Homology, Homotopy and Applications, vol. 18(1), 2016, pp.173–203

MAPPING SPACES FROM PROJECTIVE SPACES

MITSUNOBU TSUTAYA

(*communicated by Donald M. Davis*)

Abstract

We denote the n-th projective space of a topological monoid G by B_nG and the classifying space by BG. Let G be a well-pointed topological monoid having the homotopy type of a CW complex and G' a well-pointed grouplike topological monoid. We prove that there is a natural weak equivalence between the pointed mapping space $\mathrm{Map}_0(B_nG, BG')$ and the space $\mathcal{A}_n(G, G')$ of all A_n-maps from G to G'. Moreover, if we suppose $G = G'$, then an appropriate union of path-components of $\mathrm{Map}_0(B_nG, BG)$ is delooped.

This fact has several applications. As the first application, we show that the evaluation fiber sequence $\mathrm{Map}_0(B_nG, BG) \to \mathrm{Map}(B_nG, BG) \to BG$ extends to the right. As other applications, we investigate higher homotopy commutativity, A_n-types of gauge groups, T_k^f-spaces and homotopy pullback of A_n-maps. The concepts of T_k^f-space and C_k^f-space were introduced by Iwase–Mimura–Oda–Yoon, which is a generalization of T_k-spaces by Aguadé. In particular, we show that the T_k^f-space and the C_k^f-space are exactly the same concept and give some new examples of T_k^f-spaces.

1. Introduction

In this paper, we study maps between topological monoids which preserve associativity up to higher homotopy. In homotopy theory, homomorphisms are sometimes too restrictive. Sugawara [**Sug60**] studied the condition for a map $f\colon G \to G'$ between topological monoids to be the loop of a map $BG \to BG'$ between the classifying spaces. Roughly, his answer is that f is a loop map if f preserves the multiplications on G and G' up to infinitely higher homotopy associativity. In the proof of it, he used the Dold–Lashof construction [**DL59**]. After that, Stasheff [**Sta63b**] introduced A_n-*maps*, which are maps between topological monoids preserving n-th homotopy associativity. As a generalization of Sugawara's result, he gave an equivalent condition for a map being an A_n-map using the finite stages of the Dold–Lashof construction. The

Received March 30, 2015, revised September 3, 2015; published on April 20, 2016.
2010 Mathematics Subject Classification: 54C35 (primary), 18D50, 55P45 (secondary).
Key words and phrases: mapping space, homotopy fiber sequence, A_n-space, higher homotopy commutativity, gauge group.
Article available at `http://dx.doi.org/10.4310/HHA.2016.v18.n1.a10`

n-th Dold–Lashof construction of a topological monoid G is called the *n-th projective space* B_nG since those of $\mathbb{Z}/2\mathbb{Z}$, S^1 and S^3 are the classical projective spaces $\mathbb{R}P^n$, $\mathbb{C}P^n$ and $\mathbb{H}P^n$, respectively.

We will refine Stasheff's result using mapping spaces. More precisely, our main result Theorem 6.1 is the weak equivalence

$$\mathcal{A}_n(G, G') \simeq \mathrm{Map}_0(B_nG, BG'),$$

where $\mathcal{A}_n(G, G')$ is the space of A_n-maps with A_n-forms between G and G', and $\mathrm{Map}_0(B_nG, BG')$ is the space of pointed maps between B_nG and BG'. The correspondence of the path-components was already known by Fuchs [**Fuc65**] for $n = \infty$.

As an application of this result, we prove that the evaluation fiber sequence extends as

$$\overline{\mathrm{Map}}_0(B_nG, BG) \to \overline{\mathrm{Map}}(B_nG, BG) \to BG \to B\mathcal{W}\mathcal{A}_n(G, G; \mathrm{eq}),$$

where the spaces $\overline{\mathrm{Map}}_0(B_nG, BG)$ and $\overline{\mathrm{Map}}(B_nG, BG)$ are unions of appropriate path-components in $\mathrm{Map}_0(B_nG, BG)$ and $\mathrm{Map}(B_nG, BG)$, respectively, $\mathcal{A}_n(G, G; \mathrm{eq})$ denotes the space of self-A_n-equivalences on G, and the functor $\mathcal{W}$ is a kind of "cofibrant replacement". This is a generalization of the well-known extension of the evaluation fiber sequence

$$\overline{\mathrm{Map}}_0(X, X) \to \overline{\mathrm{Map}}(X, X) \to X \to B\mathcal{W}\overline{\mathrm{Map}}_0(X, X) \to B\mathcal{W}\overline{\mathrm{Map}}(X, X),$$

where $\overline{\mathrm{Map}}_0(X, X) \subset \mathrm{Map}_0(X, X)$ and $\overline{\mathrm{Map}}(X, X) \subset \mathrm{Map}(X, X)$ are the monoids consisting of homotopy equivalences. This extension can be found in [**Got73, May80**]. Moreover, our result gives the maximum extension because the path-component $\mathrm{Map}(B_nG, BG; \iota_n)$ of the inclusion $\iota_n \colon B_nG \to BG$ cannot be delooped in general. We will give such an example.

This paper is organized as follows. In Section 2, we collect elementary facts about mapping spaces. In Section 3, some maps between cubes are defined, which will be used to describe the topological category $\mathcal{A}_n$. In Section 4, we define the topological category $\mathcal{A}_n$ of topological monoids and A_n-maps. In Section 5, we construct the continuous bar construction functor. In Section 6, we investigate the mapping spaces from projective spaces and show the above weak equivalence. The rest is devoted to applications. In Section 7, we give an extension of the evaluation fiber sequence as above. In Section 8, the relation with our result and various higher homotopy commutativities such as C_k-spaces by Williams [**Wil69**], C^k-spaces by Sugawara [**Sug60**], $C_k(n)$-spaces by Hemmi–Kawamoto [**HK11**] and $C(k, \ell)$-spaces by Kishimoto–Kono [**KK10**] are studied. In Section 9, equivalent conditions for the adjoint bundle of a principal bundle being trivial are given. In Section 10, we give some application to T_k^f-spaces introduced by Iwase–Mimura–Oda–Yoon [**IMOY12**]. In particular, we show that a pointed space is a T_k^f-space if and only if it is a C_k^f-space and, as an example, study when $B\,\mathrm{SU}(2)$ is a T_k^f-space for a map $f\colon S^4 \to B\,\mathrm{SU}(2)$. In Section 11, we make some remarks on a relation of our result and homotopy pullbacks of A_n-maps.

Acknowledgments

The author would like to express his gratitude to Professors Kouyemon Iriye, Norio Iwase, Daisuke Kishimoto and Nobuyuki Oda for fruitful discussions and encouragement for this work.

2. Preliminaries on mapping spaces

We collect elementary facts on mapping spaces. We refer to [**Hov99**, Sections 2.4, 4.2] about the category of compactly generated spaces. We will work in the categories **CG** of compactly generated spaces and $\mathbf{CG}_*$ of pointed ones. In these categories, a map $f\colon X \to Y$ is said to be a *weak equivalence* if f induces isomorphisms on homotopy groups with respect to any basepoints $x_0 \in X$ and $f(x_0) \in Y$.

Definition 2.1. For compactly generated spaces X and Y, we denote the *mapping space* between X and Y by $\mathrm{Map}(X,Y)$ which consists of continuous maps from X to Y as a set. We write the subspace of basepoint preserving maps by $\mathrm{Map}_0(X,Y)$ for pointed spaces X and Y. For a pointed map $\phi\colon X \to Y$, the path-component of $\mathrm{Map}(X,Y)$ containing ϕ is denoted by $\mathrm{Map}(X,Y;\phi)$. Similarly, we denote $\mathrm{Map}_0(X,Y;\phi) := \mathrm{Map}(X,Y;\phi) \cap \mathrm{Map}_0(X,Y)$. Do not confuse it with the path-component of $\mathrm{Map}_0(X,Y)$ containing ϕ. Unless otherwise stated, the basepoints of $\mathrm{Map}_0(X,Y)$ and $\mathrm{Map}(X,Y)$ are the constant map.

The functors Map and Map_0 satisfy the following exponential laws:

$$\mathrm{Map}(X, \mathrm{Map}(Y,Z)) \cong \mathrm{Map}(X \times Y, Z),$$
$$\mathrm{Map}_0(X, \mathrm{Map}_0(Y,Z)) \cong \mathrm{Map}_0(X \wedge Y, Z),$$

where $X \wedge Y := (X \times Y)/(X \times * \cup * \times Y)$ denotes the smash product of X and Y.

From this, the evaluation map

$$\mathrm{Map}(X,Y) \times X \to Y, \quad (f,x) \mapsto f(x)$$

is continuous since it is the adjoint map of the identity map $\mathrm{Map}(X,Y) \to \mathrm{Map}(X,Y)$. Moreover, the composition

$$\circ\colon \mathrm{Map}(Y,Z) \times \mathrm{Map}(X,Y) \to \mathrm{Map}(X,Z), \quad (g,f) \mapsto g \circ f$$

is continuous since it is the adjoint of the continuous map

$$\mathrm{Map}(Y,Z) \times \mathrm{Map}(X,Y) \times X \to Z, \quad (g,f,x) \mapsto g(f(x)).$$

Similar properties hold for Map_0.

Proposition 2.2. *Let X and X' be pointed CW complexes and Y and Y' pointed spaces. Then, pointed weak equivalences $f\colon X \to Y$ and $f'\colon X' \to Y'$ induce the following weak equivalences:*

$$f^{\#}\colon \mathrm{Map}_0(Y,X') \to \mathrm{Map}_0(X,X'),$$
$$f^{\#}\colon \mathrm{Map}(Y,X') \to \mathrm{Map}(X,X'),$$
$$f'_{\#}\colon \mathrm{Map}_0(X,X') \to \mathrm{Map}_0(X,Y'),$$
$$f'_{\#}\colon \mathrm{Map}(X,X') \to \mathrm{Map}(X,Y').$$

Remark 2.3. This proposition obviously generalizes to the case when X and X' are only assumed to have the pointed homotopy types of CW complexes.

Definition 2.4. A pointed space X is said to be *well-pointed* if the inclusion of the basepoint $* \subset X$ has the homotopy extension property.

If X is well-pointed and Y is pointed, then the evaluation $\mathrm{Map}(X, Y) \to Y$ at the basepoint has the homotopy lifting property and its fiber at the basepoint is $\mathrm{Map}_0(X, Y)$.

Example 2.5. (i) Every pointed CW complex is well-pointed.

(ii) It is well-known that a pointed space X is well-pointed if and only if the pair $(X, *)$ is an NDR pair, that is, there exist a map $u\colon X \to [0, 1]$ and a homotopy $h\colon [0, 1] \times X \to X$ such that $u^{-1}(0) = *$ and the following equalities hold:

$$\begin{aligned} H(0, x) &= x && \text{for } x \in X, \\ H(t, *) &= * && \text{for } 0 \leqslant t \leqslant 1, \\ H(1, x) &= * && \text{if } u(x) < 1. \end{aligned}$$

Suppose that there exist a map $u\colon X \to [0, 1]$ and a homotopy $h\colon [0, 1] \times X \to X$ as above. Let K be a compact pointed space. Define $u'\colon \mathrm{Map}(K, X) \to [0, 1]$ and $H'\colon [0, 1] \times \mathrm{Map}(K, X) \to \mathrm{Map}(K, X)$ by

$$\begin{aligned} u'(f) &= \max u(f(K)), \\ H'(t, f)(k) &= H(t, f(k)). \end{aligned}$$

Then u' and H' satisfy the above properties for the pairs $(\mathrm{Map}(K, X), *)$ and $(\mathrm{Map}_0(K, X), *)$. Therefore, the mapping spaces $\mathrm{Map}(K, X)$ and $\mathrm{Map}_0(K, X)$ are well-pointed.

For a pointed space X, the space $\Omega X := \mathrm{Map}_0(S^1, X)$ is called the *based loop space* of X, which has the pointed homotopy type of a CW complex if X is pointed homotopy equivalent to a CW complex. By concatenation of loops, ΩX becomes a homotopy associative H-space with homotopy unit. To make this operation associative and unital, we use the Moore path technique. The *Moore based path space* $P^{\mathrm{M}}X$ of X is defined by

$$P^{\mathrm{M}}X := \{(g, \ell) \in \mathrm{Map}([0, \infty), X) \times [0, \infty) \mid g(t) = * \text{ for } t \geqslant \ell\}.$$

There is the evaluation map

$$e\colon P^{\mathrm{M}}X \to X, \quad e(g, \ell) = g(0),$$

which is a Hurewicz fibration. The fiber over the basepoint is denoted by $\Omega^{\mathrm{M}}X$ and called the *Moore based loop space*. There is the associative concatenation operation: for $(g, \ell) \in P^{\mathrm{M}}X, (g', \ell') \in \Omega^{\mathrm{M}}X$, $g + g'\colon [0, \infty) \to X$ is defined by

$$(g + g')(t) = \begin{cases} g(t) & (t \leqslant \ell) \\ g'(t - \ell) & (t \geqslant \ell) \end{cases}$$

and $(g, \ell) + (g', \ell') := (g + g', \ell + \ell') \in P^{\mathrm{M}}X$. This operation gives continuous maps

$$\begin{aligned} +&\colon P^{\mathrm{M}}X \times \Omega^{\mathrm{M}}X \to P^{\mathrm{M}}X, \\ +&\colon \Omega^{\mathrm{M}}X \times \Omega^{\mathrm{M}}X \to \Omega^{\mathrm{M}}X \end{aligned}$$

and makes $\Omega^{\mathrm{M}} X$ a topological monoid and $P^{\mathrm{M}} X \to X$ a principal fibration.

