_{j-1} \otimes g(a_j \otimes \ldots \otimes a_{j+|g|-1}) \otimes a_{j+|g|} \otimes \ldots \otimes a_n), \end{aligned}$$

and define the bracket $[\ ,\ ]$ by

$$[f, g] = f \circ g - (-1)^{(|f|-1)(|g|-1)} g \circ f.$$

The cup product and bracket induce operations on Hochschild cohomology that enjoy many useful properties, for example,

$$[\bar{f} \smile \bar{g}, \bar{h}] = [\bar{f}, \bar{h}] \smile \bar{g} \ + \ (-1)^{|\bar{f}|(|\bar{h}|-1)} \bar{f} \smile [\bar{g}, \bar{h}], \tag{2.3}$$

where $f, g, h$ are homogeneous cocycles and $\bar{f}, \bar{g}, \bar{h}$ are their images in Hochschild cohomology. See [8] for this and other properties.

**Lemma 2.2.** *As a graded $A$-bimodule, $B \otimes_A B \cong A \otimes TA \otimes A \otimes TA \otimes A$. Under this identification, the differential is given by*

$$
\begin{aligned}
&(a_0 \otimes \ldots \otimes a_{j-1}) \otimes (a_j) \otimes (a_{j+1} \otimes \ldots \otimes a_{n+1}) \mapsto \\
&\qquad \textstyle\sum_{i<j-1} (-1)^i (a_0 \otimes \ldots \otimes a_i a_{i+1} \ldots) \otimes (a_j) \otimes (\ldots \otimes a_{n+1}) \\
&\qquad +(-1)^{j-1} (a_0 \otimes \ldots) \otimes (a_{j-1} a_j) \otimes (\ldots \otimes a_{n+1}) \\
&\qquad +(-1)^{j-1} (a_0 \otimes \ldots) \otimes (a_j a_{j+1}) \otimes (\ldots \otimes a_{n+1}) \\
&\qquad + \textstyle\sum_{k>j} (-1)^{k-1} (a_0 \otimes \ldots) \otimes (a_j) \otimes (\ldots \otimes a_k a_{k+1} \ldots \otimes a_{n+1}).
\end{aligned}
$$

*Proof.* The first portion of the statement is clear. The second is an easy check from the fact that the differential on the tensor complex $X \otimes_A Y$, of any two $A$-bimodule complexes $X$ and $Y$, is given by $d(x \otimes y) = d(x) \otimes y + (-1)^{|x|} x \otimes d(y)$. □

We will deconstruct the bracket operation, realize it as a composition of several maps, and make some changes in the apparent choices involved. We will observe that these choices do not matter at the level of cohomology, giving us some freedom in the definition. It is this freedom that will allow us, in the next section, to define the bracket independently on other cochain complexes satisfying certain conditions.

We first define a chain map $F_B \colon B \otimes_A B \to B$. By the isomorphism of Lemma 2.2, elements in the tensor product $B \otimes_A B$ may be identified with sums of elements of the form

$$a \otimes x \otimes a' \otimes y \otimes a'',$$

with $x \in A^{\otimes i}$, $y \in A^{\otimes j}$ and $a, a', a'' \in A$. We define $F_B \colon B \otimes_A B \to B$ on such monomials as follows: If $i > 0$ and $j > 0$, then

$$
\begin{aligned}
F_B(a \otimes x \otimes a' \otimes y \otimes a'') &= 0, \\
F_B(a \otimes a' \otimes y \otimes a'') &= aa' \otimes y \otimes a'', \\
F_B(a \otimes x \otimes a' \otimes a'') &= -a \otimes x \otimes a'a''.
\end{aligned}
\tag{2.4}
$$

In degree 0,

$$F_B(a \otimes a' \otimes a'') = aa' \otimes a'' - a \otimes a'a''.$$

As one can see from the definition, $F_B$ is 0 on most of the tensor complex $B \otimes_A B$, and is simply given by the actions of $A$ on $B$ for the extremal terms $B^0 \otimes_A B^j$ and $B^i \otimes_A B^0$. One may check directly that $F_B$ is a chain map. Alternatively, this follows from Proposition 2.4 or the general construction given in Section 3.2.

In the remainder of this article, we use the isomorphism of Lemma 2.2, without comment, to identify $B \otimes_A B$ with $A \otimes TA \otimes A \otimes TA \otimes A$.

**Notation 2.3.** 1. Let $\Delta^{(2)}$ denote the map

$$(\Delta \otimes id)\Delta = (id \otimes \Delta)\Delta \colon B \to B \otimes_A B \otimes_A B.$$

2. Let $G \colon B \otimes_A B \to B$ denote the map given on monomials by

$$
\begin{aligned}
&G\big((a_0 \otimes \ldots \otimes a_{j-1}) \otimes (a_j) \otimes (a_{j+1} \otimes \ldots \otimes a_{n+1})\big) \\
&\qquad = (-1)^{j-1} a_0 \otimes \ldots \otimes a_{j-1} \otimes a_j \otimes a_{j+1} \otimes \ldots \otimes a_{n+1}.
\end{aligned}
$$

Notice that the circle operation $f \circ g$ of Definition 2.1, on Hochschild cochains $f$ and $g$ is precisely the composition

$$B \xrightarrow{\Delta^{(2)}} B \otimes_A B \otimes_A B \xrightarrow{id_B \otimes_A g \otimes_A id_B} B \otimes_A B \xrightarrow{G} B \xrightarrow{f} A.$$

To be clear, in the definition of the map $id_B \otimes_A g \otimes_A id_B$ one includes "Koszul signs" so that on elements the map is given by

$$(a \otimes x \otimes a' \otimes y \otimes a'' \otimes z \otimes a''') \mapsto (-1)^{|x||g|}(a \otimes x) \otimes g(a' \otimes y \otimes a'') \otimes (z \otimes a''').$$

This observation inspires our alternate definition of brackets below (Definition 2.5). First we record a crucial property of the map $G$.

**Proposition 2.4.** *Let $d$ be the differential on the complex $\mathrm{Hom}_{A^e}(B \otimes_A B, B)$. The map $G \in \mathrm{Hom}_{A^e}(B \otimes_A B, B)$ is a contracting homotopy for $F_B$, that is, $d(G) := d_B G + G d_{B\otimes_A B} = F_B$.*

*Proof.* Take a monomial

$$(a_0 \otimes \ldots \otimes a_{j-1}) \otimes (a_j) \otimes (a_{j+1} \otimes \ldots \otimes a_{n+1}) \in B_{j-1} \otimes_A B_{n-j},$$

with $j-1, n-j > 0$. Applying the formulas given in Lemma 2.2 and Notation 2.3(2), the function $Gd_{B\otimes_A B}$ sends $(a_0 \otimes \ldots \otimes a_{j-1}) \otimes (a_j) \otimes (a_{j+1} \otimes \ldots \otimes a_{n+1})$ to the element

$$\begin{array}{l} \sum_{i<j-1} (-1)^{(i+j-2)}(a_0 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_{n+1}) \\ +(-1)^{(j-1+j-2)}(a_0 \otimes \ldots \otimes a_{j-1} a_j \otimes \ldots \otimes a_{n+1}) \\ +(-1)^{(j-1+j-1)}(a_0 \otimes \ldots \otimes a_j a_{j+1} \otimes \ldots \otimes a_{n+1}) \\ +\sum_{j<k} (-1)^{(k-1+j-1)}(a_0 \otimes \ldots \otimes a_k a_{k+1} \otimes \ldots \otimes a_{n+1}) \end{array} \tag{2.5}$$

in $B$. Now $d_B G$ will send that same element in $B \otimes_A B$ to

$$\begin{array}{l} \sum_{i<j-1} (-1)^{(j-1+i)}(a_0 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_{n+1}) \\ +(-1)^{(j-1+j-1)}(a_0 \otimes \ldots \otimes a_{j-1} a_j \otimes \ldots \otimes a_{n+1}) \\ +(-1)^{(j-1+j)}(a_0 \otimes \ldots \otimes a_j a_{j+1} \otimes \ldots \otimes a_{n+1}) \\ +\sum_{j<k} (-1)^{(j-1+k)}(a_0 \otimes \ldots \otimes a_k a_{k+1} \otimes \ldots \otimes a_{n+1}). \end{array} \tag{2.6}$$

Comparing the exponents of $-1$, we see that $Gd_{B\otimes_A B} = -d_B G$ so $d(G) = 0$ on $B_{>0} \otimes_A B_{>0}$.