For a pointed map $f\colon A \to B$ and a pointed space X, the cofiber sequence $A \to B \to C_f$ induces the homotopy fiber sequence

$$\mathrm{Map}_0(\Sigma A, X) \to \mathrm{Map}_0(C_f, X) \to \mathrm{Map}_0(B, X),$$

where ΣA is the reduced suspension of A and C_f is the reduced mapping cone of f. The canonical pinch maps

$$C_f \to C_f \vee \Sigma A, \quad \Sigma A \to \Sigma A \vee \Sigma A$$

give a homotopy associative action of $\mathrm{Map}_0(\Sigma A, X)$ on $\mathrm{Map}_0(C_f, X)$. This also can be replaced by an associative one as follows. Let us consider the pullback

$$\begin{tikzcd}
\mathrm{Map}_0^{\mathrm{M}}(C_f, X) \arrow[r] \arrow[d] & P^{\mathrm{M}} \mathrm{Map}_0(A, X) \arrow[d, "e"] \\
\mathrm{Map}_0(B, X) \arrow[r, "f^{\#}"'] & \mathrm{Map}_0(A, X).
\end{tikzcd}$$

Then we have a principal fibration

$$\Omega^{\mathrm{M}} \mathrm{Map}_0(A, X) \to \mathrm{Map}_0^{\mathrm{M}}(C_f, X) \to \mathrm{Map}_0(B, X),$$

which is naturally equivalent to the above homotopy fiber sequence.

3. Certain maps between cubes

Consider the closed interval $[0,\infty] = [0,\infty) \cup \{\infty\}$ homeomorphic to the unit interval. Define the following maps:

$$\delta_k^t\colon [0,\infty]^{\times(i-1)} \to [0,\infty]^{\times i},$$
$$\delta_k^t(t_1,\ldots,t_{i-1}) = (t_1,\ldots,t_{k-1},t,t_k,\ldots,t_{i-1}),$$
$$\sigma_k\colon [0,\infty]^{\times(i-1)} \to [0,\infty]^{\times(i-2)},$$
$$\sigma_k(t_1,\ldots,t_{i-1}) = \begin{cases} (t_2,\ldots,t_{i-1}) & (k=1) \\ (t_1,\ldots,t_{k-2},\max\{t_{k-1},t_k\},t_{k+1},\ldots,t_{i-1}) & (1<k<i) \\ (t_1,\ldots,t_{i-2}) & (k=i) \end{cases}$$

for $1 \leqslant k \leqslant i$.

We also give a cubical partition of the cube $[0,\infty]^{\times(i-1)}$. For a multi-index $\mathbf{i} = (i_1,\ldots,i_r)$ consisting of positive integers with $i_1+\cdots+i_r = i$ and $\ell \in [0,\infty)$, define the map

$$\gamma_{\mathbf{i}}^{\ell}\colon [0,\infty]^{\times(r-1)} \times [0,\ell]^{\times(i_1-1)} \times \cdots \times [0,\ell]^{\times(i_r-1)} \to [0,\infty]^{\times(i-1)},$$
$$\gamma_{\mathbf{i}}^{\ell}(\mathbf{t};\mathbf{s}_1,\ldots,\mathbf{s}_r) = (\mathbf{s}_1, t_1+\ell, \mathbf{s}_2, t_2+\ell,\ldots,t_{r-1}+\ell,\mathbf{s}_r)$$

for $\mathbf{t} = (t_1,\ldots,t_{r-1}) \in [0,\infty]^{\times(r-1)}$ and $\mathbf{s}_k \in [0,\ell]^{\times(i_k-1)}$. For example, the images on $[0,\infty]^{\times 2}$ is depicted in Figure 1.

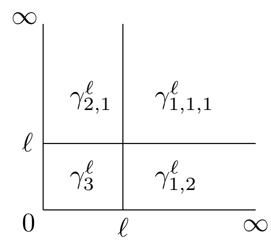

Figure 1: The cubical partition of $[0,\infty]^{\times 2}$

Lemma 3.1. *The following identities hold:*

(i) $\delta^{t}_{i_1+\cdots+i_{k-1}+m}(\gamma^{\ell}_{(i_1,\ldots,i_r)}(\mathbf{t};\mathbf{s}_1,\ldots,\mathbf{s}_r))$
$= \gamma^{\ell}_{(i_1,\ldots,i_{k-1},i_k+1,i_{k+1},\ldots,i_r)}(\mathbf{t};\mathbf{s}_1,\ldots,\mathbf{s}_{k-1},\delta^{t}_{m}(\mathbf{s}_k),\mathbf{s}_{k+1},\ldots,\mathbf{s}_r)$
for $1 \leqslant m \leqslant i_k$ *and* $t \leqslant \ell$*,*

(ii) $\delta^{\ell+t}_{i_1+\cdots+i_{k-1}+m}(\gamma^{\ell}_{(i_1,\ldots,i_r)}(\mathbf{t};\mathbf{s}_1,\ldots,\mathbf{s}_r))$
$= \gamma^{\ell}_{(i_1,\ldots,i_{k-1},m-1,i_k-m,i_{k+1},\ldots,i_r)}(\delta^{t}_{k}(\mathbf{t});\mathbf{s}_1,\ldots,\mathbf{s}_{k-1},$
$\sigma^{i_k-m}_{m}(\mathbf{s}_k),\sigma^{m-1}_{1}(\mathbf{s}_k),\mathbf{s}_{k+1},\ldots,\mathbf{s}_r)$
for $1 \leqslant m \leqslant i_k$ *and* $t \geqslant 0$*,*

(iii) $\sigma_{i_1+\cdots+i_{k-1}+m}(\gamma^{\ell}_{(i_1,\ldots,i_r)}(\mathbf{t};\mathbf{s}_1,\ldots,\mathbf{s}_r))$
$= \gamma^{\ell}_{(i_1,\ldots,i_{k-1},i_k-1,i_{k+1},\ldots,i_r)}(\mathbf{t};\mathbf{s}_1,\ldots,\mathbf{s}_{k-1},\sigma_m(\mathbf{s}_k),\mathbf{s}_{k+1},\ldots,\mathbf{s}_r)$
for $1 \leqslant m \leqslant i_k$*,*

(iv) $\gamma^{\ell+\ell'}_{(i_1,\ldots,i_r)}(\mathbf{u};\gamma^{\ell}_{(i_{1,1},\ldots,i_{1,q_1})}(\mathbf{t}_1;\mathbf{s}_{1,1},\ldots,\mathbf{s}_{1,q_1}),\ldots,\gamma^{\ell}_{(i_{r,1},\ldots,i_{r,q_r})}(\mathbf{t}_r;\mathbf{s}_{r,1},\ldots,\mathbf{s}_{r,q_r}))$
$= \gamma^{\ell}_{(i_{1,1},\ldots,i_{1,q_1},\ldots,i_{r,1},\ldots,i_{r,q_r})}(\gamma^{\ell'}_{(i_1,\ldots,i_r)}(\mathbf{u};\mathbf{t}_1,\ldots,\mathbf{t}_r);$
$\mathbf{s}_{1,1},\ldots,\mathbf{s}_{1,q_1},\ldots,\mathbf{s}_{r,1},\ldots,\mathbf{s}_{r,q_r})$.

4. The topological category of A_n-maps between topological monoids

In this section, we formulate the topological category of topological monoids and A_n-maps between them. It is known that there exists a quasicategory of A_n-spaces and A_n-maps between them by [**Tsu15**], which is justified by the results of Boardman–Vogt in [**BV73**]. Hence, using the technique of quasicategories, one can obtain the topological category of A_n-spaces from this quasicategory. But our approach is different from this and rather elementary.

As we have seen in Section 2, the source space of a mapping space should be a CW complex if one wants to consider a well-behaved mapping space. Similarly, considering maps between topological monoids, we should consider *grouplike* ones as target spaces.

Definition 4.1. A topological monoid G is said to be *grouplike* if the monoid $\pi_0(G)$ is a group with respect to the multiplication induced from the monoid structure of G.

For example, any Moore based loop space $\Omega^{\mathrm{M}}X$ is grouplike.

The following lemma is proved by induction on the dimension of the skeletons of A.

Lemma 4.2. *Let G be a grouplike topological monoid and A a pointed space of the pointed homotopy type of a CW complex. Then, for any pointed map $\alpha\colon A \to G$, there exists a map $\alpha'\colon A \to G$ such that the maps*

$$A \to G, \quad a \mapsto \alpha(a)\alpha'(a) \quad \text{and} \quad a \mapsto \alpha'(a)\alpha(a)$$

are pointed homotopic to the constant map.

This lemma leads the following proposition.

Proposition 4.3. *Let $i\colon A \to B$ be an inclusion having the homotopy extension property between spaces of the homotopy types of CW complexes and $\alpha\colon B \to G$ a map to a grouplike topological monoid. Suppose that $\mathrm{Map}(A, G)$ and $\mathrm{Map}(B, G)$ are pointed at the maps $\alpha|_A$ and α, respectively. Then the inclusion of the homotopy fiber of $i^{\#}\colon \mathrm{Map}(B, G) \to \mathrm{Map}(A, G)$ is equivalent to the map $\mathrm{Map}_0(B/A, G) \to \mathrm{Map}(B, G)$ given by the multiplication $\beta \mapsto \alpha\beta$. More precisely, the map $i^{\#}$ is a principal homotopy fibration with fiber $\mathrm{Map}_0(B/A, G)$.*

Proof. If α is the constant map 0, then the proposition is obvious. By Lemma 4.2, there is a map $\alpha'\colon B \to G$ such that $\alpha\alpha'$ and $\alpha'\alpha$ is homotopic to the constant map. Then there is a commutative diagram

$$\begin{tikzcd}
\mathrm{Map}(B, G) \arrow[r, "i^{\#}"] \arrow[d, "\alpha\cdot"'] & \mathrm{Map}(A, G) \arrow[d, "(\alpha|_A)\cdot"] \\
\mathrm{Map}(B, G) \arrow[r, "i^{\#}"'] & \mathrm{Map}(A, G)
\end{tikzcd}$$

such that the vertical maps are given by multiplying α from the left and are homotopy equivalences. By these equivalences, the proposition follows from the case when $\alpha = 0$. □

Now, we define A_n-maps. Our definition is slightly different from Stasheff's original one in [**Sta63b**]. Since we want to make the composition of A_n-maps associative and unital, the Moore type definition is adopted.

Definition 4.4. Let G and G' be topological monoids and $f\colon G \to G'$ a pointed map. A family $\{f_i\colon [0, \infty]^{\times(i-1)} \times G^{\times i} \to G'\}_{i=1}^n$ of maps is said to be an *A_n-form* of size $\ell \in [0, \infty)$ if the following conditions hold:

(i) $f_1 = f$,

(ii) $f_i(\delta_k^0(\mathbf{t}); g_1, \ldots, g_i) = f_{i-1}(\mathbf{t}; g_1, \ldots, g_k g_{k+1}, \ldots, g_i)$,

(iii) $f_i(\delta_k^t(\mathbf{t}); g_1, \ldots, g_i) = f_{k-1}(\sigma_k^{i-k}(\mathbf{t}); g_1, \ldots, g_k) f_{i-k}(\sigma_1^{k-1}(\mathbf{t}); g_{k+1}, \ldots, g_i)$ for $t \geqslant \ell$,

(iv) $f_i(\mathbf{t}; g_1, \ldots, g_{k-1}, *, g_{k+1}, \ldots, g_i) = f_{i-1}(\sigma_k(\mathbf{t}); g_1, \ldots, g_{k-1}, g_{k+1}, \ldots, g_i)$.

The triple $f = (f, \{f_i\}_{i=1}^n, \ell)$ is called an *A_n-map*. In particular, if the size ℓ of the A_n-form of f is 0, f is a homomorphism. We denote the space of A_n-maps between topological monoids G and G' by $\mathcal{A}_n(G, G')$.

We consider that an A_1-map is a pair (f, ℓ) consisting of a pointed map f and a meaningless number ℓ. Unless otherwise stated, we assign a homomorphism the A_n-form of size 0.

Definition 4.5. Let G, G' and G'' be topological monoids. For A_n-maps $f = (f, \{f_i\}_{i=1}^n, \ell)\colon G \to G'$ and $f' = (f', \{f'_i\}_{i=1}^n, \ell')\colon G' \to G''$, the composition $f' \circ f = (f' \circ f, \{F_i\}_{i=1}^n, \ell + \ell')$ is defined as follows. For a multi-index $\mathbf{i} = (i_1, \ldots, i_r)$ with $i_1 + \cdots + i_r = i$, $\mathbf{t} \in [0, \infty]^{\times(r-1)}$, $\mathbf{s}_k \in [0, \ell]^{\times(i_k-1)}$ and $\mathbf{g}_k \in G^{\times i_k}$, define

$$F_i(\gamma_{\mathbf{i}}^{\ell}(\mathbf{t}; \mathbf{s}_1, \ldots, \mathbf{s}_r); \mathbf{g}_1, \ldots, \mathbf{g}_r) = f'_r(\mathbf{t}; f_{i_1}(\mathbf{s}_1; \mathbf{g}_1), \ldots, f_{i_r}(\mathbf{s}_r; \mathbf{g}_r)).$$

Using the identities (i), (ii) and (iii) in Lemma 3.1, it is verified that $\{F_i\}$ is an A_n-form on $f' \circ f$ of size $\ell + \ell'$.

This composition defines a continuous map

$$\circ\colon \mathcal{A}_n(G', G'') \times \mathcal{A}_n(G, G') \to \mathcal{A}_n(G, G'').$$

By the identity (iv) in Lemma 3.1, it is an associative and unital operation, where the identity in $\mathcal{A}_n(G, G)$ is given by the identity map id_G. Thus we obtain a topological category as follows.

Definition 4.6. The topological category $\mathcal{A}_n$ consists of topological monoids as objects and $\mathcal{A}_n(G, G')$ as a morphism space between each pair G and G', where the composition is given as Definition 4.5.

If the underlying map f_1 of $f \in \mathcal{A}_n(G, G')$ is a weak equivalence, f is said to be a *weak A_n-equivalence*. The *homotopy category* $\pi_0\mathcal{A}_n$ of $\mathcal{A}_n$ is the category whose objects are the same as $\mathcal{A}_n$ and the morphism set between G and G' is defined by $(\pi_0\mathcal{A}_n)(G, G') = \pi_0(\mathcal{A}_n(G, G'))$.