Now consider an element $(a_0) \otimes (a_1) \otimes (a_2 \otimes \ldots \otimes a_{n+1}) \in B_0 \otimes_A B_{n-1}$, for which $n-1 > 0$. Applying $Gd_{B\otimes_A B}$ to this element yields (2.5) where $j = 1$, minus the first two summands, and applying $d_B G$ yields (2.6) where $j = 1$, minus the first summand. So applying $d(G)$ to this element yields

$$a_0 a_1 \otimes \ldots \otimes a_{n+1} \in B.$$

Similarly, for the elements $(a_0 \otimes \ldots \otimes a_{n-1}) \otimes (a_n) \otimes (a_{n+1})$, applying $d(G)$ yields

$$(-1)^{n-1+n} a_0 \otimes \ldots \otimes a_n a_{n+1} = -a_0 \otimes \ldots \otimes a_n a_{n+1}.$$

In degree 0 we have

$$d(G)((a_0) \otimes (a_1) \otimes (a_2)) = a_0 a_1 \otimes a_2 - a_0 \otimes a_1 a_2.$$

Comparing these values in the different cases to our definition of $F_B$ above, we see that $d(G) = F_B$. $\square$

### 2.1. Alternate definition of bracket on the Hochschild complex

We call a map $\phi\colon B \otimes_A B \to B$ for which $d(\phi) := d_B\phi + \phi d_{B\otimes_A B} = F_B$ a contracting homotopy for $F_B$.

**Definition 2.5** ($\phi$-circle operation, $\phi$-bracket)**.** Let $\phi \in \mathrm{Hom}_{A^e}(B \otimes_A B, B)$ be any contracting homotopy for $F_B$. The $\phi$-circle operation $f \circ_\phi g$ on Hochschild cochains is defined as the composite

$$f \circ_\phi g := f\phi(id_B \otimes_A g \otimes_A id_B)\Delta^{(2)}.$$

The $\phi$-bracket is then defined as the graded commutator

$$[f, g]_\phi := f \circ_\phi g - (-1)^{(|f|-1)(|g|-1)} g \circ_\phi f.$$

Note that the $G$-circle operation $\circ_G$ is the standard circle operation and the $G$-bracket $[\ ,\ ]_G$ is the standard Gerstenhaber bracket.

**Lemma 2.6.** *Let $\phi$ be a contracting homotopy for $F_B$. Then $(\phi - G)\colon B \otimes_A B \to B$ is a boundary in the Hom complex.*

*Proof.* The difference is a cycle, since $d(\phi) = d(G) = F_B$. Recall that the following map is a quasi-isomorphism, where $\mathrm{proj}_*$ is induced by the canonical projection of $B$ onto $A$ (considered as a complex in degree 0, with 0 in all other degrees):

$$\mathrm{Hom}_{A^e}(B \otimes_A B, B) \xrightarrow{\mathrm{proj}_*} \mathrm{Hom}_{A^e}(B \otimes_A B, A).$$

Note that $B \otimes_A B$ is also a bimodule resolution of $A$, by the Künneth formula, and so the homology of the right hand side is $\mathrm{Ext}_{A^e}(A, A)$. Since $\mathrm{Ext}_{A^e}(A, A)$ is 0 in negative degrees, we have $\mathrm{H}_{-1}(\mathrm{Hom}_{A^e}(B \otimes_A B, B)) = 0$. So any cycle in degree $-1$ is a boundary. Consequently, the difference $\phi - G$ is a boundary. $\square$

**Proposition 2.7.** *Let $f$ and $g$ be cocycles in the complex $C(A) = \mathrm{Hom}_{A^e}(B, A)$. Let $\phi$ be a contracting homotopy for $F_B$. Then the difference*

$$f \circ_\phi g - f \circ g$$

*is a boundary, as is the difference*

$$[f, g]_\phi - [f, g].$$

*Proof.* Let $\tilde{g}$ be the function $id_B \otimes_A g \otimes_A id_B$ in $\mathrm{Hom}_{A^e}(B \otimes_A B \otimes_A B, B \otimes_A B)$. By this notation $\tilde{g}$, in degree $n$, we mean the sum of all maps $id_i \otimes_A g \otimes_A id_j$ on $(B \otimes_A B \otimes_A B)_n$, where $id_i$ is the identity map on $B_i$ and $i + j = n - |g|$. Note that the map $\tilde{g}$ is still a cocycle since $g$ is a cocycle. Then

$$f \circ_\phi g = f\phi\tilde{g}\Delta^{(2)}, \text{ while } f \circ g = fG\tilde{g}\Delta^{(2)}.$$

The difference is given by

$$\begin{aligned} f \circ_\phi g - f \circ g &= f\phi\tilde{g}\Delta^{(2)} - fG\tilde{g}\Delta^{(2)} \\ &= f(\phi - G)\tilde{g}\Delta^{(2)}. \end{aligned}$$

By Lemma 2.6, there exists some map $\psi$ with $d(\psi) = \phi - G$. Then, since $f$ and $\tilde{g}$ are cocycles,

$$\begin{aligned} (-1)^{|f|} d(f\psi\tilde{g}\Delta^{(2)}) &= fd(\psi)\tilde{g}\Delta^{(2)} \\ &= f(\phi - G)\tilde{g}\Delta^{(2)} \\ &= f \circ_\phi g - f \circ g, \end{aligned}$$

whence $f \circ_\phi g - f \circ g$ is seen to be a boundary, as claimed. The second statement follows from the first. $\square$

**Corollary 2.8.** *1. For any two cocycles $f$ and $g$, the $\phi$-bracket $[f,g]_\phi$ is yet another cocycle.*

*2. If $f$ or $g$ is a boundary, then so is $[f,g]_\phi$.*

*3. On cocycles, the $\phi$-bracket is graded anti-commutative up to a boundary and also satisfies the Jacobi identity up to a boundary.*

*Proof.* All of these statements follow from the previous proposition and the fact that these conditions are satisfied by the Gerstenhaber bracket. □

**Corollary 2.9.** *For any contracting homotopy $\phi$ for $F_B$, the $\phi$-bracket $[\ ,\ ]_\phi$ induces a graded Lie bracket on the shifted cohomology*

$$[\ ,\ ]_\phi\colon \mathrm{HH}(A)[1]\otimes \mathrm{HH}(A)[1] \to \mathrm{HH}(A)[1].$$

*This bracket agrees with the standard Gerstenhaber bracket on cohomology.*

## 3. Brackets on other cochain complexes

In this section we define brackets at the cochain level on complexes other than the Hochschild complex. We show that under some conditions, these brackets induce precisely the Gerstenhaber bracket on cohomology. Koszul algebras over $k$ will satisfy these conditions.

Let $K \to A$ be a projective $A$-bimodule resolution of $A$. For most of this section we will want $K$ to satisfy some hypotheses which we outline next.

### 3.1. Hypotheses on the bimodule resolution $K \to A$

We assume that the $A$-bimodule resolution $K \to A$ satisfies the following conditions:

(a) $K$ admits an embedding $\iota\colon K \to B$ of complexes of $A$-bimodules for which the following diagram commutes:

$$\begin{tikzcd} & B \arrow[dr] & \\ K \arrow[ur, "\iota"] \arrow[rr] & & A. \end{tikzcd}$$

(b) The embedding $\iota$ admits a section $\pi\colon B \to K$, i.e. an $A^e$-chain map $\pi$ with $\pi\iota = id_K$.

(c) The diagonal map $\Delta_B\colon B \to B\otimes_A B$ preserves $K$, and hence induces a diagonal quasi-isomorphism $\Delta_K\colon K \to K\otimes_A K$. Equivalently, $K$ comes equipped with a diagonal quasi-isomorphism $\Delta_K\colon K\to K\otimes_A K$ satisfying $\Delta_B\iota = (\iota\otimes_A\iota)\Delta_K$.

Practically speaking, the easiest way for condition (b) to be satisfied is for $K$ to be free on some graded base space $W \subset K$ with $W$ mapping to $TA \subset B$ under $\iota$. Indeed, one can verify that condition (b) holds if and only if the cokernel of each map $\iota_l\colon K_l \to B_l$ is projective over $A^e$. So we could, alternately, require that $K$ satisfy the slightly stronger condition

(b′) $K$ is free on a graded base $W \subset K$ with $\iota(W) \subset TA$ in $B$.