Definition 4.7. Let G and G' be topological monoids and $f = (f, \{f_i\}_{i=1}^n, \ell)\colon G \to G'$ an A_n-map. For a left G-space X, a left G'-space X' and a map $\phi\colon X \to X'$, a family $\{\phi_i\colon [0, \infty]^i \times G^i \to G'\}_{i=0}^n$ of maps is said to be an *A_n-form* if the following conditions hold:

(i) $\phi_0 = \phi$,

(ii) $\phi_i(\delta_k^0(\mathbf{t}); g_1, \ldots, g_i, x) = \begin{cases} \phi_{i-1}(\mathbf{t}; g_1, \ldots, g_k g_{k+1}, \ldots, g_i, x) & (k < i) \\ \phi_{i-1}(\mathbf{t}; g_1, \ldots, g_{i-1}, g_i x) & (k = i), \end{cases}$

(iii) $\phi_i(\delta_k^t(\mathbf{t}); g_1, \ldots, g_i, x) = f_k(\sigma_k^{i-k+1}(\mathbf{t}); g_1, \ldots, g_k)\phi_{i-k}(\sigma_1^{k-1}(\mathbf{t}); g_{k+1}, \ldots, g_i, x)$ for $t \geqslant \ell$,

(iv) $f_i(\mathbf{t}; g_1, \ldots, g_{k-1}, *, g_{k+1}, \ldots, g_i, x) = f_{i-1}(\sigma_k(\mathbf{t}); g_1, \ldots, g_{k-1}, g_{k+1}, \ldots, g_i, x)$.

The quintuple $\phi = (f, \{f_i\}_{i=1}^n, \ell, \phi, \{\phi_i\}_{i=1}^n)$ is called an *A_n-equivariant map* through the A_n-map $f = (f, \{f_i\}_{i=1}^n, \ell)$. In particular, if the size ℓ of the A_n-form of f is 0, ϕ is an ordinary equivariant map. We denote the space of A_n-equivariant maps between a left G-space X and a left G'-space X' by $\mathcal{A}_n^{\mathrm{L}}((G, X), (G', X'))$. Similarly, we define A_n-equivariant maps between a right G-space X and a right G'-space X', and the corresponding mapping space is denoted by $\mathcal{A}_n^{\mathrm{R}}((X, G), (X', G'))$.

Definition 4.8. The topological categories $\mathcal{A}_n^{\mathrm{L}}$ and $\mathcal{A}_n^{\mathrm{R}}$ are defined as well as $\mathcal{A}_n$ in Definition 4.6. We define the projection functors $\mathcal{A}_n^{\mathrm{L}} \to \mathcal{A}_n$ and $\mathcal{A}_n^{\mathrm{R}} \to \mathcal{A}_n$ by $(G, X) \mapsto G$ and $(X, G) \mapsto G$, respectively.

In the following, we will prove various properties about the mapping spaces of $\mathcal{A}_n$. It will be convenient to consider the deformation retract

$$\mathcal{A}_n^1(G,G') \subset \mathcal{A}_n(G,G')$$

consisting of A_n-forms of size $\geqslant 1$. Assume that G is well-pointed and of the homotopy type of a CW complex, and that G' is grouplike. Note that there is a homotopy pullback diagram

$$\begin{tikzcd}
\mathcal{A}_n^1(G,G') \arrow[r] \arrow[d] & \mathrm{Map}([0,1]^{\times(n-1)} \times G^{\times n}, G') \arrow[d] \\
\mathcal{A}_{n-1}^1(G,G') \arrow[r] & \mathrm{Map}(\partial[0,1]^{\times(n-1)} \times G^{\times n} \cup [0,1]^{\times(n-1)} \times T_nG, G'),
\end{tikzcd}$$

where the left vertical map is the forgetful map and $T_nG \subset G^{\times n}$ denotes the fat wedge of G

$$T_nG := \{(g_1,\dots,g_n) \in G^{\times n} \mid g_k = * \text{ for some } k\}.$$

Here, the homotopy fiber is determined as $\mathrm{Map}_0(\Sigma^{n-1}G^{\wedge n}, G')$ by using Proposition 4.3 and does not depend on the choice of the basepoint of $\mathcal{A}_n^1(G,G')$. Moreover, the homotopy fiber has an action

$$\mathcal{A}_n^1(G,G') \times \mathrm{Map}_0(\Sigma^{n-1}G^{\wedge n}, G') \to \mathcal{A}_n^1(G,G')$$

defined as follows, which gives the structure of a principal fibration. For $\alpha \in \mathrm{Map}_0(\Sigma^{n-1}G^{\wedge n}, G')$ and $f = (f,\{f_i\}_i,\ell) \in \mathcal{A}_n^1(G,G')$, the action is defined by $f \cdot \alpha := (f,\{f_i \cdot \alpha\}_i,\ell)$ such that

$$(f_i\cdot\alpha)(t_1,\dots,t_{i-1};\mathbf{g}) = \begin{cases} f_i(t_1,\dots,t_{i-1};\mathbf{g}) & (i<n) \\ f_n(t_1,\dots,t_{n-1};\mathbf{g})\alpha(\min\{t_1,1\},\dots,\min\{t_{n-1},1\};\mathbf{g}) & (i=n), \end{cases}$$

where α is considered as the map $[0,1]^{\times(n-1)} \times G^{\times n} \to G$.

Proposition 4.9. *Let G, G', H, H' be topological monoids. Suppose that G and H are well-pointed and homotopy equivalent to CW complexes. Then the compositions of weak A_n-equivalences $f = (f,\{f_i\},\ell)\colon G \to H$ and $f' = (f',\{f_i'\},\ell')\colon G' \to H'$ induce the following weak equivalences:*

$$f^{\#}\colon \mathcal{A}_n(H,G') \xrightarrow{\simeq} \mathcal{A}_n(G,G'),$$
$$f'_{\#}\colon \mathcal{A}_n(H,G') \xrightarrow{\simeq} \mathcal{A}_n(H,H').$$

Proof. It is sufficient to prove that the maps

$$f^{\#}\colon \mathcal{A}_n^1(H,G') \xrightarrow{\simeq} \mathcal{A}_n^1(G,G'),$$
$$f'_{\#}\colon \mathcal{A}_n^1(H,G') \xrightarrow{\simeq} \mathcal{A}_n^1(H,H')$$

are weak equivalences. For $n = 1$, it follows from Proposition 2.2. Suppose that the claim is true for $n-1$. The map $f^{\#}\colon \mathcal{A}_n^1(H,G') \to \mathcal{A}_n^1(G,G')$ is recognized as the map obtained by taking the homotopy pullback along the horizontal direction of the homotopy commutative diagram

$$\begin{tikzcd}
\mathcal{A}^1_{n-1}(H,G') \arrow[r] \arrow[d, "f^\#"'] & \mathrm{Map}(\partial[0,1]^{\times(n-1)}\times H^{\times n}\cup[0,1]^{\times(n-1)}\times T_nH, G') \arrow[d, "f_1^\#"] & \mathrm{Map}([0,1]^{\times(n-1)}\times H^{\times n}, G') \arrow[l] \arrow[d, "f_1^\#"] \\
\mathcal{A}^1_{n-1}(G,G') \arrow[r] & \mathrm{Map}(\partial[0,1]^{\times(n-1)}\times G^{\times n}\cup[0,1]^{\times(n-1)}\times T_nG, G') & \mathrm{Map}([0,1]^{\times(n-1)}\times G^{\times n}, G') \arrow[l]
\end{tikzcd}$$

such that the left square commutes only up to the homotopy defined by $\{f_i\}$. Since the vertical maps are weak equivalences, the resulting map $f^\#\colon \mathcal{A}^1_n(H,G')\to\mathcal{A}^1_n(G,G')$ is a weak equivalence as well.

For $f'_\#$, the claim similarly follows from the diagram

$$\begin{tikzcd}
\mathcal{A}^1_{n-1}(H,G') \arrow[r] \arrow[d, "f'_\#"'] & \mathrm{Map}(\partial[0,1]^{\times(n-1)}\times H^{\times n}\cup[0,1]^{\times(n-1)}\times T_nH, G') \arrow[d, "f'_{1\#}"] & \mathrm{Map}([0,1]^{\times(n-1)}\times H^{\times n}, G') \arrow[l] \arrow[d, "f'_{1\#}"] \\
\mathcal{A}^1_{n-1}(H,H') \arrow[r] & \mathrm{Map}(\partial[0,1]^{\times(n-1)}\times H^{\times n}\cup[0,1]^{\times(n-1)}\times T_nH, H') & \mathrm{Map}([0,1]^{\times(n-1)}\times H^{\times n}, H'). \arrow[l]
\end{tikzcd}$$

□

Corollary 4.10. *If G and G' are well-pointed topological monoids of the homotopy types of CW complexes, then every weak A_n-equivalence $G\to G'$ has an inverse in the homotopy category $\pi_0\mathcal{A}_n$.*

5. Subdivided bar construction functor

Denote the i-dimensional simplex by Δ^i. Let X be a right G-space and Y be a left G-space for a topological monoid G. Then, the *n-th bar construction* $B_n(X,G,Y)$ is defined as

$$B_n(X,G,Y)=\left(\coprod_{0\leqslant i\leqslant n}\Delta^i\times X\times G^{\times i}\times Y\right)\Big/\sim,$$

for an appropriate simplicial relation $\sim$.

For our use, it is convenient to replace Δ^i by the cubical subdivision $\mathcal{Q}_i$ such that

$$\mathcal{Q}_i=\{(t_0,\dots,t_i)\in[0,\infty]^{\times(i+1)}\mid t_k=\infty \text{ for some } k\}.$$

The maps δ^0_k, σ_k and $\gamma^\ell_{\mathbf{i}}$ induce the following maps:

$$\begin{aligned}
&\delta^0_k\colon \mathcal{Q}_i\to\mathcal{Q}_{i-1} && \text{for } k=0,\dots,i,\\
&\sigma_k\colon \mathcal{Q}_{i-1}\to\mathcal{Q}_i && \text{for } k=0,\dots,i,\\
&\gamma^\ell_{\mathbf{i}}\colon \mathcal{Q}_{r-1}\times[0,\ell]^{\times(i_0-1)}\times\cdots\times[0,\ell]^{\times(i_r-1)}\to\mathcal{Q}_{i_0+\cdots+i_r-1} && \text{for } \mathbf{i}=(i_0,\dots,i_r).
\end{aligned}$$

Here, we shift the parametrization to $(t_0,\dots,t_i)$. For example,

$$\delta^0_k(t_0,\dots,t_i)=(t_0,\dots,t_{k-1},0,t_k,\dots,t_i).$$

Through a homeomorphism $[0,\infty]\cong[0,1]$, a natural homeomorphism $\mathcal{Q}_i\to\Delta^i$ for each i is given by

$$(t_0,\dots,t_i)(\in[0,1]^{\times(i+1)})\mapsto(t_0/T,\dots,t_i/T)$$

such that $T=t_0+\cdots+t_i$, which commutes with the boundary and degeneracy operators. For $i=2$, see Figure 2.

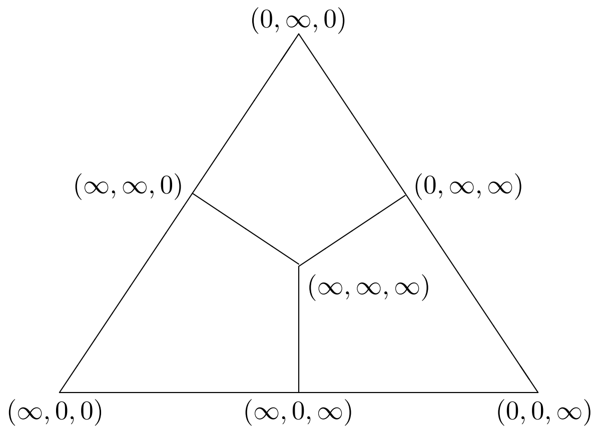

Figure 2: The cubical subdivision $\mathcal{Q}_2$

Consider the topological category $\mathcal{A}_n^{\mathrm{R}} \times_{\mathcal{A}_n} \mathcal{A}_n^{\mathrm{L}}$ defined by the fiber product.

Definition 5.1. Let $(X,G,Y) \in \mathcal{A}_n^{\mathrm{R}} \times_{\mathcal{A}_n} \mathcal{A}_n^{\mathrm{L}}$. Then the *n-th bar construction* $B_n(X, G, Y)$ is defined by

$$B_n(X,G,Y) = \left(\coprod_{0 \leqslant i \leqslant n} \mathcal{Q}_i \times X \times G^{\times i} \times Y \right) \Big/ \sim,$$

with the following identifications:

$$\text{(i)} \quad (\delta_k^0(\mathbf{t}); x, g_1, \ldots, g_i, y) \sim \begin{cases} (\mathbf{t}; xg_1, g_2, \ldots, g_i, y) & (k=0) \\ (\mathbf{t}; x, g_1, \ldots, g_{k-1}g_k, \ldots, g_i, y) & (0<k<i) \\ (\mathbf{t}; x, g_1, \ldots, g_{i-1}, g_i y) & (k=i), \end{cases}$$

$$\text{(ii)} \quad (\mathbf{t}; x, g_1, \ldots, g_{k-1}, *, g_{k+1}, \ldots, g_i, y) \sim (\sigma_k(\mathbf{t}); x, g_1, \ldots, g_{k-1}, g_{k+1}, \ldots, g_i, y).$$

The maps $\iota_{n_1}^{n_2} \colon B_{n_1}(X,G,Y) \to B_{n_2}(X,G,Y)$ for $n_1 \leqslant n_2$ and $\iota_n \colon B_n(X,G,Y) \to B(X,G,Y) := B_\infty(X,G,Y)$ denote the inclusions.