Conditions (a) and (b) can be seen as relatively mild restrictions. In contrast, condition (c) holds a great deal of significance. Indeed, it can be shown that if the minimal free bimodule resolution of a connected graded algebra can be made to satisfy (c), then the algebra is Koszul. This does not mean, however, that non-Koszul algebras have no resolutions satisfying the above conditions, or that the minimal resolution cannot be used in some way to compute the Lie bracket. We will see in Section 5 that we can still use the minimal resolution for (the group algebras of) the cyclic $p$-group in characteristic $p$ to compute the Lie bracket on Hochschild cohomology.

As the above discussion suggests, the Koszul complex of a Koszul algebra does satisfy our conditions (a)–(c). See, e.g., [**4**], [**16**], or [**17**] for a discussion of diagonal maps in the case of a Koszul algebra. Verification of the other conditions is more straightforward. We will not need the definition of a Koszul algebra however, as we work in the general setting of a complex satisfying conditions (a)–(c). In the next section we give explicitly the example of a polynomial ring, which is a Koszul algebra. One can also show that the Koszul resolution of a PBW deformation of a Koszul algebra fits into our framework ([**7**, Lemma 4.1], [**16**], [**17**]). In this case, the diagonal map on $K$ will be induced by the natural comultiplication on the base $W \subset K$. Localizations of such algebras will also fit into our scheme.

*Remark 3.1.* One actually has to replace $B$ with the reduced bar resolution to get (a)–(c) to hold in the case of a non-augmented PBW deformation of a Koszul algebra. This is, however, a straightforward process.

### 3.2. $\phi$-brackets on $\mathrm{Hom}_{A^e}(K, A)$

For this subsection, let us fix a resolution $K \to A$ satisfying the hypotheses 3.1(a)–(c). Let $\mu$ denote the given quasi-isomorphism $\mu\colon K \to A$. Then we have the two chain maps $\mu \otimes_A id_K\colon K \otimes_A K \to A \otimes_A K \cong K$ and $id_K \otimes_A \mu\colon K \otimes_A K \to K \otimes_A A \cong A$. We define the chain map $F_K$ as the difference of these two maps,

$$F_K := (\mu \otimes_A id_K - id_K \otimes_A \mu)\colon K \otimes_A K \to K.$$

(The natural isomorphisms $A \otimes_A K \cong K$ and $K \otimes_A A \cong K$ implicit in the above definition have been omitted from the notation.) In the case that $K$ satisfies (b$'$), so that $K = A \otimes W \otimes A$ and the elements in $K$ are given by sums of monomials $a \otimes x \otimes a'$, we get the elementwise definition of $F_K$ analogous to the one given in (2.4). In particular, when $K = B$ the two definitions of $F_B$ agree.

**Lemma 3.2.** *The map $F_K\colon K \otimes_A K \to K$ is a boundary in $\mathrm{Hom}_{A^e}(K \otimes_A K, K)$.*

*Proof.* It suffices to check that $F_K$ maps to 0 under the quasi-isomorphism

$$\mathrm{Hom}_{A^e}(K \otimes_A K, K) \xrightarrow{\mu_*} \mathrm{Hom}_{A^e}(K \otimes_A K, A).$$

Since $F_K$ is the difference of the two maps $\mu \otimes_A id_K$ and $id_K \otimes_A \mu$, composed with the isomorphisms $A \otimes_A K \cong K$ and $K \otimes_A A \cong K$ respectively, the image of $F_K$ is 0 if and only if the images of these two maps agree.

For any complex $M$ of $A$-bimodules, let $\varphi_M\colon A \otimes_A M \xrightarrow{\sim} M$ and $\varphi'_M\colon M \otimes_A A \xrightarrow{\sim} M$ denote the standard isomorphisms. Since these isomorphisms are natural we will have a commutative diagram

$$\begin{tikzcd}
K \otimes_A K \arrow[r, "\mu\otimes_A id_K"] & A \otimes_A K \arrow[r, "id_A\otimes_A\mu"] \arrow[d, "\varphi_K"] & A \otimes_A A \arrow[d, "\varphi_A"] \\
 & K \arrow[r, "\mu"] & A,
\end{tikzcd}$$

whence

$$\begin{aligned}
\mu_*\big(\varphi_K(\mu \otimes_A id_K)\big) &= \mu\varphi_K(\mu \otimes_A id_K) \\
&= \varphi_A(id_A \otimes_A \mu)(\mu \otimes_A id_K) \\
&= \varphi_A(\mu \otimes_A \mu).
\end{aligned}$$

Similarly, we see $\mu_*\big(\varphi'_K(id_A \otimes_A A)\big) = \varphi'_A(\mu \otimes_A \mu)$. Since there is an equality $\varphi_A = \varphi'_A$, this gives the desired equality

$$\mu_*\big(\varphi_K(\mu \otimes_A id_K)\big) = \mu_*\big(\varphi'_K(id_A \otimes_A \mu)\big),$$

and we conclude $\mu_*(F_K) = 0$. □

Recall that a contracting homotopy for $F_K$ is a map $\phi\colon K \otimes_A K \to K$ for which $d(\phi) := d_K\phi + \phi d_{K\otimes_A K} = F_K$. The lemma allows us to make the following definition.

**Definition 3.3** (General $\phi$-circle operation, $\phi$-bracket)**.** Let $\phi$ be a contracting homotopy for $F_K$, and let $\Delta_K^{(2)}$ be a chain map from $K$ to $K \otimes_A K \otimes_A K$. (Under hypothesis 3.1(c), we take $\Delta_K^{(2)} := (\Delta_K \otimes id_K)\Delta_K = (id_K \otimes \Delta_K)\Delta_K$.) The $\phi$-circle product $f \circ_\phi g$ is the composition

$$f \circ_\phi g := f\phi(id_K \otimes_A g \otimes_A id_K)\Delta_K^{(2)}.$$

The $\phi$-bracket is the graded commutator

$$[f, g]_\phi := f \circ_\phi g - (-1)^{(|f|-1)(|g|-1)} g \circ_\phi f.$$

Suppose that our resolution $K \to A$ satisfies the freeness property (b′). For example, we could take $K$ to be the Koszul resolution of a PBW deformation of a Koszul algebra. (See, e.g., [**7, 17**].) We can then express the $\phi$-circle operation and bracket on elements in the generating set $x \in W \subset K$ as

$$(f \circ_\phi g)(x) = \sum (-1)^{|g||x_1|} f\big(\phi(x_1 \otimes g(x_2) \otimes x_3)\big)$$

and

$$\begin{aligned}
[f, g]_\phi(x) = &\sum (-1)^{|g||x_1|} f\big(\phi(x_1 \otimes g(x_2) \otimes x_3)\big) \\
&-(-1)^{(|g|-1)(|f|-1)} \sum (-1)^{|f||x_1|} g\big(\phi(x_1 \otimes f(x_2) \otimes x_3)\big).
\end{aligned}$$

Here $\sum x_1 \otimes x_2 \otimes x_3$ denotes the element $\Delta^{(2)}(x)$, which lies in $W \otimes W \otimes W \subset K^{\otimes_A 3}$ by hypothesis. In the case that $K = B$ and $\phi = G$, the map $\phi$ simply inserts the apparent missing factor $(-1)^{|x_1|}$ in the above expressions.

We will see in Theorem 3.6 that the $\phi$-bracket operation preserves cocycles and coboundaries, and that the induced operation on cohomology is precisely the Gerstenhaber bracket. The following lemma will be of significance in a moment.

**Lemma 3.4.** *Let us take $G_K := \pi G(\iota \otimes_A \iota) \colon K \otimes_A K \to K$, where $G$ is the standard contracting homotopy for $F_B$ given in Notation 2.3(2). Then*

1. $F_K = \pi F_B(\iota \otimes_A \iota)$.
2. $d(G_K) = F_K$.

*Proof.* Statement (1) follows directly from the definitions of $F_K$ and $F_B$ given above, the commutative diagram

$$\begin{tikzcd} & B \arrow[dr] & \\ K \arrow[ur, "\iota"] \arrow[rr] & & A \end{tikzcd}$$

of hypothesis 3.1(a), and the fact that $\pi\iota = id_K$. Statement (2) follows from (1) since we have

$$d(G_K) = d(\pi G(\iota \otimes_A \iota)) = \pi d(G)(\iota \otimes_A \iota) = \pi F_B(\iota \otimes_A \iota) = F_K. \qquad \square$$

Since $\pi \colon B \to K$ and $\iota \colon K \to B$ are quasi-isomorphisms, they induce quasi-isomorphisms on the Hom complexes

$$\pi^* \colon \mathrm{Hom}_{A^e}(K, A) \to \mathrm{Hom}_{A^e}(B, A) \quad \text{and} \quad \iota^* \colon \mathrm{Hom}_{A^e}(B, A) \to \mathrm{Hom}_{A^e}(K, A).$$

The latter map is simply restriction to $K$.

**Proposition 3.5.** *Assume hypotheses 3.1(a)–(c). Given $f$ and $g$ in $\mathrm{Hom}_{A^e}(K, A)$ we have an equality of functions*

$$f \circ_{G_K} g = \iota^*(\pi^* f \circ \pi^* g)$$

*and subsequent equality*

$$[f, g]_{G_K} = \iota^*[\pi^* f, \pi^* g].$$

*Proof.* Let us simply expand the functions:

$$\begin{aligned} (f\pi \circ g\pi)\iota &= f\pi G(id_B \otimes_A g\pi \otimes_A id_B)\Delta_B^{(2)}\iota \\ &= f\pi G(id_B \otimes_A g\pi \otimes_A id_B)\iota^{\otimes 3}\Delta_K^{(2)} \\ &= f\pi G(\iota \otimes_A g(\pi\iota) \otimes_A \iota)\Delta_K^{(2)} \\ &= f(\pi G(\iota \otimes_A \iota))(id_K \otimes_A g \otimes_A id_K)\Delta_K^{(2)} \qquad (\text{since } \pi\iota = id_K) \\ &= f G_K(id_K \otimes_A g \otimes_A id_K)\Delta_K^{(2)} \\ &= f \circ_{G_K} g. \end{aligned}$$

Equality of brackets follows from the definition of the bracket as the graded $\circ$-commutator. $\square$

Let $\phi$ be any contracting homotopy for $F_K$. By the same proof as the one given for Proposition 2.7, the differences

$$f \circ_\phi g - f \circ_{G_K} g \quad \text{and} \quad [f, g]_\phi - [f, g]_{G_K}$$

will be boundaries whenever $f$ and $g$ are cocycles in $\mathrm{Hom}_{A^e}(K, A)$.

**Theorem 3.6.** *Suppose $K$ is a bimodule resolution of $A$ satisfying hypotheses 3.1(a)–(c), and let $\phi$ be any contracting homotopy for $F_K$. Let $f$ and $g$ be cocycles in $\mathrm{Hom}_{A^e}(K, A)$.*

1. *The bracket* $[f,g]_\phi$ *is a cocycle.*
2. *If* $f$ *or* $g$ *is a boundary, then* $[f,g]_\phi$ *is a boundary.*
3. *The induced bracket* $[\ ,\ ]_\phi\colon \mathrm{HH}(A)[1]\otimes\mathrm{HH}(A)[1]\to\mathrm{HH}(A)[1]$ *on cohomology agrees with the Gerstenhaber bracket.*

*Proof.* By the discussion preceding the statement of the theorem, we may assume without loss of generality that $\phi = G_K = \pi G(\iota\otimes_A\iota)$. Now (1) and (2) follow directly from Proposition 3.5 and the fact that $\pi^*$ and $\iota^*$ are quasi-isomorphisms. Since $id_{\mathrm{H}(\mathrm{Hom}(K,A))} = (\pi\iota)^* = \iota^*\pi^*$, we see that the induced isomorphisms on homology are mutually inverse. So we have

$$(\iota^*)^{-1} = \pi^*\colon \mathrm{H}(\mathrm{Hom}_{A^e}(K,A)) \xrightarrow{\cong} \mathrm{H}(\mathrm{Hom}_{A^e}(B,A)).$$

This isomorphism is one of graded Lie algebras since, according to Proposition 3.5, we will have an equality

$$\pi^*([f,g]_{G_K}) = \pi^*(\iota^*[\pi^*f,\pi^*g]) = (\pi^*\iota^*)[\pi^*f,\pi^*g] = [\pi^*f,\pi^*g]$$

on cohomology. Finally, the homologies $\mathrm{H}(\mathrm{Hom}_{A^e}(K,A))$ and $\mathrm{H}(\mathrm{Hom}_{A^e}(B,A))$ are precisely the Hochschild cohomology $\mathrm{HH}(A)$. □

### 3.3. Formula for $\phi$

In general, it may be difficult to find a map $\phi$ satisfying $d(\phi) := d_K\phi + \phi d_{K\otimes_A K} = F_K$. Let us give one method for constructing such a homotopy that is related to constructions of chain maps via contracting homotopies (for example, as described in Mac Lane [**15**]). Consider the extended complex $K\to A\to 0$, by which we mean the complex $\cdots\to K_1\to K_0\to A\to 0$ with $A$ in degree $-1$.

The following lemma is general, that is, it does not require hypotheses 3.1, only that $K$ be a free $A$-bimodule resolution of $A$, as well as the further hypotheses stated in the lemma.

**Lemma 3.7.** *Suppose* $K$ *is free on a graded subspace* $W\subset K$*. Let* $h$ *be any* $k$*-linear contracting homotopy for the identity map on the extended complex* $K\to A\to 0$*. Take* $\phi_{-1}=0$*. Define* $\phi$*, in each degree* $i\geqslant 0$*, as the* $A^e$*-linear map* $\phi_i\colon (K\otimes_A K)_i\to K_{i+1}$ *given inductively by the formula*

$$\phi_i|_{W\otimes A\otimes W} := h_i((F_K)_i - \phi_{i-1}(d_{K\otimes_A K})_i)|_{W\otimes A\otimes W}.$$

*Then* $d(\phi) = F_K$*.*

*Proof.* To simplify notation, take $F=F_K$ and $d=d_K$, or $d_{K\otimes_A K}$ when appropriate. Let us consider $F$ and $\phi$ as maps to the extended complex. Note that, since $F_0$ has image in the space of degree 0 cycles $Z_0(K\to A\to 0)$, this new version of $F$ will still be a chain map. Take $\phi_j=0$ for all negative $j$. Then for all negative $j$ the equality $d_{j+1}\phi_j+\phi_{j-1}d_j=F_j$ holds, since both sides are just 0. Now suppose, for a given $i$, that for all $j<i$ the formula $d_{j+1}\phi_j+\phi_{j-1}d_j=F_j$ holds. Then after restricting to the generating subspace $W\otimes A\otimes W\subset K\otimes_A K$,

$$\begin{aligned}(d_{i+1}\phi_i + \phi_{i-1}d_i) &= d_{i+1}h_i(F_i - \phi_{i-1}d_i) + \phi_{i-1}d_i \\ &= (F_i - \phi_{i-1}d_i) - h_{i-1}d_i(F_i - \phi_{i-1}d_i) + \phi_{i-1}d_i \\ &= F_i - \phi_{i-1}d_i - h_{i-1}d_iF_i + h_{i-1}d_i\phi_{i-1}d_i + \phi_{i-1}d_i \\ &= F_i - h_{i-1}d_iF_i + h_{i-1}d_i\phi_{i-1}d_i \\ &= F_i - h_{i-1}d_iF_i + h_{i-1}F_{i-1}d_i - h_{i-1}\phi_{i-2}d_{i-1}d_i \\ &= F_i - h_{i-1}d_iF_i + h_{i-1}F_{i-1}d_i \\ &= F_i,\end{aligned}$$

whence $d(\phi) = F$. □

We will use the formula of Lemma 3.7 in Sections 4 and 5.

### 3.4. $\phi$-brackets under weaker conditions

In the remainder of this section, we describe some weaker conditions under which the conclusion of Theorem 3.6 still holds. We will need this more general statement in Section 5 below. For the following lemma, we assume only hypotheses 3.1(a) and (b), and we let $\Delta_K^{(2)}$ be a chain map from $K$ to $K \otimes_A K \otimes_A K$. (We do not assume that there is a coassociative chain map $\Delta\colon K \to K \otimes_A K$ from which $\Delta_K^{(2)}$ is defined.)

Recall the canonical contracting homotopy $G_K := \pi G(\iota \otimes_A \iota) : K \otimes_A K \to K$ for $F_K$ (Lemma 3.4 does not require hypothesis 3.1(c)).