In fact, this construction induces a continuous functor

$$B_n \colon \mathcal{A}_n^{\mathrm{R}} \times_{\mathcal{A}_n} \mathcal{A}_n^{\mathrm{L}} \to \mathbf{CG}$$

as follows. Let $(\phi, f, \psi) \colon (X,G,Y) \to (X',G',Y')$ be a map of size $\ell \in [0,\infty)$ in $\mathcal{A}_n^{\mathrm{R}} \times_{\mathcal{A}_n} \mathcal{A}_n^{\mathrm{L}}$. Define a map $B_n(\phi,f,\psi) \colon B_n(X,G,Y) \to B_n(X',G',Y')$ by

$$\begin{aligned} &B_n(\phi,f,\psi)(\gamma_{\mathbf{i}}^{\ell}(\mathbf{t}; \mathbf{s}_0, \ldots, \mathbf{s}_r); x, \mathbf{g}_0, \ldots, \mathbf{g}_r, y) \\ &\qquad = [\mathbf{t}; \psi_{i_0}(\mathbf{s}_0; x, \mathbf{g}_0), f_{i_1}(\mathbf{s}_1; \mathbf{g}_1), \ldots, f_{i_{r-1}}(\mathbf{s}_{r-1}; \mathbf{g}_{r-1}), \psi_{i_r}(\mathbf{s}_r; \mathbf{g}_r, y)], \end{aligned}$$

for a multi-index $\mathbf{i} = (i_0+1, i_1, \ldots, i_{r-1}, i_r+1)$ with $i_0 + \cdots + i_r = i$, $\mathbf{t} \in \mathcal{Q}_{r-1}$, $\mathbf{s}_0 \in [0,\ell]^{\times i_0}$, $\mathbf{s}_r \in [0,\ell]^{\times i_r}$, $\mathbf{s}_k \in [0,\ell]^{\times (i_k - 1)}$ for $0<k<r$, $x \in X$, $\mathbf{g}_k \in G^{\times i_k}$ and $y \in Y$. Comparing with the definition of compositions of maps in $\mathcal{A}_n^{\mathrm{R}} \times_{\mathcal{A}_n} \mathcal{A}_n^{\mathrm{L}}$, it is straightforward to see that this construction gives a continuous functor $B_n \colon \mathcal{A}_n^{\mathrm{R}} \times_{\mathcal{A}_n} \mathcal{A}_n^{\mathrm{L}} \to \mathbf{CG}$.

Definition 5.2. For a topological monoid G, the spaces $B_nG := B_n(*, G, *)$ and $BG := B_\infty G = B(*, G, *)$ are called the *n-th projective space* and the *classifying space* of G, respectively. We denote $E_nG := B_n(*, G, G)$ and $EG := B(*, G, G)$ and often consider the canonical projection $E_nG \to B_nG$.

Remark 5.3. Projective spaces and classifying spaces are always path-connected.

Now we collect several technical lemmas. Though most of them are well-known, we give a proof using the cubical subdivision $\mathcal{Q}_i$ for consistency.

Lemma 5.4. *Let G be a topological monoid, X a right G-space, and Y a left G-space. Then the following maps are deformation retractions of the canonical inclusions $X \subset B(X, G, G)$ and $Y \subset B(G, G, Y)$, respectively:*

$$\begin{aligned} &B(X, G, G) \to X, && [\mathbf{t}; x, g_1, \ldots, g_i] \mapsto xg_1 \cdots g_i, \\ &B(G, G, Y) \to Y, && [\mathbf{t}; g_1, \ldots, g_i, y] \mapsto g_1 \cdots g_i y. \end{aligned}$$

Proof. Define a homotopy $\kappa_i \colon \mathcal{Q}_1 \times \mathcal{Q}_i \to \mathcal{Q}_{i+1}$ of $\mathcal{Q}_{i+1}$ for each i by

$$\kappa_i(s_0, s_1; t_0, \ldots, t_i) = \left(\frac{s_0 t_0}{1 + s_0 + t_0}, \ldots, \frac{s_0 t_i}{1 + s_0 + t_i}, s_1 \right),$$

which satisfies the following conditions:

$$\begin{aligned} &\kappa_i(\delta_1^0; \mathbf{t}) = \delta_{i+1}^0(\mathbf{t}), \\ &\kappa_i(\delta_0^0; \mathbf{t}) = (0, \ldots, 0, \infty), \\ &\kappa_i(\mathbf{s}; \delta_k^0(\mathbf{t})) = \delta_k^0(\kappa_{i-1}(\mathbf{s}; \mathbf{t})) && \text{for } 0 \leqslant k \leqslant i, \\ &\kappa_i(\mathbf{s}; \sigma_k(\mathbf{t})) = \sigma_k(\kappa_{i+1}(\mathbf{s}; \mathbf{t})) && \text{for } 0 \leqslant k \leqslant i. \end{aligned}$$

Note that, for any $[\mathbf{t}; x, \mathbf{g}] \in B(X, G, G)$, we have an equality

$$[\mathbf{t}; x, \mathbf{g}] = [\delta_i^0(\mathbf{t}); x, \mathbf{g}, *].$$

Then the homotopy $H \colon \mathcal{Q}_1 \times B(X, G, Y) \to B(X, G, Y)$ defined by

$$H(\mathbf{s}, [\mathbf{t}; x, \mathbf{g}]) = [\kappa_i(\mathbf{s}, \mathbf{t}); x, \mathbf{g}, *]$$

is a deformation of $B(X, G, Y)$ to X. One can prove similarly for $Y \subset B(G, G, Y)$. □

Lemma 5.5. *Let G and G' be well-pointed topological monoids, X and X' right G-spaces, and Y and Y' left G-spaces. Then the following hold:*

- *(i) The inclusion $B_{n-1}(X, G, Y) \subset B_n(X, G, Y)$ has the homotopy extension property.*
- *(ii) A map $(\phi, f, \psi) \colon (X, G, Y) \to (X', G', Y')$ in $\mathcal{A}_n^{\mathrm{R}} \times_{\mathcal{A}_n} \mathcal{A}_n^{\mathrm{L}}$ induces a weak equivalence $B_n(X, G, Y) \to B_n(X', G', Y')$ if the underlying maps of ϕ, f and ψ are weak equivalences.*

Proof. There is a pushout diagram

$$\begin{tikzcd} \partial\mathcal{Q}_n \times X \times G^{\times n} \times Y \cup \mathcal{Q}_n \times X \times T_nG \times Y \arrow[r] \arrow[d] & B_{n-1}(X, G, Y) \arrow[d, "\iota_{n-1}^n"] \\ \mathcal{Q}_n \times X \times G^{\times n} \times Y \arrow[r] & B_n(X, G, Y). \end{tikzcd}$$

Then the left vertical arrow has the homotopy extension property since so do the inclusions $\partial\mathcal{Q}_n \subset \mathcal{Q}_n$ and $* \subset G$. This implies the assertion (i). For the assertion (ii), the induced map $B_n(X,G,Y) \to B_n(X',G',Y')$ is a homotopy pushout along the horizontal direction of the diagram

$$\begin{tikzcd}
B_{n-1}(X,G,Y) \arrow[d, "B_{n-1}(\phi,f,\psi)"'] & \partial\mathcal{Q}_n \times X \times G^{\times n} \times Y \arrow[l] \arrow[r] \arrow[d, "\mathrm{id}\times\phi_0\times f_1^{\times n}\times\psi_0"] & \mathcal{Q}_n \times X \times G^{\times n} \times Y \arrow[d, "\mathrm{id}\times\phi_0\times f_1^{\times n}\times\psi_0"] \\
B_{n-1}(X',G',Y') & \partial\mathcal{Q}_n \times X' \times G'^{\times n} \times Y' \arrow[l] \arrow[r] & \mathcal{Q}_n \times X' \times G'^{\times n} \times Y'
\end{tikzcd}$$

such that the left square is given the appropriate homotopy and the right square strictly commutes. Then the assertion follows by induction on n. □

Lemma 5.6. *Let G be a well-pointed topological monoid, X a right G-space, and Y a left G-space. If the action $g\colon Y \to Y$ is a weak equivalence for any $g \in G$, then the projection $B_n(X,G,Y) \to B_n(X,G,*)$ is a quasifibration. Moreover, if G is a topological group, then the projection $B_n(X,G,Y) \to B_n(X,G,*)$ is a fiber bundle.*

Proof. The first half follows from applying the well-known criterion [**May90**, Theorem 2.7]. For the proof of the latter half, see [**May75**, Theorem 8.2]. □

Proposition 5.7. *Let $f\colon G \to H$ be a homomorphism between well-pointed grouplike topological monoids. Then the following sequence of maps is a homotopy fiber sequence*

$$G \xrightarrow{f} H \to B(*,G,H) \to BG \xrightarrow{Bf} BH.$$

Proof. The sequence of the left three terms is equivalent to the sequence

$$G \to B(G,G,H) \to B(*,G,H),$$

which is a homotopy fiber sequence by Lemma 5.6. Again by Lemma 5.6, the sequence of the middle three terms is also a homotopy fiber sequence. For the right middle terms, they also constitute a homotopy fiber sequence since the topological pullback square

$$\begin{tikzcd}
B(*,G,H) \arrow[r, "B(*,f,\mathrm{id}_H)"] \arrow[d] & B(*,H,H) \arrow[d] \\
B(*,G,*) \arrow[r, "B(*,f,*)"] & B(*,H,*)
\end{tikzcd}$$

is a homotopy pullback as well by Lemma 5.6. □

We note that there is a natural homeomorphism $B_1G \cong \Sigma G$. By the exponential law

$$\mathrm{Map}_0(\Sigma G, \Sigma G) \cong \mathrm{Map}_0(G, \Omega\Sigma G),$$

we have a natural map $E\colon G \to \Omega\Sigma G$.

Lemma 5.8. *For a grouplike topological monoid G, the composite*

$$G \xrightarrow{E} \Omega\Sigma G \cong \Omega B_1 G \xrightarrow{\Omega\iota_1} \Omega BG$$

is a weak equivalence.

Proof. Using the contracting homotopy of $B(*,G,G)$ in the proof of Lemma 5.4, we obtain the following commutative diagram:

$$\begin{tikzcd} G \arrow[r] \arrow[d] & EG \arrow[r] \arrow[d] & BG \arrow[d, equal] \\ \Omega BG \arrow[r] & PBG \arrow[r] & BG. \end{tikzcd}$$

By Lemma 5.6, the horizontal lines are homotopy fiber sequences. Then, by the definition of the homotopy κ_0, the map $G \to \Omega BG$ in this diagram is equal to the above composite and hence is a weak equivalence. □

For $n < \infty$, we define a functor $D_n \colon \mathcal{A}_n^{\mathrm{R}} \to \mathbf{CG}$ by

$$D_n(X,G) = (B_n(X,G,G) \cup (\mathcal{Q}_{n+1} \times X \times G^{\times(n+1)}))/{\sim},$$

with the following identifications:

(i) $(\delta_k^0(\mathbf{t}); x, g_1, \ldots, g_{n+1})$
$$\sim \begin{cases} [\mathbf{t}; xg_1, g_2, \ldots, g_{n+1}, *] \in B_n(X,G,G) & \text{for } k=0 \\ [\mathbf{t}; x, g_1, \ldots, g_k g_{k+1}, \ldots, g_{n+1}, *] \in B_n(X,G,G) & \text{for } 0<k<n \\ [\mathbf{t}; x, g_1, \ldots, g_{n+1}] \in B_n(X,G,G) & \text{for } k=n, \end{cases}$$

(ii) $(\mathbf{t}; x, g_1, \ldots, g_{k-1}, *, g_{k+1}, \ldots, g_{n+1})$
$\sim [\sigma_k(\mathbf{t}); x, g_1, \ldots, g_{k-1}, g_{k+1}, \ldots, g_{n+1}] \in B_n(X,G,G)$.

The induced maps are defined in the similar manner to B_n. Then there are natural maps

$$\begin{aligned} &D_{n-1}(X,G) \subset B_n(X,G,G) \subset D_n(X,G), && \\ &D_n(X,G) \to B_{n+1}(X,G,*), && [\mathbf{t}; x, g_1, \ldots, g_{n+1}] \mapsto [\mathbf{t}; x, g_1, \ldots, g_{n+1}, *], \\ &D_n(X,G) \to X, && [\mathbf{t}; x, g_1, \ldots, g_{n+1}] \mapsto xg_1 \cdots g_{n+1}, \end{aligned}$$

for $(\mathbf{t}; x, g_1, \ldots, g_{n+1}) \in \mathcal{Q}_{n+1} \times X \times G^{\times(n+1)}$. Like the proof of Lemma 5.4, one can see that the last map is a deformation retraction of the inclusion $X \subset D_n(X,G)$. We denote $D_nG := D_n(*,G)$. The well-known homotopy equivalence $E_nG \simeq \Sigma^n G^{\wedge(n+1)}$ comes from the homeomorphism $E_nG/D_{n-1}G \cong \Sigma^n G^{\wedge(n+1)}$. We also have a homeomorphism $D_nG/E_nG \cong \Sigma^{n+1} G^{\wedge(n+1)}$. Then we obtain the well-known homotopy cofiber sequence

$$\Sigma^n G^{\wedge(n+1)} \to B_nG \xrightarrow{\iota_n^{n+1}} B_{n+1}G \to \Sigma^{n+1} G^{\wedge(n+1)}.$$

Remark 5.9. The author could not find in the literature that the homotopy equivalence $E_nG \simeq \Sigma^n G^{\wedge(n+1)}$ can be obtained as above. For example, Stasheff proved this homotopy equivalence by induction on n and some homology Mayer–Vietoris sequence in [**Sta63a**].

Let us consider a map $(D_{n-1}G, E_{n-1}G) \to (D_{n-1}G \vee \Sigma^n G^{\wedge n}, E_{n-1}G)$ defined as follows: for an element $[\gamma^1_{\mathbf{i}}(\mathbf{t};\mathbf{s}_1,\dots,\mathbf{s}_r);*,g_1,\dots,g_i]\in E_{n-1}G$,

$[\gamma^1_{\mathbf{i}}(\mathbf{t};\mathbf{s}_1,\dots,\mathbf{s}_r);*,g_1,\dots,g_i]\mapsto[\mathbf{t};*,g_{i_1+1}\cdots g_{i_1+i_2},\dots,g_{i_1+\cdots+i_{r-1}+1}\cdots g_i]\in E_{n-1}G$,

and for an element $(\delta^{t_{n+1}}_{n+1}(\gamma^1_{\mathbf{i}}(\mathbf{t};\mathbf{s}_1,\dots,\mathbf{s}_r));*,g_1,\dots,g_n)\in\mathcal{Q}_n\times*\times G^{\times n}$,

$$[\delta^{t_{n+1}}_{n+1}(\gamma^1_{\mathbf{i}}(\mathbf{t};\mathbf{s}_1,\dots,\mathbf{s}_r));*,g_1,\dots,g_n]\mapsto$$
$$\begin{cases}[\mathbf{s}_1;g_1,\dots,g_n]\in\Sigma^nG^{\wedge n} & (r=1)\\ [\delta^{t_{n+1}}_{n+1}(\mathbf{t});*,g_{i_1+1}\cdots g_{i_1+i_2},\dots,g_{i_1+\cdots+i_{r-2}+1}\cdots g_{i_1+\cdots+i_{r-1}},*]\in E_{n-1}G & (1<r<n)\\ [\delta^{t_{n+1}}_{n+1}(\mathbf{t});*,g_1,\dots,g_n]\in D_{n-1}G & (r=n),\end{cases}$$

where $\Sigma^nG^{\wedge n}$ is considered as $\Sigma^nG^{\wedge n}=[0,1]^{\times n}\times G^{\times n}/(\partial[0,1]^{\times n}\times G^{\times n}\cup[0,1]^{\times n}\times T_nG)$. This map is depicted in Figure 3. We call this map the *pinch map*. The pinch map restricted to $E_{n-1}G$ is naturally homotopic to the identity. In the following lemma, we denote the reduced cone of a pointed space X by CX.