**Lemma 3.8.** *Assume hypotheses 3.1(a) and (b). For cocycles $f, g \in \mathrm{Hom}_{A^e}(K, A)$, the difference*

$$\iota^*(\pi^* f \circ \pi^* g) - f G_K(id_K \otimes_A g \otimes_A id_K)(\pi \otimes_A \pi \otimes_A \pi)\Delta_B^{(2)}\iota$$

*is a boundary. If $\Delta_K^{(2)} = (\pi \otimes_A \pi \otimes_A \pi)\Delta_B^{(2)}\iota$, then*

$$\iota^*(\pi^* f \circ \pi^* g) - f \circ_\phi g \quad \text{and} \quad i^*[\pi^* f, \pi^* g] - [f, g]_\phi$$

*are boundaries.*

*Proof.* We have

$$\iota^*(\pi^* f \circ \pi^* g) = f\pi G(id \otimes g\pi \otimes id)\Delta_B^{(2)}\iota$$

and

$$\begin{aligned}fG_K(id_K \otimes g \otimes id_K)(\pi \otimes \pi \otimes \pi)\Delta_B^{(2)}\iota &= f\pi G(\iota \otimes \iota)(\pi \otimes g\pi \otimes \pi)\Delta_B^{(2)}\iota \\ &= f\pi G(\iota\pi \otimes \iota\pi)(id \otimes g\pi \otimes id)\Delta_B^{(2)}\iota.\end{aligned}$$

Now one can check that $\pi F_B(\iota\pi \otimes \iota\pi) = \pi F_B$ (since $\pi\iota = id_K$ and by the definition of $F_B$). So

$$d(\pi G(\iota\pi \otimes \iota\pi)) = \pi d(G)(\iota\pi \otimes \iota\pi) = \pi F_B(\iota\pi \otimes \iota\pi) = \pi F_B = d(\pi G),$$

and, since cohomology vanishes in negative degrees, $\pi G(\iota\pi \otimes \iota\pi) - \pi G$ is a boundary. It follows that this difference

$$\begin{aligned}\iota^*(\pi^* f \circ \pi^* g) &- fG_K(id_K \otimes_A g \otimes_A id_K)(\pi \otimes_A \pi \otimes_A \pi)\Delta_B^{(2)}\iota \\ &= f\pi G(id \otimes g\pi \otimes id)\Delta_B^{(2)}\iota - f\pi G(\iota\pi \otimes \iota\pi)(id \otimes g\pi \otimes id)\Delta_B^{(2)}\iota\end{aligned}$$

is a boundary as well, since all of $f, \pi, \iota, g, \Delta_B^{(2)}$ are cycles. □

It follows that under hypotheses 3.1(a) and (b), taking $\Delta_K^{(2)} := (\pi \otimes_A \pi \otimes_A \pi)\Delta_B^{(2)}\iota$ in Definition 3.3, the conclusion of Theorem 3.6 holds. Thus $\phi$-brackets may be defined in a fairly general setting, at the expense of dealing more directly with maps $\pi$, $\iota$ comparing to the bar resolution. Note that Definition 3.3 can be used to define other versions of $\phi$-bracket, given other choices of chain map $\Delta_K^{(2)}$. At the moment, we do not know which of these other $\phi$-brackets are well-defined on cohomology, nor whether they have useful properties.

As we will see in the example of Section 5, one may be able to produce a satisfactory map $\Delta_K^{(2)}$ without any explicit reference to $\iota$ or $\pi$, and produce a subsequent candidate for the Gerstenhaber bracket. In the example of Section 5 we check that our map $\Delta_K^{(2)}$ is of the form $\Delta_K^{(2)} := (\pi \otimes_A \pi \otimes_A \pi)\Delta_B^{(2)}\iota$ for some choice of $\iota$ and $\pi$.

## 4. Recovering Schouten–Nijenhuis brackets for polynomial rings

### 4.1. Review of the Koszul resolution

Let $A = k[x_1, \ldots, x_n]$ be the polynomial ring in $n$ variables. We take $V$ to be the $k$-vector space with basis $\{x_1, \ldots, x_n\}$. As a formality, let $x_0 = 1$.

**Definition 4.1.** Let $S_i$ denote the symmetric group on $i$ symbols. For any $v_1, \ldots, v_i \in V$, let $o(v_1, \ldots, v_i)$ denote the $S_i$-orbit sum

$$o(v_1, \ldots, v_i) = \sum_{\sigma \in S_i} \operatorname{sgn}(\sigma) v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(i)}$$

in $V^{\otimes n} \subset A^{\otimes n}$. We take $o(\varnothing) := 1$.

Let $W$ denote the graded subspace $\oplus_{i \geqslant 0} o(V, \ldots, V)$ in $TA$. One can check that $W$ is a subcoalgebra of $TA$ and that $K = K(A) := A \otimes W \otimes A$ is a subcomplex of the bar resolution $B = (A \otimes TA \otimes A, d)$. (See also [**4, 16**], and (4.3) below.) It is well known that the embedding $K \to B$ is a quasi-isomorphism, i.e. that $A$ is Koszul. In the following lemma, the notation $\hat{v}_l$ indicates that $v_l$ has been removed.

**Lemma 4.2.** *The differential on $K$ is given on monomials by*

$$\begin{aligned} &a \otimes o(v_1, \ldots, v_i) \otimes a' \mapsto \\ &\quad \textstyle\sum_l (-1)^{l+1} a v_l \otimes o(v_1, \ldots, \hat{v}_l, \ldots, v_i) \otimes a' - (-1)^{l+1} a \otimes o(v_1, \ldots, \hat{v}_l, \ldots, v_i) \otimes v_l a'. \end{aligned}$$

*Proof.* This follows by direct computation and the fact that $v_l v_m - v_m v_l$ is 0 in $A$ for each $v_l, v_m \in V$. $\square$

We choose the ordering $x_1 < x_2 < \cdots < x_n$ on the generators of $A$ and call an element

$$1 \otimes o(x_{i_1}, \ldots, x_{i_s}) \otimes x_{j_1} \ldots x_{j_t} \otimes o(x_{k_1}, \ldots, x_{k_u}) \otimes 1$$

in $k \otimes W \otimes A \otimes W \otimes k$ an ordered "monomial" if $x_{i_l} < x_{i_{l+1}}$, $x_{j_l} \leqslant x_{j_{l+1}}$, and $x_{k_l} < x_{k_{l+1}}$, for all $l$. We define ordered monomials in $A \otimes W \otimes k$ and in $k \otimes W \otimes A$ similarly. The $A^e$ generating subspaces $k \otimes W \otimes k$ and $k \otimes W \otimes A \otimes W \otimes k$, of $K$ and $K \otimes_A K$ respectively, are spanned over $k$ by the respective sets of ordered monomials.

We employ a slight variation of a left $k$-linear contracting homotopy for the identity on the extended complex $K \to A \to 0$ given in [23]. In homological degrees $-1$ and 0, which are $A$ and $A \otimes A$ respectively, $h$ is given by the formula

$$h\colon \begin{cases} a \mapsto 1 \otimes a \\ x_{j_1} \dots x_{j_t} \otimes a \mapsto \sum_{1 \leqslant \nu \leqslant t} x_{j_1} \dots x_{j_{\nu-1}} \otimes o(x_{j_\nu}) \otimes x_{j_{\nu+1}} \dots x_{j_t} a, \end{cases}$$

for any ordered monomial $x_{j_1} \dots x_{j_t} \otimes 1$ in $A \otimes A = K_0$ and $a$ in $A$. In higher degrees we define $h$ to be the right $A$-linear map specified on ordered monomials by the formula

$$\begin{aligned} h\colon x_{j_1} \dots x_{j_t} \otimes o(x_{k_1}, \dots, x_{k_u}) \otimes 1 \mapsto \\ (-1)^u \sum_{x_{j_\nu} > x_{k_u}} x_{j_1} \dots x_{j_{\nu-1}} \otimes o(x_{k_1}, \dots x_{k_u}, x_{j_\nu}) \otimes x_{j_{\nu+1}} \dots x_{j_t}. \end{aligned}$$

When the indexing set $\{x_{j_\nu} : x_{j_\nu} > x_{k_u}\}$ is empty, the sum is indeed taken to be 0.

Using Lemma 3.7 and the contracting homotopy $h$ given above, one can easily construct a contracting homotopy $\phi\colon K \otimes_A K \to K$ for $F_K$ in low degrees. One then deduces from this information the following general formula.

**Definition 4.3.** We define the $A^e$-linear map $\phi\colon K \otimes_A K \to K$ on ordered monomials by the formulas

$$\begin{aligned} \phi\colon 1 \otimes x_{j_1} \dots x_{j_t} \otimes o(x_{k_1}, \dots, x_{k_u}) \otimes 1 \mapsto \\ (-1)^u \sum_{x_{k_u} < x_{j_\nu}} x_{j_1} \dots x_{j_{\nu-1}} \otimes o(x_{k_1}, \dots, x_{k_u}, x_{j_\nu}) \otimes x_{j_{\nu+1}} \dots x_{j_t}, \\ \phi\colon 1 \otimes o(x_{i_1}, \dots, x_{i_s}) \otimes x_{j_1} \dots x_{j_t} \otimes 1 \mapsto \\ \sum_{x_{j_\nu} < x_{i_1}} x_{j_1} \dots x_{j_{\nu-1}} \otimes o(x_{j_\nu}, x_{i_1}, \dots, x_{i_s}) \otimes x_{j_{\nu+1}} \dots x_{j_t} \end{aligned}$$

on $K_0 \otimes_A K_u$ and on $K_s \otimes_A K_0$ $(u, s \geqslant 0)$, and

$$\begin{aligned} \phi\colon 1 \otimes o(x_{i_1}, \dots, x_{i_s}) \otimes x_{j_1} \dots x_{j_t} \otimes o(x_{k_1}, \dots, x_{k_u}) \otimes 1 \mapsto \\ \sum_{x_{k_u} < x_{j_\nu} < x_{i_1}} (-1)^{su+u} x_{j_1} \dots x_{j_{\nu-1}} \otimes o(x_{k_1}, \dots x_{k_u}, x_{j_\nu}, x_{i_1}, \dots, x_{i_s}) \otimes x_{j_{\nu+1}} \dots x_{j_t}. \end{aligned} \tag{4.1}$$

**Proposition 4.4.** *The map $\phi\colon K \otimes_A K \to K$ satisfies $d(\phi) := d_K \phi + \phi d_{K \otimes_A K} = F_K$.*

We omit the proof of this proposition, which is a delicate, but straightforward calculation.

### 4.2. Computing the bracket directly from $\phi$ and Theorem 3.6

Before we begin let us make a remark. In general, one wants to be strategic in computing the Lie bracket. One should probably use some additional structures on Hochschild cohomology, such as the cup product in combination with (2.3), additional gradings, etc. However, we are able to recover here, essentially with a single calculation, a formula for the brackets of cocycles of arbitrary degree via Theorem 3.6.

We will employ the standard isomorphism

$$\begin{aligned} A[\partial_1, \dots, \partial_n] \to \mathrm{Hom}_{A^e}(K, A) = \mathrm{HH}(A) \\ \partial_i \mapsto (a \otimes o(x_j) \otimes a' \mapsto \delta_{ij} a a'). \end{aligned} \tag{4.2}$$

Here the generators $\partial_i$ are given degree 1, and $A[\partial_1, \dots, \partial_n]$ denotes the free graded commutative $A$-algebra on these generators. We identify the monomial $\partial_{i_1} \dots \partial_{i_s}$ with the function dual to the orbit sum $(-1)^{\sum_{\ell=1}^{s-1} \ell} o(x_{i_1}, \dots, x_{i_s})$ in $\mathrm{Hom}_{A^e}(K, A) =$

$\mathrm{Hom}_k(W, A)$. We note that the differential of this complex will vanish, from which we get the identification $\mathrm{Hom}_{A^e}(K, A) = \mathrm{HH}(A)$ of (4.2).

It will be convenient to have a bit more notation for the statement of the next proposition.

**Notation 4.5.** For any ordered set $I = \{i_1, \dots, i_s\}$ of integers satisfying $1 \leqslant i_k \leqslant n$ for all $k$, we take

$$\partial_I := \partial_{i_1} \dots \partial_{i_s} \in \mathrm{Hom}_{A^e}(K, A)$$

and

$$o(x_I) = o(x_{i_1}, \dots, x_{i_s}) \in W.$$

For $i_k \in I$, we take

$$I(k) := \{i_1, \dots, i_{k-1}\} \quad \text{and} \quad I'(k) := \{i_{k+1}, \dots, i_l\}.$$

For ordered sets $I$ and $J$ we give $I \amalg J$ the natural ordering with $i < j$ for each $i \in I, j \in J$.

In these notations we do *not* require that the ordering on $I$ is such that $i_k < i_{k+1}$ as integers. We will always let $i_k$ denote the $k$th element of $I$, as determined by $I$'s given order. If we take $\underline{n} = \{1, \dots, n\}$, then the standard $A$-basis for $A[\partial_1, \dots, \partial_n] = \mathrm{Hom}_{A^e}(K, A)$ can now be written as the set $\{\partial_I : I \text{ an ordered subset of } \underline{n}\}$.

Via indexing by ordered sets, we can give a clear expression of the comultiplication on $W$, and the corresponding map $\Delta\colon K \to K \otimes_A K$. We have

$$\Delta(1 \otimes o(x_I) \otimes 1) = \sum_{I_1, I_2} \pm (1 \otimes o(x_{I_1}) \otimes 1) \otimes (1 \otimes o(x_{I_2}) \otimes 1), \tag{4.3}$$

where the sum is indexed by all ordered disjoint subsets $I_1, I_2 \subset I$ with $I_1 \cup I_2 = I$, and $\pm$ is the sign of $\sigma$, where $\sigma$ is the unique permutation with $\{i_{\sigma(1)}, \dots, i_{\sigma(|I|)}\} = I_1 \amalg I_2$ as an ordered set.

**Proposition 4.6.** *The $\circ_\phi$ operation is given by*

$$(a\partial_I) \circ_\phi (b\partial_J) = \sum_{1 \leqslant k \leqslant |I|} (-1)^{(|I|-k)(|J|-1)} a \frac{\partial}{\partial x_{i_k}}(b) \partial_{I(k) \amalg J \amalg I'(k)}$$

*and the bracket $[\ ,\ ]_\phi$ is given by*

$$\begin{aligned}[a\partial_I, b\partial_J]_\phi = &\sum_{1 \leqslant k \leqslant |I|} (-1)^{(|I|-k)(|J|-1)} a \frac{\partial}{\partial x_{i_k}}(b) \partial_{I(k) \amalg J \amalg I'(k)} \\ &- \sum_{1 \leqslant l \leqslant |J|} (-1)^{(l-1)(|I|-1)} b \frac{\partial}{\partial x_{j_l}}(a) \partial_{J(l) \amalg I \amalg J'(l)}.\end{aligned}$$

Note that if $I$ and $J$ share some indices, many of the terms $a\frac{\partial}{\partial x_{i_k}}(b)\partial_{I(k)\amalg J \amalg I'(k)}$ may be 0.

*Proof.* Take $f = (a\partial_I)$ and $g = (b\partial_J)$ with $I$ and $J$ ordered subsets of $\underline{n}$. We may assume $b$ is an ordered monomial $b = x_{j_1} \dots x_{j_t}$. We first provide a computation with symbols $(-1)^{\epsilon_i}$ in place of significant signs. We will then go back and provide the appropriate signs.

Suppose we have a nonzero monomial $1\otimes o(x_{I(k)\amalg J\amalg I'(k)})\otimes 1$. This implies, in particular, that $J$ and $I(k)\cup I'(k)$ share no indices. Then

$$\begin{aligned}\Delta^{(2)}&(1\otimes o(x_{I'(k)\amalg J\amalg I(k)})\otimes 1) =\\ &(1\otimes o(x_{I'(k)})\otimes 1)\otimes(1\otimes o(x_J)\otimes 1)\otimes(1\otimes o(x_{I(k)})\otimes 1)\\ &+\textstyle\sum_{\mathscr{S}}\pm(1\otimes o(x_{I_2})\otimes 1)\otimes(1\otimes o(x_J)\otimes 1)\otimes(1\otimes o(x_{I_1})\otimes 1)\\ &+\textstyle\sum_{J'\neq J}\pm(1\otimes o(x_{I'_2})\otimes 1)\otimes(1\otimes o(x_{J'})\otimes 1)\otimes(1\otimes o(x_{I'_1})\otimes 1),\end{aligned}$$

where $\mathscr{S}$ is the set of all pairs of subsets $I_1, I_2\subset I(k)\amalg I'(k)=I\setminus\{i_k\}$ with $I_1\cup I_2 = I\setminus\{i_k\}$, *minus* the pair $\{I(k), I'(k)\}$. We do not specify the indexing set of the final sum, except to say that $J'\neq J$. Since $b\partial_J(1\otimes o(x_{J'})\otimes 1)=0$ whenever $J'\neq J$, the above expression gives

$$\begin{aligned}&(1\otimes(b\partial_J)\otimes 1)\Delta^{(2)}\big(1\otimes o(x_{I'(k)\amalg J\amalg I(k)})\otimes 1\big) =\\ &(-1)^{\epsilon_1}1\otimes o(x_{I'(k)})\otimes b\otimes o(x_{I(k)})\otimes 1+\sum_{\mathscr{S}}\pm(1\otimes o(x_{I_2})\otimes 1)\otimes b\otimes(1\otimes o(x_{I_1})\otimes 1).\end{aligned}$$