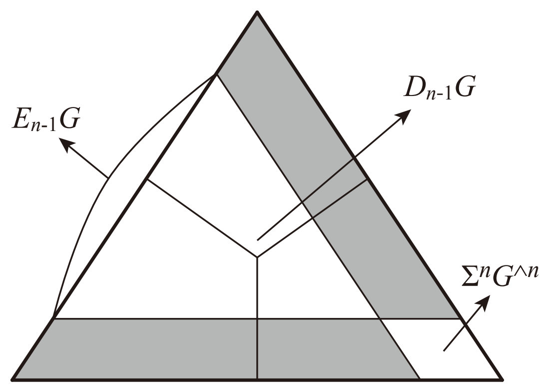

Figure 3: The pinch map

Lemma 5.10. *Let G be a well-pointed topological monoid and X a right G-space.*

(i) Then the pushout of the diagram

$$D_{n-1}(X,G)\leftarrow B_{n-1}(X,G,G)\to B_{n-1}(X,G,*)$$

is naturally homeomorphic to $B_n(X,G,*)$ by the map induced from the map $D_{n-1}(X,G)\to B_n(X,G,*)$ as above and the inclusion $\iota^n_{n-1}\colon B_{n-1}(X,G,*)\to B_n(X,G,*)$.

(ii) There exists a natural map $(CE_{n-1}G,E_{n-1}G)\to(D_{n-1}G,E_{n-1}G)$ that restricts to the identity on $E_{n-1}G$. This map induces a diagram

$$\begin{tikzcd} CE_{n-1}G \arrow[r] \arrow[d] & CE_{n-1}G\vee\Sigma^nG^{\wedge n} \arrow[d] \\ D_{n-1}G \arrow[r] & D_{n-1}G\vee\Sigma^nG^{\wedge n} \end{tikzcd}$$

commutative up to homotopy that restricts to the natural homotopy on $E_{n-1}G$, *where the horizontal arrows represent the pinch maps and the right vertical map is the identity on* $\Sigma^n G^{\wedge n}$.

Proof. It is not difficult to observe that the naturally induced map from the pushout in (i) to $B_n(X, G, *)$ is a homeomorphism. The inverse map is given by the pushout square as in the proof of Lemma 5.5. Then the assertion (i) follows.

We regard the reduced cone $CE_{n-1}G$ as

$$CE_{n-1}G = \mathcal{Q}_1 \times E_{n-1}G/(\mathcal{Q}_1 \times * \cup \delta_1^0 \times E_{n-1}G).$$

Then the map κ_{n-1} in the proof of Lemma 5.4 induces a natural map

$$(CE_{n-1}G, E_{n-1}G) \to (D_{n-1}G, E_{n-1}G).$$

For this map, it is straightforward to see the desired homotopy commutativity of the diagram in (ii). □

For a space X, the space $\mathcal{W}X := |\mathrm{Sing}(X)|$ defined to be the realization of the simplicial complex of singular simplices of X is naturally a CW complex. There is a natural weak equivalence $\mathcal{W}X \to X$. In particular, if G is a topological monoid or group, then $\mathcal{W}G$ is a CW complex equipped with the natural structure of a topological monoid or group, respectively, which is given by a cellular map $\mathcal{W}G \times \mathcal{W}G \to \mathcal{W}G$. Moreover, the natural weak equivalence $\mathcal{W}G \to G$ is a homomorphism. If G is a topological group, so is $\mathcal{W}G$. If X is a right G-space, then $\mathcal{W}X$ is a right $\mathcal{W}G$-space with the cellular action $\mathcal{W}X \times \mathcal{W}G \to \mathcal{W}X$ and the natural maps $\mathcal{W}X \to X$ and $\mathcal{W}G \to G$ preserve the action.

Lemma 5.11. *Let G be a well-pointed topological monoid, X a right G-space, and Y a left G-space, all of which have the homotopy types of CW complexes. Then $B_n(X, G, Y)$ has the homotopy type of a CW complex. In particular, the natural map $B_n(\mathcal{W}X, \mathcal{W}G, \mathcal{W}Y) \to B_n(X, G, Y)$ is a homotopy equivalence.*

Proof. Using the homotopy invariance of homotopy pushout, this lemma is similarly proved as in Lemma 5.5. □

6. Mapping spaces from projective spaces

Now, let us prove our main theorem.

Theorem 6.1. *Let G be a well-pointed topological monoid having the homotopy type of a CW complex and G' a well-pointed grouplike topological monoid. Then the following composite is a weak equivalence:*

$$\mathcal{A}_n(G, G') \xrightarrow{B_n} \mathrm{Map}_0(B_nG, B_nG') \xrightarrow{(\iota_n)_\#} \mathrm{Map}_0(B_nG, BG').$$

Proof. We prove this theorem by induction. For $n = 1$, the composite

$$\mathrm{Map}_0(G, G') \xrightarrow{\Sigma} \mathrm{Map}_0(\Sigma G, \Sigma G') \xrightarrow{(\iota_1)_\#} \mathrm{Map}_0(\Sigma G, BG') \cong \mathrm{Map}_0(G, \Omega BG')$$

is equal to $\zeta_\#$, where $\zeta\colon G' \to \Omega BG'$ is as in Lemma 5.8. Then, the composite $(\iota_1)_\# \circ B_1$ is a weak equivalence.

Suppose that the composite $(\iota_{n-1})_\# \circ B_{n-1}$ is a weak equivalence. Consider the following homotopy commutative diagram of homotopy fiber sequences (see Section 4 for the definition of $\mathcal{A}^1_n$):

$$\begin{tikzcd}
\mathrm{Map}_0(\Sigma^{n-1}G^{\wedge n}, G') \arrow[r] \arrow[d, "(\iota_1)_\#\circ\Sigma"] & \mathcal{A}^1_n(G,G') \arrow[r] \arrow[d] & \mathcal{A}^1_{n-1}(G,G') \arrow[d, "\simeq"] \\
\mathrm{Map}_0(\Sigma^{n}G^{\wedge n}, BG') \arrow[r] & \mathrm{Map}_0(B_nG, BG') \arrow[r, "(\iota^n_{n-1})^\#"] & \mathrm{Map}_0(B_{n-1}G, BG'),
\end{tikzcd}$$

where the homotopy fibers and maps between them are determined by the observation in Section 4 and Lemma 5.10. Moreover, one can see that the vertical maps preserve principal actions up to homotopy. Since the central vertical arrow induces an surjection on π_0 by Proposition 6.3, which will be proved later, then it is a weak equivalence.

For $n = \infty$, the result follows since the projections $\mathcal{A}^1_n(G,G') \to \mathcal{A}^1_{n-1}(G,G')$ and $\mathrm{Map}_0(B_nG, B_nG') \to \mathrm{Map}_0(B_nG, BG')$ have the homotopy lifting property, and $\mathcal{A}^1_\infty(G,G')$ and $\mathrm{Map}_0(BG, BG')$ are the limits along the sequences of these projections, respectively. □

To complete the proof, all we have to show is that the composite $(\iota_n)_\# \circ B_n$ induces the surjection on π_0 for $n < \infty$. First, we consider the case when G' is a topological group.

Lemma 6.2. *Under the setting in Theorem 6.1, suppose that G' is a topological group and $n < \infty$. Then the composite $(\iota_n)_\# \circ B_n$ induces the surjection on π_0.*

Proof. We show this by induction on n. Let $F\colon B_nG \to BG'$ be a pointed map. Suppose that there exists an A_{n-1}-map $f = (f, \{f_i\}_{i=1}^{n-1}, \ell)$ such that F restricts to $\iota_{n-1} \circ B_{n-1}f$ on $B_{n-1}G$. Since $EG' \to BG'$ is a fiber bundle by Lemma 5.6, there exists a dotted arrow h in the diagram

$$\begin{tikzcd}
D_{n-2}G \arrow[r, "D_{n-2}f"] \arrow[d] & EG' \arrow[d] \\
D_{n-1}G \arrow[r] \arrow[ur, dotted, "h"] & BG'
\end{tikzcd}$$

such that this diagram commutes, where the arrow $D_{n-1}G \to BG'$ is the composite $D_{n-1}G \to B_nG \xrightarrow{F} BG'$. Then one can extend the A_{n-1}-form of f to the A_n-form $\{f_i\}_{i=1}^n$ uniquely such that the restriction of h to $E_{n-1}G \to E_{n-1}G'$ coincides with $E_{n-1}f$. Note that $h\colon D_{n-1}G \to EG'$ is homotopic to the composite $D_{n-1}G \xrightarrow{D_{n-1}f} D_{n-1}G' \to EG'$ rel $E_{n-1}G$. Therefore, by (i) of Lemma 5.10, F is homotopic to B_nf. □

Proposition 6.3. *Under the setting in Theorem 6.1, the composite $(\iota_n)_\# \circ B_n$ induces the surjection on π_0 for general G' and $n < \infty$.*

Proof. Considering the weak equivalences $\mathcal{W}G' \to G'$ and $B\mathcal{W}G' \to BG'$, we may assume that G' has the homotopy type of a CW complex by Proposition 4.9. By the results on simplicial homotopy theory (for example, see [**May67**]), we can find a topological group $\tilde{G}$ which is a CW complex such that the classifying space $B\tilde{G}$ is homotopy equivalent to BG', where we use the fact that BG' has the homotopy type of a CW complex by Lemma 5.11. Then, by Theorem 6.1 for G' and $\tilde{G}$ and Corollary 4.10, there exists an A_∞-equivalence $G' \to \tilde{G}$. Combining Proposition 4.9 and Lemma 5.5 and 6.2, we obtain the desired surjectivity. $\square$

This completes the proof of Theorem 6.1.

Let X be a path-connected well-pointed space of the homotopy type of a CW complex. Then one can show that the Moore based loop space $\Omega^{\mathrm{M}}X$ is a well-pointed grouplike topological monoid having the homotopy type of a CW complex.

Remark 6.4. Let X be a well-pointed space of the homotopy type of a CW complex. Then there is a natural homotopy equivalence into the path-component containing the basepoint

$$B\Omega^{\mathrm{M}}X = B(*, \Omega^{\mathrm{M}}X, *) \xleftarrow{\simeq} B(P^{\mathrm{M}}X, \Omega^{\mathrm{M}}X, *) \to X,$$

where the left arrow is induced by the map $P^{\mathrm{M}}X \to *$ and the right arrow is induced by the evaluation $e\colon P^{\mathrm{M}}X \to X$. This homotopy equivalence is checked by the similar argument to the proof of the following corollary.

Through this homotopy equivalence, Theorem 6.1 is recognized as the adjunction

$$\mathcal{A}_n(G, \Omega^{\mathrm{M}}X) \simeq \mathrm{Map}_0(B_nG, X)$$

in certain sense. We will call the correspondence of the homotopy classes through this weak equivalence or that of Theorem 6.1 as the *adjoint*. With respect to this adjunction, we consider the unit map as in the next corollary.

Remark 6.5. Let G be a well-pointed topological monoid and X a pointed space. Norio Iwase pointed out to the author that the weak equivalence

$$\mathcal{A}_n(G, \Omega^{\mathrm{M}}X) \simeq \mathrm{Map}_0(B_nG, X)$$

stated above is in fact a *homotopy equivalence*. The inverse map of this equivalence is given as in Stasheff's lifting-extension argument in the proof of [**Sta63b**, Theorem 4.5], which can be done continuously.

Remark 6.6. As in [**Sta63a**] and [**IM89**], projective spaces and A_n-maps are defined for A_n-spaces as well. Then it is natural to ask whether Theorem 6.1 can be generalized for an A_n-space G and an A_∞-space G' or not. This might be carried out but needs many preparations about A_n-spaces. This problem will be postponed for now.

Corollary 6.7. *Let G be a well-pointed topological monoid having the homotopy type of a CW complex. Then there exists an A_n-map $\eta\colon G \to \Omega^{\mathrm{M}}B_nG$ such that the adjoint $\iota_n \circ B_n\eta\colon B_nG \to B\Omega^{\mathrm{M}}B_nG$ is a homotopy equivalence and there is a homotopy commutative diagram*

$$\begin{tikzcd} B_nG \arrow[rr, "\iota_n \circ B_n\eta"] \arrow[d, equal] & & B\Omega^{\mathrm{M}}B_nG \arrow[d, equal] \\ B_nG & B(P^{\mathrm{M}}B_nG, \Omega^{\mathrm{M}}B_nG, *) \arrow[l, "e_*"'] \arrow[r, "q"] & B\Omega^{\mathrm{M}}B_nG, \end{tikzcd}$$

where the map e_* is induced from the evaluation $e\colon P^{\mathrm{M}}B_nG \to B_nG$ and q from the map $P^{\mathrm{M}}B_nG \to *$. Moreover, the composite

$$\mathcal{A}_\infty(\Omega^{\mathrm{M}}B_nG, G') \to \mathcal{A}_n(\Omega^{\mathrm{M}}B_nG, G') \xrightarrow{\eta^\#} \mathcal{A}_n(G, G')$$

is a weak equivalence for any grouplike topological monoid G'.