Now, one can conclude from the description of $\mathscr{S}$ that the maximal element of each $I_1$ is greater than the minimal element of $I_2$. So

$$\phi\big((1\otimes o(x_{I_1})\otimes 1)\otimes b\otimes(1\otimes x_{I_2}\otimes 1)\big)=0$$

and

$$\begin{aligned}\phi(1&\otimes(b\partial_J)\otimes 1)\Delta^{(2)}(1\otimes o(x_{I'(k)\amalg J\amalg I(k)})\otimes 1)\\ &=(-1)^{\epsilon_1}\phi(1\otimes o(x_{I'(k)})\otimes b\otimes o(x_{I(k)})\otimes 1)\\ &=(-1)^{\epsilon_1}\phi(1\otimes o(x_{I'(k)})\otimes x_{j_1}\dots x_{j_t}\otimes o(x_{I(k)})\otimes 1)\\ &=(-1)^{\epsilon_2}\textstyle\sum_{x_{i_{k-1}}<x_{j_\nu}<x_{i_{k+1}}} x_{j_1}\dots x_{j_{\nu-1}}\otimes o(x_{I(k)\amalg\{j_\nu\}\amalg I'(k)})\otimes x_{j_{\nu+1}}\dots x_{j_t}.\end{aligned}$$

Finally, since $\partial_I(1\otimes o(x_{I(k)\amalg\{j_\nu\}\amalg I'(k)})\otimes 1)=0$ whenever $I(k)\amalg\{j_\nu\}\amalg I'(k)\neq I$, i.e. whenever $j_\nu\neq i_k$, we have

$$\begin{aligned}\big((a\partial_I)\circ_\phi(b\partial_J)\big)&(1\otimes o(x_{I'(k)\amalg J\amalg I(k)})\otimes 1)\\ &=a\partial_I\phi(1\otimes(b\partial_J)\otimes 1)\Delta^{(2)}(1\otimes o(x_{I'(k)\amalg J\amalg I(k)})\otimes 1)\\ &=(-1)^{\epsilon_3}a\big(|\{\nu : x_{j_\nu}=x_{i_k}\}|\tfrac{b}{x_{i_k}}\big)\\ &=(-1)^{\epsilon_3}a\tfrac{\partial}{\partial x_{i_k}}(b).\end{aligned}$$

One can check that $(a\partial_I)\circ_\phi(b\partial_J)$ vanishes on all monomials $1\otimes o(x_L)\otimes 1$ with $L$ not of the form $I'(k)\amalg J\amalg I(k)$, after some permutation. So we have

$$(a\partial_I)\circ_\phi(b\partial_J)=\sum_k(-1)^{\epsilon_4}a\frac{\partial}{\partial x_{i_k}}(b)\partial_{I'(k)\amalg J\amalg I(k)},$$

and, after switching the position of $I(k)$ and $I'(k)$,

$$(a\partial_I)\circ_\phi(b\partial_J)=\sum_k(-1)^{\epsilon_5}a\frac{\partial}{\partial x_{i_k}}(b)\partial_{I(k)\amalg J\amalg I'(k)}.$$

As for the signs, we have

$$
\begin{aligned}
&\epsilon_1 = (|I|-k)|J| + \textstyle\sum_{\ell=1}^{|J|-1} \ell && \epsilon_4 = \epsilon_3 - \textstyle\sum_{\ell''=1}^{|I|+|J|-2} \ell'' \\
&\epsilon_2 = \epsilon_1 + (|I|-k)(k-1) + (k-1) && \quad = \epsilon_3 - (|I|-1)(|J|-1) - \textstyle\sum_{\ell'=1}^{|I|-1} \ell' - \textstyle\sum_{\ell=1}^{|J|-1} \ell \\
&\epsilon_3 = \epsilon_2 + \textstyle\sum_{\ell'=1}^{|I|-1} \ell' && \epsilon_5 = \epsilon_4 + (|I|-1)|J| + (|I|-k)(k-1) \\
& && \quad = (|I|-k)(|J|-1) \mod 2.
\end{aligned}
$$

The formula for the bracket now follows by the definition of $[f,g]_\phi$. □

For elements $a\partial_i$ and $b\partial_j$ the above formula gives

$$[a\partial_i, b\partial_j]_\phi = a\frac{\partial}{\partial x_i}(b)\partial_j - b\frac{\partial}{\partial x_j}(a)\partial_i. \tag{4.4}$$

One can also verify the identity

$$[a\partial_I, b_1\partial_{J_1}b_2\partial_{J_2}]_\phi = [a\partial_I, b_1\partial_{J_1}]_\phi b_2\partial_{J_2} + (-1)^{(|I|-1)|J_1|}b_1\partial_{J_1}[a\partial_I, b_2\partial_{J_2}]_\phi. \tag{4.5}$$

So the bracket given in Proposition 4.6 is seen to recover the Schouten–Nijenhuis bracket, which is generally expressed using formula (4.4) for the bracket on degree 1 cocycles and extended to all of $A[\partial_1, \dots, \partial_i] = \mathrm{HH}(A)$ using the graded derivation identity (4.5).

## 5. Recovering Gerstenhaber brackets for groups of prime order

Assume in this section that the characteristic of the field $k$ is $p > 0$. Let $G$ be a cyclic group of order $p$, with generator $g$, and $A = kG$, the group algebra. Let $x := g - 1$ in $A$, so that $A \cong k[x]/(x^p)$. The Hochschild cohomology of $A$ is well-known. See [**6**] for the algebra structure of $\mathrm{HH}(kG)$ in the more general case that $G$ is abelian. In particular, in that case, $\mathrm{HH}(kG) \cong \mathrm{H}(G,k) \otimes kG$ as algebras, where $\mathrm{H}(G,k)$ denotes group cohomology. See Sanchez-Flores [**19**] or Yang [**24**] for the Gerstenhaber brackets when $G$ is cyclic; using our new techniques, we will recover their results in our case (i.e. $G$ has order $p$). While the minimal resolution, with its embedding into the bar resolution, that we use here does not satisfy all the hypotheses 3.1(a)–(c) assumed in Theorem 3.6, it does satisfy 3.1(a) and (b) (the weaker conditions assumed in Subsection 3.4). We will show that our alternative approach yields the Gerstenhaber bracket for these examples.

We will use the following $A^e$-module resolution of $A$ (see, e.g., [**22**, Exer. 9.1.4]):

$$K\colon \qquad \cdots \xrightarrow{\cdot v} A^e \xrightarrow{\cdot u} A^e \xrightarrow{\cdot v} A^e \xrightarrow{\cdot u} A^e \xrightarrow{m} A \to 0,$$

where $u = x \otimes 1 - 1 \otimes x$, $v = x^{p-1} \otimes 1 + x^{p-2} \otimes x + \cdots + 1 \otimes x^{p-1}$, and $m$ denotes multiplication. For each $i$, let $\xi_i$ denote the element $1 \otimes 1$ of $A^e$ in degree $i$.

The following maps $h_n\colon K_n \to K_{n+1}$ constitute a contracting homotopy for $K$, as may be verified by direct calculation:

$$
\begin{aligned}
h_{-1}(x^i) &= \xi_0 x^i, \\
h_0(x^i\xi_0x^j) &= \sum_{l=0}^{i-1} x^l\xi_1x^{i+j-1-l}, \\
h_1(x^i\xi_1x^j) &= \delta_{i,p-1}x^j\xi_2,
\end{aligned}
$$

$$h_{2n}(x^i\xi_{2n}x^j) = -\sum_{l=0}^{j-1} x^{i+j-1-l}\xi_{2n+1}x^l \quad (n \geqslant 1),$$
$$h_{2n+1}(x^i\xi_{2n+1}x^j) = \delta_{j,p-1}x^i\xi_{2n+1} \quad (n \geqslant 1).$$

Applying Lemma 3.7, we may obtain maps $\phi_n\colon (K \otimes_A K)_n \to K_{n+1}$, from the maps $h_n$, for which $d(\phi) = F_K$. We only need these maps in degrees 0 and 1:

$$\begin{aligned}
\phi_0(\xi_0 \otimes x^i\xi_0) &= \sum_{l=0}^{i-1} x^l\xi_1 x^{i-1-l},\\
\phi_1(\xi_1 \otimes x^i\xi_0) &= -\delta_{i,p-1}\xi_2,\\
\phi_1(\xi_0 \otimes x^i\xi_1) &= \delta_{i,p-1}\xi_2.
\end{aligned}$$

Next we record a diagonal map $\Delta\colon K \to K \otimes_A K$. It may be checked directly that the following map is a chain map:

$$\begin{aligned}
\Delta_0(\xi_0) &= \xi_0 \otimes \xi_0,\\
\Delta_1(\xi_1) &= \xi_1 \otimes \xi_0 + \xi_0 \otimes \xi_1,\\
\Delta_2(\xi_2) &= \xi_2 \otimes \xi_0 + \xi_0 \otimes \xi_2 + \sum_{a+b+c=p-2} x^a\xi_1 \otimes x^b\xi_1 x^c,\\
\Delta_3(\xi_3) &= \xi_3 \otimes \xi_0 + \xi_2 \otimes \xi_1 + \xi_1 \otimes \xi_2 + \xi_0 \otimes \xi_3,
\end{aligned}$$

and generally for $n \geqslant 1$,

$$\Delta_{2n}(\xi_{2n}) = \sum_{i=0}^{n} \xi_{2i} \otimes \xi_{2n-2i} + \sum_{i=0}^{n-1} \sum_{a+b+c=p-2} x^a\xi_{2i+1} \otimes x^b\xi_{2n-2i-1}x^c,$$
$$\Delta_{2n+1}(\xi_{2n+1}) = \sum_{i=0}^{2n+1} \xi_i \otimes \xi_{2n+1-i}.$$

Let

$$\Delta_K^{(2)} := (id \otimes \Delta)\Delta, \tag{5.1}$$

for the purpose of computing $\phi$-brackets under Definition 3.3. (Note that $\Delta$ is coassociative if and only if $p = 2$.) We will see later that $\Delta_K^{(2)} = (\pi \otimes \pi \otimes \pi)\Delta_B^{(2)}\iota$ for some choice of $\pi$ and $\iota$, in accordance with Lemma 3.8.

We will compute $\phi$-brackets on cohomology in low degrees. Applying $\mathrm{Hom}_{A^e}(-, A)$ to $K$, the differentials all are 0. In each degree, the cohomology is the free $A$-module $A$. Let $x^j\xi_i^* \in \mathrm{Hom}_{A^e}(A^e, A)$ denote the function that takes $\xi_i$ to $x^j$. Cup products are known: If $p = 2$, then $\xi_1^*$ generates the Hochschild cohomology as an $A$-algebra (recall $\mathrm{HH}(A) \cong \mathrm{H}(G, k) \otimes A$), while if $p > 2$, it is generated by $\xi_1^*$ and $\xi_2^*$. By applying the identity (2.3), we need only compute brackets of pairs of elements of degrees 1 and 2. The $\phi$-circle product of $x^i\xi_1^*$ and $x^j\xi_1^*$ in degree 1 is given by

$$\begin{aligned}
(x^i\xi_1^* \circ_\phi x^j\xi_1^*)(\xi_1) &= x^i\xi_1^*(\phi_0(x^j\xi_0 \otimes \xi_0 + \xi_0 \otimes x^j\xi_0 + \xi_0 \otimes \xi_0 x^j))\\
&= x^i\xi_1^*(\xi_1 x^{j-1} + x\xi_1 x^{j-2} + \cdots + x^{j-1}\xi_1)\\
&= jx^{i+j-1}.
\end{aligned}$$

Therefore, by symmetry, we obtain

$$[x^i\xi_1^*, x^j\xi_1^*]_\phi = (j-i)x^{i+j-1}\xi_1^*.$$

The $\phi$-circle product of elements in degrees 1 and 2 is given similarly by

$$(x^i\xi_1^* \circ_\phi x^j\xi_2^*)(\xi_2) = jx^{i+j-1},$$

while in the reverse order we have

$$\begin{aligned}(x^j\xi_2^* \circ_\phi x^i\xi_1^*)(\xi_2) &= x^j\xi_2^*\big(\sum_{a+b+c=p-2} \phi_1(-x^a\xi_1 x^{b+i}\otimes \xi_0 x^c) + \phi_1(\xi_0\otimes x^{a+b+i}\otimes \xi_1 x^c)\big)\\ &= \sum_{a+c=i-1} x^{i+j-1} - \sum_{a+b=p-1-i} x^{i+j-1}\\ &= (i+p-i)x^{i+j-1} \quad = \quad 0.\end{aligned}$$

So $[x^i\xi_1^*, x^j\xi_2^*]_\phi = jx^{i+j-1}\xi_2^*$. Finally, the $\phi$-circle product of two such elements of degree 2 is

$$\begin{aligned}(x^i\xi_2^* \circ_\phi x^j\xi_2^*)(\xi_3) &= x^i\xi_2^*(\phi_1(\xi_0\otimes x^j\xi_1) + \phi_1(\xi_1\otimes x^j\xi_0))\\ &= x^i\xi_2^*(\delta_{j,p-1}\xi_2 - \delta_{j,p-1}\xi_2)\\ &= 0.\end{aligned}$$

So $[x^i\xi_2^*, x^j\xi_2^*]_\phi = 0$. To summarize, in degrees 1 and 2, we have

$$[x^i\xi_1^*, x^j\xi_1^*]_\phi = (j-i)x^{i+j-1}\xi_1^*, \quad [x^i\xi_1^*, x^j\xi_2^*]_\phi = jx^{i+j-1}\xi_2^*, \quad [x^i\xi_2^*, x^j\xi_2^*]_\phi = 0.$$

Brackets in higher degrees are determined by these and the identity (2.3), since the Hochschild cohomology is generated as an $A$-algebra under cup product in degrees 1 and 2.

These computations agree with direct computations of the Gerstenhaber bracket using standard chain maps $\iota$, $\pi$. See Sanchez-Flores [**19**] and Yang [**24**] for different techniques. Next we will verify the conditions of Lemma 3.8 to explain why these $\phi$-brackets agree with Gerstenhaber brackets.

Let $\iota\colon K\to B$ and $\pi\colon B\to K$ be defined as follows. (See [**11**] for a more general setting and [**5**, Section 3] for the maps as below in this specific case.) The chain map $\iota$ is given by

$$\iota_{2l}(\xi_{2l}) = 1\otimes\alpha_l \quad \text{ and } \quad \iota_{2l+1}(\xi_{2l+1}) = 1\otimes x\otimes \alpha_l,$$

where $\alpha_0 = 1$ and if $l\geqslant 1$,

$$\alpha_l = \sum_{\{i_1+i_2+\cdots+i_{l+1}=lp-l \mid i_1,i_2,\ldots,i_l\geqslant 1\}} x^{i_1}\otimes x\otimes x^{i_2}\otimes x\otimes\cdots\otimes x\otimes x^{i_{l+1}}.$$

(Note that in the above sum, $i_{l+1}$ can take on the value 0 while each of $i_1,\ldots,i_l$ must be at least 1.) The chain map $\pi$ is given by

$$\begin{aligned}&\pi_{2l}(1\otimes x^{i_1}\otimes x^{i_2}\otimes\cdots\otimes x^{i_{2l}}\otimes 1) = \xi_{2l}x^{i_1+i_2-p}x^{i_3+i_4-p}\cdots x^{i_{2l-1}+i_{2l}-p},\\ &\pi_{2l+1}(1\otimes x^{i_1}\otimes x^{i_2}\otimes\cdots\otimes x^{i_{2l+1}}\otimes 1)\\ &\qquad\qquad = \sum_{m=0}^{i_1-1} x^m\xi_{2l+1}x^{i_1-m-1}x^{i_2+i_3-p}x^{i_4+i_5-p}\cdots x^{i_{2l}+i_{2l+1}-p}.\end{aligned}$$

(In the above expressions, any term involving a negative exponent of $x$ should be interpreted as 0.) Using these maps, we may check directly that the map $\Delta_K^{(2)}$ defined by (5.1) satisfies $\Delta_K^{(2)} = (\pi \otimes \pi \otimes \pi)\Delta_B^{(2)}\iota$. Consequently, Lemma 3.8 implies that $\phi$-brackets, defined as above, coincide with Gerstenhaber brackets, as we have observed.

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