Proof. The first half follows from the following commutative diagram and Theorem 6.1:

$$\begin{tikzcd} \Omega^{\mathrm{M}}B_nG \arrow[r, equal] \arrow[d] & \Omega^{\mathrm{M}}B_nG \arrow[r, equal] \arrow[d] & \Omega^{\mathrm{M}}B_nG \arrow[d] \\ P^{\mathrm{M}}B_nG \arrow[d] & B(P^{\mathrm{M}}B_nG, \Omega^{\mathrm{M}}B_nG, \Omega^{\mathrm{M}}B_nG) \arrow[l] \arrow[r] \arrow[d] & B(*, \Omega^{\mathrm{M}}B_nG, \Omega^{\mathrm{M}}B_nG) \arrow[d] \\ B_nG & B(P^{\mathrm{M}}B_nG, \Omega^{\mathrm{M}}B_nG, *) \arrow[l] \arrow[r] & B(*, \Omega^{\mathrm{M}}B_nG, *). \end{tikzcd}$$

The latter half can be checked by the following commutative diagram:

$$\begin{tikzcd} \mathcal{A}_\infty(\Omega^{\mathrm{M}}B_nG, G') \arrow[r] & \mathcal{A}_n(\Omega^{\mathrm{M}}B_nG, G') \arrow[r, "\eta^\#"] & \mathcal{A}_n(G, G') \\ \mathcal{A}_\infty(\Omega^{\mathrm{M}}B_nG, \mathcal{W}G') \arrow[r] \arrow[u, "\simeq"'] \arrow[d, "\simeq"] & \mathcal{A}_n(\Omega^{\mathrm{M}}B_nG, \mathcal{W}G') \arrow[r, "\eta^\#"] \arrow[u, "\simeq"'] \arrow[d, "\simeq"] & \mathcal{A}_n(G, \mathcal{W}G') \arrow[u, "\simeq"'] \arrow[d, "\simeq"] \\ \mathrm{Map}_0(B\Omega^{\mathrm{M}}B_nG, B\mathcal{W}G') \arrow[r, "(\iota_n)^\#"] & \mathrm{Map}_0(B_n\Omega^{\mathrm{M}}B_nG, B\mathcal{W}G') \arrow[r, "(B_n\eta)^\#"] & \mathrm{Map}_0(B_nG, B\mathcal{W}G'), \end{tikzcd}$$

where the composite of the arrows in the bottom row is a weak equivalence. □

Remark 6.8. The A_n-map $\eta\colon G \to \Omega^{\mathrm{M}}B_nG$ has been studied by Stasheff in [**Sta70**]. Later, McGibbon [**McG82**] proved that η is never an A_{n+1}-map for any connected non-contractible CW complex G.

7. Application: evaluation fiber sequences

For simplicity, we discuss only about topological groups rather than general topological monoids. But, using the technique of simplicial homotopy theory as in the proof of Proposition 6.3, our result may admit some generalization.

For the fundamental facts on the space of bundle maps, see Gottlieb's paper [**Got72**]. Let G be a well-pointed topological group, B a well-pointed space of the homotopy type of a CW complex and P a principal G-bundle over B classified by $\epsilon\colon B \to BG$. The *gauge group* $\mathcal{G}(P)$ of P is the topological group consisting of the

G-equivariant self maps on P that induces the identity on the quotient $P/G \cong B$. Denote the space of G-equivariant maps $P \to EG$ by $\mathcal{E}(P, EG)$. The gauge group $\mathcal{G}(P)$ acts on $\mathcal{E}(P, EG)$ from the right by composition. Then there is a Serre fibration $\beta\colon \mathcal{E}(P, EG) \to \mathrm{Map}(B, BG; \epsilon)$ which assigns the induced map on the quotient $P/G = B \to EG/G = BG$. Moreover, it is known that β is a principal $\mathcal{G}(P)$-fibration and $\pi_i(\mathcal{E}(P, EG)) = 0$ for all $i \geqslant 0$. Let $\rho\colon \mathcal{E}(P, EG) \to EG$ be the evaluation at the basepoint. This map is equivariant through the homomorphism $\rho\colon \mathcal{G}(P) \to \mathcal{G}(G) \cong G$ defined by the evaluation at the basepoint. Define the subspace $\mathcal{E}_0(P, EG) := \rho^{-1}(*) \subset \mathcal{E}(P, EG)$ and the closed subgroup $\mathcal{G}_0(P) := \rho^{-1}(*) \subset \mathcal{G}(P)$. One can check that $\beta\colon \mathcal{E}_0(P, EG) \to \mathrm{Map}_0(B, BG; \epsilon)$ is a Serre fibration with a principal action by $\mathcal{G}_0(P)$ and $\pi_i(\mathcal{E}_0(P, EG)) = 0$ for all $i > 0$.

For $g \in G$, the conjugation and the left translation

$$\begin{aligned} &\alpha_g\colon G \to G, &&\alpha_g(x) = gxg^{-1}, \\ &L_g\colon G \to G, &&L_g(x) = gx \end{aligned}$$

induce a G-equivariant map $E\alpha_g := B(*, \alpha_g, L_g)\colon EG \to EG$ and a map $B\alpha_g\colon BG \to BG$. These maps satisfy $E\alpha_{gg'} = E\alpha_g \circ E\alpha_{g'}$ and $B\alpha_{gg'} = B\alpha_g \circ B\alpha_{g'}$, and the following diagram commutes:

$$\begin{tikzcd} EG \arrow[r, "E\alpha_g"] \arrow[d] & EG \arrow[d] \\ BG \arrow[r, "B\alpha_g"] & BG. \end{tikzcd}$$

From now on, we use the notation $\mathcal{E} := \mathcal{E}(P, EG)$, $\mathcal{E}_0 := \mathcal{E}_0(P, EG)$, $\mathcal{G} := \mathcal{G}(P)$, $\mathcal{G}_0 := \mathcal{G}_0(P)$, $M := \mathrm{Map}(B, BG; \epsilon)$ and $M_0 := \mathrm{Map}_0(B, BG; \epsilon)$ for simplicity of diagrams. Following [**KK10**, Section 6], consider free actions

$$\begin{aligned} &\mathcal{G} \times (EG \times \mathcal{E}_0) \to EG \times \mathcal{E}_0, &&(\varphi, (u, f)) \mapsto (u\rho(\varphi)^{-1}, E\alpha_{\rho(\varphi)} \circ f \circ \varphi^{-1}), \\ &G \times (EG \times M_0) \to EG \times M_0, &&(g, (F, u)) \mapsto (ug^{-1}, B\alpha_g \circ F). \end{aligned}$$

Here, the induced map from $\mathcal{G}$ into each fiber of the Serre fibration $\mathcal{E}_0 \times EG \to M_0 \times_G EG$ is a homeomorphism.

Let us consider the commutative diagram

$$\begin{tikzcd} \mathcal{G} \arrow[r] & EG \times \mathcal{E}_0 \arrow[r] & EG \times_G M_0 \\ \mathcal{W}\mathcal{G} \arrow[u] \arrow[r] \arrow[d, equal] & B(\mathcal{W}EG \times \mathcal{W}\mathcal{E}_0, \mathcal{W}\mathcal{G}, \mathcal{W}\mathcal{G}) \arrow[u] \arrow[r] \arrow[d] & B(\mathcal{W}EG \times \mathcal{W}\mathcal{E}_0, \mathcal{W}\mathcal{G}, *) \arrow[u] \arrow[d] \\ \mathcal{W}\mathcal{G} \arrow[r] \arrow[d, equal] & B(*, \mathcal{W}\mathcal{G}, \mathcal{W}\mathcal{G}) \arrow[r] & B(*, \mathcal{W}\mathcal{G}, *) \\ \mathcal{W}\mathcal{G} \arrow[r] \arrow[d] & B(\mathcal{W}\mathcal{E}, \mathcal{W}\mathcal{G}, \mathcal{W}\mathcal{G}) \arrow[u] \arrow[r] \arrow[d] & B(\mathcal{W}\mathcal{E}, \mathcal{W}\mathcal{G}, *) \arrow[u] \arrow[d] \\ \mathcal{G} \arrow[r] & \mathcal{E} \arrow[r] & M. \end{tikzcd}$$

Then each row is a Serre fibration and each vertical arrow is a weak equivalence. By the similar argument, we obtain the following commutative diagram:

$$\begin{tikzcd}
EG \times M_0 \arrow[r] \arrow[dddd, bend right=60] & EG \times_G M_0 \arrow[r] & BG \\
B(\mathcal{W}EG \times \mathcal{W}\mathcal{E}_0, \mathcal{W}\mathcal{G}_0, *) \arrow[r] \arrow[u] \arrow[d] \arrow[dd, bend right=50] & B(\mathcal{W}EG \times \mathcal{W}\mathcal{E}_0, \mathcal{W}\mathcal{G}, *) \arrow[r] \arrow[u] \arrow[d] & B(\mathcal{W}EG, \mathcal{W}G, *) \arrow[u] \arrow[d] \\
B(*, \mathcal{W}\mathcal{G}_0, *) \arrow[r] & B(*, \mathcal{W}\mathcal{G}, *) \arrow[r] & B(*, \mathcal{W}G, *) \\
B(\mathcal{W}\mathcal{E}_0, \mathcal{W}\mathcal{G}_0, *) \arrow[r] \arrow[u] \arrow[d] & B(\mathcal{W}\mathcal{E}, \mathcal{W}\mathcal{G}, *) \arrow[r] \arrow[u] \arrow[d] & B(\mathcal{W}EG, \mathcal{W}G, *) \arrow[u] \arrow[d] \\
M_0 \arrow[r] & M \arrow[r] & BG,
\end{tikzcd}$$

where all the vertical arrows are weak equivalences. From this diagram, we obtain the following.

Lemma 7.1. *Let G be a well-pointed topological group, B a well-pointed space of the homotopy type of a CW complex, and P a principal G-bundle over B. Then there exist CW complexes X and X' and the following homotopy commutative diagram:*

$$\begin{tikzcd}
\mathrm{Map}_0(B, BG; \epsilon) \arrow[r] \arrow[dd, bend right=60, "\mathrm{id}"'] & EG \times_G \mathrm{Map}_0(B, BG; \epsilon) \arrow[r] & BG \arrow[d, equal] \\
X \arrow[r] \arrow[u, "\simeq"] \arrow[d, "\simeq"'] & X' \arrow[r] \arrow[u, "\simeq"] \arrow[d, "\simeq"'] & BG \arrow[d, equal] \\
\mathrm{Map}_0(B, BG; \epsilon) \arrow[r] & \mathrm{Map}(B, BG; \epsilon) \arrow[r] & BG,
\end{tikzcd}$$

where the vertical arrows are weak equivalences.

Remark 7.2. Intuitively, this result states that the top and bottom rows are equivalent as a fiber sequence. But we cannot expect the existence of a direct weak equivalence between $EG \times_G \mathrm{Map}_0(B, BG; \epsilon)$ and $\mathrm{Map}(B, BG; \epsilon)$ in general.

The conjugation defines a homomorphism

$$G \to \mathcal{A}_n(G, G), \quad g \mapsto \alpha_g$$

and a left G-action

$$G \times \mathcal{A}_n(G, G) \to \mathcal{A}_n(G, G), \quad (g, f) \mapsto \alpha_g \circ f.$$

On the other hand, we have a homomorphism

$$G \to \mathrm{Map}_0(BG, BG), \quad g \mapsto B\alpha_g$$

and a left G-action

$$G \times \mathrm{Map}_0(B_nG, BG) \to \mathrm{Map}_0(B_nG, BG), \quad (g, F) \mapsto B\alpha_g \circ F.$$

Lemma 7.3. *For a well-pointed topological group G of the homotopy type of a CW complex, the map*

$$(\iota_n)_\# \circ B_n \colon \mathcal{A}_n(G,G) \to \mathrm{Map}_0(B_nG, BG),$$

is a G-equivariant weak equivalence with respect to the above left G-action.

Proof. This immediately follows from Theorem 6.1. □

Let us denote the subspace of weak A_n-equivalences by $\mathcal{A}_n(G,G;\mathrm{eq}) \subset \mathcal{A}_n(G,G)$, whose basepoint is the identity $\mathrm{id}_G \in \mathcal{A}_n$. A subspace

$$\overline{\mathrm{Map}}_0(B_nG, BG) \subset \mathrm{Map}_0(B_nG, BG)$$

is defined as follows: for a map $F \in \mathrm{Map}_0(B_nG, BG)$, F is contained in $\overline{\mathrm{Map}}_0(B_nG, BG)$ if and only if the adjoint A_n-map $G \to G$ is a weak A_n-equivalence. We also denote the union of path-components in $\mathrm{Map}(B_nG, BG)$ that intersect nontrivially with $\overline{\mathrm{Map}}_0(B_nG, BG)$ by $\overline{\mathrm{Map}}(B_nG, BG)$. The basepoint of $\overline{\mathrm{Map}}_0(B_nG, BG)$ and $\overline{\mathrm{Map}}(B_nG, BG)$ is the inclusion $\iota_n \colon B_nG \to BG$.

Theorem 7.4. *Let G be a well-pointed topological group having the homotopy type of a CW complex. Then there is a homotopy fiber sequence*

$$\overline{\mathrm{Map}}_0(B_nG, BG) \to \overline{\mathrm{Map}}(B_nG, BG) \to BG \to B\mathcal{W}\mathcal{A}_n(G,G;\mathrm{eq})$$

such that the map $BG \to B\mathcal{W}\mathcal{A}_n(G,G;\mathrm{eq})$ is induced from the homomorphism $G \to \mathcal{A}_n(G,G;\mathrm{eq})$ giving the conjugation.

Proof. By Proposition 5.7, the sequence

$$\mathcal{A}_n(G,G;\mathrm{eq}) \to B(*,G,\mathcal{A}_n(G,G;\mathrm{eq})) \to BG \to B\mathcal{W}\mathcal{A}_n(G,G;\mathrm{eq}),$$

with respect to the conjugation action of G on $\mathcal{A}_n(G,G;\mathrm{eq})$ is a homotopy fiber sequence. Then, combining Lemma 7.1 and 7.3, we obtain the desired homotopy fiber sequence. □

Remark 7.5. As remarked in Section 1, if $n = \infty$, the extension in Theorem 7.4 coincides with the well-known fiber sequence

$$\begin{aligned} G \to \overline{\mathrm{Map}}_0(BG, BG) \to \overline{\mathrm{Map}}(BG, BG) \to BG \\ \to B\mathcal{W}\overline{\mathrm{Map}}_0(BG, BG) \to B\mathcal{W}\overline{\mathrm{Map}}(BG, BG), \end{aligned}$$

where the monoid structures on $\overline{\mathrm{Map}}_0(BG, BG)$ and $\overline{\mathrm{Map}}(BG, BG)$ are given by compositions.

The next example shows that the extension in Theorem 7.4 is the maximum.

Example 7.6. Kishimoto–Kono–Theriault [**KKT13**, Theorem 1.3] showed that the loop space $\Omega\,\mathrm{Map}(S^4, B\,\mathrm{SU}(2)_{(5)}; \iota_1)$ is not homotopy commutative. This implies that $\mathrm{Map}(S^4, B\,\mathrm{SU}(2); \iota_1)$ is never delooped.

But this is not always the case.

Example 7.7. Let T be the m-dimensional compact torus. Then the maps

$$\overline{\mathrm{Map}}_0(BT, BT) \to \overline{\mathrm{Map}}_0(B_nT, BT)$$
$$\overline{\mathrm{Map}}(BT, BT) \to \overline{\mathrm{Map}}(B_nT, BT)$$

are weak equivalences for $n \geqslant 1$. Moreover, the evaluation fiber sequence

$$\overline{\mathrm{Map}}_0(BT, BT) \to \overline{\mathrm{Map}}(BT, BT) \to BT \to B\mathcal{W}\overline{\mathrm{Map}}_0(BT, BT) \to B\mathcal{W}\overline{\mathrm{Map}}(BT, BT)$$

is equivalent to the sequence

$$\mathrm{GL}(m, \mathbb{Z}) \to BT \rtimes \mathrm{GL}(m, \mathbb{Z}) \to BT \to B\,\mathrm{GL}(m, \mathbb{Z}) \to B(BT \rtimes \mathrm{GL}(m, \mathbb{Z})).$$

This no longer extends. This can be seen by observing the action of $\pi_1(B(BT \rtimes \mathrm{GL}(m, \mathbb{Z})))$ on $\pi_3(B(BT \rtimes \mathrm{GL}(m, \mathbb{Z})))$.

As a step to observe the non-extendability, we conjecture as follows.

Conjecture 7.8. For a non-commutative compact connected Lie group G, the space $\mathrm{Map}(B_nG, BG; \iota_n)$ is never delooped for $1 \leqslant n < \infty$.

Remark 7.9. Related to this conjecture, an upper bound of the homotopy nilpotency of $\mathcal{G}(E_nG) \simeq \Omega\,\mathrm{Map}(B_nG, BG; \iota_n)$ was given by Crabb–Sutherland–Zhang [**CSZ99**] for general Lie groups G. But the author does not know any result implying the homotopy non-commutativity of $\mathcal{G}(E_nG)$.

8. Application: higher homotopy commutativity

There are several notions of higher homotopy commutativity. Sugawara introduced the strong homotopy commutativity in [**Sug60**], which is naturally generalized to Sugawara C^n-spaces [**McG89**]. Williams introduced another higher homotopy commutativity called Williams C_n-spaces in [**Wil69**]. Hemmi and Kawamoto defined $C_k(n)$-spaces in [**HK11**]. Kishimoto and Kono also considered certain higher commutativity called $C(k, \ell)$-spaces in [**KK10**]. First we compare these commutativities. Recall that they are described by using projective spaces as follows.

Proposition 8.1. *Let G be a well-pointed grouplike topological monoid having the homotopy type of a CW complex. Then the following statements hold:*

(i) G is a Williams C_n-space if and only if the map $(\iota_1, \dots, \iota_1)\colon (\Sigma G)^{\vee n} \to BG$ extends over the product $(\Sigma G)^{\times n}$,

(ii) G is a $C(k, \ell)$-space if and only if the map $(\iota_k, \iota_\ell)\colon B_kG \vee B_\ell G \to BG$ extends over the product $B_kG \times B_\ell G$,

(iii) G is a $C_k(n)$-space if and only if the map $(\iota_k, \iota_n)\colon B_kG \vee B_nG \to BG$ extends over the union $\bigcup_{i+j=n, i\leqslant k} B_iG \times B_jG$,

(iv) G is a Sugawara C^n-space if and only if G is a $C_n(n)$-space.

See [**Saü95**] for the proof of (i), [**KK10**] for (ii), and [**HK11**] for (iii) and (iv). Obviously, any Sugawara C^n-space is a $C_k(n)$-space for $k \leqslant n$, and any $C_k(n)$-space is a $C(k, n-k)$-space. The Williams C_n-space is related with other homotopy commutativities as the following lemma.

Lemma 8.2. *Let G be a well-pointed grouplike topological monoid having the homotopy type of a CW complex. If G is a $C(k,\ell)$-space, then G is a $C_{k+\ell}$-space.*

Proof. It is sufficient to prove that the map $j\colon(\iota_1,\dots,\iota_1)\colon(\Sigma G)^{\vee(k+\ell)}\to BG$ can be extended over the product $(\Sigma G)^{\times(k+\ell)}$ by Proposition 8.1 (i). Since G is a $C(k,\ell)$-space, for connected CW complexes A and B such that $\operatorname{cat}A\leqslant k$ and $\operatorname{cat}B\leqslant\ell$, any map $A\vee B\to BG$ extends over the product $A\times B$. Then the map j can be extended since the n-fold product of the suspension spaces have the L–S category less than or equal to n. □

Now we let G be a well-pointed topological group having the homotopy type of a CW complex. In Section 7, we saw that the connecting map $\delta\colon G\to\operatorname{Map}_0(B_nG,BG)$ in the evaluation fiber sequence

$$G\xrightarrow{\delta}\overline{\operatorname{Map}}_0(B_nG,BG)\to\overline{\operatorname{Map}}(B_nG,BG;\iota_n)\to BG$$

is $\delta(g)=B\alpha_g\circ\iota_n$ and is identified with the homomorphism $G\to\mathcal{A}_n(G,G)$ induced by the conjugation.

Theorem 8.3. *Let G be a well-pointed topological group having the pointed homotopy type of a CW complex. Then G is a $C(k,\ell)$-space if and only if the homomorphism $G\to\mathcal{A}_\ell(G,G)$ giving the conjugation is homotopic to the trivial map as an A_k-map.*

Proof. By Theorem 6.1, the homomorphism $G\to\mathcal{A}_\ell(G,G)$ is homotopic to the trivial map as A_k-map if and only if the composite

$$B_kG\xrightarrow{\iota_k}BG\to B\mathcal{W}\mathcal{A}_n(G,G;\mathrm{eq})$$

is null-homotopic. By the evaluation fiber sequence in Theorem 7.4, this condition is equivalent to the existence of the wedge sum $(\iota_k,\iota_\ell)\colon B_kG\vee B_\ell G\to BG$ over the product $B_kG\times B_\ell G$. By Proposition 8.1, it is equivalent to G being a $C(k,\ell)$-space. □

Corollary 8.4. *Let G be a well-pointed topological group having the pointed homotopy type of a CW complex. Then the following conditions are equivalent:*

(i) the classifying space BG is an H-space,

(ii) G is a Sugawara C^∞-space,

(iii) the map $G\to\mathcal{A}_n(G,G)$ induced by the conjugation is homotopic as an A_∞-map to the constant map to the identity.

For a pointed spaces X and Y, we denote the half-smash product by

$$X\ltimes Y:=X\times Y/X\times *.$$

Corollary 8.5. *Let G be a well-pointed topological group having the homotopy type of a CW complex. Then G is a $C(1,\ell)$-space if and only if the map*

$$G\ltimes B_\ell G\to BG,\quad(g,x)\mapsto B\alpha_g(\iota_\ell(x))$$

is homotopic rel $B_\ell G$ to the composite of the projection $G\ltimes B_\ell G\to B_\ell G$ and the inclusion $\iota_\ell\colon B_\ell G\to BG$.

Remark 8.6. When $\ell=1$, $G\ltimes\Sigma G$ is naturally homotopy equivalent to $\Sigma(G\wedge G)\vee\Sigma G$. Then it is easy to check that the composite $\Sigma(G\wedge G)\to G\ltimes\Sigma G\to BG$ is homotopic to the Whitehead product $[\iota_1,\iota_1]$. This has been known by Lang [**Lan73**].

9. Application: A_n-types of gauge groups

The classification of A_n-types ($n \geqslant 2$) of gauge groups is first considered by Crabb and Sutherland for $n = 2$ in [**CS00**] after several works on the homotopy types begun with Kono's work [**Kon91**]. The author studied the A_n-types of the gauge groups of principal SU(2)-bundles over S^4 for general n in [**Tsu12**] and [**Tsu15**]. In this section, we apply our result to the triviality of adjoint bundles. It is an important problem in the classification of A_n-types of gauge groups. The triviality we consider is defined as follows.

We refer to Section 7 for basic notions on gauge groups. Let G be a topological group and P be a principal G-bundle over a space B. The *adjoint bundle* $\operatorname{ad} P$ is the associated bundle of P induced by the conjugation on G itself. The adjoint bundle is naturally a fiberwise topological group, that is, there is a fiberwise multiplication $\operatorname{ad} P \times_B \operatorname{ad} P \to \operatorname{ad} P$ which makes the each fiber a topological group with continuous fiberwise inversion $\operatorname{ad} P \to \operatorname{ad} P$. The space of sections $\Gamma(\operatorname{ad} P)$ of $\operatorname{ad} P$ is naturally isomorphic to the gauge group $\mathcal{G}(P)$. Moreover, considering the obvious fiberwise version of A_n-map, each fiberwise A_n-map $\operatorname{ad} P \to \operatorname{ad} P$ induces an A_n-map $\mathcal{G}(P) \to \mathcal{G}(P')$. If the underlying map of the fiberwise A_n-map is a homotopy equivalence, then the induced map on the gauge groups is also a homotopy equivalence.

Definition 9.1. A fiberwise topological monoid $E \to B$ is said to be *A_n-trivial* if there exist a topological monoid G and a fiberwise A_n-map $B \times G \to E$ which restricts to a homotopy equivalence on each fiber.

Though the following proposition is partially proved in [**KK10**], we give another proof.

Proposition 9.2. *Let G be a well-pointed topological group and B be a pointed space, both of which have the pointed homotopy type of CW complexes. For a principal G-bundle P over B classified by $\epsilon\colon B \to BG$, the following conditions are equivalent:*

(i) $\operatorname{ad} P$ *is A_n-trivial,*

(ii) the composite $B \xrightarrow{\epsilon} BG \to B\mathcal{W}\mathcal{A}_n(G, G; \mathrm{eq})$ is null-homotopic,

(iii) the map $(\epsilon, \iota_n)\colon B \vee B_nG \to BG$ extends over the product $B \times B_nG$.

Proof. By the evaluation fiber sequence in Theorem 7.4

$$\overline{\mathrm{Map}}(B_nG, BG) \to BG \to B\mathcal{W}\mathcal{A}_n(G, G; \mathrm{eq}),$$

the conditions (ii) and (iii) are equivalent. Now we check the equivalence between (i) and (ii). Consider the associated bundle $E = P \times_G \mathcal{A}_n(G, G; \mathrm{eq})$ induced from the homomorphism $G \to \mathcal{A}_n(G, G; \mathrm{eq})$ giving the conjugation. By construction, the existence of a section of E and that of a fiberwise A_n-map $B \times G \to \operatorname{ad} P$ which restricts to a homotopy equivalence on each fiber are equivalent. The former is equivalent to the condition (ii). The latter is equivalent to the condition (i). This completes the proof. □

Remark 9.3. By an obstruction argument, this proposition extends to the classification theorem of fiberwise A_n-equivalence class. For details, see [**Tsu12**] and [**Tsu15**].

Proposition 9.4. *Let G be a well-pointed topological group and B be a pointed space, both of which have the pointed homotopy types of CW complexes. For a principal G-bundle P over the suspension ΣB classified by $\epsilon\colon \Sigma B \to BG$, the adjoint bundle* $\operatorname{ad} P$ *is trivial as a fiberwise A_n-space if and only if the map*

$$B \ltimes B_nG \to BG, \quad (b,x) \mapsto B\alpha_{\epsilon'(b)}(\iota_n(x))$$

is homotopic rel B_nG to the composite of the projection $B \ltimes B_nG \to B_nG$ and the map $\iota_n\colon B_nG \to BG$, where $\epsilon'\colon B \to G$ is the adjoint of ϵ.

Proof. This immediately follows from the evaluation fiber sequence

$$G \to \overline{\mathrm{Map}}_0(B_nG, BG) \to \overline{\mathrm{Map}}(B_nG, BG) \to BG \to B\mathcal{W}\mathcal{A}_n(G,G;\mathrm{eq})$$

and Proposition 9.2. □

10. Application: T_k^f-spaces

Iwase–Mimura–Oda–Yoon defined C_k^f*-spaces* and T_k^f*-spaces* in [**IMOY12**]. Since the terminology "C_k" is now confusing, we consider T_k^f-spaces.

Definition 10.1. For a pointed map $f\colon A \to X$ between the well-pointed spaces of the homotopy types of CW complexes, X is said to be a T_k^f*-space* if there exists a map $f_k\colon A \times B_k\Omega^{\mathrm{M}}X \to X$ such that the following diagram commutes up to homotopy:

$$\begin{tikzcd} A \vee B_1\Omega^{\mathrm{M}}X \arrow[d] \arrow[dr, "{(f,\iota_1)}"] & \\ A \times B_k\Omega^{\mathrm{M}}X \arrow[r, "f_k"'] & X. \end{tikzcd}$$

In particular, a $T_k^{\mathrm{id}_X}$-space is what Aguadé [**Agu87**] defined as a T_k-space.

Proposition 10.2. *Let A and X be well-pointed spaces, which have the pointed homotopy types of CW complexes. For a pointed map $f\colon A \to X$, X is a T_k^f-space if and only if there exists a map $f_k\colon A \times B_k\Omega^{\mathrm{M}}X \to X$ such that the following diagram commutes up to homotopy:*

$$\begin{tikzcd} A \vee B_k\Omega^{\mathrm{M}}X \arrow[d] \arrow[dr, "{(f,\iota_k)}"] & \\ A \times B_k\Omega^{\mathrm{M}}X \arrow[r, "f_k"'] & X. \end{tikzcd}$$

Remark 10.3. Iwase–Mimura–Oda–Yoon defined C_k^f-spaces by the condition in this proposition. Therefore, it states that the C_k^f-space and the T_k^f-space are exactly the same concept.

The if part of Proposition 10.2 is trivial. The only if part immediately follows from the next lemma.

Lemma 10.4. *Let G be a well-pointed grouplike topological monoid having the homotopy type of a CW complex. For a pointed map $F\colon B_nG \to BG$, if the adjoint $G \to G$ of the composite $F \circ \iota_1^n\colon B_1G \to BG$ is a homotopy equivalence, then there exists a homotopy equivalence $F'\colon B_nG \to B_nG$ such that the composite $F \circ F'\colon B_nG \to BG$ is pointed homotopic to ι_n.*

Proof. By Theorem 6.1, there exists an A_n-map $f \in \mathcal{A}_n(G,G)$ such that $\iota_n \circ B_nf$ is homotopic to F and the underlying map of f is a homotopy equivalence. Then by Corollary 4.10, there exists an A_n-map $f' \in \mathcal{A}_n(G,G)$ which gives the inverse of f in the homotopy category $\pi_0\mathcal{A}_n$. Thus, for $F' := B_nf'$, we obtain that $F \circ F'$ is homotopic to ι_n. □

There are certain relations between T_k^f-space and the objects in the preceding two sections as follows. One can prove them by straightforward argument.

Proposition 10.5 (Cf. [**HK11**])**.** *For a well-pointed space X of the homotopy type of a CW complex, X is a T_k-space if and only if $\Omega^{\mathrm{M}}X$ is a $C(\infty,k)$-space.*

Proposition 10.6. *Let G be a well-pointed topological group and B be a well-pointed space, both of which have the homotopy types of CW complexes. Then, for a principal G-bundle P over B classified by $\epsilon\colon B \to BG$, the adjoint bundle $\operatorname{ad} P$ is A_n-trivial if and only if BG is a T_n^ϵ-space.*

Let us denote the Moore free loop space of X by $L^{\mathrm{M}}X$. The next proposition is a generalization of Aguadé's definition of T-space in [**Agu87**].

Proposition 10.7. *Let A and X be well-pointed spaces of the homotopy types of CW complexes and $f\colon A \to X$ a pointed map. Then the following two conditions are equivalent:*

(i) X is a T_k^f-space,

(ii) the pullback $f^*L^{\mathrm{M}}X$ is A_k-trivial.

Proof. Take a group model $G_X \xrightarrow{\simeq} \Omega^{\mathrm{M}}X$ which is an A_∞-equivalence such that G_X is a CW complex. Then, by the following Lemma 10.8, $f^*L^{\mathrm{M}}X$ is A_k-trivial if and only if $f^* \operatorname{ad} EG_X$ is A_k-trivial. Combining this with Proposition 9.2, we obtain the desired result. □

Lemma 10.8. *Let G be a well-pointed topological group having the homotopy type of CW complex. Then, the adjoint bundle $\operatorname{ad} EG$ and the free loop space $L^{\mathrm{M}}BG$ are fiberwise A_∞-equivalent as fiberwise topological monoids over BG.*

Proof. We just outline the proof. As in [**Tsu12**, Lemma 7.1], one can show that $\operatorname{ad} EG$ is fiberwise A_∞-equivalent to $EG \times_G \Omega^{\mathrm{M}}BG$. Using the map $EG \to PBG \subset P^{\mathrm{M}}BG$ in the proof of Lemma 5.8, one can construct a fiberwise A_∞-equivalence

$$EG \times_G \Omega^{\mathrm{M}}BG \to L^{\mathrm{M}}BG.$$

Composing these equivalences, we have the fiberwise A_∞-equivalence $\operatorname{ad} EG \to L^{\mathrm{M}}BG$. □

We consider a natural family of subgroups of homotopy groups.

Definition 10.9. For a well-pointed space X of the homotopy type of a connected CW complex, define

$$G_{n,k}(X) := \{f \in \pi_n(X) \mid X \text{ is a } T_k^f\text{-space}\}.$$

Equivalently, $G_{n,k}(X)$ is the image of the induced map of the evaluation map

$$\pi_n(\mathrm{Map}_0(B_k\Omega^{\mathrm{M}}X, X; \iota_k)) \to \pi_n(X).$$

For $k = \infty$, $G_n(X) := G_{n,\infty}(X)$ is the *n-th Gottlieb group* introduced by Gottlieb [**Got69**]. If $k_1 > k_2$, there is the inclusion $G_{n,k_1}(X) \subset G_{n,k_2}(X)$. The space X is said to be the *Gottlieb space* if $G_{n,\infty}(X) = \pi_n(X)$ for any n.

Gottlieb [**Got69**] also introduced the subgroup called the *n-th Whitehead center*

$$P_n(X) := \{f \in \pi_n(X) \mid [f, g] = 0 \text{ for any } g \in \pi_*(X)\},$$

where $[f, g]$ denotes the Whitehead product of f and g. By Remark 8.6, $P_n(X) \supset G_{n,1}(X)$. Then we have the sequence of inclusions

$$\begin{aligned} G_n(X) = G_{n,\infty}(X) \subset \cdots \subset G_{n,k}(X) \subset G_{n,k-1}(X) \subset \cdots \subset G_{n,1}(X) \\ \subset P_n(X) \subset \pi_n(X). \end{aligned}$$

Now we interpret the result obtained by the author in [**Tsu15**] in the language of T_k^f-spaces.

Example 10.10. Let p be an odd prime. Denote the map $S^4 \cong B_1\,\mathrm{SU}(2) \xrightarrow{\iota_1} B\,\mathrm{SU}(2)$ and their localizations by ϵ. The author proved in [**Tsu15**] that $B\,\mathrm{SU}(2)_{(p)}$ is a $T^{p^r\epsilon}_{\frac{(r+1)(p-1)}{2}-1}$-space but not a $T^{p^r\epsilon}_{\frac{(r+1)(p-1)}{2}}$-space. Then

$$G_{4,k}(B\,\mathrm{SU}(2)_{(p)}) \subset \pi_4(B\,\mathrm{SU}(2)_{(p)}) \cong \mathbb{Z}_{(p)}$$

is the subgroup of index p^r if $\frac{r(p-1)}{2} \leqslant k < \frac{(r+1)(p-1)}{2}$ and $G_{4,\infty}(B\,\mathrm{SU}(2)_{(p)}) = 0$.

Each element $\alpha \in G_{4,\frac{r(p-1)}{2}}(B\,\mathrm{SU}(2)_{(p)}) - G_{4,\frac{r(p-1)}{2}-1}(B\,\mathrm{SU}(2)_{(p)})$ has a non-trivial image in

$$\pi_3(\mathrm{Map}_0(B_{\frac{r(p-1)}{2}}\,\mathrm{SU}(2)_{(p)}, B\,\mathrm{SU}(2)_{(p)}; \iota_{\frac{r(p-1)}{2}})),$$

but trivial in

$$\pi_3(\mathrm{Map}_0(B_{\frac{r(p-1)}{2}-1}\,\mathrm{SU}(2)_{(p)}, B\,\mathrm{SU}(2)_{(p)}; \iota_{\frac{r(p-1)}{2}-1})).$$

Thus, considering the fiber sequence

$$\mathrm{Map}_0(\Sigma^k G^{\wedge k}, BG; \iota_k) \to \mathrm{Map}_0(B_kG, BG; \iota_k) \to \mathrm{Map}_0(B_kG, BG; \iota_{k-1}),$$

it lifts to a non-trivial element in

$$\pi_3(\mathrm{Map}_0(\Sigma^{\frac{r(p-1)}{2}}\,\mathrm{SU}(2)_{(p)}^{\wedge\frac{r(p-1)}{2}}, B\,\mathrm{SU}(2)_{(p)}; \iota_{\frac{r(p-1)}{2}})) \cong \pi_{2r(p-1)+2}(S^3_{(p)}).$$

11. Application: some remarks on homotopy pullback of A_n-maps

Let G_1, G_2, G_3, G be well-pointed grouplike topological monoids of the homotopy types of CW complexes. Consider the following homotopy commutative diagram in $\mathcal{A}_\infty$:

$$\begin{tikzcd} G \arrow[r] \arrow[d] & G_1 \arrow[d] \\ G_2 \arrow[r] & G_3. \end{tikzcd}$$

Take another well-pointed topological monoid H of the homotopy types of CW complex. Then, for the induced two homotopy commutative diagrams:

$$\begin{tikzcd} \mathcal{A}_n(H,G) \arrow[r] \arrow[d] & \mathcal{A}_n(H,G_1) \arrow[d] \\ \mathcal{A}_n(H,G_2) \arrow[r] & \mathcal{A}_n(H,G_3), \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Map}_0(B_nH,BG) \arrow[r] \arrow[d] & \mathrm{Map}_0(B_nH,BG_1) \arrow[d] \\ \mathrm{Map}_0(B_nH,BG_2) \arrow[r] & \mathrm{Map}_0(B_nH,BG_3), \end{tikzcd}$$

the left square is a homotopy pullback if and only if so is the right by Theorem 6.1. In particular, considering the homotopy pullback of A_n-maps $H \to G_i$ for $i = 1, 2, 3$ is equivalent to that of the induced maps $B_nH \to BG_i$ for $i = 1, 2, 3$. Moreover, if $n = \infty$ and the left square is pullback for any H, then BG is the homotopy pullback of the diagram

$$BG_1 \to BG_3 \leftarrow BG_2.$$

References

[Agu87] J. Aguadé, *Decomposable free loop spaces*, Canad. J. Math. **39** (1987), 938–955.

[BV73] J. M. Boardman and R. M. Vogt, Homotopy invariant structures on topological spaces, Lecture Notes in Mathematics, vol. 347, Springer-Verlag, Berlin, 1973.

[CS00] M. C. Crabb and W. A. Sutherland, *Counting homotopy types of gauge groups*, Proc. London Math. Soc. **81** (2000), 747–768.

[CSZ99] M. C. Crabb, W. A. Sutherland and P. Zhang, *Homotopy nilpotency*, Quart J. Math. Oxford **50** (1999), 179–196.

[DL59] A. Dold and R. Lashof, *Principal quasifibrations and fibre homotopy equivalence of bundles*, Illinois J. Math. **3** (1959), 285–305.

[Fuc65] M. Fuchs, *Verallgemeinerte Homotopie-Homomorphismen und klassifizierende Raüme*, Math. Ann. **161** (1965), 197–230.

[Got69] D. H. Gottlieb, *Evaluation subgroups of homotopy groups*, Amer. J. Math. Soc. **91** (1969), 729–756.

[Got72] D. H. Gottlieb, *Applications of bundle map theory*, Trans. Amer. Math. Soc. **171** (1972), 23–50.

[**Got73**] D. H. Gottlieb, *The total space of universal fibrations*, Pacific J. Math. **46** (1973), 415–417.

[**HK11**] Y. Hemmi and Y. Kawamoto, *Higher homotopy commutativity and the resultohedra*, J. Math. Soc. Japan **63** (2011), 443–471.

[**Hov99**] M. Hovey, Model categories, Mathematical Surveys and Monographs, vol. 63, American Mathematical Society, Providence, RI, 1999.

[**IM89**] N. Iwase and M. Mimura, *Higher homotopy associativity*, Algebraic topology, Proceedings of an international conference held in Arcata, California, July 27–August 2, 1986, Lecture Notes in Mathematics, vol. 1370, 1989, 193–220.

[**IMOY12**] N. Iwase, M. Mimura, N. Oda and S. Yoon, *The Milnor–Stasheff filtration on spaces and generalized cyclic maps*, Canad. Math. Bull. **55** (2012), 523–536.

[**KK10**] D. Kishimoto and A. Kono, *Splitting of gauge groups*, Trans. Amer. Math. Soc. **362** (2010), 6715–6731.

[**KKT13**] D. Kishimoto, A. Kono and S. Theriault, *Homotopy commutativity in p-localized gauge groups*, Proc. Roy. Soc. Edinburgh: Sect. A **143** (2013), 851–870.

[**Kon91**] A. Kono, *A note on the homotopy type of certain gauge groups*, Proc. Roy. Soc. Edinburgh: Sect. A **117** (1991), 295–297.

[**Lan73**] G. E. Lang, *The evaluation map and EHP sequences*, Pacific J. Math. **44** (1973), 201–210.

[**May67**] J. P. May, Simplicial objects in algebraic topology, University of Chicago Press, 1967.

[**May75**] J. P. May, Classifying spaces and fibrations, Memoirs of the American Mathematical Society, vol. 155, 1975.

[**May80**] J. P. May, *Fibrewise localization and completion*, Trans. Amer. Math. Soc. **258** (1980), 127–146.

[**May90**] J. P. May, *Weak equivalences and quasifibrations*, Groups of self-equivalences and related topics, Lecture Notes in Mathematics, vol. 1425, 1990, 91–101

[**McG82**] C. A. McGibbon, *Multiplicative properties of power maps II*, Trans. Amer. Math. Soc. **274** (1982), 479–508.

[**McG89**] C. A. McGibbon, *Higher forms of homotopy commutativity and finite loop spaces*, Math. Z. **201** (1989), 363–374.

[**Saü95**] L. Saümell, *Higher homotopy commutativity in localized groups*, Math. Z. **219** (1995), 339–347.

[**Sta63a**] J. D. Stasheff, *Homotopy associativity of H-spaces. I*, Trans. Amer. Math. Soc. **108** (1963), 275–292.

[**Sta63b**] J. D. Stasheff, *Homotopy associativity of H-spaces. II*, Trans. Amer. Math. Soc. **108** (1963), 293–312.

[**Sta70**] J. Stasheff, H-spaces from a homotopy point of view, Lecture Notes in Mathematics, vol. 161, 1970.

[Sug60] M. Sugawara, *On the homotopy-commutativity of groups and loop spaces*, Mem. College Sci. Univ. Kyoto Ser. A Math. **33** (1960), 257–269.

[Tsu12] M. Tsutaya, *Finiteness of A_n-equivalence types of gauge groups*, J. London Math. Soc. **85** (2012), 142–164.

[Tsu15] M. Tsutaya, *Homotopy pullback of A_n-spaces and its applications to A_n-types of gauge groups*, Topology Appl. **187** (2015), 1–25.

[Wil69] F. D. Williams, *Higher homotopy-commutativity*, Trans. Amer. Math. Soc. **139** (1969), 191–206.

Mitsunobu Tsutaya `tsutaya@math.kyushu-u.ac.jp`

Faculty of Mathematics, Kyushu University, Fukuoka 819-0395, Japan